WHEN IS A STOCHASTIC INTEGRAL
A TIME CHANGE OF A DIFFUSION?

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Abstract

We give a necessary and sufficient condition (in terms of \(u, v, b, \sigma\)) that a time change of an \(n\)-dimensional Ito stochastic integral \(X_t\) on the form

\[
dX_t = u(t, \omega)dt + v(t, \omega)dB_t
\]

has the same law as a diffusion \(Y_t\) on the form

\[
dY_t = b(Y_t)dt + \sigma(Y_t)dB_t.
\]

As an application we prove a change of time formula for \(n\)-dimensional Ito integrals.
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$\S 1$. The Main Result

In the following we will let $Y_t = Y_t^x$ denote an Ito diffusion, i.e. a (weak) solution in an open set $U \subset \mathbb{R}^n$ of the Ito stochastic differential equation

$$(1.1) \quad dY_t = b(Y_t)dt + \sigma(Y_t)dB_t, \quad Y_0 = x$$

where the functions $b: \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma: \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are continuous and $(B_t, \Omega, \mathcal{F}_t, P^x)$ denotes $m$-dimensional Brownian motion. And we will let $X_t = X_t^x$ denote an Ito stochastic integral

$$(1.2) \quad dX_t = u(t,\omega)dt + v(t,\omega)dB_t, \quad X_0 = x,$$

where $u(t,\omega) \in \mathbb{R}^n$, $v(t,\omega) \in \mathbb{R}^{n \times m}$ satisfy the usual conditions for existence of the stochastic integral: $u(t,\omega)$ and $v(t,\omega)$ are $\mathcal{F}_t$-adapted and

$$P^\omega\{ \omega; \int_0^t |u(s,\omega)| + \int_0^t \sum_{i,j} |v_{ij}(s,\omega)|^2 ds < \infty \} \quad \text{for all } t = 1.$$

(See e.g. [4] or [7]). The time changes will consider are of the following form:

Let $c(t,\omega) > 0$ be an $\mathcal{F}_t$-adapted process. Define

$$(1.3) \quad \beta_t = \beta(t,\omega) = \int_0^t c(s,\omega)ds$$
We will say that $\beta_t$ is a time change with time change rate $c(t,\omega)$. Note that $\beta_t$ is also $\mathcal{F}_t$-adapted and for each $\omega$ the map $t \to \beta_t$ is nondecreasing. Let $\alpha_t = \alpha(t,\omega)$ be the right continuous inverse of $\beta_t$:

$$ a_t = \inf\{s; \beta_s > t\} \tag{1.4} $$

Then $\omega + a_t(\omega)$ is an $\mathcal{F}_s$-stopping time for each $t$, since

$$ \{\omega; a_t(\omega) < s\} = \{\omega; t < \beta(s,\omega)\} \in \mathcal{F}_s. $$

We now ask the question: When does there exist a time change $\beta_t$ as above such that $X_{\alpha_t} = Y_t$, i.e. $X_{\alpha_t}$ is identical in law to $Y_t$? In §1 we give an answer to this question (Theorems 1-3) and in §2 we use this to prove a change of time formula for stochastic integrals.

Note that $\beta(\alpha_t) = t$ for all $(t,\omega)$, so that

$$ a'(t) = \frac{1}{c(\alpha_t,\omega)} \text{ for a.a } t > 0, \omega \in \Omega. \tag{1.5} $$

Moreover,

$$ \int_0^t c(\alpha_t,\omega) d\hat{a}_t^\omega = \int_0^t c(s,\omega) ds = \int_0^t dr $$

or

$$ c(\alpha_t,\omega) d\hat{a}_t^\omega = dt, \text{ for each } \omega \in \Omega, \tag{1.6} $$

where $d\hat{a}_t^\omega$ denotes the measure $d\alpha_t$ with the point masses corresponding to the discontinuities of $\alpha_t$ taken out.

First we establish a useful measurability result. We let $\mathcal{M}_t$ and $\mathcal{N}_t$ denote the $\sigma$-algebras generated by $\{X_s; s \leq t\}$ and $\{Y_s; s \leq t\}$, respectively, and we define $\mathcal{M}_{\alpha_t}$ to be the $\sigma$-algebra in $\Omega$ generated by the functions $\omega \to X_{\alpha_t}$, $s \leq t$. 


We let \( C^2_0(U) \) denote the twice continuously differentiable functions with compact support in \( U \), and \( v^T \) denotes the transposed of the matrix.

**Lemma 1**

Let \( dX_t = u(t, \omega) dt + v(t, \omega) dB_t \), \( c(t, \omega), \alpha_t \) be as above. Then \( (vv^T)(\alpha_t, \omega) a'_t \) is \( \mathcal{M}_{\alpha_t} \)-adapted.

**Proof.**

By Itô's formula we have

\[
x(i)_t = x(i)_0 + \int_0^t x(i) \, dx(j) + \int_0^t x(j) \, dx(i) + \int_0^t (vv^T)_{ij}(s, \omega) \, ds
\]

Hence, if we put

\[
H_{ij}(t, \omega) = x(i)_t x(j) - x(i)_0 x(j) - \int_0^t x(i) \, dx(j) - \int_0^t x(j) \, dx(i)
\]

then \( H(t, \omega) \) is \( \mathcal{M}_t \)-adapted and we have

\[
\int_0^t (vv^T)(s, \omega) \, ds = H(\alpha_t, \omega)
\]

Therefore

\[
(vv^T)(\alpha_t, \omega) a'_t = \lim_{r \to 0} \frac{H(\alpha_t, \omega) - H(\alpha_{t-r}, \omega)}{r},
\]

which shows that \( (vv^T)(\alpha_t, \omega) a'_t \) is \( \mathcal{M}_{\alpha_t} \)-adapted.

**Remarks**

1) One may ask if it is also true that \( u(\alpha_t, \omega) a'_t \) is \( \mathcal{M}_{\alpha_t} \)-adapted.

However, the following example, which was pointed out to me by the referee, shows that this fails even in the case when \( \alpha_t = t, \nu = 1, m = n = 1 \).
Put
\[
\begin{cases}
B - B_t & \text{if } t < 1 \\
\frac{1 - t}{1 - t} & \\
0 & \text{if } t > 1
\end{cases}
\]
and define
\[
\tilde{B}_t = \int_0^t u(s, \omega) ds + B_t
\]
Then \( \tilde{B}_t \) is a Brownian motion and
\[
B_t = \int_0^t u(s, \omega) ds + \tilde{B}_t
\]
but \( u(t, \omega) \) is not \( \mathcal{F}_t \)-adapted.

2) The next example shows that it need not be the case that \( v(x_t, \omega) x_t \) is \( \mathcal{M}_{x_t} \)-adapted, even if \( x_t = t \): Choose \( v(t, \omega) \)
non-constant with the values \( \pm 1 \) and independent of \( \{B_t\}_{t>0} \) (\( m=n=1 \)). Define
\[
d\tilde{B}_t = v(t, \omega) dB_t
\]
Then \( \tilde{B}_t \) is a Brownian motion (see McKean [4], §2.9 and also Corollary 1 later in this article). Hence we have
\[
dB_t = v(t, \omega) d\tilde{B}_t,
\]
but \( v(t, \omega) \) is not \( \mathcal{F}_t \)-adapted.

Let \( \mathcal{B} \) denote the Borel \( \sigma \)-algebra of subsets of \( [0, \omega) \). For \( t>0 \)
we define a measure \( Q_{x_t} \) on \( \mathcal{B} \times \mathcal{F} \) by setting
\[
Q_{x_t}(f) = E^X \left[ \int_0^t f(s, \omega) ds \right]
\]
if \( f(s, \omega) \) is bounded and \( \mathcal{B} \times \mathcal{F} \)-measurable. Let \( \mathcal{X} \) denote the \( \sigma \)-
algebra in \( [0, \omega) \times \mathcal{B} \) generated by the function \( (s, \omega) \rightarrow X_s(\omega) \) and
let \( \mathbb{E}_t^\alpha [g|X] = \mathbb{E}_t^\alpha [g|X] \) denote the conditional expectation of \( g(s,\omega) \) wrt. \( X \) and wrt. the measure \( Q_t^\alpha \).

We can now state and prove the main result. First we consider the case when

\[(1.9) \quad \beta_t = 0 \text{ a.s. (i.e. } \alpha_t \beta_t = 0 \text{ for all } t<\infty \text{ a.s.).} \]

The general case will considered later in this section (Theorem 2).

**Theorem 1.**

Assume that (1.9) holds. Then the following 3 statements, (I), (II) and (III), are equivalent:

(I) (i) \( \mathbb{E}_t^\alpha [u|X] = b(X)\mathbb{E}_t^\alpha [c|X] \) for all \( t>0 \) and

(ii) \( (vv^T)(t,\omega) = c(t,\omega)(\sigma^T)X_t \) for \( a.a. \ t \in (0,\infty), \omega \in \Omega \).

(II) (i) \( \mathbb{E}_t^\alpha [u|X] = b(X)\mathbb{E}_t^\alpha [c|X] \) for all \( t>0 \) and

(iii) \( \mathbb{E}_t^\alpha [vv^T|X] = \sigma^T(X)\mathbb{E}_t^\alpha [c|X] \) for all \( t>0 \)

(III) \( X_t \sim Y_t \)

**Proof.**

(I) \(\Rightarrow\) (II): This follows by noting that (i) and (iii) state that

\[(1.10) \quad \mathbb{E}_t^\alpha \left[ \int_0^\infty u(s, \omega)g(X_s)ds \right] = \mathbb{E}_t^\alpha \left[ \int_0^\infty b(X_s)g(X_s)c(s, \omega)ds \right] \]

and

\[(1.11) \quad \mathbb{E}_t^\alpha \left[ \int_0^\infty (vv^T)(s, \omega)g(X_s)ds \right] = \mathbb{E}_t^\alpha \left[ \int_0^\infty (\sigma^T)(X_s)g(X_s)c(s, \omega)ds \right] \]

for all bounded functions \( g \).
(II) $\Rightarrow$ (III):

For $0 < t < \infty$ we define a bounded linear functional $W_t$ on $C_b(U)$ (the bounded real continuous functions on $U$ equipped with the sup norm) by

$$W_t f = E^x[f(X_{\alpha_t})]; f \in C_b(U).$$

Since $\alpha_t$ is a stopping time we have by Ito's formula (see e.g. [7], Lemma 7.8) if $f \in C^2_0(U)$:

$$W_t f = E^x[f(X_0)] + \int_0^\infty \{ \sum_{i=1}^n b_i(X_s) \cdot \frac{\partial f}{\partial x_i}(X_s) + \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{ij}(X_s) \cdot \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s) \} ds$$

So if (II) holds we obtain, using (1.10), (1.11) and (1.6)

$$W_t f = f(x) + \int_0^\infty \{ \sum_{i=1}^n b_i(X_\alpha_r) \cdot \frac{\partial f}{\partial x_i}(X_\alpha_r) + \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{ij}(X_\alpha_r) \cdot \frac{\partial^2 f}{\partial x_i \partial x_j}(X_\alpha_r) \} dr$$

where $A = \sum_i b_i(\partial / \partial x_i) + \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{ij}(\partial^2 / \partial x_i \partial x_j)$ is the generator of $Y_t$. Therefore

$$\frac{d}{dt} W_t f = W_t(Af); t > 0$$

$$W_0 f = f(x)$$

for all $f \in C^2_0(U)$. Similarly we obtain, if we put

$$V_t f = E^x[f(Y_t)]; t > 0$$

that
\[
\begin{align*}
\frac{d}{dt} V_t f &= V_t(Af), \quad t > 0 \\
V_0 f &= f(x)
\end{align*}
\]
for all \( f \in C^2_0(U) \). Since the solution of the equations (1.12) and (1.13) is unique (see [6], Lemma 2.5) we conclude that

\[
W_t f = V_t f \quad \text{for all} \quad t > 0, \quad f \in C^2_0(U).
\]

Similarly we prove by induction on \( k \) that

\[
E_\alpha^X [f(X_{t_k}^\alpha) g_1(X_{t_1}^\alpha) \cdots g_k(X_{t_k}^\alpha)] = E_\alpha^X [f(Y_{t_k}^\alpha) g_1(Y_{t_1}^\alpha) \cdots g_k(Y_{t_k}^\alpha)]
\]
for all \( t, t_1, \ldots, t_k > 0 \) and \( f, g_1, \ldots, g_k \in C^2_0(U) \) by applying the above argument to the \( n(k+1) \)-dimensional processes

\[
(X_{t_1}^\alpha, \ldots, X_{t_k}^\alpha) \quad \text{and} \quad (Y_{t_1}^\alpha, \ldots, Y_{t_k}^\alpha).
\]

\((III) \Rightarrow (I)\). Suppose \( X_t^\alpha \sim Y_t^\alpha \). Since \( Y_t^\alpha \) is a Markov process w.r.t. \( \mathcal{M}_t^\alpha \) it follows that \( X_t^\alpha \) is a Markov process w.r.t. \( \mathcal{M}_t^\alpha \) and with generator \( A \). Therefore, using Dynkin's formula (see e.g. [7], Th. 7.10) and (1.6) we have, for \( f \in C^2_0(U) \):

\[
(1.14) \quad E_\alpha^X [f(X_{t+h}^\alpha)] = E_\alpha^X [f(X_t^\alpha)] = f(X_t^\alpha) + \int_0^{t+h} \left[ \sum_i b_i(X_s^\alpha) \cdot \frac{\partial f}{\partial x_i}(X_s^\alpha) + \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{ij}(X_s^\alpha) \cdot \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s^\alpha) \right] ds + \int_0^{t+h} \left[ \sum_i b_i(X_s^\alpha) \cdot \frac{\partial f}{\partial x_i}(X_s^\alpha) + \frac{1}{2} \sum_{i,j} (\sigma \sigma^T)_{ij}(X_s^\alpha) \cdot \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s^\alpha) \right] ds (c(s, \omega) ds)
\]
On the other hand, from Ito's formula we get as before

\[(1.15) \quad E^X[f(X_{\alpha_t+h}) | M_{\alpha_t}] = f(X_{\alpha_t}) + E^X[f(X_{\alpha_t}) - f(X_{\alpha_t})]_{\alpha_t+h} \]

\[= f(X_{\alpha_t}) + E^X\left[ \int_{\alpha_t}^{\alpha_t+h} \sum_{i} u_i(s, \omega) \cdot \frac{\partial f}{\partial x_i}(X_s) + \right. \]

\[\left. \frac{1}{2} \sum_{i,j} (vv^T)_{ij}(s, \omega) \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s) \right] ds | M_{\alpha_t}, \]

and a similar formula, denoted by \[(1.15)'\], if we replace \(\alpha_t\) by \(0\).

Comparing \((1.14)\) and \((1.15)'\) for \(f(x_1, \ldots, x_n) = \exp(i(\lambda_1 x_1 + \cdots + \lambda_n x_n))\) (where \(i = \sqrt{-1}\)) we see that \((1.10)\) and \((1.11)\) holds by putting \(t=0\). Thus it remains to prove property (ii).

From \((1.14)\) and \((1.15)\) we conclude that if we fix \(i,j\) and put

\[F_t(\omega) = \int_0^{\alpha_t} (vv^T)_{ij}(s, \omega)\] \(ds\)

then

\[(\sigma \sigma^T)_{ij}(X_{\alpha_t}) = \lim_{h \to 0} \frac{1}{h} E^X[\int_0^{\alpha_t} (\sigma \sigma^T)_{ij}(X_{\alpha_t}) dr] \]

\[(1.16) \quad = \lim_{h \to 0} \frac{1}{h} E^X[F_t+h - F_t | M_{\alpha_t}] \text{ for all } t, \omega. \]

Choose a \(t > 0\) such that \(F'_t\) exists a.s. Let \(N\) be an integer. Define, for \(h > 0\),

\[G_h(\omega) = \frac{1}{h} (F_{t+h}(\omega) - F_t(\omega)) \]

\[H_h(\omega) = \begin{cases} 
G_h(\omega) & \text{if } |G_h(\omega)| < N \\
-N & \text{if } G_h(\omega) < -N \\
N & \text{if } G_h(\omega) > N
\end{cases} \]
and put

\[
H_0(\omega) = \begin{cases} 
F'_h(\omega) & \text{if } |F'_h(\omega)| < N \\
-N & \text{if } F'_h(\omega) < -N \\
N & \text{if } F'_h(\omega) > N,
\end{cases}
\]

Then \( H_0 \) is measurable wrt. \( \mathcal{M}_{\alpha_t} \) by Lemma 1. By bounded convergence we have

\[
\lim_{h \to 0} E^X[H_h|\mathcal{M}_{\alpha_t}] = E^X[\lim_{h \to 0} H_h|\mathcal{M}_{\alpha_t}] = H_0 \quad \text{a.s.}
\]

Put \( W = \{ \omega; |F'_t(\omega)| < hN \} \in \mathcal{M}_{\alpha_t} \).

Choose \( \omega \in W \). Then there exists \( h(\omega) > 0 \) such that

\[
h < h(\omega) \Rightarrow |G'_h(\omega)| < N \quad \text{i.e. } G'_h(\omega) = H'_h(\omega).
\]

We want to conclude that

\[
\lim_{h \to 0} E^X[G_h|\mathcal{M}_{\alpha_t}] = \lim_{h \to 0} E^X[H_h|\mathcal{M}_{\alpha_t}]
\]

for a.a. \( \omega \in W \).

To obtain this write

\[
E^X[f|\mathcal{M}_{\alpha_t}](\omega) = \int f(\eta)dQ_{\omega}(\eta), \text{ for a.a. } \omega \in \Omega.
\]

where \( Q_{\omega} \) is a conditional probability distribution of \( P \) given \( \mathcal{M}_{\alpha_t} \). (See Stroock and Varadhan [8], Theorem 1.16)

Let

\[
V(\omega) = \cap \{ V \in \mathcal{M}_{\alpha_t} ; \omega \in V \} \in \mathcal{M}_{\alpha_t}
\]

be the \( \mathcal{M}_{\alpha_t} \)-atom containing \( \omega \).

Since

\[
Q_{\omega}(V(\omega)) = 1 \quad \text{for a.a. } \omega
\]
([8], Theorem 1.18) and \( V(\omega) \subseteq W \) for all \( \omega \in W \) (since \( \omega \in \mathcal{M}_t \)), we have for a.a. \( \omega \in W \) and \( h < h(\omega) \)

\[
E^X[G_h | \mathcal{M}_t](\omega) = \int W G_h \, d\mathcal{Q}_\omega = \int W h(\omega) \, d\mathcal{Q}_\omega = E^X[H_h | \mathcal{M}_t]
\]

and (1.18) follows.

Combining (1.17) and (1.18) we obtain that

\[
\lim_{h \to 0} E^X[G_h | \mathcal{M}_t] = F'_t \quad \text{a.s. in } W
\]

And since \( N \) was arbitrary we conclude from (1.16)

\[(1.19) \quad (\sigma \sigma^T)_{ij}(X_{\alpha_t}) = (\nu \nu^T)_{ij}(\alpha_t, \omega)\alpha'_t \quad \text{for a.a. } t, \omega\]

or

\[(1.20) \quad (\nu \nu^T)_{ij}(\alpha_t, \omega) = c(\alpha_t, \omega)(\sigma \sigma^T)_{ij}(X_{\alpha_t}) \quad \text{for a.a. } t, \omega.
\]

Moreover, if we define

\[(1.21) \quad F'_t(\omega) = \lim_{h \to 0} \frac{1}{h} (F_{t+h} - F_t) \quad \text{for all } t, \omega,
\]

then using (1.15) and Fatou's lemma we get

\[
F'_t(\omega) = E^X[F'_t | \mathcal{M}_t] < \lim_{h \to 0} \frac{1}{h} E^X[F_{t+h} - F_t | \mathcal{M}_t]
\]

\[(1.22) \quad = (\sigma \sigma^T)_{ij}(X_{\alpha_t}) \quad \text{for all } t, \omega
\]

Thus \( t + F'_t(\omega) \) is absolutely continuous for each \( \omega \). Therefore

\[(\nu \nu^T)_{ij}(s, \omega) = 0 \quad \text{a.e. on each } s \text{-interval where } s + \beta(s, \omega) \text{ is constant i.e. where } s + c(s, \omega) \text{ is 0 a.e. and, by (1.6)}
\]

\[(\nu \nu^T)_{ij}(\alpha_t, \omega) d\alpha_t = (\sigma \sigma^T)_{ij}(X_{\alpha_t}) d\tau = (\sigma \sigma^T)_{ij}(X_{\alpha_t}) c(\alpha_t, \omega) d\alpha_t
\]
This is equivalent to saying that
\[ \int_0^t (\omega^T v_i(s, \omega)) ds = \int_0^t (\sigma^T j_i) c(s, \omega) ds \]
for all \( t, \omega \). Thus (ii) holds and the proof of Theorem 1 is complete.

Remark. Consider the more general situation when \( Y_t \) is not assumed to be a diffusion, but just a stochastic integral of the same type as \( X_t \):

\[ dY_t = e(t, \omega) dt + f(t, \omega) dB_t, \quad Y_0 = x. \]

It is natural to ask if one can find conditions on the coefficients in order that \( X_{\tilde{\tau}_t} \sim Y_t \) in case.

We end this section by considering the case when we do not assume that (1.9) holds, i.e. we allow \( \beta_\infty < \infty \). This case will be a special case of the following situation: Let

\[ X_t = X_t^X(\omega) = x + \int_0^t u(s, \omega) ds + \int_0^t v(s, \omega) dB_t; 0 < t < \tau \]

be a stochastic integral in an open set \( W \subset U \subset R^n \), where \( \tau \) is an \( \bar{\tau}_t \)-stopping time such that \( \tau < \tau_W \), the first exit time from \( W \) of \( X_t \). The probability law of \( X_t \) starting at \( x \), \( \bar{P}^X \), is defined by

\[ \bar{P}^X[X_{\tau_t} \in F_1, \ldots, X_{\bar{\tau}_t} \in F_k] = P^0[X_{\tau_t} \in F_1, \ldots, X_{\bar{\tau}_t} \in F_k], \]

and \( \bar{P}^X \) denotes integration wrt. \( \bar{P}^X \). Suppose \( Y_t \) is as before and let \( \beta^X \) denote the probability law of \( X_t \) starting at \( x \). Then we say that \( X_t \) is a time change of \( Y_t \) (with time change rate \( c(t, \omega) \)) if the process \( Z_t \) defined by
(1.23) \[ z_t = \begin{cases} x_{t}^\alpha & ; \ 0 < t < \beta_t \\ y_{t-\beta_t} & ; \ t > \beta_t \end{cases} \]

with probability law \( \tilde{\Pi}^x \) defined by

\[ \tilde{E}^x[f](z_t) \ldots f_k(z_k) \cdot \chi_{\{t_k < \beta_t < t_{k+1}\}} = \tilde{E}^x[f_1(x_{t_1}^\alpha) \ldots f_j(x_{t_j}^\alpha) \cdot \chi_{\{t_j < \beta_t < t_{j+1}\}}] \]

(1.24) \[ f_{j+1}(y_{t_j+1-\beta_t}) \ldots f_k(y_{t_k-\beta_t}) \cdot \chi_{\{t_j < \beta_t < t_{j+1}\}} \]

coincide in law with \( Y_t \) for every \( x \in \mathcal{W} \).

(For simplicity we suppress the superscript \( x \) in what follows)

Then question when \( X_t \) is a time change of \( Y_t \) can now be given an answer similar to Theorem 1, except that in this case the measure \( \tilde{Q}_t \) must be modified to the measure \( \tilde{Q}^\alpha_t \) defined by

\[ \tilde{Q}^\alpha_t(f) = \tilde{E}^x[\int_0^{\alpha_t^\tau} f(s,\omega) ds] \]

if \( f \geq 0 \) is \( \mathcal{F}_\tau \)-measurable. The corresponding conditional expectation is denoted by \( \tilde{E}^{\alpha_t^\tau}[\cdot] \).

**Theorem 2.** The following are equivalent:

(A) \[ \tilde{E}^x_{\alpha_t^\tau}[u|X] = b(X) \tilde{E}^x_{\alpha_t^\tau}[c|X] \] for all \( t > 0 \) and

(1) \[ (vv^T)(t,\omega) = c(t,\omega)(\sigma^T)(X_t) \] for a.a. \( t,\omega \) such that \( t < \beta_t \).

(B) \( X_t \) is a time change of \( Y_t \) with time change rate \( c(t,\omega) \).

**Proof.** (A) \( \Rightarrow \) (B): We proceed as in the proof of (II) \( \Rightarrow \) (III) in Theorem 1, except that now we put

\[ W_t f = \tilde{E}[f(z_t)]; \quad f \in \mathcal{C}_0^2(U), \ t > 0. \]

Then by Itô's formula we get
\[ \tilde{E}[f(Z_t) \cdot \chi_{\{t<\beta\}}] \equiv \tilde{E}[f(x) \cdot \chi_{\{t<\beta\}}] + \tilde{E}\left[\int_0^T (\nabla f)^T(x_s)v(s,\omega)dB_s \cdot \chi_{\{t<\beta\}}\right] + \tilde{E}\left[\int_0^T \left(\sum_{i} u_i(s,\omega) \frac{\partial f}{\partial x_i}(x_s) + \sum_{i,j} (vv^T)_{ij}(s,\omega) \frac{\partial^2 f}{\partial x_i \partial x_j}(x_s)\right)ds \cdot \chi_{\{t<\beta\}}\right] \]

Similarly
\[ \tilde{E}[f(Z_t) \cdot \chi_{\{t>\beta\}}] = \tilde{E}[f(Y_{\tau-\beta}) \cdot \chi_{\{t>\beta\}}] = \tilde{E}[f(X_\tau) \cdot \chi_{\{t>\beta\}}] + \tilde{E}\left[\int_0^{\tau-\beta} (Af)(Y_{\tau})du \cdot \chi_{\{t>\beta\}}\right] \]

(1.26)
\[ = \tilde{E}[f(X_\tau) \cdot \chi_{\{t>\beta\}}] + \tilde{E}\left[\int_0^\tau (Af)(Y_{\tau})dv \cdot \chi_{\{t>\beta\}}\right] \]

By Ito's formula we get
\[ \tilde{E}[f(X_\tau) \cdot \chi_{\{t>\beta\}}] = \tilde{E}[f(x) \cdot \chi_{\{t>\beta\}}] + \tilde{E}\left[\int_0^\tau (\nabla f)^T(x_s)v(s,\omega)dB_s \cdot \chi_{\{t>\beta\}}\right] + \tilde{E}\left[\int_0^\tau \left(\sum_{i} u_i(s,\omega) \frac{\partial f}{\partial x_i}(x_s) + \sum_{i,j} (vv^T)_{ij}(s,\omega) \frac{\partial^2 f}{\partial x_i \partial x_j}(x_s)\right)ds \cdot \chi_{\{t>\beta\}}\right] \]

(1.27)

so by adding (1.26) and (1.27) we obtain
\[
\tilde{E}[f(Z_t)] = f(x) + \tilde{E}\left[\int_0^\tau (\nabla f)^T(x_s)v(s,\omega)dB_s \right] + \tilde{E}\left[\int_0^\tau \left(\sum_{i} u_i(s,\omega) \frac{\partial f}{\partial x_i}(x_s) + \sum_{i,j} (vv^T)_{ij}(s,\omega) \frac{\partial^2 f}{\partial x_i \partial x_j}(x_s)\right)ds \cdot \chi_{\{t>\beta\}}\right] + \tilde{E}\left[\int_0^\tau (Af)(Y_{\tau})dv \cdot \chi_{\{t>\beta\}}\right].
\]

(1.28)

Since \( \alpha_t^{\wedge \tau} \) is a stopping time the second term on the right of (1.28) is 0 and by (A) the third term is the same as
\[
\begin{align*}
\alpha_t^\tau 
E\left[ \int_0^\tau (Af)(X_s)c(s,\omega)ds \right] &= E\left[ \int_0^\tau (Af)(X_s)c(s,\omega)ds \cdot \chi_{\{t<\beta^\tau\}} \right] \\
&+ E\left[ \int_0^\tau (Af)(X_s)c(s,\omega)ds \cdot \chi_{\{t>\beta^\tau\}} \right] \\
(1.29) &= E\left[ \int_0^\tau (Af)(X_s)dr \cdot \chi_{\{t<\beta^\tau\}} \right] + E\left[ \int_0^\tau (Af)(X_s)dr \cdot \chi_{\{t>\beta^\tau\}} \right] \\
(1.30) &= \int_0^\tau (Af)(X_s)c(s,\omega)ds = \int_0^\tau (Af)(X_s)c(s,\omega)ds,
\end{align*}
\]

(Note that)

\[
(1.31) = f(x) + \tilde{E}\left[ \int_0^\tau (Af)(X_s)c(s,\omega)ds \right] + E\left[ \int_0^\tau (Af)(X_s)c(s,\omega)ds \right] + E\left[ \int_0^\tau (Af)(X_s)c(s,\omega)ds \cdot \chi_{\{t>\beta^\tau\}} \right]
\]

Comparing \((1.28)\) and \((1.31)\) we conclude that

\[
E[f(Z_t)] = f(x) + E\left[ (Af)(Z_s)ds \right].
\]

Thus we have obtained \((1.11)\) and the rest of the proof of \((i) \Rightarrow (ii)\) follows the proof of \((II) \Rightarrow (III)\) in Theorem 1.

\((B) \Rightarrow (A)\): We reverse the argument just given. If \(Z_t\) is a Markov process with generator \(A\) we get by the Dynkin formula

\[
\tilde{E}(f(Z_t)) = f(x) + \tilde{E}\left[ (Af)(Z_s)ds \right]
\]

\[
\begin{align*}
&= f(x) + \tilde{E}\left[ \int_0^\tau (Af)(X_s)ds \right] + \tilde{E}\left[ \int_0^\tau (Af)(Z_s)ds \right] \\
&= f(x) + E\left[ \int_0^\tau (Af)(X_s)dr \right] + \tilde{E}\left[ \int_0^\tau (Af)(Z_s)ds \right] \chi_{\{t>\beta^\tau\}} \\
&= f(x) + E\left[ \int_0^\tau (Af)(X_s)c(s,\omega)ds \right] + E\left[ \int_0^\tau (Af)(Y_{v-\beta^\tau})dv \right] \chi_{\{t>\beta^\tau\}}.
\end{align*}
\]

Comparing \((1.28)\) and \((1.31)\) we conclude that
\[ \alpha_t^{\wedge \tau} \]
\[ E[ \int_0^\tau u(s,\omega)g(X_s)ds] = E[ \int_0^\tau b(X_s)c(s,\omega)g(X_s)ds] \]

and
\[ \alpha_t^{\wedge \tau} \]
\[ E[ \int_0^\tau (vv^T)(s,\omega)g(X_s)ds] = E[ \int_0^\tau (\sigma\sigma^T)(X_s)c(s,\omega)g(X_s)ds] \]

for all bounded functions g.

This proves the first identity in (A). To obtain the second identity we proceed as in the proof of (III) \( \Rightarrow (I) \) in Theorem 1: Let \( \mathcal{M}_t \)
denote the \( \sigma \)-algebra generated by \( \{Z_s; s<\tau \} \). Then by the strong Markov property we have for all \( t,\omega \)
\[ \lim_{h \to 0} \frac{1}{h} E[f(Z_{t+h})-f(Z_t)|\mathcal{M}_t] = \lim_{h \to 0} \frac{1}{h} E[f(Z_{t+h})-f(Z_t)|\mathcal{M}_t] = \]

On the other hand, from the general calculation in (1.28) we get
\[ \lim_{h \to 0} \frac{1}{h} E[f(Z_{t+h})-f(Z_t)|\mathcal{M}_t] = \]

Applying this to the function \( f(x_1,\ldots,x_n) = x_i x_j \) we get by combining (1.32) and (1.33):
\[ (\sigma\sigma^T)_{ij}(Z_t) = \lim_{h \to 0} \frac{1}{h} E[ \int_0^\tau (vv^T)_{ij}(s,\omega)ds \cdot \chi_{\{t<\beta_{\tau} \}}|\mathcal{M}_t] \]
\[ + \lim_{h \to 0} \frac{1}{h} E[ \int_t^{t+h} (\sigma\sigma^T)_{ij}(Z_v)dv \cdot \chi_{\{t<\beta_{\tau} \}}|\mathcal{M}_t] \]
\[ = (vv^T)_{ij}(\alpha_t,\omega)\alpha_t^{\wedge \tau} E[\chi_{\{t<\beta_{\tau} \}}|\mathcal{M}_t] + (\sigma\sigma^T)_{ij}(Z_t) \cdot E[\chi_{\{t<\beta_{\tau} \}}|\mathcal{M}_t], \]
by the same argument as in the proof of (1.19).
Hence

\[(1.34) \quad (\sigma^T \alpha^T)_{ij}(Z_t) E[X_t | \mathbf{\mathcal{F}}_t] = (\nu^T \alpha^T)_{ij}(Z_t) E[X_t | \mathbf{\mathcal{F}}_t] \]

Put \( B = \{ \omega; \ t < \beta_\tau \} \) and let

\[ A_0 = \{ \omega; \ E[X_B | \mathbf{\mathcal{F}}_t] = 0 \} \in \mathcal{F}_t. \]

Then

\[ P(B \cap A_0) = \int A_0 \chi_B \ dP = \int A_0 E[X_B | \mathbf{\mathcal{F}}_t] \ dP = 0, \]

so

\[ E[X_B | \mathbf{\mathcal{F}}_t] > 0 \quad a.s. \ on \ B \]

Therefore we can conclude from (1.34) that for all \( t > 0 \)

\[ (\nu^T \alpha^T)_{ij}(\alpha_t, \omega) = c(\alpha_t, \omega)(\sigma^T \alpha^T)_{ij}(X_{\alpha_t}) \]

for a.a. \( \omega \) s.t. \( t < \beta_\tau(\omega) \).

Thus we obtain the same conclusion (I) as in Theorem 1, except that it is only valid for a.a. \( t, \omega \) such that \( t < \beta_\tau(\omega) \). That completes the proof of Theorem 2.

**Corollary 1.** Suppose

\[ u(t, \omega) = c(t, \omega)b(X_{\alpha_t}) \quad \text{and} \quad (\nu^T \alpha^T)(t, \omega) = c(t, \omega)(\sigma^T)(X_{\alpha_t}) \]

for a.a. \( t, \omega \) such that \( t < \beta_\tau \).

Then \( X_{\alpha_t} \) is a time change of \( Y_{\alpha_t} \), with time change rate \( c(t, \omega) \).

Theorem 2 allows us to extend the characterization of Markovian path-preserving functions given in Csink and Øksendal [1] to the case when the time change \( \beta_t \) is not necessarily strictly increasing:
Theorem 3. Let \( dS_t = a(S_t)dt + \gamma(S_t)dB_t \) and \( dY_t = b(Y_t)dt + \sigma(Y_t)dB_t \) be Ito diffusions on open sets \( G \subseteq \mathbb{R}^p \) and \( U \subseteq \mathbb{R}^n \), respectively. Denote the generators of \( S_t \) and \( Y_t \) by \( \bar{A} \) and \( A \), respectively. Let \( \phi : G + U \) be a \( C^2 \) function. Then the following are equivalent:

1. There exists a continuous function \( \lambda > 0 \) on \( G \) such that

\[
\bar{A}[f\phi] = \lambda A[f] \circ \phi \quad \text{for all} \ f \in C^2(U)
\]

2. For each open set \( D \) with \( \bar{D} \subseteq G \) the stochastic integral

\[
\int_0^\tau \phi(S_t) \, d\bar{B}_t \ 	ext{is a time change of} \ X_t, \ \text{with time change rate} \ \lambda(S_t) \ \text{(in the sense of (1.23)-(1.24)).}
\]

Proof. By the Ito formula we have that \( X_t = \phi(S_t), t < \tau \), satisfies

\[
dx_t^{(k)} = (A_\phi)(S_t) \, dt + \nabla \phi^T(S_t) \gamma(S_t) \, dB_t, \quad k = 1, \ldots, m,
\]

where \( x_t^{(k)} \) is component no. \( k \) of \( X_t \). Therefore by Theorem 2 (2) holds if and only if

\[
E_{\alpha_t}^{\tau}[A_{\phi}(S_t)|X] = b_k(X)E_{\alpha_t}^{\tau}[\lambda(S_t)|X]
\]

and

\[
(\nabla \phi^T_k \gamma \nabla \phi_{\lambda})(S_t) = \lambda(S_t) (\sigma \sigma^T)_{k\lambda}(X_t) \quad 1 \leq k, \lambda \leq m,
\]

for a.a. \( t, \omega \) such that \( t < \beta \). Letting \( t \to 0 \) we see that equation (1.37) is equivalent to

\[
\nabla \phi^T_k \gamma \nabla \phi_{\lambda}(x) = \lambda(x)(\sigma \sigma^T)_{k\lambda}(\phi(x)), \quad 1 \leq k, \lambda \leq m
\]

for all \( x \in G \).

Similarly we claim that (1.36) is equivalent to
(1.39) \( A\phi_k(x) = \lambda(x)b_k(\phi(x)) \quad 1 < k < m, x \in G. \)

It is clear that (1.39) implies (1.36). Conversely, if (1.36) holds we consider two cases:

Case 1: \( x \) belongs to the \( S \)-fine interior \( D \) of \( N = \{ z; \lambda(z) = 0 \} \); i.e. \( \tau_N = \inf\{ t > 0; S_t \notin N \} > 0 \) a.s. Since \( \alpha^+ = \lim_{t \to 0^+} \tau_t = \tau_N \) we then get from (1.36) that

\[
K(x) = \mathbb{E}_x \left[ \int_0^{\tau_N \wedge \tau} (A\phi_k)(S_t) \, dt \right] = 0 \quad \text{for all } x \in D.
\]

Applying the characteristic operator \( \mathcal{O} \) of \( S_t \) to the function \( K \) we get (see [7], p.138)

\[
0 = \mathcal{O} \mathcal{K}(x) = (A\phi_k)(x) \quad \text{for all } x \in D,
\]

so (1.39) holds in this case.

Case 2: \( \tau_N = 0 \) a.s. Then we have \( \alpha = 0 \) a.s. and therefore from (1.36)

\[
A\phi_k(x) = \lim_{t \to 0} \frac{1}{\tau^x[\alpha_t \wedge \tau]} \mathbb{E}_x \left[ \int_0^{\alpha_t \wedge \tau} (A\phi_k)(S_r) \, dr \right]
\]

\[
= \lim_{t \to 0} \frac{1}{\tau^x[\alpha_t \wedge \tau]} \mathbb{E}_x \left[ \int_0^{\alpha_t \wedge \tau} \lambda(S_r)b_k(\phi(S_r)) \, dr \right] = \lambda(x)b_k(\phi(x)),
\]

as claimed.

We now note that (1.38) and (1.39) are equivalent to requiring that

\[
A[f \circ \phi] = \lambda A[f] \circ \phi
\]

for all polynomials

\[
f(x_1, \ldots, x_n) = \sum c_{i} x_i + \sum d_{i,j} x_i x_j
\]

of degree \( < 2 \), and hence that (1.35) holds for all \( f \in C^2(U) \).
Remark. It is natural to ask what happens if we allow a more general time change rate \( c(t, \omega) \) (not necessarily of the form \( \lambda(S_t) \)) which makes \( \phi(S_t) \) a time change of \( X_t \). However, the argument above gives that if such a \( c(t, \omega) \) exists, then as in (1.37)

\[
(\nabla \phi^T_{k \gamma} \nabla \phi^T_{l \lambda})(S_t) = c(t, \omega) (\sigma^T)_{k \lambda}(X_t) \quad \text{for } 1 \leq k, l \leq m,
\]

and so

\[
c(t, \omega) = \lambda(S_t)
\]

with

\[
\lambda(x) = \frac{(\nabla \phi^T_{k \gamma} \nabla \phi^T_{l \lambda})(x)}{(\sigma^T)_{k \lambda}(\phi(x))},
\]

i.e. we have a time change of the type discussed in Theorem 3.

§2. A TIME CHANGE FORMULA FOR ITO INTEGRALS

As an illustration we first use Theorem 1 to characterize the stochastic integrals which are time changes of Brownian motion. If \( u=0 \) the corresponding result without time change (and with time change if \( n=1 \)) was first proved by McKean ([4], §2.9). The sufficiency of condition (2.1) has been proved by F. Knight [3] (in a martingale setting).

Corollary 2. Let \( X_t \) be the n-dimensional stochastic integral in (1.2). Then there exists a time change \( \alpha_t \) as above with time change rate \( c(t, \omega) > 0 \) such that

\[
X_{\alpha_t} \sim B_t \text{ (n-dimensional Brownian motion)}
\]

if and only if
(2.1) \( E_{\alpha_t}[u|X] = 0 \) for all \( t \) and \( (vv^T)(t,\omega) = c(t,\omega)I_n \) for all \( a.a \) \( t > 0 \), \( a.a. \omega \in \Omega \)

where \( I_n \) is the \( n \times n \) identity matrix.

**Example 1.** If \( X_t \) is a 2-dimensional process the form

\[
dX_t = v(t,\omega)dB_t
\]

where \( v \in \mathbb{R}^{2 \times 2} \) and \( B_t \) is 2-dimensional Brownian motion, then \( X_t \) is a conformal martingale if and only if

\[
(vv^T)(t,\omega) = \eta(t,\omega)I_2 \quad \text{for some } \eta(t,\omega) > 0.
\]

(See [2]). Thus it follows from Corollary 2 that a conformal martingale is a change of time of Brownian motion (in \( \mathbb{R}^2 \)). This was proved by Getoor and Sharpe ([2], p. 292-293) and it follows from the result by Knight in [3].

A special case of Corollary 2 is the following:

**Corollary 3.** Let \( c(t,\omega) > 0 \) be given and let \( \alpha_t \) correspond to \( c \) as before. Put

\[
X_{\alpha_t} = \int_0^{t} \sqrt{c(s,\omega)}dB_s
\]

where \( B_s \) is \( n \)-dimensional Brownian motion. Then \( X_{\alpha_t} \) is also an \( n \)-dimensional Brownian motion.

We now use this to prove that a time change of a stochastic integral is again a stochastic integral, but driven by a different Brownian motion \( \tilde{B}_t \). First we construct \( \tilde{B}_t \):

**Lemma 2.** Suppose \( t + \alpha(t,\omega) \) is continuous, \( \alpha(0,\omega) = 0 \) for \( a.a \)
ω. Fix \( t > 0 \). For \( k = 1, 2, \ldots \) put

\[
t_j = \begin{cases} 
  j \cdot 2^{-k} & \text{if } j \cdot 2^{-k} < t \\
  t & \text{if } j \cdot 2^{-k} > t
\end{cases}
\]

and choose \( r_j \) such that \( r_j = t_j \).

Suppose \( f(s, \omega) > 0 \) is \( \mathcal{F}_s \)-adapted and satisfies

\[
p^0 \left[ \int_0^t f(s, \omega)^2 ds \right] = 1
\]

Then

\[
\lim_{k \to \infty} \sum_{j} f(\alpha_j, \omega) \Delta B \alpha_j = \int_0^t f(s, \omega) dB_s \quad \text{a.s.},
\]

where \( \alpha_j = r_j \), \( \Delta B \alpha_j = B \alpha_{j+1} - B \alpha_j \) and the limit is in \( L^2(\Omega, \mathbb{P}) \).

**Proof.** For all \( k \) we have

\[
E \left[ (\sum_{j} f(\alpha_j, \omega) \Delta B \alpha_j - \int f(s, \omega) dB_s)^2 \right] = \sum_{j} E \left[ (\int f(\alpha_j, \omega) - f(s, \omega)) dB_s)^2 \right] = \sum_{j} E \left[ \int (f(\alpha_j, \omega) - f(s, \omega))^2 ds \right] = E \left[ \int (f-f_k)^2 ds \right],
\]

where \( f_k(s, \omega) = \sum_{j} f(t_j, \omega) \chi_{[t_j, t_{j+1})}(s) \) is the elementary approximation to \( f \). (See [7], Ch. III). This implies (2.2) in the case when \( f \) is bounded and \( t + f(t, \omega) \) is continuous, for a.a. \( \omega \).

The proof in the general case follows by approximation in the usual way. (See Ch. III, Steps 1-3 in [7]).

The following result extends a 1-dimensional time change formula proved by Mckean ([4], §2.8).
Theorem 4. (Time change formula for Ito integrals)

Let \((B_s, \mathcal{F}_s)\) be \(m\)-dimensional Brownian motion and \(v(t,\omega) \in \mathbb{R}^{n \times m}\) as before. Suppose \(\alpha_t\) satisfies the conditions in Lemma 2. Define

\[ (2.3) \quad \tilde{B}_t^\alpha = \lim_{k \to \infty} \int_0^t \frac{1}{\sqrt{c(\alpha_j,\omega)}} \Delta B_{\alpha_j} = \int_0^t \frac{1}{\sqrt{c(s,\omega)}} dB_s \]

Then \(\tilde{B}_t^\alpha\) is an \((m\text{-dimensional})\) \(\mathcal{F}_t^\alpha\)-Brownian motion (i.e. \(\tilde{B}_t^\alpha\) is a Brownian motion and \(\tilde{B}_t^\alpha\) is a martingale wrt. \(\mathcal{F}_t^\alpha\)) and

\[ (2.4) \quad \int_0^t v(s,\omega) dB_s = \int_0^t v(\alpha_r, \omega) \sqrt{\alpha_r} dB_r, \text{ a.s. } P^0. \]

where \(\alpha'_r(\omega)\) is the derivative of \(\alpha_r\) wrt. \(r\), so that

\[ (2.5) \quad \alpha'_r(\omega) = \frac{1}{c(\alpha_r, \omega)} \text{ for a.a. } r>0, \omega \in \Omega. \]

Proof. The existence of the limit in (2.3) and the second identity in (2.3) follows by applying Lemma 2 to the function

\[ f(s,\omega) = \sqrt{c(s,\omega)}. \]

Then by Corollary 2 we have that \(\tilde{B}_t^\alpha\) is an \(\mathcal{F}_t^\alpha\)-Brownian motion. It remains to prove (2.4):

\[ \int_0^t v(s,\omega) dB_s = \lim_{k \to \infty} \int_0^t v(\alpha_j,\omega) \Delta B_{\alpha_j} = \lim_{k \to \infty} \int_0^t \frac{1}{\sqrt{c(\alpha_j,\omega)}} \sqrt{\frac{1}{c(\alpha_j,\omega)}} \Delta B_{\alpha_j} = \lim_{k \to \infty} \int_0^t \frac{1}{\sqrt{c(\alpha_j,\omega)}} \Delta \tilde{B}_j \]
and the proof is complete.

We now apply Theorem 4 to the case when the stochastic integral $X_t$ is an Ito diffusion

$$dX_t = a(X_t)dt + \gamma(X_t)dB_t$$

where $a: \mathbb{R}^n \to \mathbb{R}^n$, $\gamma: \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are continuous.

**Corollary 4.** Let $X_t$ be the Ito diffusion given by (2.6) and let $t \to \alpha(t, \omega)$ be absolutely continuous, $\alpha(0, \omega) = 0$ for a.a. $\omega$. Then

$X_{\alpha_t}$ is a Markov process wrt. $\mathcal{M}_{\alpha_t}$ if and only if there exists a function $q: \mathbb{R}^n \to [0, \infty)$ such that

$$c_t(\omega) = q(X_t(\omega))$$

for a.a. $t < \alpha_t$, $\omega \in \Omega$, and in that case

$$d(X_{\alpha_t}) = \frac{a(X_{\alpha_t})}{q(X_{\alpha_t})} dt + \frac{\gamma(X_{\alpha_t})}{q(X_{\alpha_t})} dB_t$$

where $B_t$ is the $\int_{\alpha_t}^\tau$-Brownian motion from Theorem 4.

**Proof.** If (2.7) holds then (2.8) follows from Theorem 4. Hence $X_{\alpha_t}$ is a weak solution of the stochastic differential equation (2.8) and therefore $X_{\alpha_t}$ is a Markov process. Conversely, if $X_{\alpha_t}$ is a Markov process wrt. $\mathcal{M}_{\alpha_t}$ then by the proof of $(III) \Rightarrow (I)(ii)$ in Theorem 1 we obtain

$$= \int_0^t v(\alpha_r, \omega) \sqrt{\frac{1}{c(\alpha_r, \omega)}} dB_r,$$
\begin{equation}
(\gamma^T) (X_t) = c(t, \omega) (\sigma^T) (X_t) \quad \text{for a.a. } t < \alpha, \omega \in \Omega
\end{equation}
i.e.
\[ c(t, \omega) = q(X_t) \]
with
\[ q(x) = \frac{(\gamma^T)(x)}{(\sigma^T)(x)}. \]

Remark. The last part of this proof does not require that \( \alpha_t \) is absolutely continuous.

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