THE MILNOR NUMBER FOR CURVES ON TORIC SURFACES

by

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It is proved in [D] that the topological type of an isolated curve singularity on a toric surface depends only on its Newton filtration. In this note we give as a corollary of Kouchnirenko's theorem I in [K] a similar formula for the Milnor number of singularities defined by one analytic function on a normal affine two dimensional toric variety. I would like to thank James Damon for his helpful remarks.

1. The toric surface as quotient singularity

Any complex normal affine toric surface is isomorphic to one of the type $X = \text{Spec}(C[A])$ where $A = \sigma \cap \mathbb{Z}^2$, $\sigma$ is a rational polyhedral cone in $\mathbb{R}^2$, and $C[A]$ is the semigroup ring of $\Lambda$ over $C$. (See [O])

Let $\sigma$ be generated by $M_1 = (m_1, n_1)$ and $M_2 = (m_2, n_2)$, i.e. $M_1, M_2 \in \mathbb{Z}^2_+$, $\gcd(m_1, n_1) = 1$, $\det(M_1, M_2) > 0$ and $\sigma = \{\alpha_1 M_1 + \alpha_2 M_2 | \alpha_i \in \mathbb{R} \}$. It is well known that $X = \mathbb{C}^2 / G$ where $G$ is a finite cyclic group acting freely outside the origin. One way of seeing this is as follows: Let $G = \{\alpha_1 M_1 + \alpha_2 M_2 | \alpha_i \in \mathbb{R}, 0 \leq \alpha_i < 1\} \cap \mathbb{Z}^2$. Then $G$ is a cyclic group of order $d = \det(M_1, M_2)$; the group addition is vector addition modulo $\langle M_1, M_2 \rangle$.

Define two linear maps $\nu_1, \nu_2: \mathbb{Z}^2 \to \mathbb{Z}$ by $\nu_1(x) = \det(x, M_2)$, $\nu_2(x) = \det(M_1, x)$. These restrict to $\nu_i: G \to \{0, 1, \ldots, d-1\}$ and induce group isomorphisms $\overline{\nu}_i: G \to \mathbb{Z}_d$, $\overline{\nu}_i(a) = \nu_i(a) \pmod{d}$. We have an action of $G$ on $C[x, y]$; for $a \in G$
Let \( T: \mathbb{Z}^2 \to \mathbb{Z}^2 \) be the linear transformation \( T(m,n) = (v_1(m,n), v_2(m,n)) \). Its matrix is \( \begin{pmatrix} n_2 & -m_2 \\ -n_1 & m_1 \end{pmatrix} \). Of course \( \mathfrak{c}[A] = \mathfrak{c}[T(A)] \subseteq \mathfrak{c}[x,y] \) and one checks that \( \mathfrak{c}[T(A)] = \mathfrak{c}[x,y]^G \).

On the other hand, given a cyclic group \( G \subseteq \text{GL}_2(\mathfrak{c}) \) and an action on \( \mathfrak{c}^2 \) one can easily construct a cone \( \sigma \subseteq \mathbb{R}^2_+ \) such that \( \text{Spec} \mathfrak{c}[\sigma \cap \mathbb{Z}^2] = \mathfrak{c}^2 / G \).

2. The Milnor number of a function on \( X \)

Assume \( f \) is an analytic function \( f: X \to \mathfrak{c} \) with an isolated critical point at \( 0 \). We know that \( f \) has a Milnor fibration \([L]\), i.e. a \( C^\infty \) fibration

\[
\begin{array}{c}
B_\varepsilon \cap X \cap f^{-1}(D^*_\eta) \to D^*_\eta \\
\end{array}
\]

induced by \( f \), where \( B_\varepsilon \) is a closed regular ball in \( \mathfrak{c}^\varepsilon \) with radius \( \varepsilon > 0 \) and \( D^*_\eta \) is a punctured disc of radius \( \eta \) in \( \mathfrak{c} \).

(Here \( e \) denotes the embedding dimension of \( X \).) If \( F \) is the typical Milnor fiber we can define \( \mu(f) = \text{rk} \, H_1(F) \) to be the Milnor number of the curve \( f^{-1}(0) \) at \( 0 \).

We are now in the following situation:

\[
\begin{array}{ccc}
\mathfrak{F} & \subseteq & \mathfrak{c}^2, \ 0 \\
\pi & \dashv & \mathfrak{F} \\
F & \subseteq & X, 0 \\
\mathfrak{f} & \dashv & \mathfrak{c}, 0 \\
\end{array}
\]

\( \pi \) is the natural map \( \mathfrak{c}^2 \to \mathfrak{c}^2 / G \). Here \( \mathfrak{f} = f \circ \pi \) and \( \mathfrak{F} \) is the Milnor fiber of \( \mathfrak{f} \) at the origin.
The embedding $\phi: X \to \C^e$ is given by monomials in $x$ and $y$ generating $\C[x,y]^G$. Call these generators for $\phi_1, \ldots, \phi_e$ and put $w_i = \deg \phi_i$. Then $\phi = \psi \circ \pi: \C^2 \to \C^e$ is the map $\phi(x,y) = (\phi_1(x,y), \ldots, \phi_e(x,y))$. If $N$ is any positive number let

$$B'_\varepsilon,N = \{ z \in \C^e : \sum_{i=1}^e |z_i|^{2b_i} < \varepsilon, b_i \cdot w_i = N \}.$$ 

The Milnor fiber constructed from regular balls in $\C^e$ is diffeomorphic to the Milnor fiber using "weighted" balls of the form $B'_\varepsilon,N$ for suitable choice of $\varepsilon$.

Notice that:

1) The unit sphere in $\C^2$ is compact so,

$$\max \{ |\phi_1(x,y)|^2, \ldots, |\phi_e(x,y)|^2 : |(x,y)| = 1 \} \text{ exists.}$$

2) $$\sum_{i=1}^e |\phi_i(x,y)|^{2b_i} = \sum_{i=1}^e |(x,y)|^{2b_iw_i} \cdot |\phi_i((x,y))|^{2b_i}$$

$$= |(x,y)|^{2N} \sum_{i=1}^e |\phi_i((x,y))|^{2b_i}$$

It follows that for appropriate $\varepsilon$ and $N$, $\phi$ maps a regular ball of $\C^2$ onto $X \cap B'_\varepsilon$. This means we can assume that $F = \overline{F}/G$;

hence $\chi(\overline{F}) = (\text{ord } G) \cdot \chi(F)$, where $\chi$ is the Euler characteristic. But since we are dealing with curves, $\chi(\overline{f}) = 1 - \mu(\overline{f})$ and $\chi(F) = 1 - \mu(f)$, so

$$\mu(f) = 1 + \frac{\mu(\overline{f}) - 1}{d}.$$ 

Recall Kouchnirenko's formula for the Milnor number of a plane curve at the origin. A power series $g \in \C[[x_1,x_2]]$ is "commode" if the monomials $x_1^m$ and $x_2^n$, $m, n > 1$, appear in $g$ with non zero coefficients. If $m$ and $n$ are the minimal such numbers, then the
Newton number $v(g)$ is defined as $2A-m-n+1$ where $A$ is the area bounded by the Newton polygon $NP(g)$ of $g$. If $g$ is not "commode" then $v(g)$ is defined by $v(g) = \sup_{m \in N} v(g + \sum x_i^m)$ where the sum is now taken over the variables, a power of which does not appear alone in $g$. The theorem then states that if $g$ has an isolated critical point at $0$ and if $g$ is nondegenerate (see [K], page 7, for the definition) then $\mu(g) = v(g)$.

Identifying an analytic $f: X \rightarrow \mathbb{C}$ with a power series in the monomials of $\mathbb{C}[A]$, we can define its Newton polygon as follows. If $f = \sum_{(i,j) \in \Lambda} c_{ij} x^i y^j$, let $\Gamma$ be the convex hull in $\sigma$ of $\cup \{(i,j) + \sigma\}$ where $(i,j) \in \{(i,j) | c_{ij} \neq 0\}$. Call $f$ commode if the monomials $(x^m y^n)^m$ and $(x^{m_2} y^{n_2})^n$ appear with non-zero coefficients. If $f$ is commode then define the Newton polygon $NP(f)$ to be the polygon given by the boundary of $\Gamma$ and the rays through $M_1$ and $M_2$.

Define the Newton number as $v(f) = 2S - m - n + 1$ where $S$ is the area bounded by $NP(f)$ and $m$ and $n$ are chosen minimally as above. If $f$ is not commode, then define $v(f)$ as in the case of plane curves above.

To make the definitions simple we will say that $f$ is non-degenerate if $\tilde{f} = f o \tau$ is non-degenerate.

**Theorem** Let $f: X \rightarrow \mathbb{C}$ be an analytic function with an isolated critical point at the origin. If $f$ is nondegenerate, then $\mu(f) = v(f)$. 
Proof. Each monomial in $\bar{f}$ is just the image of the corresponding monomial in $f$ by the mapping $\mathbb{C}[A] \xrightarrow{\pi} \mathbb{C}[T(A)] \xrightarrow{\pi} \mathbb{C}[x,y]$. So $NP(\bar{f})$ is just the image of $NP(f)$ by

$$T = \begin{pmatrix} n_2 & -m_2 \\ -n_1 & m_1 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2.$$

If $f$ is commode, $f=(x^m y^n) + (x^{m_2} y^{n_2}) + \ldots$, then the area bounded by $NP(\bar{f})$ is $(\det T) \cdot S = d \cdot S$. Since $T(mM_1) = (md,0)$, $T(nM_2) = (0,nd)$,

$$\mu(\bar{f}) = \nu(f) = 2Sd - md - nd + 1$$

Consequently

$$\mu(f) = 1 + \frac{\mu(\bar{f})-1}{d} = 2S - m - n+1 = \nu(f)$$

The same argument obviously holds in the non-commode case. \(\square\)

References


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