Extension of positive maps into $B(\mathcal{A})$. 

by 

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§1. Introduction.

Let $\mathcal{K}$ and $\mathcal{H}$ be complex Hilbert spaces and let $A$ be a norm closed self-adjoint linear subspace of the bounded operators $B(\mathcal{K})$ on $\mathcal{K}$ containing the identity operator. If we denote by $B(A,\mathcal{H})^+$ the positive maps in the set $B(A,\mathcal{H})$ of bounded linear maps of $A$ into $B(\mathcal{H})$ we shall in the present paper study the extension problem for maps in $B(A,\mathcal{H})^+$ to maps in $B(B(\mathcal{H}),\mathcal{H})^+$.

In order to do this we shall introduce cones of maps with special positivity properties. The cones are roughly described as follows, for details see the definitions in §2. We start with a cone $K$ in $B(B(\mathcal{K}),\mathcal{H})^+$ satisfying some natural topological and invariance conditions. If $\mathcal{T}$ denotes the trace class operators on $\mathcal{K}$ then $B(A,\mathcal{H})$ is isometric to $(A \hat{\otimes} \mathcal{T})^*$, and $B(A,\mathcal{H})^+$ corresponds to the functionals which are positive on the cone $A^+ \hat{\otimes} \mathcal{T}^+$ generated by tensors $\sum a_i \otimes b_i$ with $a_i \in A^+$, $b_i \in \mathcal{T}^+$. The (self-adjoint elements $x \in A \hat{\otimes} \mathcal{T}$ such that $1 \otimes x(a)$ is a positive operator in $B(\mathcal{K} \hat{\otimes} \mathcal{H})$ for all $a \in K$, where $1$ denotes the identity map, form a cone $P(A,K)$ in $A \hat{\otimes} \mathcal{T}$. We say a map $\phi \in B(A,\mathcal{H})$ is K-positive if its image $\hat{\phi} \in (A \hat{\otimes} \mathcal{T})^*$ is positive on $P(A,K)$. Our extension theorem for positive maps now states that if $\phi$ is K-positive then it has a K-positive extension in $B(B(\mathcal{K}),\mathcal{H})^+$. This result subsumes Arveson's extension theorem for completely positive maps [2], because the completely positive maps are those which are K-positive with $K$ the cone of completely positive maps in $B(B(\mathcal{K}),\mathcal{H})$. There is another characterization which is perhaps more interesting, namely $\phi$ is completely posi-
Following [4] an **operator system** is a norm closed self-adjoint linear set \( \mathbb{A} \) of bounded operators on a Hilbert space \( \mathcal{H} \) containing the identity operator on \( \mathcal{H} \). We denote by \( \mathbb{A} \otimes \mathcal{B}(\mathcal{H}) \) the norm closed subspace of the \( C^* \)-algebraic tensor product \( \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \) generated by tensors \( a \otimes b, a \in \mathbb{A}, b \in \mathcal{B}(\mathcal{H}) \), using the spatial \( C^* \)-cross norm. We denote by \( (\mathbb{A} \otimes \mathcal{B}(\mathcal{H}))^+ \) (resp. \( (\mathbb{A} \otimes \mathcal{B}(\mathcal{H}))_{sa} \)) the positive (resp. self-adjoint) operators in \( \mathbb{A} \otimes \mathcal{B}(\mathcal{H}) \). We denote by \( \mathcal{T} \) or \( \mathcal{T}^+(\mathcal{H}) \) the set of trace class operators on \( \mathcal{H} \) with the norm \( \|h\|_1 = \text{Tr}((h^*h)^{1/2}) \), where \( \text{Tr} \) is the usual trace on \( \mathcal{B}(\mathcal{H}) \) normalized to be one on minimal projections. Then \( \mathcal{T} \) is the predual \( \mathcal{B}(\mathcal{H})^* \) of \( \mathcal{B}(\mathcal{H}) \) under the identification \( \langle x, h \rangle = \text{Tr}(xh) \) for \( x \in \mathcal{B}(\mathcal{H}), h \in \mathcal{T} \). We denote by \( \mathbb{A} \circ \mathcal{T} \) the algebraic tensor product of \( \mathbb{A} \) and \( \mathcal{T} \) and denote by \( \mathbb{A} \hat{\otimes} \mathcal{T} \) its Banach space closure under the projective norm defined by

\[
\|x\| = \inf \left\{ \sum_{i=1}^n \|a_i\| \|b_i\| : x = \sum_{i=1}^n a_i \otimes b_i, a_i \in \mathbb{A}, b_i \in \mathcal{T} \right\}.
\]

We denote by \( (\mathbb{A} \hat{\otimes} \mathcal{T})^+ \) (resp. \( (\mathbb{A} \hat{\otimes} \mathcal{T})_{sa} \)) the closure of \( (\mathbb{A} \circ \mathcal{T}) \cap (\mathbb{A} \otimes \mathcal{B}(\mathcal{H}))^+ \) (resp. \( (\mathbb{A} \circ \mathcal{T}) \cap (\mathbb{A} \otimes \mathcal{B}(\mathcal{H}))_{sa} \)), and by \( \mathbb{A}^+ \hat{\otimes} \mathcal{T}^+ \) the norm closed cone in \( \mathbb{A} \hat{\otimes} \mathcal{T} \) spanned by tensors \( a \otimes b \) with \( a \in \mathbb{A}^+ (=\mathbb{A} \cap \mathcal{B}(\mathcal{H})^+) \) and \( b \in \mathcal{T}^+ = \mathcal{T} \cap \mathbb{B}(\mathcal{H})^+ \). We denote by \( x \mapsto x^t \) the transpose map of \( \mathcal{B}(\mathcal{H}) \) with respect to some orthonormal basis. Then \( t \) is an anti-automorphism of order 2.

We denote by \( \mathcal{B}(\mathbb{A}, \mathcal{H}) \) the set of bounded linear maps of \( \mathbb{A} \) into \( \mathcal{B}(\mathcal{H}) \). \( \mathcal{B}(\mathbb{A}, \mathcal{H})^+ \) shall denote the positive linear maps in \( \mathcal{B}(\mathbb{A}, \mathcal{H}) \), i.e. those \( \phi \) for which \( a \in \mathbb{A}^+ \) implies \( \phi(a) \in \mathcal{B}(\mathcal{H})^+ \). If \( \mathbb{A}=\mathcal{B}(\mathcal{H}) \) we simplify notation and write \( \mathcal{B}^2(\mathcal{H}) \) for \( \mathcal{B}(\mathcal{B}(\mathcal{H}), \mathcal{H}) \) and \( \mathcal{B}^2(\mathcal{H})^+ \) for \( \mathcal{B}(\mathcal{B}(\mathcal{H}), \mathcal{H})^+ \). Following Arveson [2] the **BW-topology**
Proof. Let \( x = \sum a_i \otimes b_i \in (A \otimes \mathcal{T})^+ \). In order to show \( x \in A^+ \otimes \mathcal{T}^+ \) it suffices to show \( (1 \otimes e)x(1 \otimes e) \in A^+ \otimes \mathcal{T}^+ \) for all finite dimensional projections \( e \in B(\mathcal{H}) \). We may thus assume \( \mathcal{T}^+ \) is finite dimensional, say \( \mathcal{T}^+ = M_n \). We also consider \( A \) as a contained in its second dual \( A^{**} \). Since \( A^{**} \) is an abelian von Neumann algebra we can for each \( \varepsilon > 0 \) find projections \( e_1, \ldots, e_m \in A^{**} \) with sum \( 1 \) and \( \lambda_{ik} \in \mathbb{C} \) such that \( \| a - \sum_{k=1}^{m} \lambda_{ik} e_k \| < \varepsilon \). Rearranging the sum \( x = \sum a_i \otimes b_i \) and adding a small multiple of \( 1 \) if necessary, we can approximate \( x \) arbitrarily well by positive operators of the form \( \sum_{k=1}^{m} e_k \otimes c_k \in (A^{**} \otimes M_n)^+ \). Since the \( e_k \) are mutually orthogonal \( c_k \in M_n^+ \) for all \( k \), so \( \sum e_k \otimes c_k \in (A^{**})^+ \otimes M_n^+ \). It follows that
\[
x \in ((A^{**})^+ \otimes M_n^+) \cap (A \otimes M_n^+).
\]
But this intersection equals \( A^+ \otimes M_n^+ \) since \( A^+ \otimes M_n^+ \) is a norm closed convex cone in \( A \otimes M_n^+ \) which is weak-* dense in \( (A^{**})^+ \otimes M_n^+ \). Thus \( x \in A^+ \otimes \mathcal{T}^+ \), and we have shown \( (A \otimes \mathcal{T})^+ \subset A^+ \otimes \mathcal{T}^+ \). Since the opposite inclusion is obvious the proof is complete.

Definition 2.3. A mapping cone is a BW-closed subcone \( K \neq \{0\} \) of \( B^2(\mathcal{H})^+ \) which has a BW-dense subset of ultraweakly continuous maps and which is invariant in the sense that if \( \alpha \in K \) and \( a, b \in B(\mathcal{H}) \) then the map
\[
x \to \alpha(bxb^*)a^*
\]
belongs to \( K \).
n x n matrices. If V is a vector space with a cone V⁺ we consider (V, V⁺) as a partially ordered vector space. Let A₁ and A₂ be the partially ordered vector spaces

\[ A₁ = (B(\mathcal{H}) \otimes M_n, B(\mathcal{H})⁺ \otimes M_n⁺), \]

\[ A₂ = (B(\mathcal{H}) \otimes M_n, (B(\mathcal{H}) \otimes M_n)⁺). \]

Note that by Lemma 2.1 A₁ represents B(B(\mathcal{H}), C^n)⁺, and by definition A₂ represents C_n. By [6, Lem.1.1] A₁ and A₂ are order-unit spaces with 1 ⊗ 1 as order-unit, cf. [1]. Let B₁ be the subspace of A₁ defined by

\[ B₁ = (C(\mathcal{H}) \otimes M_n, C(\mathcal{H})⁺ \otimes M_n⁺), \]

\[ B₂ = (C(\mathcal{H}) \otimes M_n, (C(\mathcal{H}) \otimes M_n)⁺), \]

where C(\mathcal{H}) denotes the compact operators on \( \mathcal{H} \). Let S₁ be the set of states of B₁. Since each norm continuous linear functional on C(\mathcal{H}) has an ultraweakly continuous extension to B(\mathcal{H}) the states in S₁ all extend to ultraweakly continuous states of A₁. By the ultraweak density of B₁ in A₁ an operator x ∈ A₁ is positive if and only if \( \rho(x) > 0 \) for all \( \rho \in S₁ \). Thus S₁ is what is called a full set of states and thus is w*⁻dense in the set of states of A₁, see proof of [7, 4.3.9]. Since the maps \( \alpha \in B(B(\mathcal{K}), C^n) \) with \( \tilde{\alpha} \in S₁ \) are ultraweakly continuous, and the w*⁻topology and the BW-topology coincide, it follows that both B²(\mathcal{K})⁺ and CP(\mathcal{K}) are mapping cones.

Finally let K be a mapping cone. Let \( \alpha \in K \) be ultraweakly continuous. Let p be a one dimensional projection in B(\mathcal{H}) with \( \alpha(p) \neq 0 \). If q is any one dimensional projection in B(\mathcal{H}) let v
the map
\[ \beta(x) = \sum_{i,j=1}^{n} \omega_{ij}(x)a_{ij} \]
is a linear isomorphism of \( M_n \) onto itself belonging to \( S(C^n) \).
Let \( \varepsilon > 0 \) and \( \alpha \in K \). Scaling \( \beta \) we may assume \( \| \beta \| < \varepsilon / 2 \). If \( \alpha(x) + \beta(x) = 0 \) for some \( x \neq 0 \) then \(-1 \in \text{Sp}(\beta^{-1}\alpha)\) — the spectrum of \( \beta^{-1}\alpha \). Since \( \text{Sp}(\beta^{-1}\alpha) \) is finite there is \( \lambda \in [\frac{1}{2}, \frac{3}{2}] \) such that \(-1 \in \lambda \text{Sp}(\beta^{-1}\alpha) = \text{Sp}(\lambda \beta^{-1}\alpha) \). Thus \( \lambda \beta^{-1}\alpha(x) \neq -x \) for all \( x \neq 0 \), and \( \gamma = \alpha + \lambda^{-1} \beta \) is a linear isomorphism of \( M_n \) onto itself satisfying \( \| \alpha - \gamma \| = \lambda^{-1} \| \beta \| < \varepsilon \). Since \( \beta \in S(C^n) \subset K \) by Lemma 2.4, \( \gamma \in K \), completing the proof.

Definition 2.7. Let \( K \) be a mapping cone in \( B^2(K)^+ \) and let \( A \) be an operator system. Then we denote by \( P(A, K) \) the set
\[ P(A, K) = \{ x \in (A \otimes \mathcal{T})_{sa} : \otimes x(x) \in (A \otimes B(\mathcal{K}))^+ \forall x \in K \} \]
where \( \otimes \) denotes the identity map. It is implicit in the next lemma that \( P(A, K) \) is well defined.

Lemma 2.8. In the above notation \( P(A, K) \) is a proper norm closed convex cone in \( A \otimes \mathcal{T} \) containing the cone \( A^+ \otimes \mathcal{T}^+ \).

Proof. Since \( \| b \| < \| b \| _1 \) for all \( b \in \mathcal{T} \), if \( \alpha \in B^2(\mathcal{K}) \) and \( \sum a_i \otimes b_i \in A \otimes \mathcal{T} \), we have
\[ \| \otimes x(\sum a_i \otimes b_i) \| = \| \sum a_i \otimes x(b_i) \| < \| a \| \sum \| a_i \| \| b_i \| \]
\[ < \| a \| \sum \| a_i \| \| b_i \| _1. \]
finite dimensional projections in $B(\mathcal{H})$ converging strongly to 1 and $e_\gamma \geq e$ for all $\gamma$. Then $\iota \otimes (A \circ e_\gamma)(x) + \iota \otimes a(x)$ strongly. Fix $\gamma$ and let $f$ denote the range projection of $A \circ e_\gamma a(e)$. Since $f$ is finite dimensional there exists $a > 0$ in $fB(\mathcal{H})f$ such that $a(A \circ e_\gamma a(e)) = f$. Let $a_\gamma = Ada \circ Ade_\gamma a$. Then we have

$$(\iota \otimes a_\gamma)(1 \otimes e + x) = 1 \otimes f + \iota \otimes a_\gamma(x).$$

Since $a_\gamma(e) = f$, $\|a_\gamma\| = 1$, hence $\|\otimes a_\gamma\| < 1$. In particular $\|\otimes a_\gamma(x)\| < 1$. Since $\iota \otimes a_\gamma(x) \in (1 \otimes f)(A \otimes B(\mathcal{H}))(1 \otimes f)$, it follows that

$$1 \otimes f + \iota \otimes a_\gamma(x) > 0.$$ 

Since $a$ is invertible in $fB(\mathcal{H})f$,

$$\iota \otimes A \circ e_\gamma a(1 \otimes e + x) > 0.$$ 

Since $A \circ e_\gamma a + a$ point-strongly $\iota \otimes a(1 \otimes e + x) > 0$. Now $a$ was an arbitrary map in $K$. Thus $1 \otimes e + x \in P(A,K)$. Since $x$ was an arbitrary element in $(1 \otimes e)(A \otimes \mathcal{F})_{sa}(1 \otimes e)$ with $\|x\| < 1$, $1 \otimes e$ is an interior point of $(1 \otimes e)P(A,K)(1 \otimes e)$. Q.E.D.

Let $B \supset A$ be another operator system. Since $\mathcal{F}$ considered as a Banach space has the approximation property, we may consider $A \otimes \mathcal{F}$ as a Banach subspace of $B \otimes \mathcal{F}$. Thus the next lemma makes sense.

**Lemma 2.10.** Let $A \subset B$ be operator system on the same Hilbert space. Let $K$ be a mapping cone in $B(\mathcal{H})^+$. Suppose $e$ is a projection in $B(\mathcal{H})$ with $e^* = e$. Then

$$(1 \otimes e)P(A,K)(1 \otimes e) = (1 \otimes e)P(B,K)(1 \otimes e) \cap (A \otimes \mathcal{F}).$$
$K_2 = \{ \tau \circ \beta \circ \tau^{-1} : \beta \in K_1 \}$. Then $K_2$ is a mapping cone and a map
\( \phi \in B(A_2, \mathcal{K}_2) \) is $K_2$-positive if and only if \( \tau^{-1} \circ \phi \circ \beta \in B(A_1, \mathcal{K}_1) \) is $K_1$-positive.

Proof. Clearly $K_2$ is a mapping cone. We have
\[
P(A_2, K_2) = \{ x \in A_2 \otimes \mathcal{J}(\mathcal{K}_2) : \iota \otimes \alpha(x) \in (A_2 \otimes B(\mathcal{K}_2))^+ \forall \alpha \in K_2 \}
\]
\[
= \{ \beta \circ \tau(y) : \gamma \in A_1 \otimes \mathcal{J}(\mathcal{K}_1), (\beta \circ \tau) \circ (\iota \otimes \tau^{-1} \circ \alpha)(y)
\]
\[
\in \beta \circ \tau(A_1 \otimes B(\mathcal{K}_1))^+ \forall \alpha \in K_2 \}
\]
\[
= \{ \beta \circ \tau(y) : y \in A_1 \otimes \mathcal{J}(\mathcal{K}_1), (\beta \circ \tau) \circ (\iota \otimes \alpha)(y)
\]
\[
\in \beta \circ \tau(A_1 \otimes B(\mathcal{K}_1))^+ \forall \rho \in K_1 \}
\]
\[
= \beta \circ \tau(P(A_1, K_1)).
\]

If \( \phi \in B(A_2, \mathcal{K}_2) \) the assumption $\tau \circ \iota = \iota \circ \tau$ implies
\[
(\tau^{-1} \circ \phi \circ \beta) \circ (\iota \circ \alpha) = \text{Tr}(\tau^{-1} \circ \phi \circ \beta \circ (\iota \circ \alpha))^{\dagger}
\]
\[
= \text{Tr}(\phi \circ \beta \circ (\iota \circ \alpha))^{\dagger}
\]
\[
= \tilde{\phi}(\beta \circ \tau(\iota \circ \alpha)),
\]
for all $\iota \circ \alpha \circ \beta \in A \circ \mathcal{J}$. By the first paragraph then, $\phi$ is $K_2$-positive if and only if $\tau^{-1} \circ \phi \circ \beta$ is $K_1$-positive.

Q.E.D.

§3. Extension of positive maps.

In this section we prove the extension theorems for positive maps alluded to in the introduction. The first result shows that $K$-positive maps have $K$-positive extensions.
Let $A$ be an operator system and $\phi \in B(A, \mathcal{H})$. Then $\phi$ is said to be completely positive if $\phi \otimes \mathbf{1}_n \in B(A \otimes \mathcal{H}^n, \mathcal{H}^n \otimes \mathcal{H}^n)^+$ for all $n \in \mathbb{N}$, where $\mathbf{1}_n$ is the identity map on $M_n$. We denote the set of completely positive maps by $CP(A, \mathcal{H})$.

**Theorem 3.2.** Let $A$ be an operator system and $\mathcal{H}$ a Hilbert space. Then the set of completely positive maps in $B(A, \mathcal{H})^+$ equals the mapping cone $CP(\mathcal{H})$ of Lemma 2.4. Furthermore, if $\phi \in B(A, \mathcal{H})$ the following three conditions are equivalent:

(i) $\phi$ is completely positive.

(ii) $\phi$ is positive on the cone $(A \otimes \mathcal{H})^+$ in $A \otimes \mathcal{H}$.

(iii) $\phi$ is $CP(\mathcal{H})$-positive.

**Proof.** We first show $(i) \implies (ii)$ under the assumption that $\mathcal{H}$ is finite dimensional. Say $B(\mathcal{H}) = M_n$. We define a bilinear map $\pi : M_n \otimes M_n \rightarrow M_n$ by

$$\pi(\sum a_i \otimes b_i) = \sum a_i^t b_i.$$ 

Then $\text{Tr}_\pi$ is a positive linear functional. Indeed

$$\text{Tr}_\pi((\sum a_i \otimes b_i)(\sum a_j \otimes b_j)^*) = \sum_{ij} \text{Tr}_\pi(a_i^* a_j^t b_i^* b_j)$$

$$= \sum_{ij} \text{Tr}(a_i^* a_j^t b_i^* b_j) = \sum_{ij} \text{Tr}((b_i^* a_i)(b_j^* a_j)^*)$$

$$= \text{Tr}((\sum b_i^* a_i)(\sum b_j^* a_j)^*) > 0.$$
Since $\tilde{\phi} > 0$ on $(A \otimes M_n)^+$ we thus have from (1), (2) and (3)

$$(\tilde{\phi} \otimes 1_m)(x) = \sum \tilde{\phi}(b_{i'i'kk''r'r'} \phi_{kk'r'}) \text{Tr}_m(e_{ii'}e_{r'r'})
= \sum \tilde{\phi}(b_{i'i'kk''r'} \phi_{kk'r'}) > 0.$$ 

In particular, by Lemma 2.1 $\phi \otimes 1_m$ is a positive map, so that $\phi$ is completely positive.

We have thus shown (i) $\iff$ (ii) under the assumption that $\mathcal{H}$ is finite dimensional. Let $(e_\gamma)$ be an increasing net of finite dimensional projections in $B(\mathcal{H})$ converging strongly to $1$ and satisfying $e_\gamma^* = e_\gamma$. Then $\phi$ is completely positive if and only if $A e_\gamma \phi$ is completely positive for all $\gamma$, hence by the first part of the proof if and only if $(A e_\gamma \tilde{\phi}) > 0$ on $(A \otimes e_\gamma \mathcal{H})^+$ for all $\gamma$. If $x = \sum a_i \otimes b_i \in (A \otimes J e_\gamma)^+$ then

$$(A e_\gamma \tilde{\phi})(x) = \sum \text{Tr}(\phi(a_i)(e_{\gamma i} b_{\gamma i}^t)).$$

Since the right side clearly converges to $\tilde{\phi}(x)$, it follows that $\tilde{\phi} > 0$ on $(A \otimes J)^+$ if and only if $(A e_\gamma \tilde{\phi}) > 0$ on $(A \otimes e_\gamma \mathcal{H})^+$ for all $\gamma$, hence if and only if $\tilde{\phi}$ is completely positive. Thus (i) $\iff$ (ii).

Note that in the special case when $A = B(\mathcal{H})$ we have shown that

$$\text{CP}(B(\mathcal{H}), \mathcal{H}) = \{ \phi \in B^2(\mathcal{H}), \tilde{\phi} > 0 \text{ on } (B(\mathcal{H}) \otimes J)^+ \}$$

so by Lemma 2.4 $\text{CP}(B(\mathcal{H}), \mathcal{H})$ is the mapping cone $\text{CP}(\mathcal{H})$. By an easy application of Stinespring's theorem [11] each ultraweakly continuous map in $\text{CP}(\mathcal{H})$ is a countable sum of maps of the form $A a$, $a \in B(\mathcal{H})$. Thus

$$P(A, \text{CP}(\mathcal{H})) = \{ x \in (A \otimes J)_\text{sa} : 1 \otimes \text{Ada}(x) \in (A \otimes B(\mathcal{H}))^+ \forall a \in B(\mathcal{H}) \} = (A \otimes J)^+$$

Thus (ii) $\iff$ (iii). Q.E.D.
proving the assertion. By Theorem 3.2 \( \phi \) is completely positive if and only if \( \tilde{\phi} > 0 \) on \( (M_m \otimes M_n)^+ \), hence if and only if \( h > 0 \).

Since

\[
h^t = \sum e_{ji} \otimes (e_{ji})
\]

is the matrix which Choi associated with \( \phi \), we recover his result that \( \phi \) is completely positive if and only if \( h^t > 0 \).

Let \( \alpha \in B^2(\mathcal{H})^+ \) be a map of finite type, cf. Def. 2.5, and let \( e \) be a finite dimensional projection such that \( \alpha(1) = \alpha(e) \otimes 1 \).

Then \( \alpha \) restricts to a map in \( B^2(e\mathcal{H}) \), i.e. \( \alpha \) can be considered as a bounded operator on the Hilbert space \( B(e\mathcal{H}) \) with inner product defined by the trace. Thus \( \alpha \) has an adjoint \( \alpha^* \) as an operator on \( B(e\mathcal{H}) \), which is easily seen to be positive. Replacing \( \alpha^* \) by \( \alpha^* \circ \alpha \otimes e, \alpha^* \) becomes a map of finite type in \( B^2(\mathcal{H})^+ \).

We denote by \( \alpha^d \) the map \( \alpha \circ \alpha^* \otimes \alpha \), and note that \( \alpha^d = \alpha \).

If \( K \) is a mapping cone in \( B^2(\mathcal{H})^+ \) the maps of finite type are BW-dense in \( K \) by Lemma 2.6. Let \( K^d \) be the BW-closure of the maps \( \alpha^d \) with \( \alpha \) of finite type in \( K \). Then \( K^d \) is a mapping cone such that \( K^{dd} = K \). \( K^d \) is called the adjoint cone of \( K \).

**Theorem 3.6.** Let \( A \) be an operator system and \( K \) a mapping cone in \( B^2(\mathcal{H})^+ \). Let \( C_K \) denote the BW-closed cone in \( B(A,\mathcal{H}) \) generated by all maps of the form \( \alpha \circ \phi \) with \( \alpha \) in the adjoint cone \( K^d \) of \( K \) and \( \phi \in CP(A,\mathcal{H}) \). Then a map \( \phi \in B(A,\mathcal{H}) \) is \( K \)-positive if and only if \( \phi \in C_K \).

To prove the theorem we shall need two preliminary results.
Since N ∩ J is weakly dense in N, \( \phi(a)eB(\mathcal{M})e = B(e\mathcal{M}) \) and both A and N are self-adjoint, Tr(\( \phi(a)b^t \)) = 0 for all \( b \in N \). Now \( b^t \) belongs to the null space of \( \alpha ot \) for all \( b \in N \). Thus by Lemma 3.7

\[
\phi(A) \subset (\alpha ot)^*(B(\mathcal{M})) = \text{to}^*(B(\mathcal{M})) = \alpha^d(B(\mathcal{M})) = B(e\mathcal{M}).
\]

Since \( \alpha \) is an isomorphism on \( B(e\mathcal{M}) \), we define \( \alpha^{-1} \) in \( B^2(\mathcal{M}) \) to be the inverse of \( \alpha \) on \( B(e\mathcal{M}) \) composed with \( A \text{de} \). Let \( \psi = \alpha^{-1} \circ \phi \). Then \( \phi \in B(A, \mathcal{M}) \) since \( \phi(A) \subset B(e\mathcal{M}) \). We show \( \phi \in \text{CP}(A, \mathcal{M}) \). For this let \( x = \sum a_i \otimes b_i \in (A \otimes \mathcal{M})^+ \). Since \( e^t = e \),

\[
\otimes \alpha^{-1}(\sum a_i \otimes (\alpha^{-1})^d(b_i)) = (\otimes \text{de})(x) \in (A \otimes B(\mathcal{M}))^+
\]

so that \( \sum a_i \otimes (\alpha^{-1})^d(b_i) \in P_\alpha \). Since \( \tilde{\phi} \) is positive on \( P_\alpha \) we have

\[
\tilde{\phi}(x) = \sum \text{Tr}(\phi(a_i)b_i^t)
\]

\[
= \sum \text{Tr}(\phi(a_i)(\alpha^{-1})^* ot(b_i))
\]

\[
= \sum \text{Tr}(\phi(a_i)(\alpha^{-1})^d(b_i)^t)
\]

\[
= \tilde{\phi}(\sum a_i \otimes (\alpha^{-1})^d(b_i)) > 0.
\]

By Theorem 3.2 \( \psi \) is completely positive. Since \( \phi = \alpha \circ \psi \) the proof is complete.

**Proof of Theorem 3.6.** Suppose \( \phi \in \text{C}_K \). In order to show \( \phi \) is K-positive it suffices by density of maps of finite type in \( K^d \) to consider \( \phi \) of the form \( \phi = \alpha \circ \psi \) with \( \alpha \in K^d \) of finite type and \( \psi \in \text{CP}(A, \mathcal{M}) \). Note that since \( \alpha^d \in K \), if \( x = \sum a_i \otimes b_i \in P(A, K) \) then

\[
\sum a_i \otimes (\alpha^d(b_i)) \in (A \otimes \mathcal{M})^+.
\]

Thus we have
Lemma 3.9. \( P(B(\mathcal{X}), B^2(\mathcal{X})^+) = B(\mathcal{X})^+ \otimes J \).

Proof. By definition if \( \alpha \in B^2(\mathcal{X}) \) then \( \tilde{\alpha} > 0 \) on \( P(B(\mathcal{X}), B^2(\mathcal{X})^+) \) if and only if \( \alpha \) is \( B^2(\mathcal{X})^+ \)-positive, which by Theorem 3.6 is equivalent to \( \alpha \in B^2(\mathcal{X})^+ \), which by Lemma 2.1 is equivalent to \( \tilde{\alpha} > 0 \) on \( B(\mathcal{X})^+ \otimes J^+ \). Since \( B^2(\mathcal{X}) = (B(\mathcal{X}) \otimes J)^* \) and \( P(B(\mathcal{X}), B^2(\mathcal{X})^+) = B(\mathcal{X})^+ \otimes J^+ \) the Hahn-Banach theorem thus implies the conclusion of the lemma. Q.E.D.

Definition 3.10. Let \( A \) be an operator system acting on \( \mathcal{X} \). A map \( \phi \in B(A, \mathcal{X})^+ \) is said to be extendible if \( \phi \) has an extension in \( B(B(\mathcal{X}), \mathcal{X})^+ \).

Theorem 3.11. Let \( A \) be an operator system acting on a Hilbert space \( \mathcal{X} \). Let \( \mathcal{K} \) be a Hilbert space such that \( \dim \mathcal{K} < \dim \mathcal{X} \). Let \( \phi \in B(A, \mathcal{X})^+ \). Then the following three conditions are equivalent:

(i) \( \phi \) is extendible.

(ii) \( \phi \) is \( B^2(\mathcal{X})^+ \)-positive.

(iii) \( \phi \) belongs to the BW-closed cone spanned by all maps of the form \( \alpha \circ \phi \in B(A, \mathcal{X})^+ \) with \( \alpha \in B^2(\mathcal{X})^+ \), \( \phi \in \text{CP}(A, \mathcal{X}) \).

Proof. (i)\( \Rightarrow \) (ii). Let \( v \) be an isometry of \( \mathcal{K} \) into \( \mathcal{X} \) and let \( \tilde{A} = vAv^* + C1 \), and let \( \beta \) be the map of \( B(\mathcal{X}) \) onto \( B(\mathcal{K}) \) given by \( \beta(x) = v^* xv \). Suppose \( \tilde{\phi} \) is an extension of \( \phi \) in \( B(B(\mathcal{X}), \mathcal{X})^+ \). Then \( \tilde{\phi} \circ \beta \in B^2(\mathcal{K}) \) and is an extension of \( \phi \circ \beta \) considered as a map in \( B(\tilde{A}, \mathcal{K})^+ \). By Lemma 3.9 \( (\tilde{\phi} \circ \beta) > 0 \) on \( P(B(\mathcal{X}), B^2(\mathcal{K})^+) \), hence so is by Lemma 2.10 its restriction \( (\tilde{\phi} \circ \beta) \) to

\[ P(\tilde{A}, B^2(\mathcal{K})^+) = P(B(\mathcal{X}), B^2(\mathcal{K})^+) \cap (\tilde{A} \otimes J) \]
Remark 3.13. It is important that $\dim K < \dim L$ in Theorem 3.11.
Indeed it was pointed out by Woronowicz [13] that all maps in $B^2(\mathbb{C}^2)^+$ are decomposable, i.e. they are sums of maps of the form $\alpha$ and $\beta \circ t$, with $\alpha, \beta \in \text{CP}(\mathbb{C}^2)$. Since it was shown by Woronowicz [13] that not all maps in $B(M_4, \mathbb{C}^2)^+$ are decomposable, and each $B^2(\mathbb{C}^2)^+$-positive maps is decomposable by an easy application of Theorem 3.6, the implication (i)$\Rightarrow$(ii) of Theorem 3.11 does not hold.

Recall that a C*-algebra $A$ is called nuclear if $A \otimes B$ has a unique C*-cross norm for all C*-algebras $B$. By a theorem of Effros and Lance [4] the second dual of a nuclear C*-algebra is injective, i.e. there exists a projection of norm one of the bounded operators onto it. We next show that positive maps from nuclear C*-algebras into $B(H)$ are extendible. We leave the problem open for general C*-algebras, but we incline to believe it is false. It was pointed out to us by U. Haagerup that the proof below shows that the result is true for the so-called WEP-algebras of Lance [8].

Theorem 3.14. Let $A$ be a nuclear C*-algebra acting on a Hilbert space. Then all maps in $B(A,H)^+$ are extendible.

Proof. Let $A$ act on the Hilbert space $H$, and assume $A^{**}$ acts on the Hilbert space $L$. Then $\dim K < \dim L$. If $H'$ is a Hilbert space containing $H$ we may consider a map $\phi \in B(A,H)^+$ as a map in $B(A,H')^+$. If $\phi \in B(B(H,H'),H')^+$ is an extension of $\phi$ then with $p$
Conversely suppose $\tilde{\phi}: C^*(A) \to C^*(B)$ is a positive extension of $\phi$. Since $B$ is simple there is by [5, Thm. 2.1] a positive projection $Q: C^*(B) \to C^*(B)$ such that $Q(C^*(B)) = B$. Considering $Qo\tilde{\phi}$ instead of $\tilde{\phi}$ we may assume $\phi: C^*(A)_sa \to B$. Let $\phi = \phi^1 o \phi$. Then $\psi: C^*(A)_sa \to A$ is positive. Furthermore, if $x \in A$ then $\phi(x) = \phi^1 (\phi(x)) = \phi^1 (\phi(x)) = x$, so $\psi: C^*(A) \to C^*(A)$ is a positive projection such that $\psi(C^*(A)_sa) = A$. But then it follows from [5, Lem. 1.2 (2)] that $\psi$ is faithful, considering the extension of $\phi$ to $C^*(A)**$ if necessary. By [5, Thm. 1.4 (2)] $A$ is a JC-subalgebra of $C^*(A)$. Q.E.D.

Example 3.16. We give an example of an operator system $A \subset M_3$ and a map in $B(A, C^2)^+$ which has no extension in $B(M_3, C^2)^+$. A similar example has been given by Arveson [2, Appendix 2] when $A \subset C(T)$ - the continuous complex functions on the circle group. Let

$$A = \left\{ \begin{bmatrix} x & y & z \\ y & x & 0 \\ z & 0 & x \end{bmatrix} \in M_3 : x, y, z \in \mathbb{R} \right\}.$$ 

Then $A$ (or $A+iA$) is an operator system. It is easy to see that $C^*(A) = M_3$ and that

$$\begin{bmatrix} x & y & z \\ y & x & 0 \\ z & 0 & x \end{bmatrix} > 0 \iff x > 0 \text{ and } x^2y^2+z^2.$$ 

Let $S_2$ denote the real symmetric $2 \times 2$ matrices, and let $\phi:A \to S_2$ by

$$\phi\left( \begin{bmatrix} y & x & 0 \\ z & 0 & x \end{bmatrix} \right) = \begin{bmatrix} x & y & z \\ x-y & z & x+y \end{bmatrix}.$$ 

Then it is straightforward to show $\phi$ is a unital order-isomorphism of $A$ onto $S_2$. Since $A$ is not a JC-algebra $\phi$ has no extension in $B(M_3, C^2)^+$ by Proposition 3.15.