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NOTATIONS

$k$ field of characteristic $0$ and algebraically closed
$k^*$ the multiplicative group of $k$
$k[x] = k[x_1, \ldots, x_n]$
$\mathbb{P}^n$ the projective $n$-space
$\mathcal{L}$ the category of local artinian $k$-algebras with residue field $k$
$\mathcal{L}_H^\wedge$ the category of local artinian $H^\wedge$-algebras with residue field $k$
$\mathcal{G}$ the category of groups
$\text{aut}_S(X \otimes_k S) = \{ \phi \in \text{Aut}_S(X \otimes_k S) | \phi \otimes_k 1 \}$, $S$ in $\mathcal{L}$
$\text{aut}_R(X \otimes_{H^\wedge} R) = \{ \phi \in \text{Aut}_R(X \otimes_{H^\wedge} R) | \phi \otimes_R 1 \}$, $R$ in $\mathcal{L}_H^\wedge$
$\text{aut}_R(H^\wedge \otimes_k R) = \{ \phi \in \text{Aut}_R(H^\wedge \otimes_k R) | \phi(m \otimes_R R) \subseteq m \otimes_R R \}$
Introduction. The purpose of this paper is to contribute towards a better understanding of the local moduli problem in algebraic geometry. Let $k$ be field and let $X$ be an algebraic object, say a projective $k$-scheme.

The local moduli problem may then be phrased as follows. Describe the set of isomorphism classes of objects $X'$ occuring as "arbitrary small deformations" of $X$.

In practice this means to define a natural filtration on the set of these isomorphism classes, such that each subset of the filtration may be given an algebraic structure, say as a $k$-scheme or, more generally, as an algebraic space. We shall refer to any such, natural, filtration $\{M_\tau\}$ as the locale moduli suite of $X$. This done, one would like to find the local structure of these new objects, their dimensions etc.

Our approach, which is rather general in scope, starts with a study of the infinitesimal automorphisms of the formal moduli of $X$, leaving the formal versal family invariant, see §1 and §2.

Coupled with a close look at the properties of the kernel $V$ of the Kodaira-Spencer map of a formally versal family $\tilde{X} \times \mathbb{H}$ of $X$, which we shall assume exists, this leads to a proof of the existence of a "fine" local moduli suite in the category of algebraic spaces, see (3.18), provided the objects $X$ of our study, satisfy a set of rather strict conditions, see §3, $(A_1)$, $(A_2)$ and $(V')$. In particular these conditions imply the existence of an algebraization of the formal versal family, see (3.6) and (3.7).

One of the main results of this paper is the theorem (3.24) which asserts that the fine local moduli suite $\{M_\tau\}$ is, generically the quotient of a canonical filtration $\{S_\tau\}$ of the base space $\mathbb{H}$ by the $k$-Lie-algebra $V$. 
In the process we obtain useful criteria for smoothability and non-smoothability of singularities, see (3.10).

We also show that for every $\tau$ there is a flat family of Lie-algebras the fibers of which are the Lie-algebras of non liftable infinitesimal automorphisms $L(X')$ of those $X'$ representing the classes of $M_\tau$. In §4 we specialize to the case of a hypersurface singularity in the algebroid sense.

Finally in §5 and §6 we treat the case of quasihomogenous plane curve singularities.

To illustrate the main ideas of this paper, let us consider a simple example, that of the cusp $X' = \text{Spec}(k[x,y]/(x^3+y^2))$. As an affine scheme $X$ admits an affine formally versal family, (see §4), the obvious algebraization of the formal versal family, given by $H = k[t_0,t_1]$, $F = x^3+y^2+t_1x+t_0$, $\bar{X} = \text{Spec}(H[x,y]/(F))$, $\bar{H} = \text{Spec}(H)$ and $\pi:\bar{X} \to \bar{H}$.

Refering to (4.2) we find that the Kodaira-Spencer map

$$g: \text{Der}_k(H) \to A^1(H,\bar{X},O_{\bar{X}})$$

maps $\frac{\partial}{\partial t_0}$ to 1 and $\frac{\partial}{\partial t_1}$ to $x$, 1 and $x$ considered as representing classes of

$$A^1(H,\bar{X},O_{\bar{X}}) = H[x,y]/(F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y})$$.

The kernel $V$ of $g$ is easily seen to be generated as an $H$-module by the derivations

$$\delta_0 = t_0 \frac{\partial}{\partial t_0} + \frac{2}{3} t_1 \frac{\partial}{\partial t_1}$$

$$\delta_1 = -\frac{3}{9} t_1^2 \frac{\partial}{\partial t_0} + t_0 \frac{\partial}{\partial t_1}$$.
Obviously, the generic fiber of $\pi$ is rigid, and by (3.26) the discriminant of $\pi$ is therefore the determinant of $V$, i.e.

$$\Delta = (27t_0^2 + 4t_1^3)$$

as it should. The closed points of the open subscheme $S_0 = H \Delta$ correspond to the elliptic curves $x^3 + y^2 + t_1 x + t_0$, $\Delta \neq 0$ which form a one-dimensional family of isomorphism-classes of affine curves, $M_0$.

However the quotient $S_0/V$ is easily seen to be reduced to a point. This contradicts the assertion of (3.24), and points to one of the main difficulties of the theory of local moduli. In fact the condition $(A_2)$ of §3 is not satisfied for affine schemes. Therefore the $k$-Lie-algebra $V$ does not, in general, define the correct equivalence relation on $S_0$.

If however, we choose to consider the cusp as a cone, or as a graded $k$-algebra, $C = k[x,y,z]/(x^3 + y^2 z)$ we find that there is a formally versal family of graded $k$-algebras with, as before, $H = k[t_0, t_1]$, and $C = H[x,y,z]/(x^2 + y^2 z + t_1 x z^2 + t_0 z^3)$.

In this new category, the kernel $V$ of the corresponding Kodaira-Spencer map is generated by $\delta_0$ and the quotient $M_0 = S_0/V$ is now $\text{Spec}(k[t_0, t_1]^\delta) = \text{Spec}(k[-t_1^3])$ where $\frac{t_1^3}{\Delta}$ is a modular function for the elliptic curves in the family $\pi$, in the classical sense. The point here is, of course, that for finite type graded $k$-algebras, the condition $(A_2)$ is satisfied, so that we may apply (3.18) and (3.24).

However, we cannot, as the above example might lead us to believe, always reduce the local moduli problem of an affine scheme to the corresponding problem for a graded $k$-algebra, as is easily seen in the example $X = \text{Spec}(k[x,y]/(x^n + y^n))$. 
To treat the local moduli problem for affine schemes, it seems therefore that we have to find a replacement for the Kodaira-Spencer kernel $V$ as a means to defining the correct equivalence relation on the $S_i$'s. This may not be impossible, but we shall in this paper concentrate on applications of the general theory of §3, to those categories of objects that satisfy (A$_2$). This, in particular, includes the category of isolated hypersurface singularities in the algebroid sense, see §4, §5 and §6.

The main results are summed up in the introductions to each paragraph.

This paper is the outgrowth of a collaboration between the two authors during the last 3 years. A first, very sketchy version appeared in 1983 [La-Pf].

Many authors have previously treated the same subject. In particular Zariski in his lecture notes [Z], published in 1973, laid the foundations to the study of hypersurface singularities in the algebroid sense. Obviously, his results together with those of Delorme [Del], Washburn [Wash], etc. have influenced upon our work, even though our methods are quite different, and our goals seemingly somewhat wider.

It should be explicitly mentioned that Palamodov in [Pal] studied the notion of prrepresenting substratum (§1) and that Saito in [S] has the same calculation as we obtain in §4, of the kernel of the Kodaira-Spencer map. These ideas are, however, part of the folklore of the last 15 years, and were at the origin of one of the authors interest in this subject, see [La].

Finally, we are grateful for the financial support by the Humboldt-Universität of Berlin, DDR, by the University of Oslo, Norway and by the Norwegian Research Council, NAVF.
Given an algebraically closed field \( k \) of characteristic zero, we shall, throughout §1-§3 be concerned with an algebraic object \( X \) such as,

**Example 1.** \( X = \mathcal{C} \), a small category of \( k \)-schemes.

Put \( A^i = A^i(k, \mathcal{C}, O_\mathcal{C}) \), \( i > 0 \), see [La].

**Example 2.** \( X = \text{Spec}(A) \), \( A \) any \( k \)-algebra with isolated singularities. In particular, we shall be interested in the case where \( A = (k[[X_1, \ldots, X_n]]/(f)) \) is the local ring of an isolated hypersurface singularity. In this case \( A^0 = \text{Der}_k(k[[X_1, \ldots, X_n]]/(f)) \), and \( A^1 = (x) \cdot k[[X_1, \ldots, X_n]]/(x) \cdot \left( \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n}, f \right) \).

**Example 3.** \( X = \mathcal{E} \), a small category of \( \mathcal{O}_Y \)-Modules where \( Y \) is some \( k \)-scheme. Here \( A^i = \text{Ext}^i_{\mathcal{O}_Y}(\mathcal{O}_\mathcal{E}, \mathcal{O}_\mathcal{E}) \), \( i > 0 \) are defined as in [La] with \( \text{Hom} \) replacing \( \text{Der} \). See the concluding remark, loc.cit. p. 150, and see [La 2].

**Example 4.** \( X = E \), a coherent \( \mathcal{O}_{\mathbb{P}^n} \)-Module. \( A^i = \text{Ext}^i_{\mathcal{O}_{\mathbb{P}^n}}(E, E) \), \( i > 0 \). Of particular interest is the case where \( E \) is a vector bundle on \( \mathbb{P}^n \).

Assume now that \( \dim_k A^i = \infty \) for \( i = 1, 2 \). Then, see e.g. [La], (4.2.4), there exist in all these cases a formal moduli \( \mathbb{H}^\wedge \) (a prorepresenting hull for the deformation functor) of \( X \), and a formal versal family

\[ \pi^\wedge: X^\wedge \to \text{Spf}(\mathbb{H}^\wedge) = \mathbb{H}^\wedge \]

The first part of this paper is devoted to the study of \( \pi^\wedge \) in this generality.
§1. THE PROREPRESENTING SUBSTRATUM OF THE FORMAL MODULI

Introduction. Let $X$ be an object of the type we are considering, see the main Introduction above.

The basic notion in the study of local moduli of $X$ is the notion of prorepresenting substratum of the formal moduli. If $H^\wedge$ is the formal moduli of $X$, then the prorepresenting substratum $H^\wedge_0$ is the unique maximal quotient of $H^\wedge$ for which the obvious composition

$$\text{Mor}(H^\wedge_0, -) \rightarrow \text{Mor}(H^\wedge, -) \rightarrow \text{Def}_X$$

is injective.

This quotient exists in all generality, and the object of this § is its construction, see (1.3). Later, in §3, we shall want to extend this notion to the algebraization of the formal moduli. It turns out that this is facilitated by the introduction of the concept of the $n$th equicohomological substratum $H^\wedge(n)$ of $H^\wedge$ and by proving that $H^\wedge_0$ coincides with the 0-th equicohomological substratum $H^\wedge(0)$.

Let $X$ be any algebraic object of the type discussed in the Introduction, and consider the deformation functor

$$\text{Def}_X: \mathbb{A} \rightarrow \text{Sets},$$

the corresponding cohomology $A^i = A^i(k, X; O_X)$, $i \geq 0$ and the universal obstruction morphism

$$O_X: T^2 \rightarrow T^1$$

where $T^i = \text{Sym}_k(A^{i*})^\wedge$, (see [Lal] (4.2.4)). Denote by

$$H^\wedge = T^1 \otimes_k T^2$$
the formal moduli of $X$, i.e. the prorepresentable hull of the deformation functor $\text{Def}_X$, and put

$$H^\wedge = \text{Spf}(H^\wedge).$$

In general there are lots of infinitesimal automorphisms of $X$, and obstructions for lifting these (see [Sch]). Therefore $H^\wedge$ does not necessarily prorepresent $\text{Def}_X$. However, as we shall see there is a universal prorepresenting substratum $H^\wedge_0$ of $H^\wedge$, corresponding to a quotient

$$H^\wedge_0 = H^\wedge / \sigma_l$$

of $H^\wedge$.

In fact, let us consider the category $\mathcal{A}_H$ of all artinian local $H^\wedge$-algebras with residue field $k$.

Let $X^\wedge$ be the formal versal family on $H^\wedge$ defined by the identity $\text{Mor}(H^\wedge, H^\wedge)$ and consider the functor $\text{aut}_{X^\wedge}$ defined by:

$$\text{aut}_{X^\wedge}(S) = \{ \phi \in \text{Aut}_S(X^\wedge \otimes S) | \phi \otimes k = 1_X \} = \text{aut}_S(X^\wedge \otimes S).$$

**Theorem (1.1).** Assume $\text{dim}_k A^i$ is countable $i = 0, 1$. Then there exists a morphism of complete local $H^\wedge$-algebras

$$o_a : H^\wedge \otimes T^1 \rightarrow H^\wedge \otimes T^0$$

such that

(i) $o_a(m^1_{H^\wedge \otimes T^1}) \subseteq m^2_{H^\wedge \otimes T^1}$

(ii) $a^\wedge : (H^\wedge \otimes T^0) \otimes H^\wedge \otimes T^1 \otimes k \rightarrow H^\wedge \otimes T^1 \otimes k$

is a prorepresenting hull for the functor $\text{aut}_{X^\wedge}$. 
Proof. This follows from the proof of [Lal], (4.2.4) with \( \text{aut}_X^\wedge \) replacing \( \text{Def}_X \) and \( A_i^{-1} \) replacing \( A_i \), \( i = 1, 2 \). Q.E.D.

Recall that there is the usual automorphism functor of \( X \),

\[ \text{Aut}_X : \text{sch}/k \to \mathcal{G} \]

defined by:

\[ \text{Aut}_X(S) = \text{Aut}_S(X \otimes S) \]

Assume \( \text{Aut}_X \) is represented by the \( k \)-scheme \( \text{Aut}(X) \) and let \( 1 \in \text{Aut}(X) \) be the identity element. Then the completion \( O_{\text{Aut}(X)}^\wedge, 1 \)

of the local ring of \( \text{Aut}(X) \) at \( 1 \), represents the fiber-functor of \( \text{Aut}_X \) at \( 1 \in \text{Aut}_k(X) \), i.e. the functor

\[ \text{aut}_X : 1 + \mathcal{G} \]

defined by

\[ \text{aut}_X(S) = \{ \psi \in \text{Aut}_S(X \otimes S) | \psi \circ k = 1 \} = \text{aut}_S(X \otimes X) \]

Let \( a_X \) be the prorepresentable hull of \( \text{aut}_X \), such that with the assumption above

\[ a_X = O_{\text{Aut}(X)}^\wedge, 1 \]

Notice that if \( \text{Aut}_X \) is smooth, then \( a_X = \text{Sym}_X(\Lambda^0)^\wedge \) (see [Lal] Ch. 4).

Definition (1.2). Let the ideal \( \mathcal{O} \subseteq H^\wedge \) be generated by the coefficients of the elements of \( O_{\mathcal{A}}(m) \subseteq H \otimes_k T^0 \), \( m \) being the maximal ideal of \( H \otimes T^1 \). Then the prorepresenting substratum

\[ H^\wedge_0 \subseteq H^\wedge \]

is the formal sub-scheme defined by \( \mathcal{O} \).
Put \( H^\wedge_0 = H^\wedge /\mathfrak{m} \). Then \( H^\wedge_0 = \text{Spf}(H^\wedge_0) \) and we shall, mildly abusing the notations, also speak about the prorepresenting substratum \( H^\wedge \).

By construction of \( \mathfrak{o}_2 \) it is clear that \( H^\wedge_0 \) is the maximal quotient of \( H^\wedge \) for which

\[
\begin{align*}
\mathfrak{o}_2 \times H^\wedge_0 \\
\text{is } H^\wedge_0\text{-smooth.}
\end{align*}
\]

**Proposition (1.3).** \( H^\wedge_0 \) is the maximal quotient of \( H^\wedge \) for which the canonical morphism of functors on \( \mathfrak{m} \),

\[
\rho_0 : \text{Mor}(H^\wedge_0, -) \to \text{Def}_X
\]

is injective.

**Proof.** Let \( H^\wedge_1 \) be a quotient of \( H^\wedge \), and assume \( \psi_1, \psi_2 \in \text{Mor}(H^\wedge_1, R) \) are mapped onto the same element \( \bar{\psi}_1 = \bar{\psi}_2 \) in \( \text{Def}_X(R) \). This, of course, means that there exists an \( R \)-isomorphism \( X^\wedge \otimes R \cong X^\wedge \otimes R \)

where at the left side \( R \) is considered as \( H^\wedge \)-module via \( \psi_1 \) and at the right hand side \( R \) is considered as \( H^\wedge \)-module via \( \psi_2 \).

We may assume, by induction, \( \psi_1 \equiv \psi_2 \pmod{\mathfrak{n}} \) where \( \mathfrak{n} \) is some ideal of \( R \) killed by the maximal ideal \( \mathfrak{m}_R \). Then \( \phi \otimes R/\mathfrak{n} \) is an automorphism of \( X^\wedge \otimes R/\mathfrak{n} \), corresponding to a morphism \( \phi \otimes H^\wedge_1 \to R/\mathfrak{n} \). If \( a \otimes H^\wedge_1 \) is formally \( H^\wedge_1 \)-smooth, then obviously this morphism may be lifted to a morphism \( a \otimes H^\wedge_1 \to R \), proving that \( \phi \otimes R/\mathfrak{n} \) is liftable as an automorphism to some \( \phi : X^\wedge \otimes R = X^\wedge \otimes R \). But then

\[
\phi \circ \phi^{-1} : X^\wedge \otimes R \to X^\wedge \otimes R
\]

is an isomorphism extending the identity of \( \phi_1 \otimes \phi_2 \)

\[
X^\wedge \otimes R/\mathfrak{n} \). Thus \( \psi_1 = \psi_2 \). From this follows that \( \rho_0 : \text{Mor}(H^\wedge_0, -) \to \text{Def}_X \) is injective.
Conversally assume $H_1^\wedge$ is a quotient of $H^\wedge$ such that $\rho_1: \text{Mor}(H_1^\wedge,-) + \text{Def}_X$ is injective. If $R$, an object of $\ell_H$, is any $H_1^\wedge$-algebra, then any automorphism $\phi_n$ of $X^\wedge \otimes R/n = (X^\wedge \otimes H_1^\wedge) \otimes R/n$ may always be lifted to an automorphism of $X^\wedge \otimes R$. It follows that $a \otimes H_1^\wedge X^\wedge H_1^\wedge$ has to be formally smooth, which proves the proposition. Q.E.D.

Remark (1.4). Recall that $H/m^2$ represents the restriction of the deformation functor $\text{Def}_X$ to the subcategory $\ell_2 = \{R \in \ell | m^2 R = 0\}$ of $\ell$. Notice that, nevertheless, $H/m^2$ is rarely a quotient of $H_0^\wedge$, see §4 for lots of examples.

Consider for any $n > 0$ the subfunctors $\text{Def}_X^n$ of $\text{Def}_X$ defined by:

$$\text{Def}_X^n(R) = \{X_R \in \text{Def}_X(R) | A^n(R, X_R; O_{X_R}) \text{ is a deformation of } A^n(k, X; O_X)\}$$

Then one may prove that $\text{Def}_X^n$ has a prorepresentable hull $H(n)$ which is a quotient of $H^\wedge$.

Definition (1.5). The formal subscheme $\text{Spf } H(n)$ is called the $n$-th equicohomological substratum of $H^\wedge$, and is denoted by $H(n)$.

Proposition (1.6). The prorepresenting substratum $H_0^\wedge$ coincides with the $0$-th equicohomological substratum $H_0^\wedge$.

Proof. For any object $R$ of $\ell_H$, there exists a bijective map

$$\exp: \ker \{ A^0(R, X^\wedge \otimes R; O_X^\wedge \otimes R) + A^0(k, X; O_X) \} \rightarrow \text{aut}_R(X^\wedge \otimes R)$$

the inverse of which is log.

In fact, any element $\sigma$ of $\ker \{ A^0(R, X^\wedge \otimes R; O_X^\wedge \otimes R) + A^0(k, X; O_X) \}$
is a section of $\text{Der}_R(O_X \otimes_R; O_X \otimes_R)$ (resp. of $\text{End}_R(O_X \otimes_R; O_X \otimes_R)$ in the module case) mapping any local section $x$ of $O_X \otimes_R$ into $\mathbb{H} \cdot O_X \otimes_R$. Since for some $n$, $m^n \cdot O_X \otimes_R = 0$ exp $\sigma$ is defined. Given a homomorphism $\nu: \mathbb{H}(0) \to \mathbb{R}$, then the map

$$\eta^R_S: A^0(R, X^\otimes_R) \to A^0(S, X^\otimes_S; O_X \otimes_S)$$

induced by some surjective morphism $\pi: \mathbb{R} \to \mathbb{S}$ in $\mathbb{A}$ is surjective. This implies that the corresponding

$$\delta^R_S: \text{aut}_R(X^\otimes_R) \to \text{aut}_S(X^\otimes_S)$$

is surjective, thus by definition of $\mathbb{H}(0)$ there exists a unique morphism $\mathbb{H}(0) \to \mathbb{H}(0)$, such that the diagram

$$\begin{array}{ccc}
\mathbb{H} & \to & \mathbb{H}(0) \\
\downarrow & & \downarrow \nu \\
\mathbb{H}(0) & \mu & \to \mathbb{R}
\end{array}$$

commutes.

On the other hand, given a homomorphism $\mu: \mathbb{H}(0) \to \mathbb{R}$ we may compose it with the canonical homomorphism $\mathbb{R} \to \mathbb{R}[\varepsilon]$ to obtain a homomorphism $\mathbb{H}(0) \to \mathbb{R}[\varepsilon]$. By definition of $\mathbb{H}(0)$ it follows that for any surjective $\pi: \mathbb{R} \to \mathbb{S}$ in $\mathbb{A}$ the horizontal maps in the following diagram are surjective,

$$\begin{array}{ccc}
\text{aut}_R(\mathbb{X} \otimes_R) & \longrightarrow & \text{aut}_S(\mathbb{X} \otimes_S) \\
\downarrow \delta^R_S & & \downarrow \delta^S_S \\
\text{aut}_R(\mathbb{H} \otimes \mathbb{R}) & \longrightarrow & \text{aut}_S(\mathbb{H} \otimes \mathbb{S})
\end{array}$$
Since the vertical maps have sections, we find that
\[ A^0(R, X^\wedge \otimes R; O_{H^\wedge}^\wedge R) = \ker \delta_R^\wedge \mathbb{Z} \] maps surjectively onto \[ A^0(S, X^\wedge \otimes S; O_{H^\wedge}^\wedge S) = \ker \delta_S^\wedge \mathbb{Z}. \] By definition of \( H(0) \) there exists a unique morphism \( H(0) \to H^\wedge_0 \) such that the diagram
\[
\begin{array}{ccc}
H^\wedge & \to & H(0) \\
\downarrow & & \downarrow \nu \\
H^\wedge_0 & \to & \mu + R
\end{array}
\]
commutes. Consequently we find \( H^\wedge_0 = H(0) \). Q.E.D.

Remark (1.7). Let \( H^\wedge_{-i}(1) \) be the \( i \)-th equicohomological substratum of \( H^\wedge \), and consider the subfamily \( \mu(i): X^\wedge_{-i}(1) \to H^\wedge_{-i}(1) \) of \( \mu \). Let \( H^\wedge_{-i} \) be the intersection of the \( H^\wedge_{-i}(1) \)'s, and let \( X^\wedge_{-i} \) be the restriction of \( X^\wedge \) to \( H^\wedge_{-i} \). Then \( A^i(H^\wedge_{-i}, X^\wedge_{-i}; O_{X^\wedge_{-i}}) \) is \( H^\wedge_{-i} \)-flat for all \( i>0 \).

Suppose that \( A^i(H^\wedge_{-i}, X^\wedge_{-i}; O_{X^\wedge_{-i}}) \) is of finite type over \( H^\wedge_{-i} \). Then, in particular, \( A^i(H^\wedge_{-i}/m^n; X^\wedge_{-i}/m^n; O_{X^\wedge_{-i}}/m^n) \) is reflexive as an \( H^\wedge_{-i}/m^n \)-module, for all \( n>0 \).

Now assuming we have a flat family \( \eta: \check{Y} \to \text{Spec}(S) \) such that \( A^1(S, \check{Y}; O_{\check{Y}}) \) is reflexive as an \( S \)-module for \( i = 1, 2 \), there exist a morphism of complete \( S \)-algebras
\[
T^2_S = \text{Sym}_S(A^2_{\wedge})^\wedge \to T^1_S = \text{Sym}_S(A^1_{\wedge})^\wedge
\]
such that the \( S \)-algebra
\[
(T^1_S \otimes S) \otimes k(s)
\]
is the formal moduli of $\tilde{Y}(s) = \eta^{-1}(s)$ for all closed points $s \in \text{Spec}(S)$. The proof of this parallels the proof of [La] (4.4.2).
§2. AUTOMORPHISMS OF THE FORMAL MODULI

Introduction. The prorepresentable substraum constructed in §1 is a closed subproscheme $H_0^\wedge$ of $H^\wedge$. In this § we shall show that $H_0^\wedge$ is the fixed proscheme of $H^\wedge$ under the action of a subgroup functor $i_X$ of $\text{aut}_H^\wedge$ contained in the covering automorphism group functor of the morphism $\rho: \text{Mor}(H^\wedge, -) \rightarrow \text{Def}_X$.

Notice that this does not imply that $H_0^\wedge$ is the fixed proscheme of some natural subgroup of $\text{aut}_H^\wedge$, see (1.4).

As we shall see in §3, the group-functor $i_X$ does not, in general, extend to a group functor of automorphisms of an algebraization $H$ of $H^\wedge$. To remedy this we consider the Lie-algebra-functor $\lambda(\pi^\wedge)$ of $i_X$.

The main result of this § is then (2.5), where we, in particular, prove that $\lambda(\pi^\wedge)$ is an $H^\wedge$-submodule and a sub $k$-Lie-algebra of $\text{Der}_k(H^\wedge)$, such that $\lambda(\pi^\wedge) \otimes k = A^0(k, X; O_X)/A_0^\wedge_{\pi^\wedge}$ where $A_0^\wedge$ is the Lie-ideal in $A^0(k, X; O_X)$ of those infinitesimal automorphisms of $X$ that can be lifted to $X^\wedge$. Moreover $\lambda(\pi^\wedge) \otimes H_0^\wedge$ is an $H_0^\wedge$-Lie-algebra defining a deformation of the Lie-algebra $L(X) = A^0(k, X; O_X)/A_0^\wedge_{\pi^\wedge}$ to $H_0^\wedge$.

Consider any formal deformation of $X$,

$\pi: Y^\wedge \rightarrow S^\wedge = \text{Spf}(S^\wedge)$.

In particular we shall be interested in the formal versal family

$\pi^\wedge: X^\wedge \rightarrow H^\wedge = \text{Spf}(H^\wedge)$.

Let $\text{aut}^\wedge_{S^\wedge}$ be the subfunctor of $\text{Aut}^\wedge_{S^\wedge} \rightarrow \text{Sets}$ such that for
every object $R$ of $\mathcal{I}$, $\underline{\text{aut}}_{S^\wedge}(R)$ is the subset of those $\psi \in \text{Aut}_{S^\wedge}(R)$ for which the following diagram commutes

$$
\begin{array}{ccc}
S^\wedge \otimes_R \phi & \xrightarrow{\phi} & S^\wedge \otimes_R \phi \\
\downarrow & & \downarrow \\
R & \xrightarrow{id} & R
\end{array}
$$

Consider the subfunctors $I_\pi, i_\pi$ and $i_\pi^*$ of $\underline{\text{aut}}_{S^\wedge}$ defined by

$$
\begin{align*}
I_\pi(R) &= \{ \psi \in \text{aut}_{S^\wedge}(R) \mid Y^\wedge \otimes_R \phi = Y^\wedge \phi(S^\wedge \otimes R) \} \\
i_\pi(R) &= \{ \psi \in I_\pi(R) \mid x \otimes_k \phi = id_X \} \\
i_\pi^*(R) &= \{ \psi \in i_\pi(R) \mid x \otimes_k \phi = id_{X \otimes R} \}
\end{align*}
$$

where we have written $Y^\wedge \phi(S^\wedge \otimes R)$ for the pull-back of $Y^\wedge \times \text{Spec}(R)$ by the morphism

$$
\text{Spf}(\phi) : \text{Spf}(S^\wedge \otimes R) \to \text{Spf}(S^\wedge \otimes R).
$$

In particular, corresponding to the formal versal family, we put

$$
I_X = I_\pi, \quad i_X = i_\pi
$$

$$
I(X) = I_X(k) \quad i(X) = i_X(k).
$$

Notice that $i(X) = i_X(k) = i_\pi^*(k)$, by definition, consists of those auto-morphisms $\psi$ of $H^\wedge$ which leaves

$$
\rho : \text{Mor}_X(\hat{A}, \cdot) \to \text{Def}_X(\cdot)
$$

fixed, i.e. s.t. the diagram

$$
\begin{array}{ccc}
\text{Mor}_X(H^\wedge, \cdot) & \xrightarrow{\psi^*} & \text{Mor}_X(H^\wedge, \cdot) \\
\downarrow \rho & & \downarrow \rho \\
\text{Def}_X(\cdot)
\end{array}
$$
commutes. The group-functor $i_X^{-1}$ thus measures the extent of non prorepresentability of $\text{Def}_X$.

Recall, from §1, that if $\text{Aut}_X$ is smooth, then $\text{aut}_X$ is prorepresented by $T^0$.

Now it is easy to show that $\text{aut}_X$, restricted to $\mathfrak{a}$, is smooth, in all generality. Let us prove it when $X$ is a $k$-algebra.

Consider surjective morphisms $\rho: T \to R$ and $\eta: R \to S$ of $\mathfrak{a}$ such that $m_R \cdot \ker \eta = 0$.

Suppose $\theta_R \in \text{aut}_X(R)$ is such that $\theta_R \otimes S = \text{id} \circ \text{aut}_X(S)$. Then

$\theta_R = \text{id}_{X_R} + D$ where $D \in \text{Der}_k(X_X, X \otimes \ker \eta)$. We may write $D = \sum_{i=1}^m r_i D_i$, $r_i \in \ker \eta$, $D_i \in \text{Der}_k(X)$. Pick $t_i \in T$ such that $\rho(t_i) = r_i$ and consider the derivation $D' = \sum_{i=1}^m t_i D_i \in \text{Der}_k(X_X, X \otimes T)$. $D'$ defines in an obvious way a derivation $D' \in \text{Der}_T(X \otimes T)$. Let $\theta' = \exp D' = \text{id} + D' + \frac{1}{2!} D'^2 + \frac{1}{3!} D'^3 + \ldots$. Then $\theta'$ is an element of $\text{Aut}_T(X \otimes T)$ such that $\theta' \otimes R = \theta_R$. An easy induction argument then shows that $\text{aut}_X(T) \to \text{aut}(R)$ is surjective, thus $\text{aut}_X$ is smooth.

**Theorem (2.1).** There is a (non-canonical) morphism of the underlying set-theoretical functors

$$\phi: \text{aut}_X(H^\wedge \otimes -) \to \text{Aut}_{H^\wedge}$$

such that

(i) $\langle \text{im} \phi \rangle = i_X^{-1}$, as subfunctors, of $\text{Aut}_{H^\wedge}$.

(ii) $H^\wedge_0$ is the maximal quotient of $H^\wedge$ trivializing $i(X)$.

**Proof.** Consider the prorepresenting hull $\mathfrak{a}_X$, of the group functor $\text{aut}_X \wedge$, see §1. The identity $\text{id}: \mathfrak{a}_X \to \mathfrak{a}_X$ corresponds to the univer-
Let $\theta \in \text{aut}_{X^\wedge}(X^\wedge \otimes a_X) = \text{aut}_{X^\wedge}(a_X)$.

By $(1,1) a_X = H \otimes k T^0 / \alpha$, where $\alpha$ is an ideal contained in the square of the maximal ideal of $H \otimes k T^0$.

Consider now the trivial lifting $X^\wedge \otimes a_X$ to $H \otimes k T^0$ defined in terms of the quotient morphism $q: H \otimes k T^0 \to a_X$, i.e. the lifting corresponding to the canonical homomorphism

$$i: H^\wedge + H \otimes k T^0.$$ 

The automorphism $\theta$ of $X^\wedge \otimes a_X$ may be lifted to an isomorphism $\overline{\theta}$ making the following diagram commutative

$$\begin{array}{ccc}
X^\wedge \otimes k T^0 & \xrightarrow{\overline{\theta}} & X^\wedge \otimes (H \otimes k T^0) \\
\uparrow & & \uparrow \\
X^\wedge \otimes H^\wedge a_X & \xrightarrow{\theta} & X^\wedge \otimes H^\wedge a_X
\end{array}$$

where $\phi: H^\wedge + H \otimes k T^0$ is some homomorphism such that $\phi \circ q = \text{id}_R$.

By definition $H^\wedge_0$ is the maximal quotient of $H^\wedge$ for which

$$i \otimes 1^\wedge = \phi \otimes 1^\wedge.$$ 

Consider the map which associates to every $\alpha \in \text{Mor}_k(T^0, H \otimes k R) = \text{Mor}_R(T^0 \otimes k R, H \otimes k R)$ the composition

$$\phi(\alpha): H \otimes k R \xrightarrow{\phi \otimes 1_R} H \otimes k T^0 \otimes k R \xrightarrow{1_H \otimes \alpha} H \otimes k R.$$ 

Since $1 \otimes \alpha$ maps $\alpha$ into the square of the maximal ideal of $H \otimes k R$, it follows from $\phi \circ q = \text{id}_R$ that $\phi(\alpha)$ reduces to the identity on the tangent level. Therefore $\phi(\alpha) \in \text{Aut}_R(H \otimes k R)$, and we
obtain a map

\[ \phi_R : \text{aut}_X(H^\otimes R) \to \text{Aut}_{\text{H}^\otimes R}(R) \]

which, as one easily checks, is functorial. Furthermore, by construction,

\[ \delta \otimes (H \otimes R) : X^\otimes R \to X^\otimes H \otimes R \]

\[ 1_{H \otimes R} \quad \phi(\alpha) \]

is an \( H \otimes R \)-isomorphism, so we know \( \phi(\alpha) \in \text{i}_X(R) \). We need only prove that \( \text{i}_X(R) \) is generated by \( \phi_R(\text{aut}_X(H \otimes R)) \). The rest is clear. Therefore the proof is reduced to proving the next Proposition.

Q.E.D.

**Proposition (2.2).**

(i) Let \( \delta \in \text{Mor}(H^\otimes S) \) correspond to the deformation \( X_S \in \text{Def}_X(S) \).

Then for every \( \psi \in i(X) \), the morphism \( \phi \circ \delta \in \text{Mor}(H, S) \) corresponds to the same deformation \( X_S \).

(ii) If the surjections \( \rho_i : H \otimes R \twoheadrightarrow S, i=1,2, \) correspond to the same deformation \( X_S \), then there exists a sequence of automorphism \( \alpha_n \in \text{aut}_X(H \otimes R) \) such that \( \rho_2 = \lim_{n \to \infty} \phi(\alpha_n) \circ \phi(\alpha_{n-1}) \circ \cdots \circ \phi(\alpha_2) \circ \rho_1 \).

**Proof.** (i) is obvious. To prove (ii) consider the morphisms \( \rho_i^2 : H \otimes R \twoheadrightarrow S/m^2 = S_2 \). By assumption, we have a commutative diagram

\[
\begin{array}{ccc}
(X^\otimes_k R) \otimes S & \xrightarrow{\tau} & (X^\otimes_k R) \otimes S \\
\downarrow & & \downarrow \\
(X^\otimes_k R) \otimes S_2 & \xrightarrow{\tau_2} & (X^\otimes_k R) \otimes S_2 \\
\downarrow \rho_1^2 & & \downarrow \rho_2^2 \\
X & = & X
\end{array}
\]

where \( \tau \) and \( \tau_2 \) are isomorphisms.
Since $H_2$ represents the deformation functor $\text{Def}_X$ restricted to the subcategory $L_2$ of $L$, $\rho_1^2 = \rho_2^2$ and $\tau_2$ is therefore an automorphism. As such it corresponds to a morphism $\mu_2: a_X \to S_2$, which, composed with the canonical morphism $H \hat{\otimes} T^0 \to a_X$, gives us an $H^\wedge$-morphism $\bar{\mu}_2: H \hat{\otimes} T^0 \to S_2$. Lift $\bar{\mu}_2$ to an $H^\wedge$-morphism $\mu_2: H \hat{\otimes} T^0 + H \hat{\otimes} R$, and consider the composition

$$\phi_2: H \hat{\otimes} R \to H \hat{\otimes} T^0 \otimes R \to H \hat{\otimes} R$$

By construction there is a commutative diagram

\[
\begin{array}{c}
\overline{\delta \otimes (H \hat{\otimes} R)} \\
X \hat{\otimes} R \to (X \hat{\otimes} R) \otimes (H \hat{\otimes} R) \\
k \phi_2 \end{array}
\]

\[
\begin{array}{c}
\downarrow \downarrow \\
(X \hat{\otimes} R) \otimes S_2 \ni (X \hat{\otimes} R) \otimes S_2 \\
\rho_1^2 \tau_2 \rho_1^2
\end{array}
\]

Now, consider $\rho_1' = \phi_2 \circ \rho_1$. Then, put $S_3 = S/m^3$, and consider the commutative diagram

\[
\begin{array}{c}
(X \hat{\otimes} R) \otimes S \ni (X \hat{\otimes} R) \otimes S \\
\rho_1' \tau_1' \rho_2
\end{array}
\]

\[
\begin{array}{c}
\downarrow \downarrow \\
(X \hat{\otimes} R) \otimes S_3 \ni (X \hat{\otimes} R) \otimes S_3 \\
\rho_1^2 \tau_3 \rho_2^2
\end{array}
\]

\[
\begin{array}{c}
\downarrow \downarrow \\
(X \hat{\otimes} R) \otimes S_2 = (X \hat{\otimes} R) \otimes S_2 \\
\rho_1^2 \rho_2^2
\end{array}
\]

It follows from the commutativity of the lower square that $\rho_1'^3 = \rho_2^3$, therefore that $\tau_3$ is an automorphism.

Now, copy the procedure above, get $\alpha_3: H \hat{\otimes} T^0 + H \hat{\otimes} R$ such that
if \( \rho_1^" = \phi(\alpha_3) \circ \rho_1 \) then \( \rho_1^{"4} = \rho_2^4 \) etc. See now that the corresponding \( \alpha_n \in \text{aut}_X(H \hat{\otimes} R) \) \( n \geq 2 \), have exactly the properties of (ii).

Q.E.D.

There is an obvious homomorphism of group functors

\[
\sigma: \text{Aut}_X \rightarrow \text{I}_X/I_X^X
\]

In fact, to each automorphism \( \alpha \in \text{Aut}_X(R) = \text{Aut}_R(X \otimes R) \) there exists an automorphism \( \phi(\alpha) \in \text{I}_X(R) \) such that the following diagram commutes

\[
\begin{array}{ccc}
X^\otimes R & \overset{\sim}{\rightarrow} & (X^\otimes R) \otimes (H \hat{\otimes} R) \\
\downarrow & & \downarrow \phi(\alpha) \\
X \otimes R & \overset{\sim}{\rightarrow} & X \otimes R
\end{array}
\]

just as above. It is clear that the class of \( \phi(\alpha) \) in \( \text{I}_X(R)/I_X^X(R) \) is unique, and one checks easily that the map \( \alpha \rightarrow \sigma(\alpha) = \text{class of } \phi(\alpha) \) is a group homomorphism.

Let \( \text{Aut}_X^1 \) (resp. \( \text{aut}_X^1 \)) be the subgroup functor of \( \text{Aut}_X \) (resp. \( \text{aut}_X \)) consisting for each \( R \) of those automorphisms of \( X \otimes R \) that lifts to \( X^{\otimes \hat{\otimes}} R \).

With these notations we have the following:

\textbf{Corollary (2.3).} (i) There is a canonical action of \( \text{I}(X)/I(X) \) on \( H_0 \). (ii) \( \text{Aut}_X/I_X^X = \text{I}_X/I_X^X \) as group functors.

\textbf{Proof.} By (2.1) \( H_0^\wedge \) is the maximal quotient of \( H^\wedge \) such that for all \( R \) in \( \text{I} \) \( H_0^\wedge \otimes R \) is a quotient of \( H^\wedge \otimes R/\{h-ih | h \in H^\wedge \otimes R, i \in \text{I}_X(R) \} \).

Let \( i' \in \text{I}_X(R) \). Since \( i_X(R) \) is normal in \( \text{I}_X(R) \), \( i'(h-ih) = i'h-i'ih = (i'h)-(i'h) \) where \( i \in \text{I}_X(R) \) and where \( j \in \text{I}_X(R) \) is defined by \( i'i = ji' \). Thus \( \text{I}(X) \), which is contained in the group functor \( \text{I}_X \), operates on \( H_0 \). The rest is clear. Q.E.D.
Remark (2.4). (i) Notice that although $\text{aut}_X^\wedge$ is smooth on $H_0^\wedge$, the group functor $\text{Aut}_X^\wedge$ is not in general, smooth. An example is furnished by any hyperelliptic curve $X$ of genus $> 3$. In this case we have $H = H_0$ but the involution is never liftable to $H^\wedge/m^2$, see [La-Lø].

(ii) $I(X)/i(X)$ does not, in general, operate effectively on $H_0$. In fact if $X = \text{Spec}(k[x]/(f))$ where $f$ is quasihomogeneous, the torus action $\tau \in \text{Aut}_k(X)$ is not, in general, liftable to $X^\wedge$, but $\tau$ operates trivially on $H^\wedge$. (iii) The subgroup of $\text{Aut}_k(H_0^\wedge)$ consisting of those $\phi$ for which there exists an isomorphism $\chi_\phi: X_0^\wedge = X_0^\wedge \otimes_{H_0} H_0$, is a quotient of $I(X)/i(X)$. In fact let $\phi$ be any such automorphism and consider the restriction of $\chi_\phi$ to the special fiber, $\phi: X \to X_0$. Consider further any representative $\sigma(\phi^{-1}) \in I(X)$ of $\sigma(\phi^{-1}) \in I(X)/i(X)$. Let $\psi'$ be the restriction of $\sigma(\phi^{-1})$ to $H_0^\wedge$. Then there is an isomorphism $\tau_\psi: X_0^\wedge \to X_0^\wedge \otimes_{H_0} H_0$ and the composition

$$X_0^\wedge \xrightarrow{\chi_\phi} X_0^\wedge \otimes_{H_0} H_0 \xrightarrow{\chi_\phi \otimes 1_{H_0}} (X_0^\wedge \otimes_{H_0} H_0) \otimes_{H_0} H_0$$

reduces to the identity on the special fiber. It follows that $\phi \psi' \in \text{Aut}_k(H_0^\wedge)$ conserves the universal family $X_0^\wedge$, thus $\phi \psi' = 1_{H_0}$. But then $\phi$ is induced by $\sigma(\phi) \in I(X)/i(X)$.

(iv) The action of $I(X)$ on $H^\wedge$ induces a linear action on the tangent space $A^1(k, X; O_X)$. Since $i(X)$ acts trivially on the tangent space of $H^\wedge$, we obtain a linear action of $I(X)/i(X)$ on $A^1(k, X; O_X)$. This action is well understood and has been used in many instances, see e.g. [La-Lø]. It is, in view of (2.3) (ii), given in terms of the action

$$\sum: \text{Aut}(X) \to GL(A^1(k, X; O_X))$$

defined by $\sum(\phi) = \phi^{-1} \circ \phi$. 
The above picture is better understood if we look at it at the Lie-algebra level.

Given a functor $F$ of groups on $\mathcal{A}$. Recall the definition of the Lie-algebra $\operatorname{lie}_F$, see [D-G],

$$\operatorname{lie}_F(R) = \ker \{ F(R) \otimes k[\varepsilon] \to F(R) \}.$$

Notice that

$$\operatorname{lie}_{\operatorname{aut}}(k) = \operatorname{Der}_*(S^\wedge)$$

and that by definition

$$\operatorname{lie}_*= (k) = \{ D \in \operatorname{Der}_*(S^\wedge) | \forall \varepsilon \in k[\varepsilon] \exists (Y^\wedge \otimes k[\varepsilon]) \circ (S^\wedge \otimes k[\varepsilon]) $$

where $\phi = \operatorname{id} + \varepsilon \cdot D$ and $\chi_k[\varepsilon] \cdot k = \operatorname{id}_{Y^\wedge}.$

As a short-hand we shall write

$$\lambda(\pi) = \operatorname{lie}_*(k).$$

This notion has the advantage, that it is readily relativized, and that it functions well with respect to functorality. In fact, consider together with the formal family

$$\pi: Y \to \operatorname{Spf}(S^\wedge) = S$$

a morphism of complete local $k$-algebras

$$\rho: S^\wedge \to T^\wedge.$$ 

Let

$$\pi': Y' \to \operatorname{Spf}(T^\wedge) = T$$

be the pull-back, and put

$$\lambda(\pi, \rho) = \{ D \in \operatorname{Der}_*(S^\wedge, T^\wedge) | \forall \varepsilon \in k[\varepsilon] \exists (Y' \otimes k[\varepsilon]) \circ (T^\wedge \otimes k[\varepsilon]) $$

where $\phi = \rho + \varepsilon \cdot D$ and $\chi_k[\varepsilon] \cdot k = \operatorname{id}_{Y'}.$
Obviously \( l(\pi, \rho) \) is the value at \( k \) of a functor \( l(\pi, \rho) \) that the reader may want to explicite. Now the restrictions of the natural morphisms

\[
\begin{array}{ccc}
\text{Der}_* (S^*, S^*) & \to & \text{Der}_* (T^*, T^*) \\
\downarrow & & \downarrow \\
\text{Der}_* (S^*, T^*) & \to & \text{Der}_* (T^*, T^*)
\end{array}
\]

defines maps,

\[
l(\pi) \quad l(\pi') \\
\downarrow \downarrow \\
l(\pi, \rho).
\]

Before we state the main result of this §, let us put as another short-hand

\[
A^0_\pi = \text{im} \{ A^0_\pi (T', Y; O_Y), + A^0 (k, X; O_X) \},
\]

and let us recall the canonical action

\[
\otimes: \text{Aut}_X + \frac{\text{Aut}(\pi)}{\text{Aut}_X} + \mathfrak{gl}(A^1 (k, X; O_X))
\]

and the isomorphism \( \frac{\text{Aut}_X}{\text{Aut}_X} = I_X/\tilde{I}_X^* \), see (2.3) (ii) and (2.4) (iv), above noticing that \( A^1 (k, X; O_X) \otimes_R A^1 (R, X \otimes_R O_X \otimes_R) \). One checks that \( \text{lie} \frac{\text{Aut}_X}{\text{Aut}_X} (k) = A^0_\pi \) is a Lie-ideal of \( \text{lie} \text{Aut}_X (k) = A^0 (k, X; O_X) \) and that we therefore obtain a morphism of Lie-algebras \( \sigma: A^0_\pi (k, X; O_X) \to \End (A^1 (k, X; O_X)) \) that factors via

\[
A^0_\pi (k, X; O_X)/A^0_\pi + \text{lie} (I_X/\tilde{I}_X^*) (k).
\]

Lemma (2.5). \( \text{lie} I_X = \text{lie} \tilde{I}_X \).
Proof. Let $\delta \in \text{Lie } \mathfrak{l}_\mathcal{X}(R)$, then $\delta \in \text{Lie } \text{aut } \mathcal{H}_A^*(R)$. Since $\text{Lie } \text{aut } \mathcal{H}_A^*(R) = \text{Der}_R(H^*_\mathcal{X} R_\mathcal{X})$, $\delta$ corresponds to a $D \in \text{Der}_R(H^*_\mathcal{X} R_\mathcal{X})$ such that $\delta = \text{id} + \varepsilon \cdot D \in \text{aut } R[\varepsilon](H^*_\mathcal{X} R_\mathcal{X})$. Moreover there exist isomorphisms $\omega$ and $\chi'$ such that

$$X^\wedge \otimes_k R[\varepsilon] \xrightarrow{\chi} (X^\wedge \otimes_k R[\varepsilon]) \otimes_\delta (H^*_\mathcal{X} R_\mathcal{X})$$

commutes. Let $\chi = (\omega^{-1} \otimes \text{id}_k[\varepsilon]) \chi'$ and notice that the following diagram is commutative

$$\begin{array}{ccc}
X^\wedge \otimes_k R[\varepsilon] & \xrightarrow{\chi} & (X^\wedge \otimes_k R[\varepsilon]) \otimes_\delta (H^*_\mathcal{X} R_\mathcal{X}) \\
\downarrow \hspace{2cm} \downarrow \hspace{2cm} \downarrow \\
X^\wedge \otimes_k R
\end{array}$$

This shows that $\delta \in \text{Lie } \mathfrak{l}_\mathcal{X}(R)$ and we are through. Q.E.D.

From this it follows that there exists an action

$$\sigma_0 : \text{End}(\pi^\wedge) \rightarrow \text{End}(A^1(k, X; O_X))$$

which we shall use extensively.

Now let us prove the main result of this §.

**Theorem (2.6).** (i) $\lambda(\pi, \rho) = \{D \in \text{Der}_R(S^\wedge, T^\wedge) \mid \exists \ E \in A^0(k, Y, O_Y, ) \text{ st.} \\
\text{for any local section } y \text{ of } O_Y \text{ and any } s \in S^\wedge, \\
E(sy) = \rho(s) \cdot E(y) + D(s) \cdot y\}$. 

(ii) \( \lambda(\pi,\rho) \) is a \( T^\wedge \)-sub-module of \( \text{Der}_*(S^\wedge, T^\wedge) \).

(ii') If \( \rho = \text{id}_{S^\wedge} \) then \( \lambda(\pi) \) is a sub \( k \)-Lie-algebra of \( \text{Der}_*(S^\wedge) \).

(iii) There is a \( T^\wedge \to k \) semi-linear map

\[
\lambda: \lambda(\pi,\rho) \to A^0(k, X; O_X)/A^0_{\pi},
\]

which to any \( D\in \lambda(\pi,\rho) \) associates the class of \( E\otimes_k A^0(k, X; O_X) \).

(iv) Suppose \( \pi = \pi^\wedge \), i.e. \( S^\wedge = H^\wedge \) and \( Y = X^\wedge \), then \( \lambda \) induces an isomorphism

\[
\lambda(\pi,\rho) \otimes_k = A^0(k, X; O_X)/A^0_{\pi},
\]

(v) \( H^\wedge_0 \) is the maximal quotient of \( H^\wedge \) trivializing \( \lambda(\pi) \).

Moreover \( \lambda(\pi) \otimes H^\wedge_0 \) is an \( H^\wedge_0 \)-Lie algebra and a deformation of the Lie-algebra \( L(X) = A^0(k, X; O_X)/A^0_{\pi} \).

**Proof.** Let \( D\in \lambda(\pi,\rho) \) then \( \phi = \rho + \varepsilon \cdot D \) is a morphism of complete local \( k \)-algebras. Put \( S = S^\wedge \) and \( T = T^\wedge \) then

\[
\phi: S \to T \otimes k[\varepsilon] = T[\varepsilon]
\]

is such that there exists an \( T \otimes k[\varepsilon] \)-isomorphism

\[
\chi_D: Y \otimes T[\varepsilon] \cong Y \otimes T[\varepsilon]
\]

lifting the identity on \( Y \otimes T \).

There are commutative diagrams of morphisms of \( k \)-schemes,
The difference $E' = \phi' \circ \chi_{D'} - \rho'$ corresponds to an element $E$ of $A^0(k, X, O_X \otimes T)$. If $y$ is a local section of $O_Y$ and $s \in S$, then $\rho_{\varepsilon^*}$ maps $s y$ to $\rho(s) y$ in $O_Y \otimes T[\varepsilon]$, and $\phi^*_{\varepsilon^*}$ maps $s y$ to $(\rho(s) + \varepsilon \cdot D(s)) y$ in $O_Y \otimes T[\varepsilon]$. Since $\chi_D$ is the identity on $Y \otimes T$, $\chi_{D^*}(\rho(s) + \varepsilon \cdot D(s)) y = (\rho(s) + \varepsilon \cdot D(s))(y + \varepsilon E(y))$. Thus $E^*(s y) = \varepsilon(\rho(s) E(y) + D(s) y)$ and consequently

(*) $E(s y) = \rho(s) E(y) + D(s) y$.

If on the other hand $E \in A^0(k, X, O_X \otimes T)$ is such that (*) holds, for some derivation $D \in \text{Der}_k(S, T)$ then for $\phi = \rho + \varepsilon D$ and $\chi_{D^*}(\varepsilon y \otimes 1) = y + \varepsilon E(y)$, we find that $\chi_D$ is an isomorphism between $Y \otimes T[\varepsilon]$ and $Y \otimes T[\varepsilon]$ lifting the identity on $Y \otimes T$, therefore $D \in \mathfrak{l}(\pi, \rho)$, and we have proved (i). Since $A^0(k, X, O_X \otimes T)$ is a $T$-module (ii) follows. If $E_i \in A^0(k, Y, O_Y)$ correspond to $D_i$ as in (i), for $i = 1, 2$, then one checks that

$$[E_1, E_2](s y) = s[E_1, E_2](y) + [D_1, D_2](s) y$$

therefore $\mathfrak{l}(\pi)$ is a sub Lie-algebra of $\text{Der}_k(S, S)$ and (ii') follows. (iii) is a consequence of (i), as $\chi_{D^*} \otimes k = \chi = \text{id} + \varepsilon \lambda(D)$, compare the right hand diagram above. Now, for the proof of (iv), we first notice that $\lambda$ is surjective. In fact if $E \in A^0(k, X, O_X)$, then $\chi = \text{id}_X + \varepsilon \tilde{E} \in \text{Aut}_k[\varepsilon](X \otimes k[\varepsilon])$. As above (see (2.3) (ii)), there is a $\tilde{\phi} : H \otimes k[\varepsilon] \to H$ such that $X^\wedge \otimes k[\varepsilon]$ and $X^\wedge \otimes k[\varepsilon]$ are isomorphic, the isomorphism lifting $\text{id}_X + \varepsilon \tilde{E}$. Tensorise with $T$ and obtain $\phi : H \otimes T[\varepsilon]$ such that $X^\wedge \otimes T[\varepsilon]$ and $X^\wedge \otimes T[\varepsilon]$ are isomorphic, the isomorphism $H \otimes T[\varepsilon]$ lifting $\text{id}_X + \varepsilon \tilde{E}$. Since $\text{id}_X + \varepsilon \tilde{E} \in \text{ker}(\text{Aut}_k[\varepsilon](X \otimes k[\varepsilon]) \cdot \text{Aut}_k(X))$ we may arrange $\tilde{\phi}$ such that $\tilde{\phi} = \text{id} + \varepsilon \tilde{D}$, with $\tilde{D} \in \text{Der}_k(H)$. In fact the composition $\theta : \tilde{H} \otimes H[\varepsilon] \otimes H$ is such that $X^\wedge \otimes H$ and $X^\wedge \otimes H$ are isomor-
phic, the isomorphism lifting the identity on $X$. Then $\theta$ is necessarily an automorphism. Consider the composition $\tilde{\phi} : H^0 + H\rightarrow H[\varepsilon]$ and see that $\tilde{\phi}$ has the required property. But then $\phi = \rho + \varepsilon \cdot D$, with $D \in \lambda(\pi, \rho)$, and we have proved that $\lambda$ is surjective.

To complete the proof of (iv) we shall prove that for any basis $\{E_i\}_{i=1}^N$ of $A^0/A_0$, with $E_i = \lambda(D_i)$, and $D_i \in \lambda(\pi^\wedge, \rho)$ corresponding to $E_i \in A^0(k, X^\wedge; 0_X \wedge \Omega_T)$, the $D_i$'s generate $\lambda(\pi^\wedge, \rho)$ as a $T$-module.

Now pick any such basis $\{E_i\}_{i=1}^N$ and corresponding $D_i$'s, and $E_i$'s. Let $h_i \in T$, then $\overline{h_i D_i} \in \lambda(\pi^\wedge, \rho)$ corresponds to $\overline{h_i E_i}$.

Consequently we need only prove that for any $D \in \lambda(\pi^\wedge, \rho)$, the corresponding $E$ is a sum of the form $\sum_i h_i E_i$, modulo $A^0(H, X^\wedge; 0_X \wedge \Omega_T)$.

But this is easily achieved. In fact, the image of $E$ in $A^0(k, X, 0_X)$ can be written as $\sum_{i=1}^N h_i^0 E_i + E(0)$, $E(0) \in A^0$. 

Let $E(0) \in A^0(H, X^\wedge; 0_X \wedge \Omega_T)$ be the preimage of $\overline{E(0)}$, then $D$ corresponds to $E - E(0)$ as well, and $\overline{E - E(0)} = \sum_{i=1}^N h_i^0 E_i$. Therefore $E - E(0) - \sum_{i=1}^N h_i E_i$ maps to zero in $A^0(k, X, 0_X)$, hence also in $A^0(k, X^\wedge; 0_X)$. Notice that this is a consequence of the fact that $D$ and the $D_i$'s map the maximal ideal $m \subset H^\wedge$ into the maximal ideal $n \subset T$.

Consider the exact sequence

$$0 \rightarrow A^0(k, X^\wedge; 0_X \wedge \Omega_{H^\wedge n/n^2}) \rightarrow A^0(k, X^\wedge; 0_X \wedge \Omega_{H^\wedge T/n^2}) \rightarrow A^0(k, X^\wedge; 0_X \wedge \Omega_{H^\wedge}, k)$$

Obviously:

$$A^0(k, X^\wedge; 0_X \wedge \Omega_{H^\wedge n/n^2}) = A^0(k, X^\wedge; 0_X) \otimes_{\mathcal{O}} n/n^2$$

$$A^0(k, X^\wedge; 0_X \wedge \Omega_{H^\wedge k}) = A^0(k, X^\wedge; 0_X).$$
Therefore the image of \((E-E(0)-\sum_{i=1}^{N} h_i E_i)\) in \(A^0(k,X^\wedge;0_X^\wedge\otimes T/n^2)\) sits in \(A^0(k,X^\wedge;0_X^\wedge\otimes H^\wedge n/n^2)\) and in fact in the sub vector space \(A^0(k,X;0_X^\wedge\otimes n/n^2)\). There must exist \(h_1^n \in T\) such that the image of \((E-E(0)-\sum_{i=1}^{N} h_i E_i)\) in \(A^0(k,X^\wedge;0_X^\wedge\otimes T/n^2)\) sits in \(A^0(k,X;0_X^\wedge\otimes n/n^2)\). Thus there exists \(E(1) \in \mathfrak{n}_H A^0(H^\wedge,X^\wedge;0_X^\wedge\otimes T)\) such that \(E-E(0)+E(1) - \sum_{i=1}^{N} (h_i^1 + h_i) E_i\) maps to zero in \(A^0(k,X^\wedge;0_X^\wedge\otimes T/n^2)\).

Continue, considering the exact sequences induced by \(0 \to \mathfrak{n}_p / \mathfrak{n}_{p+1} \to T/\mathfrak{n}_{p+1} \to T/\mathfrak{n}_p \to 0\), we obtain \(E = \sum_{i=1}^{N} h_i E_i + \sum_{j=0}^{m} E(j)\) where \(h_i \in \mathfrak{t}\), \(E(j) \in A^0(H^\wedge,X^\wedge;0_X^\wedge\otimes T)\) and \(\sum_{j=0}^{m} E(j)\) converges to an element of \(A^0(H^\wedge,X^\wedge;0_X^\wedge\otimes T)\).

Now, to prove (v), notice that it follows from (iv), with \(\rho: H^\wedge \to H_0^\wedge\), and from (1.6) that \(H_0^\wedge\) is the maximal quotient of \(H^\wedge\) trivializing \(l(\pi^\wedge)\). Therefore if \(H_0^\wedge = H^\wedge / \mathfrak{a}\) any \(\delta \in l(\pi^\wedge)\) maps \(H^\wedge\) into \(\mathfrak{a}\).

Moreover, if \(k \in H, \delta_1, \delta_2 \in l(\pi^\wedge)\) then \([\delta_1, k \delta_2] = k[\delta_1, \delta_2] + \delta_1(k) \delta_2\). Consequently \(\mathfrak{a} \cdot l(\pi^\wedge)\) is a Lie-ideal of \(l(\pi^\wedge)\) and \(l(\pi^\wedge) \otimes H_0^\wedge\) is an \(H_0^\wedge\)-Lie-algebra. Q.E.D.

Remark (2.7). In particular we have proved that if \(\dim_k A^0 / A^0_{\pi^\wedge} \leq m\), then \(l(\pi)\) is of finite type as \(H^\wedge\)-module and vice versa.

Moreover the rank of \(l(\pi)\) is bounded by \(\dim_k (A^0 / A^0_{\pi^\wedge})\).

For every \(D \in l(\pi)\), \(D\) is a derivation of \(H^\wedge\) mapping the maximal ideal \(\mathfrak{m}\) into itself. Therefore \(D\) induces an endomorphism of the tangent space of \(H^\wedge\), \(D_*: A^1(k,X;0_X^\wedge) \to A^1(k,X;0_X^\wedge)\). Because of (2.6) (iv) the representation \(D + D_*\),

\(l(\pi) \to \text{End}(A^1(k,X;0_X^\wedge))\)

factors via \(A^0(k,X;0_X^\wedge)/A^0_{\pi^\wedge}\). In particular, we find a repre-
sentation, of Lie-algebras

\[ \rho : A^0(k, X; O_X) \rightarrow \text{End}(A^1(k, X; O_X)) \]

which we shall make explicite. Put for \( \tilde{E} \in A^0(k, X; O_X) \), \( \tilde{E}_x = \rho(\tilde{E}) \).

Notice first that any element \( \xi \in A^1(k, X; O_X) \) may be considered as a morphism of complete local \( k \)-algebras

\[ \xi : H^\wedge + k[\eta], \eta^2 = 0. \]

Consider

\[ l(\pi^\wedge, \xi) = \{ D \in \text{Der}_*(H^\wedge, k[\eta]) | X^\wedge \otimes k[\eta] \otimes k[\varepsilon] \otimes k[\eta, \varepsilon], \psi = 1 + \varepsilon \cdot D, \chi \otimes k = \text{id} \}. \]

By (2.2) (ii) there exists an element \( \alpha \in \text{Lie}_X(k[\varepsilon]) \) such that

\[ (\xi \otimes \text{id}_k[\varepsilon]) \alpha = \xi + \varepsilon \cdot D. \]

Since \( \alpha = \text{id} + \varepsilon \cdot E \in \text{Lie}_i(X) \) we obtain

\[ D = \xi \circ E. \]

On the other hand if \( E \in \text{Lie}_i(X) \) then certainly

\[ \xi \circ E \in (\pi^\wedge, \xi). \]

Since \( \eta \cdot \text{Der}_*(H^\wedge, k[\eta]) = 0, (2.6) (iv) implies the following

Corollary (2.8). The orbit of \( \xi \) under \( l(\pi) \) is equal to

\[ l(\pi^\wedge, \xi) = A^0(k, X; O_X)/A^0_{\xi} \]

where \( A^0_{\xi} \) is the subspace of \( A^0(k, X; O_X) \) of those elements

that lift to the family \( X_\xi \) on \( k[\eta] \).

More explicitly, we have the following.

Proposition (2.9). Let \( \tilde{E} \in A^0(k, X; O_X) \), then \( \tilde{E}_x \in \text{End}(A^1) \) is defined in the following way:

Given \( \xi \in A^1(k, X; O_X) \), let \( X_\xi \) be the corresponding lifting of \( X \)
to \( k[\eta] \), and consider the lifting situation; \( \varepsilon^2 = \eta^2 = 0 \).
Let $o(id+\varepsilon E)$ be the obstruction for lifting the automorphism $id+\varepsilon E$ as an automorphism of $X_\varepsilon \otimes k[\varepsilon]$, then $o(id+\varepsilon E) \in A^1(k[\varepsilon], X\otimes k[\varepsilon]; 0_X \otimes k[\varepsilon]) = A^1(k, X; 0_X) \otimes k[\varepsilon]$, has the form $\varepsilon \cdot E_x(\xi)$.

Proof. Let $D\in A^1(\pi)$ correspond to $E\in A^0(k, X^\wedge; 0_X^\wedge)$ as in (2.6) (i) and consider $E \in A^0(k, X; 0_X)$, $E = E \otimes k$. $D$ operates on $A^1(k, X; 0_X)$ via the automorphism $\delta = id+\varepsilon \cdot D \in Aut_{k[\varepsilon]}(H^\wedge \otimes k[\varepsilon])$. By construction, putting $\chi_\delta = id+\varepsilon \cdot E$ the following diagram commutes

\[
\begin{array}{ccc}
X^\wedge \otimes k[\varepsilon] & \xrightarrow{\chi_\delta} & (X^\wedge \otimes k[\varepsilon]) \otimes (H^\wedge \otimes k[\varepsilon]) \\

\uparrow & & \uparrow \\
X \otimes k[\varepsilon] & \xrightarrow{id+\varepsilon \cdot E} & X \otimes k[\varepsilon]
\end{array}
\]

Reduce the diagram modulo $m^2_H$ and obtain

\[
\begin{array}{ccc}
X_2^\wedge \otimes k[\varepsilon] & \xrightarrow{\chi_{\delta_2}} & (X_2^\wedge \otimes k[\varepsilon]) \otimes (H_2 \otimes k[\varepsilon]) \\

\uparrow & & \uparrow \\
X \otimes k[\varepsilon] & \xrightarrow{id+\varepsilon E} & X \otimes k[\varepsilon]
\end{array}
\]

where $\delta_2 = id + \varepsilon \cdot D_2$ and $D_2$ is the linearization of $D$, i.e. the derivation on $H_2 = H^\wedge / m^2$ induced by $D$. Notice that since $m/m^2 = A^1(k, X; 0_X)^\wedge$, $\xi$ is a linear form on $m/m^2$ inducing a $k$-algebra homomorphism $\xi : H_2 \to \kappa[\eta]$. Consider the homomorphism
The composition \( \delta_2 \circ (\xi \otimes k[x]) : H_2 \otimes k[x] + k(\eta) \otimes k[x] \) is by definition of \( E_\ast \), \( \xi \otimes k[x] + \varepsilon \cdot E_\ast \circ (\xi \otimes 1_k) \) where \( E_\ast (\xi) \in A^1(k,x;0_X) \) is considered as an algebra homomorphism \( E_\ast : H_2 \rightarrow k[\eta] \), as explained above.

But then \( (X^\ast \otimes k[x]) \circ (H_2 \otimes k[x]) \otimes (k[\eta] \otimes k[x]) \) is the lifting \( \delta_2 \circ (\xi \otimes k[x]) \) of \( X \otimes k[x] \) corresponding to \( \varepsilon \cdot E_\ast \otimes k[x] \). From the existence of the commutative diagram \((\ast)\) we deduce

\[
(id + \varepsilon \cdot E)_\ast^{-1} \circ (id + \varepsilon \cdot E) \circ (\xi \otimes k[x]) = \xi \otimes k[x] + \varepsilon \cdot E_\ast \circ (\xi \otimes k[x])
\]

where \( (id + \varepsilon \cdot E)_\ast : A^1(k[x],X \otimes k[x],0_X \otimes k[x]) \) is the automorphism induced by \( id + \varepsilon \cdot E : X \otimes k[x] + X \otimes k[x] \) and \( (id + \varepsilon \cdot E)_\ast \) the one induced by \( id + \varepsilon \cdot E : X \otimes k[x] + 0_X \otimes k[x] \). But then \( \varepsilon \cdot E_\ast \circ (\xi) \) is the obstruction for lifting \( id + \varepsilon \cdot E \) to \( X \otimes k[x] \). Q.E.D.

**Corollary (2.10).** Let \( E \in A^0(k,x;0_X) \) then \( E_\ast \in \text{End}(A^1) \) is defined by

\[
\varepsilon \cdot E_\ast = (id + \varepsilon \cdot E)_\ast \circ (id + \varepsilon \cdot E)_\ast^{-1} \circ id
\]

Thus \( \rho = \sigma \).

**Proof.** This is exactly the contention of the proof above. Q.E.D.
Corollary (2.11). Let $X = \text{Spec} \left( \frac{k[x_1, \ldots, x_n]}{J} \right)$ and let $D \in \text{Der}_K \left( \frac{k[x]}{J} \right)$. Then the action of $\rho(D)$ on $H^1(k, X; \mathcal{O}_X)$ is defined as follows. Let $\xi \in H^1(k, X; \mathcal{O}_X)$ be represented by a homomorphism $\tilde{\xi} : J + k[x]/J$ and lift $D$ to a derivation $E : k[x] + k[x]$ then $\rho(D)(\xi)$ is represented by the homomorphism $D \tilde{\xi} - \xi \circ E : J + k[x]/J$.

Proof. Use (2.10). Q.E.D.

In particular we have:

Corollary (2.12). Let $X = \text{Spec} \left( \frac{k[x]}{f} \right)$ and let $E \in \text{Der}_k \left( \frac{k[x]}{f} \right)$ be defined by $\sum_{i=1}^n \frac{\partial f}{\partial x_i} \tilde{E}_i(x_i) = q(x)f(x)$. Then if $\xi$ is the class of the polynomial $\xi(x)$ in $H^1(k, k[x]/(f), k[x]/(f)) = k[x]/(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$, the action of $E_*$ on $\xi$ is given by $E_*(\xi) = \text{class of } \left( \sum_{i=1}^n \frac{\partial \xi(x)}{\partial x_i} E(x_i) - q(x)\xi(x) \right)$.

Corollary (2.13). The rank of the linearized action of $\lambda(\pi)$ on $H^\wedge$ is equal to the dimension of the maximal orbit of $A^1(k, X; \mathcal{O}_X)$ under the action of $A^0(k, X; \mathcal{O}_X)$.

Proof. By definition the linearized action of $\lambda(\pi)$ on $H^\wedge$ is the one given in terms of the action of $\lambda/\pi$ on the tangent space $(m/m^2)^* = A^1(k, X; \mathcal{O}_X)$. The rank of $\lambda(\pi)$ is then almost by definition the dimension of the maximal orbit of $A^1(X, \mathcal{O}_X)$ under $\lambda(\pi)$. Q.E.D.
§3. THE KODAIRA-SPENCER MAP AND ITS KERNEL

Introduction. In this § we shall apply the results of §1 and §2 to study the local properties of formally versal families of objects of the type we are concerned with, see (3.6) for the definition of formal versality, and notice that the families we are talking about are algebraic families, not formal ones.

We start by defining the Kodaira-Spencer map \( g: \text{Der}_K(S) \rightarrow A^1(S, Y; \mathcal{O}_Y) \) associated to a flat family \( \eta: Y \rightarrow S \).

The first part of the § is a study of the properties of the kernel \( V_\eta \) of \( g \). As we shall see \( V_\eta \) is a sub \( k \)-Lie-algebra of \( \text{Der}_K(S) \) with nice functorial properties.

We shall not venture into the difficult problem of when formally versal algebraic families exists. At this point, we take the easy way out; assuming that the objects \( X \) we are handling are such that

\( (A_1) \) there exists an algebraization \( \pi: \tilde{X} \rightarrow \mathcal{H} = \text{Spec}(\mathcal{H}) \) of the formal versal family \( \pi^\wedge: X^\wedge \rightarrow \mathcal{H}^\wedge \)

and such that

\( (V) \) \( \pi \) is formally versal.

We then formulate a set of conditions \( (V') \) akin to the conditions studied by M. Artin, see [Ar], which imply formal versality if \( \pi \) exists. Assuming \( (V') \) we also prove that the infinitesimal notions of §1 and §2, such as the prorepresenting stratum \( \mathcal{H}_0^\wedge \) and the Lie-algebra \( \lambda(\pi^\wedge) \) are formalizations of local notions, \( \mathcal{H}_0 \) and \( V \) respectively, see (3.5).
Since the main objectiv of this § is the construction of a local moduli space, we shall have to impose the following condition on our objects $X$:

(A$_2$) If $\eta:Y+\mathcal{S}$ is a flat family with fiber $X = \eta^{-1}(s)$ for some closed point $s \in \mathcal{S}$, then there exists an étale neighbourhood $\varepsilon: E+\mathcal{S}$ of $s$ and a morphism $\rho: E+\mathcal{H}$ such that there is an isomorphism $\varepsilon^*(\eta) = \rho^*(\pi)$ between the pull-backs of $\eta$ and $\pi$.

We know, see [E], that (A$_1$) holds for affine schemes with isolated singularities. In [Gro1], Théorème (5.4.5), Grothendick gives conditions for when (A$_1$) holds for projective schemes. Moreover, (A$_2$) holds when $X$ is projective, when $X$ is a finitely generated graded $k$-algebra and we restrict to deformations within the category of graded $k$-algebras, or when $X$ is a complete local $k$-algebra of finite type, restricting this time the deformations to the category of such formal, or algebroid, schemes. This last example is going to be treated in some detail in §4-§6.

The main results of this § can now be summed up as follows. Assume (A$_1$), (A$_2$) and (V'). The flattening filtration $\{S_\tau\}_\tau$ of $H$ corresponding to the $H$-module $V$, is stable under the operations defined by $V$. There exist reasonably good quotiens $M_\tau$ of $S_\tau$, in the category of algebraic spaces, carrying families the pull-back of which are the restrictions $\pi_\tau$ of $\pi$ to $S_\tau$, see (3.18), such that the collection $\{M_\tau\}$ deserve the name, local moduli suite of $X$.

Perhaps of more interest for the applications we have in mind, we prove, see (3.24), that for every component $S_{\tau,c}$ of $S_\tau$ there is an open dense subscheme of the reduced normalization $S'_{\tau,c}$ of $S_{\tau,c}$ on which the action of the kernel of the corresponding Kodaira-Spencer map has a strict quotient in the category of schemes.
Notice that we do not know whether $V$ acts rationally on $H$, therefore we cannot invoke the general results of Rosenlicht, see [R], or Dixmier and Raynaud, see [D-R].

Assuming $(V')$ we prove, see (3.12), that $V \otimes_{H} H$ is a deformation of the Lie-algebra $L(X) = A^{0}(k, X; O_{X})/A^{0}_{X}$. In fact, see (3.17) we obtain for every $\tau$ a flat family of Lie-algebras defined on $S_{\tau}$, the fibers of which are the $k$-Lie-algebras $L(X(\tau))$, $\tau \in S_{\tau}$.

Thus we are led to the study of deformations of Lie-algebras.

Now, we might have included Lie-algebras among our objects of study from the beginning. For reasons to be explained elsewhere, we shall treat it apart. In particular we have to modify the definition of cohomology of Lie-algebras to obtain a natural setting.

In the above process we also find a necessary condition and a sufficient condition for an object satisfying $(A_{1})$, $(A_{2})$ and $(V)$ to have a rigidification, see (3.11). Moreover, we prove a result which was conjectured by Wahl, see [W], and recently also proved by Greuel and Looijenga [G-L] on the dimension of a smoothing component of $H$, see (3.10).

Let $S$ be any $k$-algebra and consider a flat family

$\eta: Y \to \text{Spec}(S)$.

Corresponding to the simplicial $k$-algebra

$$S \overset{v_{1}}{\leftarrow} S \overset{\cdot}{\leftarrow} S \overset{\cdot}{\leftarrow} S \overset{\cdot}{\leftarrow} \cdots \overset{\cdot}{\leftarrow} S \overset{\cdot}{\leftarrow} S \overset{\cdot}{\leftarrow} S \overset{\cdot}{\leftarrow} \cdots, \quad v_{1} = \text{id} \otimes 1, \quad v_{2} = 1 \otimes \text{id}$$

one may define a series of obstructions for descent of $Y$ to $k$. 
The first descent obstruction is gotten in the following way. Put $I = \ker\{S \otimes_S M \otimes S\}$, where $m$ is the multiplication, and consider the diagram

$$
\begin{array}{c}
\begin{array}{c}
S \otimes_S S/I^2 + I/I^2 \to \mathcal{O}_S/
\end{array}
\end{array}
$$

Since $\nu_1^*(Y)$ and $\nu_2^*(Y)$ are two liftings of $Y$ to $S \otimes_S S/I^2$, the difference

$$
g(Y) = \nu_1^*(Y) - \nu_2^*(Y)
$$

sits in $A^1(S, Y; O_Y \otimes S/\mathcal{O}_S)$. $g(Y)$ is the obstruction for lifting the identity morphism on $Y$ to a morphism between $Y \otimes (S \otimes S/I^2)$ and $Y \otimes (S \otimes S/I^2)$. Since for every $S$-module $M$, $\text{Der}_S(S, M) = \text{Hom}(\mathcal{O}_S/M, S)$, any $D \in \text{Der}_S(S, M)$ induces a homomorphism

$$
\delta_D: A^1(S, S; O_Y \otimes S) \to A^1(S, Y; O_Y \otimes M).
$$

Consider the $S$-module homomorphism defined by

$$
g_M: \text{Der}_S(S, M) \to A^1(S, Y; O_Y \otimes M),
$$

where

$$
g_M(D) = \delta_D(g(Y)).
$$

**Definition (3.1).** The morphism $g_S$ denoted by,

$$
g_\eta: \Theta_S \to A^1(S, Y; O_Y)
$$

is called the Kodaira–Spencer map associated to the family $\eta$. 


Proposition (3.2). Let $s \in \text{Spec}(S)$ be a $k$-point. Then the induced map

$$g \otimes_S k(s) : \theta_{S, s} \to A^1(k(s), Y(s); \mathcal{O}_Y(s))$$

where $Y(s) = \eta^{-1}(s)$, is the composition of the tangent map $g : \mathfrak{t}_S, s \to A^1(k(s), X(s), \mathcal{O}_X(s))$ of the canonical morphism $\mathcal{S}^\wedge_S S^\wedge(s)$ defined by the formal deformation $Y \otimes_S S^\wedge$ of $Y(s)$ to $S^\wedge_S$ and the canonical map $\theta_{S, s} \to \mathfrak{t}_S, s$. (Here $H^\wedge(s)$ is the formal moduli of $Y(s)$.)

Proof. Let $P_i, i = 1, 2$ be the ring $S \otimes_S S/I^2$ considered as left $S$-algebra via $\nu_1$ and $\nu_2$, respectively, and as right $S$-algebra via $\nu_2$. Then, as left $S$-algebra we have the following isomorphisms

$$P_1 \otimes_S k(s) = S/m^2_S$$
$$P_2 \otimes_S k(s) = S/m^2_S = k(s)[m_S/m^2_S]$$

By definition, we have

$$\nu_i^*(Y) = Y \otimes_{S^\wedge(s)} P_i, \quad i = 1, 2$$

therefore,

$$\nu_i^*(Y) \otimes_S k(s) = \begin{cases} Y \otimes S/m^2_S & i = 1 \\ Y(s) \otimes k(s)[m_S/m^2_S] & i = 2 \end{cases}$$

Consider the exact sequence of right $S$-modules

$$0 \to \mathfrak{t} \to P_i \to \mathcal{O}_{S/k} \to 0$$
and the commutative diagram:

\[
\begin{array}{ccc}
Y & \overset{v_i}{\to} & Y \otimes P_i \\
\downarrow & & \downarrow \\
\text{Spec}(S) & \overset{v_i}{\to} & \text{Spec}(P_i) \end{array}
\]

Obviously \( g(Y) = Y \otimes P_1 - Y \otimes P_2 \in A^1(S, Y; O_Y \otimes S/k) \) is, under the specialization map \( S \to k(s) \), mapped to \( Y \otimes S/m_s^2 - Y(s) \otimes k(s)/S/m_s^2 \in A^1(k(s), Y(s); O_{Y(s)} \otimes k(s)/m_s \). 

Therefore \( g \otimes k(s) \) is induced by the map

\[
\theta_{S, S} = (m_s/m_s^2)^* + A^1(k(s), X(s), O_X(s))
\]
defined by \( \tilde{X} \otimes S/m_s^2 \). Q.E.D.

We shall need a relativized version of the Kodaira-Spencer map.

Let \( \rho: S + T \) be a morphism of \( k \)-algebras of finite type, and let \( \eta: Y + \text{Spec}(S) \) be a flat morphism of finite type, then

\( Y' = Y \times \text{Spec}(T) = Y \otimes T \) is defined and \( \eta': Y' + \text{Spec}(T) \) is flat. Moreover, putting \( S[\varepsilon] = S \otimes k[\varepsilon], \rho' = \text{Spec}(\rho) \), we obtain a commutative cube:

Consider the set
\[ V(\eta, \rho) = \{ \phi' \mid \text{there exists an } f \text{ making the diagram above commutative} \}. \]

Now, the morphisms \( \phi' = \text{Spec}(\phi) : \text{Spec}(T[\varepsilon]) \to \text{Spec}(S[\varepsilon]) \) lifting \( \text{Spec}(\rho) : \text{Spec}(T) \to \text{Spec}(S) \), correspond to derivations \( D \in \text{Der}_K(S, T) \), such that \( \phi = \rho + \varepsilon \cdot D \). Thus,

\[ V(\eta, \rho) = \{ D \in \text{Der}_K(S, T) \mid (Y[\varepsilon])^T = Y[\varepsilon] \} \]

where \( \phi = \rho + \varepsilon \cdot D \) and \( \chi[\varepsilon] = \text{id}_Y \).

The following result is easy but not entirely trivial

**Proposition (3.3).** In the situation above

(i) \( V(\eta, \rho) = \ker \{ \text{Der}_K(S, T) \to A^1(S, \Omega_S \otimes T) \} \)

(ii) \( V(\eta, \rho) \) is a \( T \)-submodule of \( \text{Der}_K(S, T) \)

(iii) \( V(\eta) \) is a sub-Lie-algebra of \( \text{Der}_K(S) \).

(iv) The natural morphisms of \( S \)-modules

\[ \text{Der}_K(S, S) \to \text{Der}_K(S, T) \to \text{Der}_K(T, T) \]

induce morphisms of \( S \)-modules

\[ V(\eta) + V(\eta, \rho) + V(\eta') \]

Moreover \( j^{-1}(V(\eta, \rho)) = V(\eta') \).

**Proof.** Let \( d : S \to \Omega_{S/k} \) be the universal derivation, \( d(s) = s \otimes 1 - 1 \otimes s \), modulo \( I^2 \), and consider the morphisms

\[ v_i : S \to S[I^2/k] = S[\Omega_{S/k}], \quad i = 1, 2. \]

Clearly \( v_1 - v_2 : S \to \Omega_{S/k} \) is equal to \( d \). Let \( D \in \text{Der}_K(S, T) \) and consider \( D \) as an \( s \)-linear homomorphism \( D' : \Omega_{S/k} \to T \). There is an associated map of Nagata-rings

\[ \rho - \varepsilon \cdot D' : S[\Omega_{S/k}] \to T[\varepsilon] \]
such that \((p-e:D')ov_1 = iop:S + T[\varepsilon]\) and \((p-e:D')ov_2 = p+e:D:S + T[\varepsilon]\), where \(i:T + T[\varepsilon]\) is the obvious inclusion.

By definition of the Kodaira-Spencer map \(g_T\) we have

\[
g_T(D) = \sigma_D(v_1^*(Y)-v_2^*(Y)) = (p-e:D')^*v_1^*(Y)-(p+e:D)^*v_2^*(Y)
\]

\[
= (iop)^*(Y)-(p+e:D)^*(Y).
\]

From this we see immediately that (i), and (iv) follows. Since \(g_T\) is \(T\)-linear (ii) is trivial. (iii) follows from the definition of \(V(\eta)\), as \(V(\eta)\) obviously is the Lie-algebra of a corresponding group-functor, like \(\mathfrak{l}(\pi)\) in §2.

Lemma (3.4). Let \(S\) be a \(k\)-algebra (essentially) of finite type and let \(\eta:Y + \text{Spec}(S)\) be a flat morphism of finite type.

Denote by \(m\) a maximal ideal of \(S\) such that \(S/m = k\). Then for any coherent \(O_Y\)-module \(M\),

(i) There is an exact sequence \(0 + \lim_{\to}(1)\bigcap_{1}(S/m^n,Y\otimes S/m^n;M\otimes S/m^n)\to^A(S,Y,M^\wedge)\to \lim_{\to}A^{i}(S/m^n,Y\otimes S/m^n;M\otimes S/m^n)\to 0\).

(ii) \(A^i(S,Y,M^\wedge) = A^i(S,Y,M)^\wedge, i \geq 0\).

(iii) If \(S\) is regular and \(M\) is flat as \(S\)-module and \(R\) is an \(S\)-algebra of finite type, then there exists a spectral sequence given by:

\[
E_2^{p,q} = \text{Tor}^S_{-p}(A^q(S,Y,M),R)
\]

converging to

\[
A^{p+q}(S,Y;M\otimes R).
\]

Proof. Suppose that \(Y = \text{Spec}(A)\). Recall the definition of the André-cohomology,
where $C'(M)$ is a functorial complex of $A$-modules such that

$$C^m(M) = \prod \limits_{I_m} M$$

for some indexing set $I_m$.

It follows that

$$\lim \limits_n C'(M \otimes S/m^n) = C'(M^\wedge)$$

$$\lim \limits_n (1) C'(M \otimes S/m^n) = 0$$

since the projective system $C'(M \otimes S/m^n)$ is surjective. (i) follows therefore from general nonsense on $\lim$ of complexes, see e.g. [La1], and ([An] 21). (ii) follows from (21.2) of [An]. Let $L$ be a finite $S$-free resolution of $R$, then $C'(M \otimes L) = C'(M \otimes L)$ and $S$ $S$

(iii) is simply the spectral sequence associated to the double complex $C'(M \otimes L)$. The generalization to the global case presents no difficulties, see [La1], Chap 3. Q.E.D.

Suppose from now on that,

(A1) there exists an algebraization $\pi: \tilde{X} \rightarrow H = \text{Spec}(H)$

of the formal versal family $\pi^\wedge: X^\wedge \rightarrow H^\wedge$.

Consider the Kodaira-Spencer map $g_X = g_X'$

$$g_X: \theta_H \rightarrow A^1(H, \tilde{X}; \mathcal{O}_{\tilde{X}})$$

and put

$$V_X = \ker g_X.$$

Let $\rho: S \rightarrow H$ be a morphism, $s_0 \in S$ a closed point mapping to $* \in H$. 

$$A^q(S, A; M) = H^q(C'(M))$$
Denote by \( \pi': \tilde{X}' \to S \) the pull-back of \( \pi \) to \( S \), and consider \( V(\pi, \rho) \subseteq \text{Der}_k(H, S) \). Notice that \( V(\pi, \rho) \subseteq \text{Der}_*(H, S) \). This follows from the fact that \( \hat{H}_* \) prorepresents the deformation functor on the subcategory \( \mathfrak{A}_2 \) of \( \mathfrak{A} \). More precisely, if \( D( V(\pi, \rho) ) \) and if we let \( \phi \) be the composition of \( H \to H[\varepsilon] \) and \( \rho + \varepsilon D: H[\varepsilon] \to S[\varepsilon] \), then \( \tilde{X} \otimes S[\varepsilon] = \tilde{X} \otimes S[\varepsilon] \). Let \( \hat{\phi} \) be the composition of \( \hat{\phi} \) with \( S[\varepsilon] \to k[\varepsilon] \) given by the point \( s_0 \), then \( \tilde{X} \otimes k[\varepsilon] = \tilde{X} \otimes k[\varepsilon] \)

implying that \( \hat{\phi} \) is equal to the composition \( H \to k \times k[\varepsilon] \)

defined by \( \ast \), thus that \( D \) maps the maximal ideal \( m_{s_0} \) of \( H \)
into the maximal ideal \( n_{s_0} \) of \( S \).

**Corollary (3.5).** With the assumptions and notations above we have

(i) \( V(\pi, \rho)^\wedge = V(\pi^\wedge, \rho^\wedge) = \lambda(\pi^\wedge, \rho^\wedge) \). \n
where completion is with respect to the \( n_{s_0} \)- resp. \( m_{s_0} \)-adic topology.

(ii) \( V(\pi, \rho) \otimes_k \text{A}^0(k, X;\mathcal{O}_X) = \text{A}^0(\text{A}^0, X) \).

(iii) \( V(\pi) \otimes S + V(\pi, \rho) \) is surjective.

**Proof.** Since we know that \( V(\pi, \rho) \subseteq \text{Der}_*(H, S) \) it is clear by

(3.3) (i) that \( V(\pi, \rho) = \ker(\text{Der}_*(H, S) \to \text{A}^1(S, \tilde{X}' ;\mathcal{O}_{\tilde{X}' })) \). Since by assumption the \( S \)-modules involved are of finite type, completion with respect to the \( n \)-adic topology commutes with the formation of kernel. Moreover \( \text{Der}_*(H_1, S)^\wedge = \text{Der}_*(H_1^\wedge, S^\wedge) \) and \( \text{A}^1(S, \tilde{X}' ;\mathcal{O}_{\tilde{X}' })^\wedge = \text{A}^1(S^\wedge, \tilde{X}' ;\mathcal{O}_{\tilde{X}' }) \) by (3.4), (ii). But this implies that \( V(\pi_1, \rho)^\wedge = V(\pi^\wedge, \rho^\wedge) \) which is equal to \( \lambda(\pi_1^\wedge, \rho_1^\wedge) \). From this (i) follows.

(ii) is a consequence of (i) and (2.6) (iv), and (iii) follows from (ii) and Nakayama's lemma. Q.E.D.
Definition (3.6). A flat family \( \pi: \tilde{X} \to S \) is called formally versal if at every k-point \( \xi \in S \), \( g_x: T_{\xi} \to A^1(k(\xi), \mathcal{O}_{\tilde{X}(\xi)}) \) is surjective.

Corollary (3.7). If a deformation of \( X \),

\[
\begin{array}{ccc}
X & \to & \tilde{X} \\
\downarrow & & \downarrow \pi \\
\text{Spec}(k) & \to & S
\end{array}
\]

satisfies the conditions

a) \( S \) is smooth
b) \( t_{s, \pi} A^1(k, \mathcal{O}_X) \) is surjective
c) \( A^i(S, \tilde{X}; \mathcal{O}_{\tilde{X}}) \) is an \( \mathcal{O}_S \)-module of finite type for \( i = 1 \) and flat for \( i > 2 \), then in some neighbourhood of \( \ast \), \( \pi \) is formally versal.

Proof. Consider the Kodaira-Spencer map \( g \) and the diagram

\[
\begin{array}{ccc}
\theta_S & \xrightarrow{g} & A^1(S, \tilde{X}; \mathcal{O}_{\tilde{X}}) \\
\eta & \xrightarrow{t_{s, \pi}} & A^1(k(s), \mathcal{O}_X)
\end{array}
\]

By (3.4) (iii) and the conditions c), \( A^1(S, \tilde{X}; \mathcal{O}_{\tilde{X}}) \) is an \( \mathcal{O}_S \)-module of finite type for all \( s \in S \). Since moreover, by assumption, \( A^1(s, \tilde{X}; \mathcal{O}_{\tilde{X}}) \) is of finite type as \( \mathcal{O}_s \)-module, it follows from Nakayamas lemma, that \( \text{im } g \) generates \( A^1(S, \tilde{X}; \mathcal{O}_{\tilde{X}}) \) in some neighbourhood of \( \ast \). Then, by (3.2) the tangent map \( S^\wedge \to \mathcal{H}(s)^\wedge \) is surjective. Q.E.D.

Suppose from now on that
(V) $\pi$ is formally versal.

In particular it follows from (3.6) that this last assumption holds in some neighbourhood of $\ast$ if:

1. $H$ is $k$-smooth
2. $A^1(H,\tilde{X};O_X)$ is an $H$-module of finite type.
3. $A^i(H,\tilde{X};O_X)$ is a flat $H$-module for $i > 2$.

Now, let $s \in \text{Spec}(S)$ be any $k$-rational point, then by (3.2) and the condition (V), there is a commutative diagram where the top and the bottom sequences are exact:

\[
\begin{array}{ccc}
0 & \to & V(\pi, \rho) \\
& \to & \text{Der}_K(H, S) \\
& \to & A^1(H, \tilde{X}; O_X \otimes S) \\
\end{array}
\]

\[
\begin{array}{ccc}
& \to & 0 \\
& \to & V(\pi, \rho) \otimes k(s) \\
& \to & \text{Der}_K(H, S) \otimes k(s) \\
& \to & A^1(H, \tilde{X}, O_X \otimes S) \otimes k(s) \\
\end{array}
\]

\[
\begin{array}{ccc}
& \to & 0 \\
& \to & t_{H, \rho(s)} \\
& \to & A^1(k(s), X(\rho(s))^\times; O_X(\rho(s))) \\
\end{array}
\]

Remark (3.8). Observe that from this follows

\[
\dim_k A^1(k, X(t); O_X(t)) = \dim_{t_{H, \rho}} V(\pi_1, \rho)
\]

for $t = \rho(s)$, whenever $m_s$ and $\lambda_s$ are isomorphisms. Notice that $m_s$ is an isomorphism when $t$ is non-singular. In particular, we find in this case,

\[
\min_{s \in S} \{\dim_k A^1(k, X(\rho(s)); O_X(\rho(s)))\} > \dim_{H} V(\pi_1, \rho).
\]

Now, in the light of (3.5) (ii) above, we are therefore interested in knowing when $m_s$ and $\lambda_s$ are isomorphism and when $V(\pi, \rho)$ is locally free at $s_0$. The following proposition is a partial result in this direction.
Proposition (3.9). Assume either

(1) $S$ is a nonsingular curve.

(2) $S$ is nonsingular, and the conditions $(V')$ hold.

Then

(i) $l_s$ and $m_s$ are injective

(ii) $V(\pi, \rho)$ is locally free in a neighbourhood of $s_0$.

Proof. We have, by definition an exact sequence of $S$-modules

$$0 \rightarrow V(\pi, \rho) \rightarrow \text{Der}_K(H, S) \rightarrow A^1(H, S; O_X \otimes S).$$

Suppose first that $S$ is a nonsingular curve. $S$ is a Dedekind domain. Since $\text{Der}_K(H, S)$ has no $S$-torsion, $V(\pi, \rho)$ has no $S$-torsion. But then $\text{Der}_K(H, S)$ and $V(\pi, \rho)$ are $S$-flat, and therefore locally free in a neighbourhood of $s_0$.

Since $t = \rho'(s)$ is a nonsingular point, $m_s$ is an isomorphism. Let $n_s = (t)$ be the maximal ideal of $s \in S$, and consider the exact sequence $0 \rightarrow S \rightarrow S + k(s) \rightarrow 0$. The induced sequence $\cdots \rightarrow A^1(H, \tilde{X}; O_X \otimes S) \rightarrow A^1(k(s), X(s); O_X(s)) \rightarrow \cdots$ is therefore exact for all $i > 0$ and this implies that $m_s$ and $l_s$ are injective.

In case (2), observe that $g_s$ is surjective in a neighbourhood of $s$, so that for any $i$, $\text{Tor}_i^S(V(\pi, \rho), k)$ is a quotient of $\text{Tor}_{i+1}^S(A^1(H, \tilde{X}; O_X \otimes S), k)$. It suffices for our purpose to prove that $\text{Tor}_2^S(A^1(H, \tilde{X}; O_X \otimes S), k) = 0$. Now since $\pi' : \tilde{X}' \rightarrow S$ is the pull-back of $\pi$, we have $A^1(H, \tilde{X}; O_X \otimes H) = A^1(S, \tilde{X}'; O_X)$. Use the spectral sequence of (3.4) (iii). See first that the map $A^0(S, \tilde{X}'; O_X) \otimes k = A^1(\tilde{X}'; k) + A^0(k, X; O_X)$ is injective. This implies that the second differential in the spectral sequence,

$$d_2 : \text{Tor}_2^S(A^1(S, \tilde{X}'; O_X), k) \rightarrow A^0(S, \tilde{X}'; O_X) \otimes k$$
is zero. For degree reasons all other differentials
\[ d_k: \text{Tor}_2^S(A^1(S, \tilde{X}'; O_{\tilde{X}}), k) \to \text{Tor}_2^S(A^{1-k+1}(S, \tilde{X}'; O_{\tilde{X}}), k) \]
must vanish.

Now, since the abutment of the spectral sequence of total degree \(-1\) is \(A^{-1}(k, X; O_X) = 0\), so \(E^{-2,1}_\infty = 0\) we find that \(\text{Tor}_2^S(A^1(S, \tilde{X}'; O_{\tilde{X}}), k) = E^{-2,1}_2\) must be the sum of the images
\[ d_r: E_{r-2,r}^{-2} + E_{r-2,r}^{-2,1}, \quad r \geq 2. \]

However, since by the assumption \((V')\), \(\text{Tor}_p^S(A^q(S, \tilde{X}'; O_{\tilde{X}}), -) = 0\) for all \(p > 1, q > 2\) we find that \(E_{r-2,r}^{-2} = 0\) for \(r \geq 2\) therefore also \(E_{r}^{-4,2} = 0\) for \(r > 2\). Consequently \(\text{Tor}_2^S(A^1(S, \tilde{X}'; O_{\tilde{X}}), k) = 0\) and thus \(\text{Tor}_1^S(V(\pi, \rho), k) = 0\), i.e. \(V(\pi, \rho)\) is locally constant in a neighbourhood of \(s_0\).

Q.E.D.

From this follows,

**Corollary (3.10)** (Wahl's conjecture). Suppose there exists a nonsingular curve \(S\) and a morphism \(\rho': S \to H\) such that the image of \(\rho'\) contains a non-singular point of \(H\) and such that the pull-back \(\pi'\) of \(\pi\) contains a rigid fiber. Then the dimension of a rigidifying component of \(H\) is
\[ \dim_k(A^0(k, X; O_X)/A^0_{\pi'}) \]

**Proof.** This follows immediately from (3.9) (1) together with (3.5) and (3.8). Q.E.D.

This result was conjectured by J. Wahl and proved by him in some special cases, see [W], and has recently also been proved in the complex analytic case by Greuel & Loojenga [G-L].
Since by (2.9) we know how to compute the linearized action of $V$ on $H$, and since by [La1] all components of $H$ have dimension greater or equal to

$$\dim_k A^1(k, X; O_X) - \dim_k A^2(k, X; O_X)$$

we obtain a smoothing, and a non-smoothing criterion as follows.

\textbf{Proposition (3.11).} Assume (A) and (V). Then we have:

(i) Suppose there exists an unobstructed $\xi \in A^1(k, X; O_X)$ such that

$$A^0(k, X; O_X) \cdot \xi = A^1(k, X; O_X),$$

then $X$ may be rigidified.

(ii) Let $A^0_{\pi}$ be the sub-Lie-algebra of those elements of

$$A^0(k, X; O_X)$$

that may be lifted everywhere. Suppose

$$\dim_k A^1(k, X; O_X) - \dim_k A^2(k, X; O_X) > \dim_k (A^0(k, X; O_X)/A^0_{\pi})$$

then $X$ cannot be rigidified.

\textbf{Proof.} (i) Let $\rho: S \to H$ be a morphism, $S$ a non-singular curve, such that $\xi$ is in the image of the tangent map

$$t_{\rho}: t_{\overline{S}, s_0} \to t_{H, x} = A^1(k, X; O_X).$$

By (2.8) we know that the orbit of $\xi$ under $\lambda(\pi)$ or what is the same under $A^0(k, X; O_X)$, is isomorphic to $A^0(k, X; O_X)/A^0_{\xi}$. Therefore $A^0(k, X; O_X)/A^0_{\pi} \cong A^1(k, X; O_X)$. Now, obviously $A^0(k, X; O_X)/A^0_{\pi} \to A^0(k, X; O_X)/A^0_{\xi}$ is surjective. Therefore, by (3.9) $V(\pi, \rho)$ is locally free of rank $\dim(A^0(k, X; O_X)/A^0_{\pi})$ at $s_0$. Moreover, since $S$ is a nonsingular curve, there exist an open subset of points $s \in S$, such that the composition

$$V(\pi, \rho) \otimes k(s) \to \text{Der}_k(H, S) \otimes k(s) \overset{\mathfrak{m} \cdot t_{H, \rho(s)}}{\to} S$$

is injective. Since $\dim t_{H, \rho(s)} < \dim A^1(k, X; O_X)$ and since by assumption $\dim V(\pi, \rho) \otimes k(s) > \dim A^1(k, X; O_X)$ we find $V(\pi, \rho) \otimes k(s) =$
But then the map $t_{\mathbb{H}, t} \to A^1(k(t), X(t); O_X(t))$, when $t = \rho(s)$, is zero. However, by (V) this map is also surjective, therefore $X(t)$ is rigid.

(ii) is already proved. Q.E.D.

**Proposition (3.12).** (i) Suppose the conditions (V') hold. Let $H_0$ be the subscheme of $H$ defined by $V(\pi) = 0$. Then for every point $t \in H_0$, the formalization $H_0^\wedge$ of $H_0$ at $t$ is the prorepresenting substratum of the formal moduli $H^\wedge(t)$ of $X(t)$.

(ii) $V(\pi) \otimes H_0^\wedge$ is an $H_0^\wedge$-Lie-algebra and a deformation of the Lie-algebra $L(X) = A^0(k, X; O_X)/A^0_{\pi}$.

**Proof.** The conditions (V') imply that for every $t \in H_0$, $H_0^\wedge = H(t)^\wedge$. But then (3.12) follows from (3.5), with $\rho = id_H$, together with (2.6) (v). Q.E.D.

Consider the $H$-module $A^1(H, X; O_X)$, and let $\{S_t\}, t = 0, \ldots, t^1 = \dim_k A^1(k, X; O_X)$ be the flattening stratification of $A^1(H, X; O_X)$ (see [M], Lecture 8), and let $S_t = \bigcup_{c=1}^t S_{t, c}$ be the decomposition of $S_t$ into its connected components. We shall consider only those $S_{t, c}$ for which $s \in S_{t, c}$.

Notice that the $S_{t, c}$'s are locally closed subschemes of $H$ and that $S_{t, t} = H_0^\wedge$. Denote by $\pi_{t, c}$, respectively $\pi_{t, c', c'}$, the restriction of $\pi$ to $S_{t, c}$, respectively $S_{t, c'}$.

Put for every $t \in H$, with $t \in S_{t, c}$

$$T_t = S_{t, c}.$$ 

Consider also, for any $k$-point $t \in S_t \subset H$, the formal family
\[ \pi_t^\wedge: \tilde{X}_t^\wedge \rightarrow H_t^\wedge \]

where \( H_t^\wedge = \text{Spf}(H_t^\wedge), m_t \), being the maximal ideal of \( H \) corresponding to \( t \).

As above we denote by \( X(t) \) the fiber \( \pi^{-1}(t) \). Obviously the closed fiber of \( \pi_t^\wedge \) is precisely \( X(t) \).

Consider the formal moduli \( \tilde{H}(t)^\wedge \) of \( X(t) \), i.e. \( \tilde{H}(t)^\wedge = \text{Spf}(H(t)^\wedge) \), and let

\[ \pi(t): \tilde{X}(t) \rightarrow \tilde{H}(t), \]

with \( \tilde{H}(t) = \text{Spec}(H(t)) \), be an algebraization of the formal versal family

\[ \pi(t)^\wedge: X(t)^\wedge \rightarrow H(t)^\wedge. \]

Let \( H_0(t) \) be the prorepresenting substratum of \( H(t) \), and let

\[ \pi_0(t): \tilde{X}_0(t) \rightarrow H_0(t) \]

be the restriction of \( \pi(t) \). Then, by formal versality, there is a morphism

\[ \rho_t^\wedge: H_t^\wedge \rightarrow H(t)^\wedge \]

such that \( \pi_t^\wedge \) is the pullback of \( \pi(t)^\wedge \) by \( \rho_t^\wedge \). Since \( \pi \) is versal, \( \rho_t^\wedge \) has maximal rank at \( t \).

Now to proceed we have to assume that our objects \( X \) satisfy the condition:

\( (A_2) \) If \( \eta: Y \rightarrow S \) is a flat family with fiber \( X = \eta^{-1}(s) \) for some closed point \( s \in S \), then there exists an étale neighbourhood \( \varepsilon: E \rightarrow S \) of \( s \) and a morphism \( \rho: E \rightarrow H \) such that there is an isomorphism between the pull-back \( \varepsilon^*(\eta) \) of \( \eta \) and the pull-back \( \rho^*(\pi) \) of \( \pi \).
In the applications we have in mind (A₂) is a consequence of M. Artins approximation theorem, see [Ar].

Now, (A₂) implies the existence of an étale neighbourhood
\[ \eta(t) : E(t) \rightarrow \mathbb{H} \]
of \( t \in \mathbb{H} \), and a dominant morphism
\[ \rho(t) : E(t) \rightarrow \mathbb{H}(t) \]
such that \( \eta(t)^* (\pi) = \rho(t)^* (\pi(t)) \).

Put \( E_r(t) = \rho(t)^{-1}(\mathbb{H}_0(t)) \) and consider the cartesian diagram

\[
\begin{array}{ccc}
E(t) & \xrightarrow{\rho(t)} & \mathbb{H}(t) \\
\downarrow & & \downarrow \\
E_r(t) & \xrightarrow{\rho_0(t)} & \mathbb{H}_0(t)
\end{array}
\]

and let
\[ \pi'(t) : \bar{Y}(t) \rightarrow E(t) \]
\[ \pi'_r(t) : \bar{Y}(t) \rightarrow E_r(t) \]
be the corresponding pull-backs of the family \( \pi \).

It is clear that \( \rho(t) \) is of constant maximal rank at every closed point of \( E(t) \). Assume for a moment that for every \( t \in \mathbb{H} \), \( \mathbb{H}(t) \) is nonsingular and in particular that \( \mathbb{H} \) is nonsingular, then \( E(t) \) is nonsingular, and \( \rho(t) \) is smooth in a neighbourhood of every point of \( E_r(t) \).

Consequently \( \rho_r(t) \) is smooth. In particular the fiber of \( \rho_r(t) \) are smooth subschemes of \( E_r(t) \). It is clear that any such fiber is a maximal integral submanifold for \( V(\pi'_r) \), see remarks following (3.8). Let us denote by \( D(t) \) the special fiber of \( \rho_r(t) \).
Corollary (3.13). Suppose the conditions $(V')$ hold. Through every point $t \in \mathcal{H}$ there passes a smooth maximal integral submanifold $D_t$ for $V$. Moreover $D_t \subseteq T_t$ and

$$\dim H_0(t) = \dim T_t - \dim D_t.$$ 

Proof. Let $D_t = \eta(t)(D(t))$ and glue. 

Let $H_0(t) = \text{Spec}(H_0(t))$, $E(t) = \text{Spec}(E(t))$, $E_t(t) = \text{Spec}(E_t(t))$.

By shrinking the étale neighbourhood $E(t)$ of $t \in \mathcal{H}$ we may assume that $\rho(t)$ is smooth.

Lemma (3.14). In the above situation, there exists an exact sequence of $E(t)$-modules

$$0 \rightarrow \text{Der}_{H(t)}(E(t),E(t)) \rightarrow V(\pi'(t)) \rightarrow V(\pi(t)) \otimes E(t) \rightarrow 0.$$ 

Proof. Since $E(t)$ is $H(t)$-smooth, the sequence

$$0 \rightarrow \text{Der}_{H(t)}(E(t),E(t)) \rightarrow \text{Der}_k(E(t),E(t)) \rightarrow \text{Der}_k(H(t),E(t)) \rightarrow 0$$

is exact. Now, use (3.3) and (3.5) for the situation

$$\tilde{X}(t) \rightarrow \rho(t)^*(\tilde{X}(t)) = \tilde{Y}(t)$$ 

$$\tilde{X}(t) \rightarrow \rho(t)^*(\tilde{X}(t)) = \tilde{Y}(t)$$ 

By (3.3) (iv) we know that $V(\pi'(t)) = j^{-1}(V(\pi(t),\rho))$ and from (3.5) (iii) follows that $V(\pi(t)) \otimes E(t) \rightarrow V(\pi(t),\rho)$ is surjective.

However, since $E(t)$ is $H(t)$-flat and $\text{Der}_k(H(t)) \otimes E(t) = \text{Der}_k(H(t),E(t))$, $V(\pi(t)) \otimes E(t) \rightarrow V(\pi(t),\rho)$ is also injective, therefore an isomorphism. 

Q.E.D.

Notice that since $E_t(t) \rightarrow H_0(t)$ is smooth and $V(\pi_0(t)) = 0$ the above proof also yields the formula
\[ V(\pi'_t(t)) = \text{Der}_H(\pi'_t(t))(E_t(t)). \]

Observe also that \( E_t(t) \) is stable under \( V(E_t(t)) \). To see this let \( \delta \in V(E_t(t)) \), and consider the automorphism

\[ \kappa = \text{id} + \varepsilon \cdot \delta : E_t(t) \otimes k[\varepsilon] + E_t(t) \otimes k[\varepsilon]. \]

Since the corresponding automorphism \( \kappa^{-1} \kappa^* \) of \( \text{Der}_K(E_t(t)) \otimes k[\varepsilon] \) is given by \( \kappa^{-1} \kappa^*(D) = D + \varepsilon[D, \delta] \), it maps \( V(\pi'_t(t)) \otimes k[\varepsilon] \) into itself. Consequently in \( E_t(t) \otimes k[\varepsilon] \) we must have that

\[ \text{rank}\{\delta_1, \ldots, \delta_{r+1}\} = \text{rank}\{\delta_1 + \varepsilon[\delta_1, \delta], \ldots, \delta_{r+1} + \varepsilon[\delta_{r+1}, \delta]\} \]

for all sequences \( \delta_1, \ldots, \delta_{r+1} \) from \( V(\pi'_t(t)) \). Let \( \mathfrak{m}_t \) be the ideal generated by the determinants of the form

\[
\begin{vmatrix}
\delta_1(a_1), & \delta_1(a_2), & \ldots, & \delta_1(a_{r+1}) \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{r+1}(a_1), & \delta_{r+1}(a_2), & \ldots, & \delta_{r+1}(a_{r+1})
\end{vmatrix}, \quad \delta_i \in V(E_t(t)), \quad a_j \in E_t(t), \quad r = r' - r
\]

Since \( \mathfrak{m}_t \) defines the closure of the flattening stratum \( E_t(t) \) of \( A^1(E_t(t), \overline{Y}(t), O_{\overline{Y}(t)}) \), the ideal in \( E_t(t) \otimes k[\varepsilon] \) generated by \( \mathfrak{m}_t \) must be equal to the ideal generated by the determinants of the form

\[
\begin{vmatrix}
\delta_1(a_1) + \varepsilon[\delta_1, \delta](a_1), & \ldots, & \delta_1(a_{r+1}) + \varepsilon[\delta_1, \delta](a_{r+1}) \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{r+1}(a_1) + \varepsilon[\delta_{r+1}, \delta](a_1), & \ldots, & \delta_{r+1}(a_{r+1}) + \varepsilon[\delta_{r+1}, \delta](a_{r+1})
\end{vmatrix}
\]

\[ = \varepsilon \delta \begin{vmatrix}
\delta_1(a_1), \ldots, [\delta_1, \delta](a_j), \ldots, \delta_1(a_{r+1}) \\
\delta_{r+1}(a_1), \ldots, [\delta_{r+1}, \delta](a_j), \ldots, \delta_{r+1}(a_{r+1})
\end{vmatrix} \]

\[ + \begin{vmatrix}
\delta_1(a_1), \ldots, \delta_1(a_{r+1}) \\
\delta_{r+1}(a_1), \ldots, \delta_{r+1}(a_{r+1})
\end{vmatrix}
\]
Consequently

\[
\delta \det \begin{pmatrix}
\delta_1(a_1), \delta_1(a_2), \ldots, \delta_1(a_{r+1}) \\
\vdots \\
\delta_{r+1}(a_1), \delta_{r+1}(a_2), \ldots, \delta_{r+1}(a_{r+1})
\end{pmatrix}
\]

\[
= \sum_j \det \begin{pmatrix}
\delta_1(a_1), \ldots, [\delta, \delta_1](a_j), \ldots, \delta_1(a_{r+1}) \\
\vdots \\
\delta_{r+1}(a_1), \ldots, [\delta, \delta_{r+1}](a_j), \ldots, \delta_{r+1}(a_{r+1})
\end{pmatrix}
\]

is contained in the ideal \(\mathfrak{N}_\tau\). Thus \(\delta\) maps \(\mathfrak{N}_\tau\) into itself, i.e. \(E_\tau(\tau)\) is stable under \(V(\pi'(\tau))\). In exactly the same way we prove that \(S_\tau\) is stable under \(V(\pi)\).

From this follows that the image of \(V(\pi)\) (resp. \(V(\pi'(\tau))\)) by the map \(\text{Der}_k(H, H) \to \text{Der}_k(H, S_\tau)\) (resp. \(\text{Der}_k(E_\tau(\tau), E_\tau(\tau)) \to \text{Der}_k(E_\tau(\tau), S_\tau)\)) sits in \(\text{Der}(O_{S_\tau}, O_{S_\tau}) \to \text{Der}(H, O_{S_\tau})\) (resp. \(\text{Der}(E_\tau(\tau), E_\tau(\tau)) \subseteq \text{Der}_k(E_\tau(\tau), E_\tau(\tau))\)). Moreover \(V(\pi) \otimes_{O_{S_\tau}} H \subseteq \text{Der}_k(E_\tau(\tau), E_\tau(\tau))\).

Lemma (3.15). Suppose the conditions \((V')\) hold. Then there are surjective homomorphisms of Lie-algebras

\[V(\pi'(\tau)) \to V(\pi'(\tau)), \quad V(\pi) \to V(\pi_\tau)\]
inducing surjective $E_\tau(t)$, resp. $\mathcal{O}_S$-module homomorphism

$$\left.\begin{array}{c}
\nu(t) \otimes E_\tau(t) \\ E(t) \end{array}\right\} \twoheadrightarrow \left.\begin{array}{c}
\nu(t) \otimes \mathcal{O}_{S_\tau} \\ \mathcal{H}_{S_\tau} \end{array}\right\} \twoheadrightarrow \left.\begin{array}{c}
\nu(t). \end{array}\right\}$$

**Proof.** The existence is already clear. To prove the surjectivity it suffices, using (3.3) (iv) to show that $\nu(t)$ maps surjectively onto $\nu(t,p)$ where $p: E(t) \rightarrow E_\tau(t)$ is the quotient map. But consider the diagram of exact sequences,

$$\begin{array}{ccccccc}
\nu(t) & \otimes & E_\tau(t) & + & \text{Der}_k(E(t)) & \otimes & E_\tau(t) & + & \text{A}^1(E(t), \tilde{Y}, \mathcal{O}_Y) & \otimes & E_\tau(t) & + & 0 \\
& + & s & + & r & + & q \\
0 & + & \nu(t,p) & + & \text{Der}_k(E(t), T(t)) & + & \text{A}^1(E_\tau(t), \tilde{Y}_0, \mathcal{O}_{\tilde{Y}}) & + & 0
\end{array}$$

By (3.4) (iii) $q$ is an isomorphism, $r$ is surjective (in fact, $r$ is an isomorphism) therefore $s$ is surjective. Q.E.D.

Combining the last two lemmas, we obtain the following diagram of exact sequences,

$$\begin{array}{ccccccc}
\text{Der}_H(E_\tau(t), E(t)) & + & \lambda(t) \\
0 & + & \text{ker } n(t) & + & \nu(t) \otimes E_\tau(t) & + & \nu(t) \otimes E_\tau(t) & + & 0 \\
\text{m}(t) & + & \nu(t) \otimes E_\tau(t) & + & 0
\end{array}$$

from which we deduce

**Proposition (3.16).** Suppose the conditions (V') hold. For every $t \in S_\tau$, $\lambda(t)$ is an isomorphism, implying that
(i) \( V(\pi(t)) \otimes k(t) = \ker\{n(t): V(\pi) \otimes k(t) + V(\pi) \otimes k(t)\} \)

\( H(t) \)

(ii) \( V(\pi'(t)) \otimes E(\tau(t)) \) is a semidirect product of \( E(\tau(t)) \) and \( \text{Der}_{H_0(t)}(E(\tau(t))) \), as Lie-algebras

(iii) \( V(\pi'(t)) = \text{Der}_{H_0(t)}(E(\tau(t)), E(\tau(t))) \).

(iv) The kernel of \( V(\pi) \otimes O + V(\pi) \) is a flat \( O_{\tau(S)} \)-Lie-algebra, the fibers of which are the Lie algebra \( L(t) \), \( t \in S_\tau \).

Proof. Clearly \( \text{Der}_{H(t)}(E(t), E(\tau(t))) = \text{Der}_{H_0(t)}(E(\tau(t))) \) which by the formula following (3.14) is equal to \( V(\pi'(t)) \), therefore \( \lambda(t) \) is an isomorphism. Consequently \( n(t) \) splits and (ii) follows.

Tensorization with \( k(t) \) on \( T(t) \) yields (i). Finally (iii) is already proved, and (iv) follows from (ii) since \( E(\tau(t)) \) is faithfully flat as \( O_{\tau(S)} \)-module.

Q.E.D

Remark (3.17). It follows from (3.16) that \( V(\pi'(t)) \otimes E(\tau(t)) \) is the semidirect product of the \( H_0(t) \)-Lie-algebras

\( (V(\pi(t)) \otimes E(\tau(t))) = (V(\pi(t)) \otimes H_0(t)) \otimes E(\tau(t)) \) and

\( \text{Der}_{H_0(t)}(E(\tau(t)), E(\tau(t))) \). Consequently \( V(\pi'(t)) \otimes E(\tau(t)) \) is an \( H_0(t) \)-Lie-algebra, and \( (V(\pi(t)) \otimes E(\tau(t))) \) is an \( E(\tau(t)) \)-Lie-algebra.

Proposition (3.18). Assume the conditions (V') hold. Let \( 0 < \tau < t \).

Then there is a unique way of glueing together the families

\( \pi_0(t): \tilde{X}_0(t) + H_0(t), t \in S_\tau \)

in the category of algebraic spaces, to form a family
\[ \tilde{\pi}_\tau : X_\tau \to M_\tau \]

such that,

(i) \( M_\tau \) is a quotient of \( S_\tau \), i.e. there is a natural epimorphism \( S_\tau \to M_\tau \).

(ii) if \( \pi_\tau \) restricted to some subscheme \( Y \subseteq M_\tau \), is constant, then \( Y \) is finite and reduced.

**Proof.** Take a finite covering of \( S_\tau \) in the étale topology, of the form \( E_\tau(t_i) \to S_\tau \), and consider the diagram

\[
\begin{array}{cccc}
\bigcup_{i,j} E_\tau(t_i) \times_{S_\tau} E_\tau(t_j) & \to & \bigcup_{e} E_\tau(t_e) \to S_\tau \\
\rho_1 & & & \bigcup_{e} H_0(t_e) \times_{S_\tau} H_0(t_e)
\end{array}
\]

Consider the scheme theoretic image \( R \) of \( \rho_1 \). Obviously \( R \to \bigcup_{i,j} H_0(t_i) \times_{S_\tau} H_0(t_j) = U \times U \), where we have put \( U = \bigcup_{i} H_0(t_i) \), is a closed equivalence relation.

Now we shall prove that \( R \) is étale. This is clearly the same as to say that \( R_{ij} = H_0(t_i) \times_{S_\tau} H_0(t_j) \) is étale on \( H_0(t_k) \), \( k = i,j \).

Consider this on the affine level, then

\( R_{ij} = \text{im}(H_0(t_i) \otimes_{S_\tau} H_0(t_j) \to T(t_i) \otimes_{S_\tau} T(t_j)) \), and we must show that for every \( k \)-point \( t \in R_{ij} \) mapping to the point \( t_1 \in H_0(t_i) \) and to the point \( t_2 \in H_0(t_j) \) the homomorphisms

\[
\begin{align*}
H_0(t_i)_{t_1} & \to R_{ij,t} \\quad \sim \\quad H_0(t_j)_{t_2} \to R_{ij,t}
\end{align*}
\]

are isomorphisms.

Now there exist points \( s_1 \in E_\tau(t_i) \), \( s_2 \in E_\tau(t_j) \) mapping to the same
Consider the morphism of local rings

\[
\xymatrix{
E_{\tau}(t_i)_{S_1} & H_0(t_1)_{T_1} \\
S & H_0(t_i)^* \ar[rr]^\varphi_1 \\
E_{\tau}(t_j)_{S_2} & H_0(t_j)_{T_2} \\
}
\]

with their induced families. Since the fiber of the family on \(H_0(t_1)\) at \(t_1\) is isomorphic to the one on \(H_0(t_j)\) at \(t_2\), and both are isomorphic to the special fiber of the family on \(H_0(s)\), we know that there must, after completion, exist isomorphisms \(\varphi_i\), \(i = 1,2\), and a morphism \(\psi\) as in the diagram, compatible with the families. Since we are in the prorepresentable substratum all such morphisms must be unique. Consequently the following diagram must be commutative,

\[
\xymatrix{
E_{\tau}(t_i)_{S_1} \ar[d]_S & H_0(t_1)_{T_1} \ar[rr]^\varphi_1 \\
E_{\tau}(t_j)_{S_2} & H_0(t_j)_{T_2} \ar[rr]_{\psi_2} \\
& H_0(s)^* \ar[u]_{\psi_1} \\
}
\]

But then \(H_0(t_1)_{T_1} \circ H_0(t_j)_{T_2}^*\) maps onto \(\psi(H_0(s)^*)\) in \(S^*\). Since \(R_{ij,k}\) is equal to this image, we obtain isomorphisms

\[
H_0(t_1)_{T_1} = R_{ij,k}^*
\]

\[
H_0(t_j)_{T_2} = R_{ij,k}^*
\]

Since \(R_{ij,k}\) is of finite type as \(k\)-algebra, as well as \(H_0(t_k)\), \(k = i,j\), and since \(k\) is separably closed the proposition follows from (EGA IV 17.6.3), see also [Kn] (I.4.5).

Q.E.D.
There are no reasons to expect the $M_\tau$'s to be schemes, and there are, in particular, no reasons at all to expect $M_\tau$ to be a scheme-theoretic quotient of $S_\tau$. However we shall under the conditions (V') prove that there are scheme-theoretic quotients of an open dense subset of the reduced normalized components $S_{\tau,c}$ (the union of which cover $M_\tau$), having good properties. First we need some preparation.

**Lemma (3.19).** Let $S$ be a noetherian normal domain, $K = K(S)$ its quotient field and let $T$ be a subring of $K$ containing $S$. Suppose for every prime $p$ of $S$ of height 1 that $\text{Spec}(T_p) \rightarrow \text{Spec}(S_p)$ is surjective. Then $S = T$.

**Proof.** Let $p$ be any prime ideal of height 1 in $S$, and let $q$ be any preimage in $T$. Then $S_p \subseteq T_q \subseteq K$. Since $S_p$ is a valuation ring, therefore maximal, we have $S_p = T_q$. Obviously $T \subseteq \cap T_q = \cap S_q = S$, the last equality being well-known, see for instance, [Serre, J. P.: Multiplicités (III-13, Remarque à Proposition 9)]. Q.E.D.

Notice also the following classical result,

**Lemma (3.20).** If $S$ is a normal domain with quotient field $K$, and if $K \subseteq L$ is a finite separable field extension, then the normal closure $T$ of $S$ in $L$ is a finite extension of $S$.

**Proof.** See, Serre: Loc-cit. (III-16, Proposition 11) or Zariski-Samuel: Vol. 1, Ch. V, Thm. 9. Q.E.D.
Lemma (3.21). Let \( \pi_0 : \tilde{X}_0 \rightarrow \mathcal{H}_0 \) with \( \mathcal{H}_0 \) reduced and irreducible be a flat family, \( h_0 \in \mathcal{H}_0 \), \( X = \pi_0^{-1}(h_0) \). Suppose \( h_0 \) is in the prorepresenting stratum of \( \mathcal{H}(X) \). Let \( \rho_0 : \mathcal{H}_1 \rightarrow \mathcal{H}_0 \) be the normalization of \( \mathcal{H}_0 \), and let \( \rho_1 : \mathbb{N} \rightarrow \mathcal{H}_1 \) be étale and finite. Consider an epimorphism \( \rho : T \rightarrow \mathbb{N} \) commuting with automorphisms \( g \) on \( \mathbb{N} \) and \( g' \) on \( T \). Let \( t \in T, n \in \mathbb{N}, h \in \mathcal{H}_1 \) be such that \( \rho(t) = n, \rho_1(n) = h, \rho_0(h) = h_0 \). Suppose \( n \) is a fix-point for \( g \), and assume there is an isomorphism \( \theta_g : \rho_1^*(\rho_0 \circ \rho_1 \circ \rho)^*(\tilde{X}_0) \overset{\sim}{\rightarrow} (\rho_0 \circ \rho_1 \circ \rho)^*(\tilde{X}_0) \). Then \( g = id_{\mathbb{N}} \).

Proof. Since \( \rho \circ \rho_1 \circ \rho(\tilde{X}_0) = g \circ \rho \circ \rho_1 \circ \rho(\tilde{X}_0) \) at \( t \) it is clear that the formalization of \( \rho_0 \circ \rho_1 \circ \rho \) at \( t \) resp. \( n_0 \) coincides with the formalization of \( \rho_0 \circ \rho_1 \circ \rho \circ \rho' = \rho_0 \circ \rho_1 \circ \rho \circ \rho' \) at \( t \) resp. \( n_0 \). Since \( \rho \) is epimorphic, the formalization of \( \rho_0 \circ \rho_1 \) at \( n \) resp. \( n_0 \) coincides with that of \( \rho_0 \circ \rho_1 \circ g \), but then \( \rho_1 \) and \( \rho_1 \circ g \) coincide since \( \rho_0 \) is a normalization. But then \( g \in Aut_{K(H)}(K(\mathbb{N})) \) and since \( n \) is unramified over \( h \), \( g \) cannot leave \( n \) fixed unless \( g \) is the identity. Q.E.D.

Lemma (3.22). Keep the notations and assumptions of (3.2). Assume moreover that \( \rho : T \rightarrow \mathbb{N} \) is flat. Then there is a unique isomorphism

\[ \theta_g : \rho_1^*(\rho_0 \circ \rho_1)^*(\tilde{X}_0) \overset{\sim}{\rightarrow} (\rho_0 \circ \rho_1)^*(\tilde{X}_0) \]

commuting with \( \theta'_g \).

Proof. By assumption \( \rho \) is faithfully flat and the result therefore follows from ordinary decent theory, see [Gr2] (B. Theorem 2). Q.E.D.
Remark (3.23). Instead of a family \( \pi_0: \tilde{X}_0 \to \mathbb{H}_0 \) of preschemes, we may consider any family of quasicoherent sheaves. The conclusion of (3.21) and (3.22) still hold. From now on, we shall assume that the objects \( X \) we are considering is of a class for which faithfully flat morphisms are of descent and étale morphisms are of strict descent. It follows from [Gr2] (B. Theorem 3) that this includes the class of quasicoherent sheaves and the class of quasiprojective schemes.

Now we are ready to prove the main theorem of this §.

Theorem (3.24). Suppose the conditions \((V')\) hold for all objects \( X(t) \) involved. Suppose also that these objects satisfy the assumptions of (3.23). Let \( S^n_{t,c} \) be the normalization of the reduction of the component \( S_{t,c} \) of \( S_t \), and let \( \pi_n: \tilde{X}_n \to S^n_{t,c} \) be the pull-back of \( \pi: \tilde{X}_t \to S_t \). Then there exists a "strict" scheme-theoretic quotient \( N_{t,c} \) of an open dense subscheme \( S'_{t,c} \) of \( S^n_{t,c} \) with respect to \( V'(\pi') \). Moreover there is a flat family \( \mathcal{X}_{t,c} \to N_{t,c} \) and, in the category of algebraic spaces, a cartesian diagram

\[
\begin{array}{ccc}
\tilde{X}_{t,c} & \xrightarrow{\pi'} & S'_{t,c} \\
\downarrow & & \downarrow \rho_{t,c} \\
\mathcal{X}_{t,c} & \xrightarrow{\mu_{t,c}} & N_{t,c}
\end{array}
\]

such that

(i) If \( Y \to N_{t,c} \) is a connected subscheme along which \( \mu_{t,c} \) is constant, then \( Y \) is finite.

(ii) \( N_{t,c} \) is formally in the prorepresenting stratum for all \( c \) if and only if the reduction of every component of \( H_0(s), s \in S_t \), is normal.

(iii) \( \rho_{t,c} \) is smooth.
Proof. Consider \( S_\tau, c + S_\tau, c + S_\tau \) and let \( s_0 \in S^n_\tau, c \) map onto \( s \in S_\tau, c \). Pick a connected étale neighbourhood

\[
\eta': T(t') + S_\tau
\]

with \( \eta'(t') = s \), such that there exists a morphism

\[
\rho': T(t') + H_0(s)
\]

with the property that

\[
\eta'^*(\bar{x}_\tau) = \rho'^*(\bar{x}_0(x))
\]

where as above \( \pi_\tau: x_\tau + S_\tau \) is the restriction of \( \pi \) to \( S_\tau \) and \( \pi_0(s): x_0(s) + H_0(s) \) is the restriction of \( \pi(s) \). Recall that, under our conditions, we may assume \( \rho' \) smooth. Take the pull-back of \( \eta' \) to \( S^n_\tau, c \) and get a cartesian diagram

\[
\begin{array}{c}
H_0(s) + \rho', T(t') \\
\eta' + S_\tau \\
v + \\
\eta + S^n_\tau, c
\end{array}
\]

Since \( S^n_\tau, c \) is integral, \( T(t) \) is also integral and \( v \) is the normalization of its image in \( T(t') \). \( \eta \) is an étale morphism. Restricting to an open dense subset \( S'_\tau, c \) of \( S^n_\tau, c \) we may assume \( \eta \) is finite. Therefore we may as well assume \( T(t) + S'_\tau, c = \eta \) is a Galois covering with Galois group \( G = \text{Gal}(T/S^0) \). Let \( H_0(s)_c \) be the reduced image of \( \rho' \circ v \). Since \( T(t) \) is reduced, irreducible and connected, \( H_0(s)_c \) is a reduced component of \( H_0(s) \). Put \( H_0 = H_0(s)_c \) and let \( \rho_0: H_1 + H_0 \) be the normalization of \( H_0 \). To simplify notations, we shall put \( T = T(t) \), \( S' = S'_\tau, c \). Consider now the diagram
where \( \rho \) is the pull-back of \( \rho' \). Since \( H_1 \) is normal and \( \rho \) is smooth we know by (EGA IV (7.3.8)) that \( T' \) is normal, therefore \( T' = T \).

Let \( \pi_0: X_0 \to H_0 \) be the pull-back of \( \pi_0(s) \) by \( H_0 \to H_0(s) \), and let \( \pi': X' \to S' \) be the pull-back of \( \pi_T \) by \( S' \to S_T \). Then we know (identifying \( T' \) and \( T \)) that

\[
\eta^*(X') = (\rho_0 \circ \rho)^*(X_0).
\]

Let \( \pi: X \to T \) be this common pull-back of \( \pi_0 \) and \( \pi' \). We may assume \( H_i = \text{Spec}(H_i) \), \( i = 0, 1 \), \( T = \text{Spec}(T) \) and \( S' = \text{Spec}(S') \). Then, since \( \eta \) is étale, we find

\[
V(\pi') \subseteq \text{Der}_K(S')
\]
\[
V(\pi) \subseteq \text{Der}_K(T)
\]
\[
V(\pi') \otimes_T S' = V(\pi)
\]

see (3.3), and observe that \( \text{Der}_K(S', T) = \text{Der}_K(T, T) \) and therefore by (3.3) (iv), \( V(\pi) = V(\pi', \eta) \), and since \( V(\pi) \otimes_T S' = V(\pi', \eta) \). The last statement follows from the fact that \( T \) is locally free of finite rank on \( S' \), coupled with (3.3) (i).
In particular, this implies

\[ S \cdot V \rightarrow T^V \]

and for every \( g \in \text{Gal}(T/S') \) and \( t \in T^V \), \( g(t) \in T^V \). Thus \( T^V \) is stable under \( G = \text{Gal}(T/S') \), and obviously \( (T^V)^G = S \cdot V \). We know already (see (3.16) (iii)) that

\[ V(\pi'_0(t')) = \text{Der}_{H_0}(S)(T(t')) \]

where \( \pi'_0(t') \) is the pull-back of \( \pi_t \) by \( \eta' \). From this follows easily that

\[ V(\pi) = \text{Der}_{H_1}(T) = \text{Der}_{T^V}(T). \]

In fact since \( \rho' \) is smooth \( \text{Der}_{H_0}(S)(T(t'), T(t')) \otimes T = T(t') \), \( \text{Der}_{H_1}(T, T) \subseteq V(\pi) \). Notice also that \( V(\pi_0(S)) = 0 \) therefore

\[ V(\pi_0) = 0 \]

and so \( V(\rho_0^*(\pi_0)) = 0 \) since \( \rho_0 \) is the normalization map. This implies \( H_1 \subseteq T^V \). As \( V(\pi) \) has the same rank as \( V(\pi'_0(t')) \), namely the dimension of the fibers of \( \rho' \), \( \text{Der}_{H_1}(T) \) and \( V(\pi) \) has the same rank everywhere, therefore \( \text{Der}_{H_1}(T) = V(\pi) \).

It follows that the codimension of \( \rho \) which is equal to the rank of \( V(\pi) \) is equal to the rank of \( \text{Der}_{T^V}(T) \). However, rank \( \text{Der}_{T^V}(T) = \text{tr.deg.}(K(T)/K(T^V)) \).

Consequently \( K(H_1) \subseteq K(T^V) \) has trancendence degree 0, and is therefore algebraic. We shall show that \( K(H_1) \subseteq K(T^V) \) is a finite extension.

In fact, if \( K(H_1) \subseteq K_2 \) is a finite subextension and \( H_2 \) is the integral closure of \( H_1 \) in \( K_2 \), then \( \rho_1: H_2 \rightarrow H_1 \) is finite. Since \( T \) is normal, \( T^V \) is also normal, therefore \( H_2 \subseteq T^V \). Given \( g' \in S' \),
$\eta^{-1}(s') = \{t_1, \ldots, t_r\}$ and let $h_i = \rho_2(t_i), i = 1, \ldots, r$, where $\rho_2 : T + H_s \rightarrow$ is the obvious morphism. Since all fibers of $\rho : T + H_1$ are smooth, the points $h_1, \ldots, h_r$ must be unramified for $\rho_1$. It follows from (E.G.A. IV (8.4.9)) that $\rho_1$ is étale in $h_j, j = 1, \ldots, r$. But then $\rho_2 : T + H_2$ is smooth at the points $h_j \in H_2, j = 1, \ldots, r$, since the composition $T \xrightarrow{\rho_2} H_2 \xrightarrow{\rho_1} H_1$ is smooth. Therefore $\text{im} \rho_2$ contains an open set $U_2 \subseteq H_2$ containing $h_1, \ldots, h_r$. In particular $U_2$ will contain the generic fiber of $\rho_1$.

Now suppose there exist an infinite chain of finite sub-extensions $K(H_1) \subseteq K_2 \subseteq K_3 \subseteq \ldots \subseteq K(T^V)$. Let $H_i$ be the integral closure of $H_1$ in $K_i$, then $\rho$ splits up into an infinite composition $T + T^V + \ldots + H_{i+1} + H_i + \ldots + H_1$. The generic fiber of $\rho$ would then have an infinite number of connected components, which is nonsense.

Therefore $K(H_1) \subseteq K(T^V)$ is a finite extension. Put $K_2 = K(T^V)$ and let $U_2 \subseteq H_2$ be as above. Since $H_2$ is affine, there exists an $m \in H_2$ such that the open set $D(m)$ is contained in $U_2$ and contains $h_1, \ldots, h_r$. In particular $m(h_i) \neq 0, i = 1, \ldots, r$. Let $N(m) = \prod g(m)$ be the norm of $m$. Obviously $N(m) \in S^V$ and since the elements of $G$ permutes the points $t_1, \ldots, t_r$ we have $N(m)(t_1) \neq 0, i = 1, \ldots, r$, and thus $N(m)(s') \neq 0$. Therefore $h_1, \ldots, h_r \in D(N(m)) \subseteq D(m) \subseteq U_2$. By construction the restriction of $\rho_2$ is surjectiv on $D(N(m))$. Since $H_2[N(m)] \subseteq T^V[N(m)] \subseteq T[N(m)]$ and $D(N(m)) = \text{Spec}(H_2[N(m)])$ it follows that $\text{Spec}(T^V_{[N(m)]}) \rightarrow \text{Spec}(H_2[N(m)])$ is onto.
As $H_2\{N(m)\}$ obviously is normal, it follows from (3.19) that
$H_2\{N(m)\} = T^V\{N(m)\}$, in particular $T^V\{N(m)\}$ is of finite type.
From (3.21) we conclude that $G$ operates on $T^V\{N(m)\}$ without fix-points. Therefore
\[
S^V\{N(m)\} = (T^V\{N(m)\})^G \subseteq T^V\{N(m)\}
\]
is étale and $S^V\{N(m)\}$ is of finite type. Summing up, what we have
proved so far is.

Given any $s_0' \in S'_\tau, c$ there exists an affine neighbourhood $S' \subseteq S'_{\tau, c}$
of $s_0'$ such that for every $s' \in S'$ and any open neighbourhood $U \subseteq S'$ of $s'$ there exists an element $N(s') \in S^V\{N(m)\}$ such that
$s' \in D(N(s')) \subseteq U$, and $S^V\{N(s')\}$ is of finite type.
For every $s' \in S'$ take $N(s') = \text{Spec}(S^V\{N(s')\})$ and glue. The
 glueing is unproblematic, and we obtain a scheme $N_{\tau, c}$ the
quotient of $S'_{\tau, c}$ by $V(\pi')$. Since locally the morphism
\[
\rho_{\tau, c} : S'_{\tau, c} \to N_{\tau, c}
\]
sits in the following diagram
\[
\begin{array}{ccc}
H_2\{N(m)\} & \eta_2 \to & N_{\tau, c} \\
\rho \downarrow & \rho_{\tau, c} \downarrow & \eta_1 \downarrow \\
T^V\{N(m)\} & \eta \to & S'_{\tau, c}
\end{array}
\]
where $\rho$ is smooth, $\eta$ and $\eta_2$ are étale, it follows that $\rho_{\tau, c}$ is
smooth. Moreover, since by (3.22) the descent-datum on $\eta^*(\pi')$
descends to a descent-datum on $(\rho_0 \circ \rho_1)^*(\pi_0)$ for $\eta_2$, this family
descends to $S^V\{N(m)\}$. Glue these families and obtain a family
\[
\mu_{\tau, c} : \tau, c \to N_{\tau, c}
\]
which, must pull back by $\rho_{\tau, c} \circ \eta$ to $\eta^*\pi'$. The
rest is obvious.

Q.E.D.
It is clear that there is a quasifinite dominant morphism $N_{\tau,c} \to M_{\tau}$. Moreover if there exists a coarse moduli space for the objects of the family $\pi_\tau$, then, locally, it must be a discrete quotient of $M_{\tau}$. Therefore the dimension of $N_{\tau,c}$ is the correct "number of moduli" in the sense of Riemann.

**Definition (3.25).** We shall refer to the collection $M = \{M_\tau\}$ as the local moduli suite of $X$.

The strict modality of $X$ is the integer

$$sm(X) = \dim H_0.$$ 

By the $\tau$-modality of $X$ we shall mean the integer

$$\tau m(X) = \max \{\dim N_{\tau,c} | c\}.$$ 

The modality of $X$ is defined to be

$$m(X) = \max \{\tau m(X) | 0 < \tau < \tau_1\}.$$ 

Finally, the degeneration set of $X$ is defined to be the partially ordered set

$$\Gamma(X) = \{S_{\tau,c} | 0 < \tau < \tau_1, c\}$$

where the order is defined by

$$S_{\tau,c} > S_{\tau',c'} \iff S_{\tau,c} \supset S_{\tau',c'}.$$ 

Suppose, as above, that $H$ is non singular, and suppose the generic fiber of $\pi: \tilde{X} \to H$ is rigid. Then by (3.10) $\tau(X) = \dim H$ is equal to $\dim_k (V_{\tilde{H}})$. therefore the inclusion $V \subseteq \text{Der}_k(H)$ induces an $H$-linear map $i: \tilde{V} \to \tilde{\text{Der}}_k(H) = H$, the image of which is a principal ideal.
Definition (3.26). The image $\Delta$ of $i$, a principal ideal generated by the determinant of $V$, is called the discriminant of $\pi$. The corresponding hypersurface will be denoted by $\Delta$.

Remark (3.27). By definition $\Delta = \bigcup_{\tau > 0} S_{\tau}$. 

§4 APPLICATIONS TO ISOLATED HYPERSURFACES SINGULARITIES

Introduction. Up to now we have been studying the problem of local moduli in as general a setting as possible.

In this and the following §'s we shall restrict our attention to hypersurface singularities in the algebroid sense.

If \( X = \text{Spec}(k[x_1, \ldots, x_n]/(f)) \) is a hypersurface with isolated singularities, then \( X \) satisfies the conditions \((A_1)\) and \((V')\) but not necessarily \((A_2)\), see the main Introduction for an example.

Therefore we cannot invoke the theorems (3.18) and (3.25). As we have seen the local moduli problem in this case is much more difficult. If, however, we restrict our attention to the completions \( k[[x_1, \ldots, x_n]]/(f) \), we have to be a little careful with respect to which category we are working in, and we have to change our cohomology slightly, but otherwise we find ourselves in the situation of §3.

The basic fact is, of course, that in this algebroid situation Artin's approximation theorem applies so that \((A_2)\) holds.

Thus, in particular, there is, in the category of algebraic spaces, a local moduli suite \( \{M_{\tau}\} \), see (3.18), and (3.24) holds. The problem of actual computation of the components \( M_{\tau} \) seems at this point outside of our reach (compare in this connection the quotation from Zariski in the introduction of §5).

Now, for hypersurface singularities, we have at our disposal the algebro-topological invariant \( \mu \), the Milnor number, and we may restrict our attention to the \( \mu \)-constant stratum \( H_{-\mu} \) of \( H \) and to the corresponding \( \mu \)-constant substratum \( M_{\mu \tau} \) of \( M_{\tau} \). The
structure of these substrata is still much too difficult for us, even in the quasihomogenous case. One problem is that $S_{\tau}$ may contain one component inside $H_\mu$ and another component intersecting the first one but not included in $H_\mu$, see §5 for examples.

The main result of this §, (4.5), is an "explicite" calculation of the kernel of the Kodaira-Spencer map of $\tau_\mu$. This result is the basic tool in our study of the curve case in §5 and §6.

From now on $X$ will be a hypersurface $f$, i.e. $X = \text{spec}(A)$, $A = k[x_1, \ldots, x_n]/(f)$ where $f = f(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n]$. Suppose $f$ has only isolated singularities, then $X$ satisfies the conditions (V') of §3. Moreover

$$A^1(k, X; \mathcal{O}_X) = \frac{k[x_1, \ldots, x_n]}{(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)}.$$ 

Pick a base for $A^1(k, X; \mathcal{O}_X)$ represented by $\left\{\lambda_i\right\}_{i=1}^{\tau_1}$, $\lambda_i \in k[x]$, where as above,

$$\tau_1 = \tau(f) = \dim_k A^1(k, X; \mathcal{O}_X).$$

Put, $H = k[t_1, \ldots, t_{\tau_1}]$, $F = f + \sum_{i=1}^{\tau_1} \lambda_i t_i \in H[x]$ and $\tilde{A} = H[x]/(F)$.

Then $\tilde{X} = \text{Spec}(\tilde{A}) + H = \text{Spec}(H)$ is the versal family with which we shall have to work.

Remark (4.1). Since (A_2) of §3 is not satisfied for affine schemes in general, see the example of appendix 2, we cannot use the theorems (3.18) and (3.24). However there is a prorepresenting substratum $H_0$ of $H$. Moreover we have
Proposition (4.2). The Kodaira-Spencer map

\[ g: \Theta_H \to A^1(H, \tilde{A}; \tilde{A}) \]

is given by

\[ g(\frac{\partial}{\partial t_i}) = \text{class of } \frac{\partial F}{\partial t_i} \text{ in } H[x_1, \ldots, x_n]/(F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}). \]

Proof. By definition of \( g \), it is the obstruction for lifting \( 1 \in \text{Aut}_H(\tilde{A}) \) to an isomorphism \( \phi: i_0^*(\tilde{A}) \to i_1^*(\tilde{A}) \) where \( i_K: H \to H \otimes \mathbb{H}/I^2 \), \( K = 1, 2 \) and \( I = \ker(H \otimes H + H) \) are defined as the \( \nu_K \)'s in §3.

Now \( H \otimes H/I^2 = k[[\zeta, \eta]]/(\zeta^2 - \zeta \eta)^2, \ i_0^*(\tilde{A}) = k[[\zeta, \eta]][\zeta]/F_\zeta, \ i_1^*(\tilde{A}) = k[[\zeta, \eta]][\zeta]/F_\zeta \), where \( F_\zeta = f + \epsilon \lambda_1 \zeta, \ F_\eta = f + \epsilon \lambda_1 \eta \). Obviously this obstruction is simply given by the difference \( (F_\zeta - F_\eta) = \epsilon \lambda_1(\zeta^2 - \zeta \eta) \) in \( A^1(H, \tilde{A}; H \otimes I/I^2) = H[[x]]/(F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}) \otimes I/I^2 \). But then

\[ g = \Sigma(\text{class of } \lambda_1) \cdot dt_1. \]

Q.E.D

Now, in this § we shall concentrate on the deformations of the completion \( \hat{A}(\x) = k[[\x]]/(f) \) as a complete augmented \( k \)-algebra, see [Z]. (We shall refer to \( A^\wedge(\x) \) or to \( f \) as the hypersurface singularity). By a deformation of \( \hat{A}(\x) \to k \), to an augmented \( k \)-algebra \( R \to k \) we shall understand any commutative diagram

\[
\begin{array}{ccc}
R & \to & A_R \\
\downarrow & & \downarrow \\
\hat{A}(\x) & \to & k
\end{array}
\]

where \( \rho: A_R \to R \) is a flat augmented \( R \)-algebra such that

\[ A_R = \lim_{n} A_R/(\ker \rho)^n, \text{ and where } A_R \otimes k = \hat{A}(\x). \]
We may develop a cohomology theory for this case, with a corresponding obstruction theory, and we may prove the existence of a formal moduli. Moreover, for any flat augmented $R$-algebra $\rho : A_R \to R$ such that $A_R = \varprojlim A_R/(\ker \rho)^n$, there exists a Kodaira-Spencer map just as above.

The cohomology is given by

$$A^1(k, \hat{A}(x) \to k, \hat{A}(x)) = H^1(k, \hat{A}(x); (x)\hat{A}(x)) = \frac{(x)}{(x)}(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$$

$$= \ker \left[ k[[x]]/(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \to k \right]$$

$$\text{coker} \left[ \text{Der}_k(k[[x]]/(f)) \to \text{Der}_k(k[[x]]/(f), k) \right]$$

$$A^1(R, A_R \to R; A_R) = H^1(R, A_R; \ker \rho).$$

If we put, as customary, $\tau(f) = \dim_k k[[x]]/(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$, usually called the Tjurina number of the singularity, then picking any basis $\{t_1^*, \ldots, t_s^*\}$ for $A^1(k, \hat{A}(x) \to k, \hat{A}(x))$, we find $s = \tau - 1 + n$ and the formal moduli of $\hat{A}(x) \to k$ is $H^\wedge = k[[t_1, \ldots, t_s]]$. The formal versal family $X^\wedge = H.[[x]]^\wedge/(F.)$, is given by $F. = f + \sum_{i=1}^s g_i * t_i^*$, where $g_i$ are representatives in $(x)k[[x]]$ of $t_i^*$, and where $H.[[x]]^\wedge$ means completion in the $t$-adic topology. From now on we shall always pick a basis $\{t_1^*, \ldots, t_s^*\}$ for which we may choose the representatives $g_i$ to be monomials in $k[[x]]$. Put $g_i = \frac{a_i}{x^i}$, $i = 1, \ldots, s$, then $F. = f + \sum_{i=1}^s \frac{a_i}{x^i} * t_i$. Now let us consider the complete augumented $H.$-algebra $H.[[x]]/(F.) \to H.$ where $H. = k[t_1, \ldots, t_s]$ and put $\tilde{H.} = \text{Spec}(H.)$; $\check{X.} = \text{Spec}(H.[[x]]/(F.))$. Then

$$\tilde{H.} \to \check{X.} \to H.$$ is a versal deformation of the pointed complete local scheme

$\text{Spec}(k) \to \text{Spec}(\hat{A}(x)) \to \text{Spec}(k)$. 
The corresponding Kodaira-Spencer map

$$g: \theta_H \to A^1(H;H[[x]]/(F.)) \to H;H[[x]]/(F.)$$

is given by

$$g(\frac{\partial}{\partial t}) = \text{class of } \frac{\partial F_i}{\partial t_i} = \text{class of } X_i,$$

in \( (x)^*H;H[[x]]/(F.,\frac{\partial F_i}{\partial x_1},...,\frac{\partial F_i}{\partial x_n})(x).$$

Copying the proof of (3.18) and (3.24) we prove the following result.

**Theorem (4.3).** Let \( f \in k[x] \) be an isolated singularity with

\( \tau_0 = \tau(f) \). Then there exists a local moduli suite for \( f \).

Moreover there exists a finite collection of families of augmented algebraic schemes,

\[ \mu_{\tau,c}: X_{\tau,c} \to N_{\tau,c}, \quad 0 < \tau < \tau_0 \]

such that

(i) \( N_{\tau,c} \) is the quotient of an open dense subscheme of \( S_{\tau,c} \) by \( V \) (see (3.24)).

(ii) There exists a quasifinite dominant morphism \( N_{\tau,c} \to M_{\tau} \) such that \( \mu_{\tau,c} \) is the pull back of \( \pi_{\tau} \).

There seems at present to be little hope getting much further in the study of the local moduli suite \( M = \{ M_{\tau} \} \) for general hypersurface singularities.

However, if we restrict ourselves to quasihomogenous singularities \( f \), and if we fix the Milnor number we are in a much better situation. The rest of this § is therefore devoted to the study of
topologically constant deformations of isolated hypersurface singularities \( f \), where \( f \) is quasihomogenous of weights \( w_1, \ldots, w_n \) and degree 1.

Recall the definition of the Milnor number of an isolated hypersurface singularity \( f \),
\[
\kappa[[x_1, \ldots, x_n]] = \dim_k \kappa[[x_1, \ldots, x_n]]/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n),
\]
and let from now on \( H_\mu \subseteq H. \) denote the closed substratum of \( H. \) where \( \mu \) is constant and equal to \( \mu = \mu(f) \).

Assume that:
\[
f = \sum_{i=1}^{n} a_i x_i, \quad w_i = \frac{1}{a_i}, \quad i = 1, \ldots, n.
\]

In this case \( A^1(k, \kappa[[x]](f) \to k, \kappa[[x]]/(f)) = \)
\[
(x) \cdot \kappa[[x]]/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n) \cdot (x)
\]
has a monomial basis \( \{x^\alpha | \alpha \in I.\} \)
where if \( I = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^+|0 < \alpha_i < a_i - 2\} \) then
\[
I. = \bigcup\{(0, \ldots, a_i - 1, 0, \ldots) | i = 1, \ldots, n\} \{0, \ldots, 0\}. \text{ Put}
\]
\[
I. = \{\alpha \in I. | |\alpha| > 1\}
\]
where \( |\alpha| = \sum_{i=1}^{n} w_i \alpha_i \), then
\[
H. = k[t_\alpha | \alpha \in I.].
\]
\[
H. = H/(t_\alpha / \alpha \in I. \setminus I. \mu) = k[t_\alpha | \alpha \in I. \mu].
\]

Moreover \( H. \) is a graded \( k \)-algebra with \( \deg t_\alpha = 1 - |\alpha| \). In particular \( H. \mu \) is therefore a negatively graded \( k \)-algebra.

The versal family \( F. \) restricted to \( \text{Spec}(H. \mu) = H. \mu \subseteq H. \), is then given as
\[
F. \mu = f + \sum_{\alpha \in I. \mu} x^\alpha t_\alpha, \text{ with } \deg F. \mu = 1.
\]
Notice that $H_\mu[[x]]/(\frac{\partial F}{\partial x_i})$ is a graded $k$-algebra, with $\deg x_i = w_i$, locally free as an $H_\mu$-module, with basis $\{x^a\}_{a \in \mathcal{O}}$. We shall have to compute the kernel $V_\mu = V(\pi_\mu)$ of the Kodaira-Spencer map

$$g_\mu : \text{Der}(H_\mu) \to H_\mu[[x]]/(F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n})$$

given by $g_\mu(\frac{\partial}{\partial t_a}) = \text{class of } \frac{\partial F}{\partial t_a}$, see (4.2).

But before this, let's consider the Lie-algebra $\text{Der}_k(k[[x]]/(f))$. Since $f$ is quasihomogeneous, there is one special derivation $D_0 \in \text{Der}_k(k[[x]]/(f))$, defined by

$$D_0(x_i) = w_i \cdot x_i, \ i = 1, \ldots, n.$$  

Moreover, for every $i, j = 1, \ldots, n$, there is a derivation $E_{ij} \in \text{Der}_k(k[[x]]/(f))$ defined by

$$E_{ij}(x_k) = \begin{cases} 0, & k \neq i, j \\ \frac{\partial f}{\partial x_j}, & k = i \\ -\frac{\partial f}{\partial x_i}, & k = j \end{cases}$$

Clearly $E_{ij} \in \text{Der}_k$, for all $i, j = 1, \ldots, n$, where $\text{Der}_k \subseteq \text{Der}_k(k[[x]]/(f))$, as in §3, is the Lie-ideal consisting of those derivations that may be lifted everywhere. Given any $D \in \text{Der}(k[[x]]/(f))$ there exist representatives $\xi_i \in k[[x]]$ of $D(x_i)$ such that

$$\sum_{i=1}^n \frac{\partial f}{\partial x_i} \cdot \xi_i = q \cdot f$$

for some $q \in k[[x]]$. Recalling the Euler identity $\sum_{i=1}^n w_i x_i \frac{\partial f}{\partial x_i} = f$, we find the relation
Since \( f \) has an isolated singularity at the origin, the only relations among the \( \frac{\partial f}{\partial x_i} \)'s are the trivial ones.

It follows that \( D - q \cdot D_0 = \sum r_{ij} E_{ij}, \ r_{ij} \in k[[x]], \) i.e.

\[
D - q \cdot D_0 \in \text{Der}_\pi
\]

thus the map

\[
\phi: k[[x]]/\left( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) \to \text{Der}_k(k[[x]]/(f))/\text{Der}_\pi
\]

defined by: \( \phi(q) = q \cdot D_0 \) is surjective.

It is not difficult to see that it is an isomorphism, but we shall not need it so we leave it as an exercise.

For every \( \alpha \in I_\pi \), let's denote by \( D_\alpha \) the image by \( \phi \) of \( x^\alpha \), i.e.

\[
D_\alpha = x^\alpha \cdot D_0.
\]

One may easily prove the following

**Lemma (4.4).** With the notations above, we have

1. \([D_\alpha, D_\beta] = (|\alpha| - |\beta|) D_{\alpha + \beta}\)
2. \( L = \text{Der}_k(k[[x]]/(f))/\text{Der}_1 \) is solvable
3. \([L, L] = L_0 \) is nilpotent and \( L/L_0 \) is generated by \( D_0 \).

Now, returning to the Kodaira-Spencer map, we observe that \( g_\mu \) may be factorized into

\[
g_\mu: \text{Der}_k(H_\mu) \to k[[x]]/\left( \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n} \right)
\]
and the natural map

\[ H_\mu[[x]] \frac{\partial F}{\partial x^\alpha} / (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}) \rightarrow H_\mu[[x]] / (F, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}) \]

where \( g_0(\frac{\partial}{\partial x^\alpha}) = x^\alpha \) for \( \alpha \in \mathbb{I}_\mu \).

Since \( H_\mu[[x]] / (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}) \) is locally free with basis \( \{ x^\alpha \}_{\alpha \in \mathbb{I}} \),

\( g_0 \) is clearly injective. Moreover let \( \{ x^{\alpha*} \}_{\alpha \in \mathbb{I}} \) be the dual basis, then \( x^{\alpha*} \), restricted to \( \text{Der}_k(H_\mu) \), may be identified with the map \( \lambda_\mu : \text{Der}_k(H_\mu) \rightarrow H_\mu \).

Consider now the Euler-relation

\[
E = \sum_{i=1}^n w_i x^i \frac{\partial F}{\partial x_i} - F = \sum_{\alpha \in \mathbb{I}_\mu} (|\alpha|-1)x^\alpha \cdot t^\alpha
\]

as an element of \( H_\mu[[x]] / (\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}) \). For every \( \alpha \in \mathbb{I} \), we may write

\[
x^{\alpha*}E = \sum_{\beta \in \mathbb{I}_\mu} k_{\alpha \beta} x^\beta, \quad k_{\alpha \beta} \in H_\mu
\]

where \( k_{\alpha \beta} \) are homogeneous of degree \( 1+|\alpha|-|\beta| \).

**Proposition (4.5).** With the notations above,

(i) \( V_\mu \) is a graded Lie-algebra locally generated as \( H_\mu \)-module by the elements

\[
\delta_\alpha = \sum_{\beta \in \mathbb{I}_\mu} k_{\alpha \beta} \frac{\partial}{\partial t_\beta}, \quad \alpha \in \mathbb{I}, \text{with deg } \delta_\alpha = |\alpha|.
\]

(ii) The canonical morphism

\[
\lambda_\mu : V_\mu \otimes k H_\mu = \text{Der}(k[[x]]/(f))/\text{Der}_\mu
\]

where \( \text{Der}_\mu \) is the Lie-ideal of those derivations that may
be lifted to $H_\mu$, is given by

$$\lambda_\mu(\delta) = -D_\delta.$$

(iii) $[\delta_\alpha, \delta_\beta] = (|\alpha| - |\beta|)\delta_{\alpha+\beta} + \sum_{|\gamma| > |\alpha| + |\beta|} h_{\alpha, \beta, \gamma} \delta_\gamma$ where $h_{\alpha, \beta, \gamma} \in \mathfrak{I}_\mu \subset H_\mu$.

Proof. Obviously $x^\alpha E$ is zero in $H_\mu[[x]]/(F_\mu, \delta F|^E_\mu)$, therefore $\delta \in \mathcal{E}_\mu$. Conversely, suppose $\delta = \sum_{\beta \in \mathfrak{I}_\mu} h_{\beta} \frac{\delta}{\delta t^\beta} \in \mathcal{E}_\mu$, then $g_0(\delta) =$

$$\sum_{\beta \in \mathfrak{I}_\mu} h_{\beta} \frac{\delta}{\delta t^\beta} = \sum_{i=1}^n p_i \frac{\delta F}{\delta x_i} - p F_\mu = \sum_{i=1}^n (p_i - w_i x_i \cdot p) \frac{\delta F}{\delta x_i} + p \cdot E.$$

Now $p \cdot E = \sum_{\alpha \in \mathfrak{I}} k_{\alpha} x^\alpha E$ $k_{\alpha} \in \mu$. In $H_\mu[[x]]/(\delta F_{x_1}, \ldots, \delta F_{x_n})$ we have therefore

$$\sum_{\beta \in \mathfrak{I}_\mu} h_{\beta} \frac{\delta}{\delta t^\beta} = \sum_{\alpha \in \mathfrak{I}} k_{\alpha} \delta_{\alpha}.$$

Since $\{x^\alpha\}_{\alpha \in \mathfrak{I}_\mu}$ is part of a basis, it follows

$$\sum_{\beta \in \mathfrak{I}_\mu} h_{\beta} \frac{\delta}{\delta t^\beta} = \sum_{\alpha \in \mathfrak{I}} k_{\alpha} \delta_{\alpha}$$

proving (i).

(ii) Let $\delta \in \mathcal{E}_\mu$, then by (2.6), there exists a derivation

$$E_\alpha \in \text{Der}_k(H_\mu[[x]])/(F_\mu))$$

such that for all $k \in \mu$, $E_\alpha(k \cdot x_i) = k E_\alpha(x_i) + \delta_{\alpha}(k) \cdot x_i$, $i = 1, \ldots, n$. Moreover $\lambda(\delta_{\alpha})$ is the reduction of $E$. Now since $E_\alpha$ is a $k$-derivation we must have

$$\sum_{i=1}^n \frac{\delta F}{\delta x_i} E_\alpha(x_i) + \sum_{\beta \in \mathfrak{I}_\mu} \frac{\delta F}{\delta t^\beta} \alpha_{\beta} \delta_\beta (t^\beta) = 0 \text{ in } H_\mu[[x]]/(F_\mu).$$

Since the left side of this equation can be written

$$\sum_{i=1}^n \frac{\delta F}{\delta x_i} E_\alpha(x_i) + \sum_{\beta \in \mathfrak{I}_\mu} \frac{\delta F}{\delta t^\beta} \alpha_{\beta} \delta_\beta = \sum_{i=1}^n \frac{\delta F}{\delta x_i} E_\alpha(x_i) + \alpha_{\alpha} E$$


it is sufficient to see that, if we put $E_{\alpha}(x_i) = -x_{\alpha}w_i x_i$ this is zero modulo $\mathbb{F}_\mu$. In fact, $E_{\alpha}$ is then a $k$-derivation and for all $i = 1, \ldots, n$ $\lambda(\delta_{\alpha})(x_i) = E_{\alpha}(x_i) = x_{\alpha}w_i x_i$, i.e. $\lambda(\delta_{\alpha}) = -D_{\alpha}$.

(iii) now follows from (4.4), (i) and (ii).

Q.E.D.

Before we continue the study of $V_{\mu}$, let us pause, recalling the action $\rho$ of $\text{Der}(k[[x]]/(f))$ on $H^1(k, k[[x]]/(f); k[[x]]/(f)) = k[[x]]/(\partial f/\partial x_1), \ldots, \partial f/\partial x_n)$, see (2.12).

**Corollary (4.6).** With the notations above, we have:

(i) The action $\rho$ of $\text{Der}(k[[x]]/(f))/\text{Der}_1$ on $H^1(k, k[[x]]/(f); k[[x]]/(f))$ is given by

$$\rho(D_{\alpha})(x^\beta) = (|\beta| - 1)x^{\alpha + \beta}, \alpha, \beta \in I.$$  

(ii) The corresponding action of $V_{\mu}$ on $H_{\mu}$ has a linear part given by:

$$\delta_{\alpha}(t_{\gamma}) = (|\gamma - \alpha| - 1)t_{\gamma - \alpha} \pmod{(t)^2}$$  

$\gamma \in I_{\mu}$, $\alpha \in I$.

Recall that $t_{\alpha} = 0$ in $H_{\mu}$ if $|\alpha| < 1$.

(iii) Consider the action $\rho_0$ of $\text{Der}(k[[x]]/(f))/\text{Der}_\mu$ on the tangent space of $H_{\mu}$ at 0 given in terms of

$$\rho_0(D_{\alpha})(x^\beta) = \begin{cases} 0 & \text{if } |\beta| = 1 \\ x^{\alpha + \beta} & \text{if } |\beta| \neq 1 \end{cases}$$

then the dimensions of the maximal orbits of $\rho$ and $\rho_0$ are equal.

**Proof.** (i) and (ii) are just variations on (2.12). To prove (iii) it suffices to see that the dimension of the orbit of $\sum t_{\beta}x^\beta$ under $\rho$ is the same as the dimension of the orbit of $\sum t_{\beta}(|\beta| - 1)x^\beta$ under $\rho_0$.

Q.E.D.
Denote by $K = k(t)$ the matrix $(k_{ab})_{a \in \mathcal{J}, b \in \mathcal{I}_\mu}$. Recall the notations from (3.14).

**Proposition (4.7).** For every point $\mathfrak{t} \in \mathcal{H}_\mu$, we have

$$\tau(F(\mathfrak{t})) = \min \text{rank } K(\mathfrak{t}).$$

Put

$$\mu_{\text{sm}}(F(\mathfrak{t})) = \dim \{ \mathfrak{t} \in \mathcal{H}_\mu \mid \text{rank } K(\mathfrak{t}) = \text{rank } K(\mathfrak{t}) \} - \text{rank } K(\mathfrak{t})$$

$$\mu_{\text{m}}(f) = \max_{\mathfrak{t} \in \mathcal{H}_\mu} \mu_{\text{sm}}(F(\mathfrak{t})).$$

Notice that $\mu_{\text{m}}(f)$ is the usual modality of $f$ with respect to the $\mu$-constant stratum, under the action of the contact groups, see [A]. If $\tau = \tau(F(\mathfrak{t}))$ then for $M_{\mu \tau} = \text{im} \{ H_{\mu} \cap M_{\tau} \}$

$$\mu_{\text{sm}}(F(\mathfrak{t})) = \dim M_{\mu \tau, C} \text{ where } M_{\mu \tau, C} \text{ is the component of } M_{\mu \tau} \text{ containing } \mathfrak{t}.$$
§5. PLANE CURVE SINGULARITIES WITH $\k^*$-ACTION

Introduction. It turns out to be a big difference between the theory of hypersurface singularities in dim 1 and in higher dimension.

Even in the quasi-homogeneous case the situation in dimensions $\geq 2$ seems to be horrendously complicated, meriting Zariski's warning to those venturing into this study: "Le problème de la description complète de l'espace des modèles $M$ d'une classe d'équisingularité donnée est entièrement ouvert et les quelques exemples du chapitre V montrent que $M$ a une structure trop complexe pour espérer répondre totalement à la question". [Z]

In dimension 1 however there is some light to be seen. In this § we study the dimensions of the components $M_{\mu, \tau^{\min} \leq \tau \leq \tau(f)}$ of the $\mu$-constant stratum of the local moduli suite for a quasi homogenous curve singularity $f$.

The main results are (5.2), where we prove that $M_{\mu, \tau} \neq \emptyset$ for all $\tau$, $\tau^{\min} \leq \mu(f) = \mu$, $\dim M_{\mu, \tau} > \dim M_{\mu, \tau+1}$ for $\tau^{\min} \leq \tau \leq \tau(f)-1$, and (5.13), (5.14) and (5.16) where we give formulas for $\dim M_{\mu, \tau^{\min}}$ generalizing results of Zariski [Z] and Delorme [Del].

Theorem (5.1). Suppose $f \in k[x_1, x_2]$ is a weighted homogeneous polynomial defining an isolated singularity. Then

$$\mu(m(f) = m_0(f) + \tau^{\min}_{\mu}.\mu.$$
(i) \( t + \text{sm}(F(t)) \) is upper semicontinuous.

(ii) \( \tau(F(t)), t \in H_\mu \) takes every possible value between \( \tau_{\text{min}} \) and \( \tau(f) = \mu \).

Theorem 5.2 fails in dimension 2. In fact let \( f = x_1^3 + x_2^{10} + x_3^{19} \) and then \( \mu = 324 \), \( \tau_{\text{min}} = 246 \). However \( \tau = 247 \) does not occur and the \( \tau = 248 \)-stratum is an open set in a hypersurface in the \( \mu \)-constant stratum.

We shall prove the theorems for the case \( f = x_1^{a_1} + x_2^{a_2}, a_1 < a_2 \). Clearly Theorem (5.1) is a consequence of (5.2) (i). To prove (5.2) we shall have to study the matrix \( K(t) \) more carefully.

Notice first that (4.6) (ii), together with (3.11) implies \( H_0 = H_\mu / J \), where \( J \) is the ideal generated by \( \{ t_x | x > 1 \} \). Thus \( H_0 = k[t_x], |x| = 1 \).

Now let \( \lambda_1 < \ldots < \lambda_\mu \) be the monomial basis \( \{ x^\alpha \} \alpha \in I \) ordered by degree and lexicographic order, i.e. such that

\[
\alpha < \beta \iff \text{either } |\alpha| < |\beta| \text{ or } |\alpha| = |\beta| \text{ and } \alpha = (a', a_n), \beta = (b', b_n)
\]

with \( a' < b' \). If \( \lambda_i = x^\alpha \), we put \( \deg \lambda_i = |\alpha| \).

Remark (5.3). Observe that:

(i) \( \lambda_1 = 1, \lambda_\mu = x_1^{a_1 - 2}, x_2^{a_2 - 2} \)

(ii) \( E\lambda_i = 0, \text{ if } \deg \lambda_i > \deg \lambda_\mu - \min \{ \deg \lambda_k | \deg \lambda_k > 1 \} \)

(iii) duality: \( \lambda_i^\vee \cdot \lambda_{\mu-i+1} = \lambda_\mu \). We shall write \( \lambda_i^\vee = \lambda_{-\mu+i} \).

(iv) \( E\lambda_i \) is contained in the submodule of \( H^1 \), generated by \( \{ \lambda_k | \deg \lambda_k > 1 \} \).

From this it follows that the matrix \( K \) looks like
Let $M_1$ be the sub $H_{\mu}$-module of $H_{\mu}[[x_1,x_2]]/(\partial F_1, \partial F_2)$ generated by the $\lambda_i$'s with $\deg \lambda_i < \deg \lambda_{\mu} - \min(\deg \lambda_k | \deg \lambda_k > 1)$, i.e. by the first $r$ $\lambda_i$'s, and let $M_2$ be the sub $H_{\mu}$-module generated by the $\lambda_i$'s with $\deg \lambda_i > 1$, i.e. by those $x^{\mu}$'s sitting strictly above the face of $f$.

Notice that $\lambda_i \in M_1$ iff $\lambda_i \not\in M_2$. Put $\mu_j = \lambda_{r-j+1}$ and $t_j = \frac{a_j}{a_j}$; if $\mu_j = \frac{a_j}{a_j}$. Then $\{\lambda_i\}_{i=1}^r$ is a basis for $M_1$, and $\{\mu_j\}_{j=1}^r$ is a basis for $M_2$.

Observe also that:

- (v) for all $i$, $1 \leq i \leq r$ there is a $j_i$ with the following property.
  1. if $j > j_i$ then there is a $k(i,j)$, $1 \leq k(i,j) < r$ such that $\mu_j = \nu_{k(i,j)} \lambda_i$.
  2. if $j < j_i$ then $\lambda_i \not\in \mu_j$.

With these notations we find that $K_0$ is the matrix associated to the $H_{\mu}$-linear map

$$E: M_1 \rightarrow M_2$$

defined by multiplication with $E$. Before we proceed with the general theory, let us consider an example.
Example (5.4). Let \( f = x^5 + y^{11} \), \( n = 2 \). Then

\[
F_\mu(t) = f + t_1 x y^9 + t_2 x^2 y^7 + t_3 x^3 y^5 + t_4 x^4 y^8 + t_5 x^5 y^6
\]
\[
+ t_6 x^2 y^9 + t_7 x^3 y^7 + t_8 x^3 y^8 + t_9 x^3 y^9
\]

\[
E = \frac{1}{55} (t_1 x y^9 + 2 t_2 x^2 y^7 + 3 t_3 x^3 y^5 + 7 t_4 x^4 y^8 + 8 t_5 x^5 y^6 + 12 t_6 x^2 y^9
\]
\[
+ 13 t_7 x^3 y^7 + 18 t_8 x^3 y^8 + 23 t_9 x^3 y^9)
\]

This is easily seen by inspecting the Newton diagram of \( f \).

\[
a_1 = 5 \quad a_2 = 11
\]
\[
\mu(f) = \tau(f) = 0.40
\]
\[
m_0(f) = 9
\]
\[
sm(f) = 0
\]
\[
r = 9, H_\mu = k[t_1, \ldots, t_9]
\]

Now, put

\[
A = 2 t_2 - \frac{9}{11} t_1^2,
\]
\[
B = 3 t_3 - \frac{7}{11} t_1 t_2,
\]
\[
C = 7 t_4 + \frac{1}{11} t_1 t_3 \left( \frac{9}{11} t_1^2 - \frac{25}{11} t_1 t_2 + 3 t_3 \right),
\]
\[
D = 8 t_5 - \frac{8}{11} t_1 t_4 + \frac{1}{11} t_1 t_3 \left( \frac{63}{11} t_1^2 - \frac{5}{11} t_1 t_2 - \frac{14}{11} t_2 \right)
\]
\[
E = 8 t_5 - \frac{71}{11} t_1 t_4 - \frac{5}{11} t_1 t_3^2, \quad F = 7 t_4 + \frac{3}{11} t_1 t_3
\]

then the matrix \( 55 R_0 \) looks like:
the flattening stratification \( \{ S_\tau \}_{\tau \in T} \) of \( A^1(\mathbb{H}, \tilde{\mathbb{A}}; \mathbb{A}) \), see § 3, coincides with the rank-filtration of \( K_0 \), and it is easily seen that \( T_\mu = \{ 34, 35, 36, 37, 38, 39, 40 \} \) is the set of possible Tjurina numbers with constant \( \mu (= 40) \) in the neighbourhood of \( f \).

As T. Yano observed, a suitable change of the monomial base of \( M_1 \) leads to a symmetric matrix. In our example we find that \( K_0 \) becomes,
with \[ A = 2t_2 - \frac{9}{11}t_1^2 \quad B = 3t_3 - \frac{7}{11}t_1t_2 \quad C = 7t_4 + \frac{3}{11}t_1t_3^2 \]
\[ D = 8t_5 - \frac{8}{11}t_1t_4 + \frac{2}{11}t_1^2t_3^2 \]
\[ E = 13t_7 - \frac{117}{11}t_1t_6 + \frac{16}{11}t_2t_3t_5 - \frac{7}{11}t_1t_2t_3t_4 + \frac{1}{121}t_1^2t_2^2t_3^2 \]

The flattening statification is given by:

\[ S_{34}^\min: 4t_2^2-3t_1t_3-t_1t_2\neq 0 \]
\[ S_{35}^\min: 4t_2^2-3t_1t_3-t_1t_2 = 0 \text{ and } A\neq 0 \text{ or } B\neq 0 \text{ or } \]
\[ D(2t_2C-t_1D) - C(C(3t_3+t_2t_2) - 2t_2D) \neq 0. \]

Notice that the last minor defining \( S_{35} \) is not invariant, but never the less any point in \( S_{35} \) has an open invariant affine

neighbourhood.

In fact, replacing the last minor by \((t_1D - 2t_2C)^2 - 2(A^2 + \frac{1}{11}t_1^2A)(t_1t_7 - 2t_2t_6)\) we get an open invariant covering. \( S_{35} \) is smooth.

\[ S_{36}^\min: A = B = t_1(qt_1C - 11D)^2 = 0 \text{ and } C^2 - t_1E \text{ or } CD - \frac{9}{11}t_1^2E \]
\[ \text{or } D(qt_2C - 11D) \text{ or } C(qt_1C - 11D) \neq 0. \]

\( S_{36}^\min \) is not reduced.

\[ S_{37}^\min: A = B = t_1(qt_1C - 11D)^2 = C^2 - t_1E = CD - \frac{9}{11}t_1^2E = D(qt_1(-11D) = C(qt_1C - 11D) = 0 \text{ and } C\neq 0 \text{ or } t_1\neq 0 \text{ or } D\neq 0 \text{ or } E\neq 0. \]

\[ S_{38}^\min: t_1 = \ldots = t_5 = t_7 = 0 \text{ and } t_6 \neq 0 \text{ or } t_8 \neq 0. \]
\[ S_{39}^\min: t_1 = \ldots = t_8 = 0 \text{ and } t_q \neq 0. \]
\[ S_{40}^\min: t_1 = \ldots = t_q = 0. \]

Remark. Notice that \( S_{\tau}^\min \) is not necessarily the \( \tau_{\min} \)-constant

stratum of \( H \): In fact the family
\[ x_1^5 + x_2^1 + x_2^1x_3^2 + 2t_1x_1^3x_2 + t_1x_1^3x_2 \]

is \( \tau \)-const. (\( \tau = 34 \) for
\[ 0 < |t| \ll 1 \) but not \( \mu \)-const. (\( \mu(t_1\neq 0) = 39 \)).
We find the table

<table>
<thead>
<tr>
<th>t</th>
<th>34</th>
<th>35</th>
<th>36</th>
<th>37</th>
<th>38</th>
<th>39</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>tm(f)</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Thus, in particular,

$$\mu_{m}(f) = 3.$$ 

The nice properties of the matrix $K_0$ becomes apparent if one restrict attention to the linear terms of $K_0(t)$. The corresponding matrix will be denoted $L(t)$. In fact $55 \cdot L(t)$ looks like:

$$
\begin{array}{cccccccccc}
t_1 & 2t_2 & 3t_3 & 7t_4 & 8t_5 & 12t_6 & 13t_7 & 18t_8 & 23t_9 \\
0 & 0 & 0 & 2t_2 & 3t_3 & 7t_4 & 8t_5 & 13t_7 & 18t_8 \\
0 & 0 & 0 & 0 & 0 & 2t_2 & 3t_3 & 8t_5 & 13t_7 \\
0 & 0 & 0 & 0 & 0 & 0 & t_1 & 2t_2 & 7t_4 & 12t_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3t_3 & 8t_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2t_2 & 7t_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3t_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2t_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 \\
\end{array}
$$

Notice that $L(t)$ is symmetric on the antidiagonal already with respect to the monomial base we started with.

Now this symmetry is a general feature. In fact we have the following
Proposition (5.5). With the notations above,

(i) \( \lambda_{ij} = \lambda_{r-j+1,r-i+1} \)

(ii) \( \lambda_{ij} = \partial_k(i,j) t_k(i,j) \) with \( \partial_i = \deg \mu_i - 1 \), whenever \( j > i \) \( \lambda_{ij} = \min\{j, h_{ij} + 0\} \)

(iii) \( k(i,j) < k(i,j+1) \) and \( k(i+1,j) < k(i,j) \)

(iv) \( j_i < j_i + 1 \).

Proof. (i) Since \( \lambda_{ij} = \sum_{j=1}^{r} h_{ij} \mu_j \) and since \( \lambda_{r-\lambda+1} = \mu_X \) we find

\[
\lambda_{ij} = \lambda_{r-j+1,r-i+1},
\]

so that \( \lambda_{ij} \) is homogeneous of degree \( \deg \lambda_i - \deg \lambda_j \).

(ii) The rest is a consequence of (5.3) (v).

Proposition (5.6). There is a basis \( \{\bar{\lambda}_i\}_{1 \leq i \leq v} \) of \( M_1 \) with the following properties:

1. \( \bar{\lambda}_i = \lambda_i + \sum_{j=0}^{v} e_{ij}^\lambda_{j} \) homogeneous of degree \( \deg \lambda_i - \deg \lambda_j \).
2. The matrix of \( \lambda_i \) \( \partial_{e_j} \mu \) with respect to \( \{\bar{\lambda}_i\} \) in \( M_1 \) and \( \{\mu_i\} \) in \( M_2 \) is symmetric with the same linear part \( L(t) \) as before.

Proof: Consider on \( H_\mu[x_1,x_2]/(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}) \) the following pairing defined by the coefficient of the Hessian \( t_\mu \) of the product of two elements:

Let \( h \in H_\mu[x_1,x_2]/(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}) \), define \( c_\mu(h) \) by \( h = \sum c_i(h) \lambda_i \)
\( \langle h, k \rangle = c \mu (h \cdot k) \)

\( \langle , \rangle \) is a nondegenerate symmetric bilinear form:

(i) \( \langle \lambda_i, \lambda_{\mu-i+1} \rangle = 1 \)

(ii) \( \langle \lambda_i, \lambda_j \rangle = 0 \) if \( i+j \neq \mu+1 \)

(iii) if \( \langle \lambda_i, \lambda_j \rangle \neq 0 \) then \( \langle \lambda_i, \lambda_j \rangle \) is homogeneous of degree

\[ \deg \lambda_i + \deg \lambda_j - \deg \lambda_\mu. \]

Denote by \( E_0 \) the matrix of \( \langle , \rangle \) with respect to the base \( \{ \lambda_i \} \).

Now obviously the map defined by multiplication with \( \mu \) is self-adjointed, i.e. \( \langle \lambda_i F, \lambda_j \rangle = \langle \lambda_i, \lambda_j F \rangle \). Denote by \( C \) the matrix of \( \langle F, - \rangle \) with respect to the base \( \{ \lambda_i \} \). Notice that

\[ \langle \lambda_i F, \lambda_j \rangle = 0 \] if \( \lambda_i \neq \lambda_j \in \{ \lambda_1, \ldots, \lambda_r \} \) (this is a special property of the curve case). Now

(iv) \( C = K \cdot E_0 \).

This implies that \( E_0^{-1} C E_0^{-1} = E_0^{-1} K \) is symmetric. \( E_0^{-1} \) defines a base change in \( H_\mu[x_1, x_2]/(\frac{\delta F}{\delta x_1}, \frac{\delta F}{\delta x_2}) \). The induced base change in \( M_1 \) has the required properties:

Let \( E_0^{-1} = (\bar{e}_{ij}) \) then

\[ \bar{e}_{i, \mu-i+1} = 1 \]

\[ \bar{e}_{i,j} = 0 \] if \( i+j \neq \mu+1 \)

\[ -\deg \bar{e}_{ij} = \deg \lambda_i + \deg \lambda_j - \deg \lambda_\mu. \]

\( \{ \bar{\lambda}_i \} \) is defined by

\[ \bar{\lambda}_i = \sum_{j<r} \bar{e}_{ij} \lambda_{\mu-i+1} \lambda_j. \]

Let \( (\bar{c}_{ij}) \) be the matrix of \( E_0^{-1} C E_0^{-1} \) then, because \( \lambda_j E = 0 \) if \( j=r+1 \), the matrix of \( E: M_1 \cdot M_2 \) with respect to \( \{ \bar{\lambda}_i \} \) in \( M_1 \) and \( \{ \mu_j \} \) in \( M_2 \) defined by \( \bar{\lambda}_i E = \sum_{j=1}^{r} \bar{c}_{ij} \mu_j \) satisfies \( \bar{c}_{ij} = \bar{c}_{i, r-j+1} \).

Notice that a base change in \( M_1 \) gives new generators of the kernel of the Kodaira-Spencer map as the corresponding combination of the \( \delta a_i \).
Remark: A base change in $M_2$ would require a suitable change of the family $F_{\mu}$ which is not always possible. If, however we are just interested in the flattening stratification $\{S_i\}$ we may also change the base in $M_2$.

Corollary (5.7). There is a basis $\{\bar{\mu}_i\}_{1 \leq i \leq r}$ of $M_2$ and an automorphism $\phi: H_\mu \to H_\mu$ with the following properties:

1. $\bar{\mu}_i = \mu_i + \sum_{j \neq i} e_{ij} \mu_j$, $\deg e_{ij} + \deg \mu_j = \deg \mu_i$
2. $\phi$ is homogeneous
3. the matrix of $E:M_1 \to M_2$ with respect to the basis $\{\lambda_i\}$ in $M_1$ and $\{\bar{\mu}_i\}$ in $M_2$ is $L(\phi(t))$.

Proof: Consider the base change on $M_2$ induced by $E_0$:

$$\bar{\lambda}_{i+r} = \bar{\lambda}_i = \sum_{j} e_{ij} \lambda_{r-i+1} \lambda_j$$

because of

$$e_{ij} \lambda_{r-i+1} = 0 \text{ if } r-i+1+j<\mu+1 \text{ and } \lambda_{i+r} = \mu_j.$$

The matrix of $E:M_1 \to M_2$ defined by $\lambda_i E = \sum_{j=1}^{r} \bar{h}_{ij} \bar{\mu}_j$ satisfies

$$\bar{h}_{ij} = c_{i-r+j+1}. \text{ Now consider } \lambda_i E = \sum_{j=1}^{r} \bar{h}_{ij} \bar{\mu}_j \text{ then } c_{ij} = \langle \lambda_i \mu, \lambda_j \rangle = \begin{cases} h_{i-r} & \text{if } \lambda_i \lambda_j = \lambda_i, \lambda < r \\ 0 & \text{else} \end{cases}. \text{ Notice that with the notations of (5.5)}$$

$$\lambda = k(i,r-j+1). \text{ But } h_{i-r} = (\deg \mu_{r-e+1-1}) t_{r-e+1} + h_{i-r}(t_1, \ldots, t_{r-e-1}).$$

We may choose $\phi: H_\mu \to H_\mu$ with $\phi((\deg \mu_{r-e+1-1}) t_{r-e+1}) = h_{e,r}$.

Remark: If we consider the similar situation for surfaces then $c_{ij} = \langle \lambda_i \mu, \lambda_j \rangle$ may be different from zero even when $\lambda_i \lambda_j$ is not in the monomial base. This is the reason why the flattening stratification cannot be described in terms of the
linear matrix and why (5.2) fails. In the example \( x^3 + y^{10} + z^9 \)
the minor of the matrix \( K_0 \) giving maximal rank (= 78) is

\[
\begin{array}{cccccccccc}
-1 & 2 & t & 3 & 1 & -1 & 2 & t & 2 & * & * & * & * & * & * & * \\
0 & -1 & 2 & t & 3 & 1 & * & * & * & * & * & * & & & & \\
0 & 0 & & & & & & & & & & & & & & & \\
\end{array}
\]

For \( t_1 = 0 \) the rank decreases by 2.

**Proof of (5.2).** Because of (5.7) we may consider \( L(t) \) instead of \( K_0 \). We first prove the second part. So let's compute the determinant of the minors of \( L(t) \).

We shall use (5.5). In particular, it follows that any minor of the linear matrix \( L(t) \) has the form,

\[
M(t) = (d_{a(i,j)} t^a(i,j))
\]

where \( a(i,j) \in \{0, 1, \ldots, r\} \), \( i, j = 1, \ldots, m \), \( d_{a(i,j)} = \deg \lambda^{a(i,j)} - 1 \neq 0 \), if \( a(i,j) \neq 0 \), \( d_0 = 0 \), and where

(1) \( a(i, j-1) \neq 0 \) implies \( a(i, j) < a(i, j+1) \)

(2) \( a(i-1, j) \neq 0 \) implies \( a(i, j) < a(i-1, j) \).

**Lemma (5.8).** With the notations above,

\[
\det M(t) \neq 0 \text{ if and only if } a(i, i) \neq 0 \text{ for all } i = 1, \ldots, m.
\]
Proof. Suppose \( a(i,i) = 0 \) for some \( i \), then \( M(t) \) has the form

\[
\begin{pmatrix}
\vdots & \\
* & 0 & \cdots & 0 & \cdots & * \\
0 & \cdots & 0 & \cdots & 0 & \cdots \\
\vdots & \\
\end{pmatrix}
\]

therefore \( \det M(t) = 0 \).

Assume \( a(i,i) \neq 0 \) for all \( i = 1, \ldots, m \). Use induction on the number of \( a(i,j) \neq 0 \). Let \( s = \min\{a(i,j) | a(i,j) \neq 0\} \), and assume

\[
\det M(t) \big|_{t_s=0} = 0.
\]

By induction \( M(t) \big|_{t_s=0} \) has a diagonal element \( a(i,i) = 0 \). This implies that \( M(t) \) has the form

\[
\begin{pmatrix}
\vdots & \\
A & * \\
\vdots & \\
0 & \cdots & 0 & d & t_s & \cdots & * \\
\vdots & \\
\end{pmatrix}
\]

where \( A \) and \( B \) are minors of \( L(t) \), and therefore has the same form as \( M(t) \). By induction \( \det A \neq 0 \), \( \det B \neq 0 \), therefore

\[
\det M(t) = d_s \cdot t_s \det A \cdot \det B \neq 0.
\]

Q.E.D

If we agree to call "a diagonal" any string of elements of a matrix parallel to the diagonal, we may prove the following.
Lemma (5.9). The rank of $L(t)$ is the length $\ell$ of the maximal diagonal containing no zeros. With the notations of (5.5):

$$\ell = m_0(f) - sm(f) - \max\{j_i - i\}.$$ 

Proof. From (4.14) we deduce $\ell \leq \text{rank } L(t)$. To prove the inverse inequality, let $M(t)$ be any $m \times m$ minor of $L(t)$ gotten by picking the $i_1$th, $i_2$nd, ..., $i_m$th rows and the $j_1$th, $j_2$nd, ..., $j_m$th column of $L(t)$. Using (4.10) and (4.14) we find that $\det M(t) \neq 0$ implies that $l_{ir} \neq 0$ for $1 \leq i \leq m$, $l_i, r-1 \neq 0$ for $1 \leq i < m-1$, ..., $l_i, r-m \neq 0$ for $1 \leq i < l$. Therefore there exists a diagonal of $L(t)$ of length $m$ containing no zeros. Q.E.D.

The second part of (5.2) now follows by induction on $\tau$. Let's consider the maximal diagonal of $L(t)$ containing no zeros. Among the $k(i,j)$ for which $t_k(i,j)$ occur on the corresponding diagonal of $L(t)$, let $k(i_1,j_1), l = 1, \ldots, p$ be the smallest. On the subset defined by $t_i = 0, i < k(i_1,j_1)$ the rank of $L(t)$ has decreased by 1.

To prove (5.2) (i), we need the following lemma.

Lemma (5.10). Let $0 < p < \max\{\text{rk } L(t)\} = p_0$, then

$$\dim \{ t \in H | \text{rk } L(t) < p_0 - p \} < r - p.$$ 

Proof. Consider the ideal $I^L_p$ of $H$ generated by the $(p+1)$-minors of $L(t)$. We need only prove:
 Lemma (5.11). Let \( a(i,j) \in \{0, \ldots, r\} \), \( i, j = 1, \ldots, m \) satisfy

1. \( a(i, j+1) \neq 0 \) implies \( a(i, j) < a(i, j+1) \)
2. \( a(i-1, j) \neq 0 \) implies \( a(i, j) < a(i-1, j) \)

and let \( d_0 = 0, d_s = \deg \mu_s - 1 \) as above. Consider

\[ M(t) = (d_{a(i,j)}^t a(i,j)) \]

and assume \( \det M(t) \neq 0 \). Let \( I_p \) be the ideal generated by the \((p+1)\)-minors of \( M(t) \). Then

\[ \text{ht } I_p > m-p \quad \text{i.e. dim } k[t]/I_p < r-m+p. \]

Proof. Use induction on \( m \) and on the number of different \( a(i,j)'s \) involved in the matrix.

Let \( s = \min \{a(i,j) \mid a(i,j) \neq 0\} \) and let \( U \) be a component of

\[ V(I_p) = \{ t \in \mathbb{H} \mid \forall p \in I_p, p(t) = 0 \}. \]

1. case. \( U \subseteq V(t_s) = \{ t \mid t_s = 0 \} \). Consider the sub-matrix

\[ (d_{a(i,j)}^t a(i,j))_{1 \leq i < m-1, 2 < j < m} \]

obtained from \( M(t) \) by deleting the \( m \)-th row and the 1-st column. The conditions (1) and (2) together with the assumption \( \det(d_{a(i,j)}^t a(i,j)) \neq 0 \) imply that \( s \neq a(i,i+1) \) \( \forall i, j \). Therefore \( \det(d_{a(i,j)}^t a(i,j))_{1 \leq i < m-1, 2 < j < m} \neq 0 \), and we may apply the induction hypotheses.

2. case. \( U \subseteq V(t_s) \). It follows from (1) and (2) that \( \text{rk } M(t) > \lambda \)

for all \( t \) with \( t_s \neq 0 \), where \( \lambda \) is the number of times \( t_s \) occurs in \( M(t) \), see fig. In particular this implies \( \lambda < p \). Consider the \((m-\lambda) \times (m-\lambda)\) sub-matrix \( M_0(t) \) of \( M(t) \) obtained by deleting the rows and columns in which \( t_s \) occurs.
For $t_s \neq 0$ it is easy to see that $\text{rk } M(t) = l + \text{rk } M_0(t)$, therefore

$$\{ t | \text{rk } M(t) \leq p \} = \{ t | \text{rk } M_0(t) \leq p-1 \}.$$ 

Note that since $\det M(t) \neq 0$ all $t_s$ occurring in $M(t)$ sit under the diagonal, thus $\det M_0(t) \neq 0$. Moreover $M_0(t)$ does not contain $t_s$. We may therefore apply the induction hypotheses, and the lemma is proved. Q.E.D.

This ends the proof of theorem (4.6).

To compute the maximal rank of $K_0(t)$, or what is the same, the dimension of $M_{\min}$, i.e. the number $\mu_m(f)$ defined above, it turns out that one may use (4.6). In fact the maximal rank of $K_0(t)$ is the same as the maximal rank of $L(t)$ which is the same as the dimension of the maximal orbits of the action $\rho$ of $\text{Der}(k[[x]]/(f))/\text{Der}_1$ on $H^1(k, k[[x]]/(f)+k, k[[x]]/(f))$. In the case of $f = x_1^{a_1} + x_2^{a_2}$, Hans Olav Herøy has proved the following formula, see [Her].
Proposition (5.12). (Herøy). Let $f = x_1^{a_1} + x_2^{a_2}$ with $2 | (a_1, a_2)$ and $a_1 < a_2$ and let $q = (a_1, a_2)$. Then the maximal dimension of the orbits of $\rho$ is

$$rk K_0 = \frac{(a_1-2)(a_2-2)}{4} - \frac{q}{2} + \begin{cases} 1 & \text{if } q = a_1 \\ 0 & \text{otherwise} \end{cases}$$

Corollary (5.13). With the above notations, assuming $2 | (a_1, a_2)$ we have

$$\mu_m(f) = m_0(f) - \frac{(a_1-2)(a_2-2)}{4} + \frac{q}{2} + \begin{cases} 1 & \text{if } q = a_1 \\ 0 & \text{otherwise} \end{cases}$$

$$= sm(f) + \frac{1}{4}(a_1-4)(a_2-4) - \frac{a_2}{a_1} - \begin{cases} 1 & \text{if } q = a_1, a_1 < a_2 \\ 0 & \text{otherwise} \end{cases}$$

When $(a_1, a_2) = 1$, Delorme [Del] gives a formula for rank $K_0$:

Let $\frac{a_2}{a_1} = [r_1, \ldots, r_k] = r_1 + \frac{1}{r_2 + r_3 + \ldots + \frac{1}{r_k}}$

define $l_i$ and $t_i$ inductively:

$l_k = 0, t_k = 1$

$l_{i-1} = l_i + t_i r_i$ and $t_{i-1} = \begin{cases} 0 & \text{if } t_i = 1 \text{ and } l_{i-1} \text{ even} \\ 1 & \text{otherwise} \end{cases}$

Proposition (5.14): (1) The maximal dimension of the orbits of $\rho$ is

$$rk K_0 = \frac{(a_1-2)(a_2-2)}{4} - \frac{1}{4} t_0 + \frac{1}{2} t_1 (r_1 + t_2 - 2) + \frac{1}{2}$$

(2) $\mu_m(f) = \frac{1}{4} (a_1-4)(a_2-4) + \frac{1}{4} t_0 + \frac{1}{2} (2-t_1)(r_1 - 2) - \frac{1}{2} t_1 t_2$

(3) $\frac{1}{4} (a_1-4)(a_2-4) \leq \mu_m(f) \leq \frac{1}{4} (a_1-2)(a_2-4)$

except the trivial case $a_1 = 2, a_2 = 3$.

Remark: The left hand side of the inequality is sharp: $(8, 11)$.

The right hand side can be replaced by $\frac{1}{4} (a_1-3)(a_2-3)$. 
Corollary (5.15):

1. If \( a_2 = ra_1 +1 \) or \( a_2 = ra_1 -1 \), then the maximal dimension of the orbits is
   \[
   \frac{1}{4}(a_1^2-2)(a_2^2-3) \quad \text{a}_1 \text{ even}
   \]
   \[
   \frac{1}{4}(a_1^2-1)(a_2-r-3) \quad \text{a}_1 \text{ odd}
   \]

2. If \( a_2 = ra_1 +2 \), the maximal dimension of the orbit is
   \[
   \frac{1}{4}(a_1^2-2)(a_2^2-2)-1 \quad \text{a}_1 \text{ even}
   \]
   \[
   \frac{1}{4}(a_1^2-1)(a_2-r-2)-1 \quad \text{a}_1 \text{ odd}
   \]

3. If \( a_2 = ra_1 -2 \), the maximal dimension of the orbits is
   \[
   \frac{1}{4}(a_1^2-2)(a_2^2-2)-1 \quad \text{a}_1 \text{ even}
   \]
   \[
   \frac{1}{4}(a_1^2-1)(a_2^2-2) \quad \text{a}_2 \text{ odd}
   \]

4. If \( a_2 = ra_1 \), the maximal dimension of the orbits is
   \[
   \frac{1}{4}(a_1^2-2)(a_2^2-4) \quad \text{a}_1 \text{ even}
   \]
   \[
   \frac{1}{4}(a_1^2-1)(a_2^2-4) \quad \text{a}_1 \text{ odd}, r>2
   \]
   \[
   \frac{1}{4}(a_1^2-3)^2 \quad \text{a}_1 \text{ odd}, r=1
   \]

Remark (5.16). (5.9) gives an algorithm to compute \( \mu_m(f) \) in the cases not covered by the formulas:

\[
\mu_m(f) = \max_{j} \{j_1-i\}:
\]

1. Compute the spectral numbers corresponding to \( \{\lambda_1\}, \{\mu_1\} \) (i.e. the weighted degrees): \( s(\lambda_1), \ldots, s(\lambda_r), s(\mu_1), \ldots, s(\mu_r) \).
   Let \( \Lambda = \{s(\lambda_1), \ldots, s(\lambda_r)\}, \mu = \{s(\mu_1), \ldots, s(\mu_r)\} \).
2. \( j_1 = \min \{j, s(\mu_j)-s(\lambda_1) \in \mu\} \)
For the example $x_1^5 x_2^{11}$

$\Lambda = \{0, 5, 10, 11, 15, 16, 20, 21, 22\}$

$\mu = \{56, 57, 58, 62, 63, 67, 68, 73, 78\}$

$j_1 = 1, j_2 = 4, j_3 = 6, j_4 = 6, j_5 = 8, j_6 = 8,$

$j_7 = j_8 = j_9 = 9.$

**Remark (5.17).** In the general case, Briançon, Granger and Maisonobe, have just found a recursive formula, reducing the computation of $\mu m(f)$ to the situation of (5.12) or (5.14), (see [Br]).
§6 THE GENERIC COMPONENT OF THE LOCAL MODULI SUITE FOR IRREDUCIBLE QUASIHOMOGENEOUS ISOLATED PLANE CURVE SINGULARITIES

Introduction. In this § we shall return to the problem of constructing a moduli scheme for isolated curve singularities. As we have already pointed out, this problem has been studied by O. Zariski [Z] in the case \( f = x_1^m + x_2^{m+1} \), and before him by S. Eben [Eb]. We shall now prove that there exists a coarse moduli scheme for irreducible plane curve singularities with fixed semigroup
\[ \Gamma = \langle a_1, a_2 \rangle, \quad (a_1, a_2) = 1, \] (see [Z] chap. II for definitions), and minimal \( \tau \). Notice that the assumption \( (a_1, a_2) = 1 \) is not necessary - see [L-M-P]: As we shall see, the existence of this moduli scheme follows from proving that the \( \mu \)-constant stratum of the generic component \( M_{\tau_{\text{min}}}^\mu \) of the local moduli suite of \( f = x_1^{a_1} + x_2^{a_2} \) is a scheme. Washburn [Wash] has published a result of this type, together with a dimension formula. His proof is however incomplete, and the dimension formula is incorrect.

Washburn's method is based on the assumption that the multiplication \( E : M_1 \cdot M_2 \) (see §4) is compatible with the filtrations induced by the \( (x_1, x_2) \)-adic filtration of \( k[[x_1, x_2]] \). This is unfortunately false in general, as one may see considering the example \( f = x_1^{5} + x_2^{12} \). We shall never the less show that there exists a filtration of this type, depending on \( (a_1, a_2) \), providing us with enough invariants in \( S_{\tau_{\text{min}}}^\mu \) to prove the existence of a geometric quotient of \( S_{\tau_{\text{min}}}^\mu \) by \( V_{\tau_{\text{min}}}^\mu \). We shall assume \( k = \mathbb{C} \), the field of complex numbers.

The purpose of this § is the proof of the following

**Theorem (6.1).** Let \( \Gamma = \langle a_1, a_2 \rangle, \quad (a_1, a_2) = 1, \) be a semigroup.

There exists a coarse moduli space \( T_{\Gamma, \tau} \) parametrising all
plane curve singularities with the semigroup \( \Gamma \) and minimal Tjurina number \( \tau_{\text{min}} = \tau \), and a universal family \( \pi: X_{\Gamma, \tau} \rightarrow T_{\Gamma, \tau} \) such that

1. \( T_{\Gamma, \tau} \) is a quasismooth scheme, i.e. locally an open subset in a weighted projective space of dimension
\[
\frac{1}{4}(a_1 - 4)(a_2 - 4) + \frac{1}{2}t_0 + \frac{1}{2}(2-t_1)(r_1 - 2) - \frac{1}{2}t_1 t_2
\]
(see 5.14 for the definition of \( l_0, r_1, t_1, t_2 \)).

2. \( X_{\Gamma, \tau} \) is an algebraic space and there is an affine covering \( \{ U_i \} \) of \( T_{\Gamma, \tau} \) such that \( \pi^{-1}(U_i) \) are affine schemes.

**Proof:** We may suppose \( a_1 < a_2 \). Let \((X_0, 0)\) be a germ of an irreducible plane curve with singularity at 0, having \( \Gamma \) as semigroup. Consider \( X_{\mu} + H_{\mu} \), the versal \( \mu \)-constant deformation of the singularity defined by \( x_1^{a_1} + x_2^{a_2} \). There is a \( t \in H_{\mu} \) such that \((X_0, 0) = (X_{\mu, t}, 0)\) (cf. [A]).

**Lemma (6.2).** If for \((X_t, 0) = (X_{\mu, t}, 0)\), then \( t_1 \) and \( t_2 \) are in an analytically trivial subfamily of \( X_{\mu} + H_{\mu} \).

**Proof:** The \( C^* \)-action induces a canonical filtration on the automorphism group \( \varepsilon \) of \( C[[x_1, x_2]] \). Put,
\[
\varepsilon_\lambda = \{ \phi \in \varepsilon | \deg(\phi(x_1) - x_1) > \lambda + a_2, \deg(\phi(x_2) - x_2) > \lambda + a_1 \}
\]
and \( \deg \phi = \lambda \) iff \( \phi \in \varepsilon_\lambda \). Suppose now that \((X_{\mu, t_1}, 0) = (X_{\mu, t_2}, 0)\), i.e. suppose there is an \( \phi \in \varepsilon \) and a unit \( u(x) \in C[[x_1, x_2]] \) such that
\[
F_{\mu}(x_1, x_2, t_1) = u(x)F_{\mu}(\phi(x_1), \phi(x_2), t_2).
\]
We will prove that \( \deg \phi > 0 \).

If this is true \( t_1 \) and \( t_2 \) are in an analytically trivial
subfamily of $X + H_{\mu}$. In fact consider the trivial family of singularities

$$G(\lambda) = u(\lambda a_1 x_1, \lambda a_2 x_2) F_{\mu}(\frac{1}{\lambda a_2} \phi(x_1)(\lambda a_2 x_1, \lambda a_1 x_2), \frac{1}{\lambda a_1} \phi(x_2)(\lambda a_2 x_1, \lambda a_1 x_2, z_2))$$

as an unfolding of $F_{\mu}(x_1, x_2, z_2)$. It is induced by the universal unfolding, i.e. there exists $\phi$ and $v$ such that:

$$G(\lambda) = F_{\mu}(\phi(x_1, \lambda), \phi(x_2, \lambda), v(\lambda))$$

with $v(0) = t_2$ and $v(1) = t_1$.

Let us consider now the map between the local rings corresponding to the singularities $(x_1, 0, 0)$ and $(x_2, 0, 0)$ induced by $\phi$

$$\phi: C[[x_1, x_2]]/F_{\mu}(x_1, x_2, z_1) \to C[[x_1, x_2]]/F_{\mu}(x_1, x_2, z_2)$$

and the corresponding map $\bar{\phi}$ of their normalizations

$$C[[t]] \to C[[t]]$$

$$C[[t^{a_1} + \text{higher order}, t^{a_2} + \text{higher order}]] \to C[[t^{a_1} + \text{higher order}]]$$

$$C[[x_1, x_2]/F_{\mu}(x_1, x_2, z_1) \to C[[x_1, x_2]/F_{\mu}(x_1, x_2, z_2)$$

We must necessarily have $\bar{\phi}(t) = t \cdot h(t)$, $h(t)$ unit in $C[[t]]$.

Since $x_1 = t^{a_2} + \text{higher order terms}$ and $x_2 = t^{a_2} + \text{higher order terms}$ it is clear that $\deg \phi > 0$. Q.E.D.

Using this lemma we see that set theoretically $T_{\tau, \tau} = S_{\tau}/V_{\mu}$, where $S_{\tau}$ is the open stratum in the flattening stratification of $H_{\mu}$ and $V_{\mu}$ is the kernel of the Kodaira-Spencer map.

We will prove that $S_{\tau}/V_{\mu}$ is, locally, an open subset in a
weighted projective space.

\( V_\mu \) is a graded Lie-algebra generated as \( H_\mu \)-module by the elements \( \delta_\alpha, \deg \delta_\alpha = |\alpha| \), see (4.5).

Let \( V^+ \) be the sub Lie-algebra of all vectorfields of \( V_\mu \) of degree \( \geq 0 \). \( V^+ \) is a finite dimensional solvable C-Lie-algebra and \( [V^+, V^+] = :V \) is nilpotent. Since \( V^+ \) generates \( V_\mu \) as an \( S_\tau \)-module it is enough to study \( S_\tau/V^+ \).

Notice that the algebraic group \( G^+ = \exp V^+ \) acts rationally on \( S_\tau \). The orbits of \( G^+ \) are the maximal integral manifolds of \( V^+ \), i.e. the maximal integral manifolds of \( V_\mu \).

\( G = \exp V \) is a normal subgroup of \( G^+ \), \( G+/G = C^* \) and the \( C^* \)-action is induced by the Euler-vector field \( \delta_0 \) of \( V^+ \). \( G \) acts homogeneously (with respect to the \( C^* \)-action) on \( S_\tau \). We will prove that \( S_\tau/V = S_\tau/G \) is a smooth algebraic variety. The singularities of \( T_{G^+} = (S_\tau/G)/C^* \) are the singularities of the corresponding weighted projective space.

We will have to describe more precisely the open set \( S_\tau \) and the action of \( V^+ \) in terms of the matrix \( K_0 \) (see §5) corresponding to the generators \( \{ \delta_\alpha \} \) of \( V^+ \).

As before \( K_0 = (k_{ij}) \) is the matrix of the \( H_\mu \)-linear map \( E: M_1 \to M_2 \) with respect to the bases \( \{ \lambda_i \}_{i=1}^r \) of \( M_1 \) and \( \{ \mu_i \}_{i=1}^r \) of \( M_2 \). If \( \lambda_i = x^a_i \) then \( \delta x^a_i = \sum k_{ij} \frac{\partial}{\partial t^j} \).

Let \( r_0 \) be maximal such that \( \sum \frac{\mu_k}{x^j} \{ \mu_k \}_{k=1}^r \) for \( j \leq r_0 \). Since \( a_1 \leq a_2, \mu_1, \ldots, \mu_{r_0} \) are the monomials closest to the face defined by \( (a_1, 0) \) and \( (0, a_2) \) in \( R^r_+ \). Notice that for \( i > 1, \delta x^a_i \in \sum \lim H_\mu \frac{\partial}{\partial t^j} \).
This implies that $t_1, \ldots, t_{r_0}$ are invariant functions under the action of $V$.

We shall prove the existence of a filtration $\mathcal{F} = \{\mathcal{F}^P M_2\}_{P \in \mathbb{Z}}$ on $M_2$, induced by a filtration of the base $\{\mu_j\}$, and compatible with the degree, and also the dual filtration $\mathcal{F}^\ast$ on $M_1$ defined by $\lambda_i \in \mathcal{F}^P M_1$ iff $\lambda_i = \mu_{r-i+1} \in \mathcal{F}^P M_2$, such that:

- $\mathcal{F}^{-n} M_2 = M_2$, $\mathcal{F}^n M_2 = \{0\}$ if $n > 0$
- $\mathcal{F}^i M_2 \supset \mathcal{F}^{i+1} M_2$

Let us denote by $K_0^P$ the part of the matrix $K_0$ corresponding to the map $\text{gr}_p E : \text{gr}_p M_1 + \text{gr}_p M_2$ and by $I_0 = H_\mu$ the radical of the ideal generated by the maximal minors of $K_0^P$ with the convention that $I_0 = H_\mu$ if $\text{gr}_p M_1 = \text{gr}_p M_2 = \{0\}$.

Lemma (6.2). There is a filtration $\mathcal{F}^\ast$ of $M_2$ satisfying (P) and the following properties:

(0) $\text{rk}_H \text{gr}_p M_1, \text{rk}_H \text{gr}_p M_2$ for $p < 0$.

(1) The elements of $K_0^P$ are polynomials in $t_1, \ldots, t_{r_0}$.

(2) $I_0 = I_{-p}$.

(3) $I_0 \subseteq I_{p-2}$ if $p < -2$, $I_0 \cap I_{-1} = I_{-2}$.

(4) $\lambda_i \in \mathcal{F}_p$ implies $\lambda_i \lambda_j \in \mathcal{F}_{p+1}$ if $\lambda_i \lambda_j \in \{\lambda_k\}_{k=1, \ldots, r}$ where $r = \dim H_\mu$.

(5) $\lambda_i \in \mathcal{F}_p$ implies $\lambda_i \mu_j \in \mathcal{F}_p$.

(6) $S_r$ is the open set defined by $I_0 \cap I_{-1}$.

Consider first the example $x_1^5 + x_2^{11}$, see (§5): We shall be guided by the following facts. Consider the linear matrix $L(t)$. 
This matrix has maximal rank iff the "leading" submatrices

\[
\begin{pmatrix}
  t_1 & 2t_2 & 3t_3 \\
  2t_2 & 3t_3 & 7t_4 \\
  t_1 & 2t_2 & 7t_4 \\
\end{pmatrix},
\begin{pmatrix}
  2t_2 & 3t_3 \\
  t_1 & 2t_2 \\
  3t_3 \\
\end{pmatrix},
\begin{pmatrix}
  3t_3 \\
  2t_2 \\
  t_1 \\
\end{pmatrix}
\]

have maximal rank. Because of the symmetry we have to consider only the first three of them (up to the middle matrix). These matrices have the following property: We get the next one from the one before by, cancelling a column, adding a row or just leaving it unchanged. This means that maximal rank of the "middle" matrix implies maximal rank of the other ones. The "leading" submatrices are "small" enough such that their entries are just the obviously invariant functions $t_1, 2t_2, 3t_3$ with respect to the action of $V$.

We will construct a filtration $\mathcal{F}$ on $M_2$ such that the matrix of the corresponding graded map is given by these "leading" submatrices. For technical reasons the filtration will not be strict, in general. In our example the filtration $\mathcal{F}$ gives us the $(x_1, x_2)$-adic filtration:
$\mathcal{F}^{-3}M_2$ generated by $x_1x_2^9x_2^7x_3^5x_1^6x_2^8x_3^6x_1^2x_2^9x_3^7x_1x_2^3x_2^9$

$\mathcal{F}^{-2}M_2$ generated by $2x_1x_2^8x_3^6x_1^2x_2^9x_3^7x_1x_2^3x_2^9$

$\mathcal{F}^{-1}M_2$ generated by $x_1x_2^2x_1x_2^2x_1x_2^2x_1x_2^2$

$\mathcal{F}^0M_2 = \mathcal{F}^{-1}M_2$

$\mathcal{F}^1M_2$ generated by $x_1^2x_2^3x_3^9$

$\mathcal{F}^2M_2 = \mathcal{F}^1M_2$

$\mathcal{F}^3M_2$ generated by $x_1x_2^3$

$K_0^{-3} = \frac{1}{55}(t_1, 2t_2, 3t_3)$

$K_0^{-2} = \frac{1}{55}(A, B)$

$K_0^{-1} = 0$

$K_0^{0} = \frac{1}{55}(t_1, 2t_2)$

$K_0^{1} = 0$

$K_0^{2} = \frac{1}{55}(\frac{B-9}{11}t_1A)$

$K_0^{3} = \frac{1}{55} \left( \begin{array}{c} B-9 \\ \frac{A}{11} \end{array} \right)$

$I_{-3} = I_3 = (t_1, t_2, t_3), I_{-2} = I_2 = (A, B), I_{-1} = I_1 = k[t_1, \ldots, t_9]$

$I_0 = (2t_2A-t_1B)$

$I_{-3} \supseteq I_{-2} \supseteq I_{-1} \cap I_0$

$\tau = 34$ and $S_{34}$ is defined by $2t_2A-t_1B \neq 0$.

Remark: Because of (1) the elements of the matrix $K_0^p$ are invariant functions with respect to the action of $V^+(\tau)$. 
Lemma (6.3). Let $R$ be a commutative algebra over a field $k$, and consider $r$ derivations, $\delta_1, \ldots, \delta_r \in \text{Der}_k(R)$ with the following properties:

(i) $[\delta_i, \delta_j] = 0$ for all $i, j$

(ii) $\delta_i$ is nilpotent, i.e. for all $a \in R$ there is an integer $n$ such that $\delta_i^n(a)(a) = 0$

(iii) there are $z_1, \ldots, z_r \in R$ such that

- $\delta_i(z_j)$ is invariant with respect to the action of the Lie-algebra $\bigoplus_{i=1}^r k \cdot \delta_i = \mathfrak{L}$

- $\det(\delta_i(z_j))$ is invertible in $R$.

Then $R[[z_1, \ldots, z_r]] = R$.

We shall postpone the proof of (6.2) and (6.3) until the end of this §.

Now let us study the action of $V$ on $S_r$. Using (6) and (3) and (1) of (6.2) we can cover $S_r$ by invariant affin open sets defined by the product of a suitable family of minors of $K_0^P$, $p<0$.

Let $U = \text{Spec } C[[t]]^h$, $h = h_1, \ldots, h_r$ be one of these affine open sets, $h_i$ minor of $K_0^i$.

Let $i_0, i_1, \ldots, i_r, i_{r+1}, \ldots, i_r, \ldots, i_r$ = $r$ define the columns and $1 = j_0, j_1, \ldots, j_{s_1}, \ldots, j_{s_r}$ the rows of $K_0$ corresponding to these minors, $r = s_r = \mu - r - 1$. Denote by $\delta_j$ the vectorfields corresponding to $\lambda_j$. Let $\delta_k \in V$, and $s_1 < s < s_{i+1} - 1$. Because of (6.2) (4) and (4.5) (iii) and (P)

$$[\delta_j, \delta_k] \text{ is in the } C[[t]]^h \text{-module generated by } \delta_{j_{s_i+1}}, \ldots, \delta_{j_{s_r}}$$

Starting with $\delta_{j_{s_i+1}}, \ldots, \delta_{j_{s_r}}$ we apply (6.3) $l$ times and get

$$C[[t]]^h[t_{i_{s_i+1}}, \ldots, t_{i_{\mu - r - 1}}] = C[[t]]^h.$$

We may choose homogeneous invariant functions
\[d_j, \ldots, d_{j-r+\mu+1} \in C[t] \]

generating \(C[t]^V_h\) determined by

\[d_j / h = t_j \mod (t_1, \ldots, t_{r-1})\]

Then \(U/V = \text{Spec } C[d_j, \ldots, d_{j-r+\mu+1}]_h\) and \(x^{a_1}x^{a_2} + \sum d_{jk}h^h = 0\)

is the corresponding family.

\(U/V^+\) is then the open set defined by \(h\) in the corresponding weighted projective space.

Now it is clear that the quotient of the invariant open sets covering \(S_\tau\) by \(V^+\) glue to a quasismooth scheme \(T_\tau\). The corresponding families also glue in the étale topology.

Let us return to our example \(x_1^5 + x_2^{11}\) (see §5). We have \(\tau = \tau_{\min} = 34\), \(S_\tau = \text{Spec } C[t_1, t_2, t_3, A-t_4, B]\).

Consider \(U\) defined by \(t_1A(2t_2A-t_1B) = h\), \(i_0 = 1, i_1 = 4, r_1 = 1, i_2 = 6, i_3 = 7, r_2 = 3, i_4 = 8, r_3 = 4, i_5 = 9, r_4 = 5\). We find

\(C[t]^V_h = C[t_1, t_2, t_3, A_t^5 - B_t^4]_h\),

and the family:

\[x_1^9 + x_2^{11} + t_1x_1x_2 + t_2x_1x_2 + t_3x_1x_2 + (t_5 - A_4)x_1x_2^2\]

Now \(T_\tau = S_\tau / V^+\) is the open set \(D_+(2t_2A-t_1B)\) in \(P_3 = (1:2:3:10)\) = \(\text{Proj } C[t_1, t_2, t_3, y]\) with \(y = A_t^5 - B_t^4\).

As above we find universal families

\[x_1^5 + x_2^{11} + t_1x_1x_2 + t_2x_1x_2 + t_3x_1x_2 + A_t^3 x_1x_2\]

defined on the open set \(U_1 : A \neq 0\) in \(T_\tau\), and

\[x_1^5 + x_2^{11} + t_1x_1x_2 + t_2x_1x_2 + t_3x_1x_2 + B_t^3 x_1x_2\]

defined on the open set \(U_2 : B \neq 0\) in \(T_\tau\),

One may check, by direct computation, that these two families do not glue - algebraically - on \(U_1 \cap U_2\). However, they obviously glue in the étale topology.
Proof of (6.3). We may assume that \( \delta_i(z_j) = \delta_{ij} = \{1 \text{ if } i=j, 0 \text{ otherwise}\).

Otherwise let \( Z = (z_{ij}) \) be the inverse of the matrix \( (\delta_i(z_j)) \).

The \( z_{ij} \) are invariant under the action of \( L \).

Let \( \bar{Z}_i = \sum_j z_{ji} z_j \) then \( \delta_i(\bar{Z}_j) = \delta_{ij} \).

Denote by \( R_n = \{y|\delta_1, \ldots, \delta_r y = 0 \text{ if } v_1 + \ldots + v_r > n\} \).

\( R_1 = R^L \) and \( y \in R_n \) implies \( \delta_i(y) \in R_{n-1} \).

Assume now \( R_{n-1} \subseteq R^L[z_1, \ldots, z_r] \) and let \( y \in R_n \).

Then \( \delta_i(y) = \sum_v h_v^i z_1^v, \ldots, z_r^v, h_v^i \in R^L \).

\( \delta_k \delta_i(y) = \delta_i \delta_k(y) \) implies \( v^i h_{v_1}^i, \ldots, v^i_{v_k-1}, v^i_r = v_1^k, \ldots, v^k_{v_k-1}, v^k_r \) for all \( v \), with \( \sum v < n \). If \( v \) is given

with \( \sum v < n \) and \( v_k > 0 \) then let \( h_v^i = \frac{1}{v_k} h_{v_1}^i, \ldots, v^i_{v_k-1}, v^i_r \), and \( \bar{y} = \sum h_v^i z_1^v, \ldots, z_r^v \). Then \( y - \bar{y} \in R^L \).

Proof of (6.2). Let us first describe an effective procedure (due to B. Martin) to order the monomial bases of \( M_1 \) and \( M_2 \) by degree. Choose integers \( \alpha, \beta \) such that \( \alpha a_2 + \beta a_1 = a_1 a_2 + 1, 0 < \alpha < a_1 \) and \( 0 < \beta < a_2 \). Define for every integer \( k \),

\( k*(\alpha, \beta) = (\bar{\alpha}, \bar{\beta}) \)

with \( 0 < \bar{\alpha} < a_1, 0 < \bar{\beta} < a_2 \) and \( \bar{\alpha} = k* \alpha \mod a_1), \bar{\beta} = k* \beta \mod a_2 \).

Put \( Z = \{x|0 < k < a_1 a_2 - 2a_1 - 2a_2\} \). Then one may check that for \( k \in Z \)

the quasihomogeneous degree of the vector \( k*(\alpha, \cdot) \) is given by

\[
|k*(\alpha, \beta)| = \begin{cases} k/a_1 a_2 & \text{if } |k*(\alpha, \beta)| < 1 \\ 1 + k/a_1 a_2 & \text{if } |k*(\alpha, \beta)| > 1. \end{cases}
\]

Since for any element \( x_1 x_2 \) of our basis of \( M_1 \) (resp. \( M_2 \)) there is a unique \( k_i \in Z \) such that \( k_i* (\alpha, \beta) = (\alpha_i, \beta_i) \), we indeed obtain an effective procedure for ordering the basis of \( M_1 \) (resp. \( M_2 \)) by degree. We also get informations about the matrix of \( E:M_1 \to M_2 \) with respect to the basis \( \{l_i\}_{i \leq r} \) of \( M_1 \) and \( \{\mu_i\}_{i \leq r} \).
of $M_2$, $\lambda_i E = \sum k_{ij} \mu_j$:

Let $\lambda_i = x_i^y_i \alpha_i \beta_i$ correspond to $k_i$, i.e. $\alpha_i \equiv k_i \alpha \mod a_1$, $\beta_i \equiv k_i \mod a_2$, and $\mu_j = x_j^y_j \sigma_i \delta_j$ correspond to $k_j$, then $k_{ij} \neq 0$ iff $k_i < k_j$. In the example $x_1 + x_2$ we have $x = 1$ and $\beta = 9$

<table>
<thead>
<tr>
<th>$0 \times (1, 9)$</th>
<th>$(0, 0)$</th>
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<tr>
<td>$1 \times (1, 9)$</td>
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<td>$16 \times (1, 9)$</td>
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<td>$22 \times (1, 9)$</td>
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<td>$\lambda_9$</td>
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<tr>
<td>$23 \times (1, 9)$</td>
<td>$(3, 9)$</td>
<td>$\mu_9$</td>
</tr>
</tbody>
</table>
For \( k \in \mathbb{Z} \) we define its dual \( \check{k} \in \mathbb{Z} \) by \( \check{k} = a_1a_2 - 2a_1 - 2a_2 - k \). This definition is compatible with the duality on \( H_\mu \) defined in §5.

If \( k_1*(\alpha, \beta) = (\alpha_1, \beta_1) \) and \( \lambda_i = x_1^\alpha x_2^\beta \) then \( \chi_i = \mu_{r-i+1} \) (cf. §5) corresponds to \( \check{k}_i \), i.e. \( \check{k}_1*(\alpha, \beta) = (\check{\alpha}_i, \check{\beta}_i) \) and \( \mu_{r-i+1} = x_1^{\check{\alpha}_i}x_2^{\check{\beta}_i} \). We will use this description of the monomial basis to construct our filtration.

Let us denote by \( Z_u = \{ k \mid k \in \mathbb{Z}_1, \mu_1 = x \} \) by \( Z_0 = \mathbb{Z} \setminus (Z_u \cup Z_u) \). Then

\[ Z_0 = Z_{a_1} \cup Z_{a_2} \]

where \( Z_{a_1} = \{ k \mid k \in \mathbb{Z}, k \pm a_1 \} \) and

\[ Z_{a_2} = \{ k \mid k \in \mathbb{Z}, k \pm a_2 \} \].

**Lemma (6.4).**

1. \( k \in \mathbb{Z}_0 \) iff \( \check{k} \in \mathbb{Z}_0 \)
2. \( k \in \mathbb{Z}_1 \) implies \( k + a_1 \in \mathbb{Z}_1 \) and either \( k - a_1 \in \mathbb{Z}_1 \) or \( k - a_1 \in \mathbb{Z}_0 \)
3. \( k \in \mathbb{Z}_u \) implies \( k - a_1 \in \mathbb{Z}_u \) and either \( k + a_1 \in \mathbb{Z}_u \) or \( k + a_1 \in \mathbb{Z}_0 \)
4. \( k \in \mathbb{Z}_{a_1} \) implies \( k \pm a_1 \in \mathbb{Z}_1 \)
5. \( k \in \mathbb{Z}_{a_2} \) implies \( k \pm a_2 \in \mathbb{Z}_2 \)

**Proof:** (4) and (5) are trivial.

1. holds because of \( a_2 \alpha + a_1 \beta = a_1a_2 + 1 \)
2. let \( k \in \mathbb{Z}_1, k*(\alpha, \beta) = (\alpha_1, \alpha_2) \)

then \( 0 < \alpha_1 < a_1 - 1, 0 < \alpha_2 < a_2 - 1 \) and \( k = a_2 \alpha_1 + a_1 \alpha_2 \).

Now \( \alpha_1*(\alpha, \beta) = (0, 1) \).

Suppose \( \alpha_2 = a_2 - 2 \) then \( k = a_2 \alpha_1 + a_1 a_2 - 2a_2 \), but \( k < a_1 a_2 - 2a_2 - 2a_1 \).

Because of \( a_2 \alpha_2 - 2 \) we get \( (k + a_1)* (\alpha, \beta) = (\alpha_1, \alpha_2 + 1) \) and \( k + a_1 \in \mathbb{Z}_1 \).

If \( \alpha_2 > 0 \) then \( (k - a_1)* (\alpha, \beta) = (\alpha_1, \alpha_2 - 1) \), i.e. \( k - a_1 \in \mathbb{Z}_1 \).

If \( \alpha_2 = 0 \) then \( (k - a_1)* (\alpha, \beta) = (\alpha_1, a_2 - 1) \), i.e. \( k - a_1 \in \mathbb{Z}_0 \).

(3) is similar. Q.E.D.
Denote by \( u(k) = \# \{ h \in Z \mid h \leq k \} \) and by \( \lambda(k) = \# \{ h \in Z \mid h < k \} \). Then \( d(k) = u(k) - \lambda(k) \) gives the rank of the linear matrix. In fact, let \( d = \max \{ d(k) \mid k \leq a_1 a_2 - a_1 - a_2 \} \) then \( d \) determines the maximal diagonal containing no zeros in the linear matrix. Its length is \( r - d \).

**Lemma 6.5.** Let \( k \leq a_1 a_2 - a_1 - a_2 \) be maximal with the property \( d(k) = d \) then \( k < a_1 + 1 \).

**Proof:** Let us suppose \( k > a_1 + 1 \), i.e.

\[
\begin{align*}
& a_1 a_2 - 2a_1 - 2a_2 - 2k > a_1 + 1 \\
& k \leq a_1 a_2 - 2a_1 - a_2 - 1.
\end{align*}
\]

We shall show that for any \( k \leq a_1 a_2 - 2a_1 - a_2 - 1 \)

\((*)\): \( \# Z \cap \{ k+1, \ldots, k+a_1 \} > \# Z \cap \{ k+1, \ldots, k+a_1 \} \). If this is true let \( h < k + a_1 \) be maximal such that \( h \in Z \). Then \( d(h) = d \). Because of the choice of \( k \) we have \( h \geq a_1 a_2 - a_1 - a_2 \), i.e. \( h < a_1 a_2 - a_1 - a_2 \). Now \( d(h) = d(h) - 1 \) and \( h \in Z \). This implies \( d(h) = d \), i.e. \( h < k \).

Then we have \( h > k + 1 = k + 1 \), but \( h < k + a_1 \), i.e. \( a_1 + 1 > k \).

We have to prove \((*)\). Since \( (2, a_2) = 1 \) there exists a permutation \( k_0, \ldots, k_{a_1 - 1} \) of \( k+1, \ldots, k+a_1 \), \( k_{i}(a, b) = (i, c_i) \), \( i = 0, \ldots, a_1 - 1 \).

**Lemma 6.6.** Suppose \( k \leq a_1 a_2 - 2a_1 - a_2 \) then

\[
\begin{align*}
& (1) \quad k_1 \in Z \implies k_j \in Z \text{ for } j < i \\
& (2) \quad k_1 \in Z \text{ u implies } k_j \in Z \text{ for } i < j < a_1 - 2 \\
& (3) \quad k_1 \in Z \text{ implies } i = a_1 - 1 \text{ or } i + \frac{a_1 - 1}{2}
\end{align*}
\]

**Proof:** (1) Let us assume that \( k_j \in Z \) for some \( j < i \) then \( k_j = j \cdot a_2 + c_j \cdot a_1 - a_2 \), \( k_i = i \cdot a_2 + c_i \cdot a_1 \). But this implies \( |k_j - k_i| > a_1 \) which is a contradiction.
(2) is similar to (1)

(3) if \( k_i \in \mathbb{Z}_0 \) and \( i < a_1 - 1 \) then \( c_i = a_2 - 1 \), i.e.

\[
k_i = i \cdot a_2 + (a_2 - 1) a_1 - a_2 = i a_2 - a_1.
\]

If \( i < -\frac{1}{2} \) then \( k_i > \frac{1}{2} a_1 a_2 - \frac{1}{2} a_2 - a_1 \). But \( k_i \leq \frac{1}{2} a_1 a_2 - \frac{1}{2} a_1 - a_2 \), therefore

\[
i < -\frac{1}{2}.
\]

Q.E.D.

Now (*) is an immediate consequence of (6.6). Suppose

\[
\mathbb{Z}_u \cap \{ k+1, \ldots, k+a_1 \} = \{ k_0, \ldots, k_{t-1} \}
\]

then \( \{ k_{t+1}, \ldots, t_{a_1 - 2} \} \subseteq \mathbb{Z}_u \) and

either \( k_t \notin \mathbb{Z}_u \) or \( k_t \notin \mathbb{Z}_0 \). If \( k_t \notin \mathbb{Z}_0 \) then \( t < -\frac{1}{2} \), i.e.

\[
\# \mathbb{Z}_u \cap \{ k+1, \ldots, k+a_1 \} = t < a_1 - t - 2 = \mathbb{Z}_u \cap \{ k+1, \ldots, k+a_1 \}.
\]

If \( k_t \notin \mathbb{Z}_0 \) then \( \# \mathbb{Z}_u \cap \{ k+1, \ldots, k+a_1 \} = a_1 - t - 1 \) and we have to prove

\[
t < -\frac{1}{2}, \text{ i.e. } k_{a_1 - 1} \notin \mathbb{Z}_u.
\]

Suppose \( k_{a_1 - 1} \in \mathbb{Z}_u \) then \( k_{a_1 - 1} = a_2 [\frac{a_1 - 1}{2}] + a_1 c_{a_1 - 1} > \frac{1}{2} a_1 a_2 - a_2 \).

But \( k_{a_1 - 1} < \frac{1}{2} a_1 a_2 - a_2 \) gives a contradiction. Q.E.D.

We are now ready to construct the filtration. We start with a filtration on \( \mathbb{Z} \).

Let \( k \) be maximal with the property \( d(k) = d \) and \( k < \frac{1}{2} a_1 a_2 - a_1 - a_2 \).

By 6.5 we get \( k - k < a_1 + 1 \).

Let \( H^0 = \{ k+1, \ldots, k+1 \} \)

\[
H^{-1} = \{ k - a_1, \ldots, k \}
\]

\[
H^{-i} = \{ k - i a_1, \ldots, k - (i-1) a_1 - 1 \} \cap \mathbb{Z}, \text{ i>2}
\]

\[
H^i = H^{-i}, \text{ i>0}.
\]

Let \( \mathbb{F}^{-n} \mathbb{Z} = \mathbb{Z} \) for \( n > 0 \) and let \( \mathbb{F}^i \) be the filtration induced by the decomposition \( \mathbb{Z} = \bigcup_{i \in \mathbb{Z}} H^i \), such that \( \mathbb{F}^i \mathbb{P} \mathbb{Z} = \mathbb{F}^{i-1} \mathbb{P}^{-1} \mathbb{H}^{i-1} \mathbb{P}^{-1} \).

In our example \( x_1^5 + x_2^{11}, d = 3, k = 9, \mathbb{F} = 14 \)

\[
H^0 = \{ 10, 11, 12, 13 \}, \quad H^{-1} = \{ 9 \}, \quad H^{-2} = \{ 4, 5, 6, 7, 8 \}, \quad H^{-3} = \{ 0, 1, 2, 3 \}, \quad H^{-4} = \emptyset.
\]
Notice that it is possible that $H^0 = \emptyset$ or $H^{-1} \subseteq \mathbb{Z}_0$ (as in the example). In these cases we will get the $(x_1, x_2)$-adic filtration on $M_2$.

$\mathbb{F}$ induces filtrations on $Z_u$ and $Z_\lambda$:

$\mathbb{F}^*Z_u = \mathbb{F}^*Z \cap Z_u$, $\mathbb{F}^*Z_\lambda = \mathbb{F}^*Z \cap Z_\lambda$.

Let us denote the induced filtration on $M_2$ also by $\mathbb{F}$. Notice that $H^0 = \emptyset$ implies $\mathbb{F}^0 = \mathbb{F}^1$ and $H^{-1} \subseteq \mathbb{Z}_0$ implies $\mathbb{F}^{-1}Z_u = \mathbb{F}^0Z_u$ and $\mathbb{F}^1Z_u = \mathbb{F}^2Z_u$. Besides these exceptions the filtration is always strict in the non-trivial region. To avoid these equalities would complicate the proof.

$\mathbb{F}$ has the properties required in (6.2):

(P) is obvious

(0) $gr_p M_1$ is generated by $\{x^i_1 x^j_2 | (a_i, b_i) = k_i \ast (a, b), k_i \in gr_p Z_\lambda = H^p \cap Z_\lambda \}$

$gr_p M_2$ is generated by $\{x^i_1 x^j_2 | (a_i, b_i) = k_i \ast (a, b), k_i \in gr_p Z_u = H^p \cap Z_u \}$

By definition $\#H^0 \cap Z_u = \#H^0 \cap Z_\lambda$ and $\#H^{-1} \cap Z_u = \#H^{-1} \cap Z_\lambda$.

By 6.6 $\#H^{-i} \cap Z_u \ni \#H^{-i} \cap Z_\lambda$ if $i > 2$.

(1) holds because the difference of the minimal and the maximal element in $H^p$ is always smaller or equal to $a_1 - 1$: If an element of $K^0_0$ would depend on $t_a$ for some $a > r_0$ (we may assume its linear part is $x \ast t_a, x \neq 0$) and this element is in the column corresponding to $h \in H^p$ then $h - a_1 \in H^0$.

(2) is a consequence of the duality and the fact that the change of the matrix $K$ to the symmetric matrix (cf. 5.6) does not change $I_p$.

(3) is a consequence of (6.4): $gr_p Z = H^p$, because of (6.4)

(i) $H^{p-1} \cap Z_u = \{k - a_1, k \in H^p \cap Z_u \} \cup L \quad p < 2$

$H^{-2} \cap Z_u = \{k - a_1, k \in (H^{-1} \cap H^0) \cap Z_u \} \cup L$ and $L$ is empty or contains just one element.
(ii) $H^{p-1} \cap Z^* = \{k \in H^{p-1} | k \in H^p \cap Z^* \} \setminus T \quad p \leq -2$

$H^{-2} \cap Z^* = \{k \in H^{-1} \cup H^0 | k \in H^{-1} \cap Z^* \} \setminus T$ and $T$ is empty or contains just one element.

Further more $\# H^p \cap Z^* > \# H^p \cap Z^* \quad p < 0$.

Let $d_p = \# H^{p-1} \cap Z^* \quad p = -1$.

$I_{p-1}$ is the radical of the ideal generated by the $d_{p-1}$-minors of the matrix $K^p_{0}$, $I_{-1} \cap I_0$ is in the radical of the ideal generated by the $d_{-1}$-minors of the matrix $K^0_0 \cap K^0_0$.

Let $\lambda_{i_1}, \ldots, \lambda_{i_{d_{p-1}}}$ generate $\text{gr}_{p-1} M_1$, $p < 1$. Suppose $I_{p-1}$ vanishes at a point $t$, then the leading forms of $\lambda_{i_1} E, \ldots, \lambda_{i_{d_{p-1}}}$ with respect to the graduation, i.e. in $\text{gr}_{p-1} \cap M_2$ are dependent. Now because of (ii) the leading forms of $x_{2 \lambda_{i_1}} E, \ldots, x_{2 \lambda_{i_{d_{p-1}}}}$ define rows of the matrix $K^p_0$ (if $p < -3$) resp. $K^{-1}_0 \cap K^0_0$ (if $p = -2$). They are also dependent (i).

This implies that the corresponding $d_{p-1}$-minors of $K^p_0$ resp. $K^{-1}_0 \cap K^0_0$ vanish. But $d_p > d_{p-1}$ implies that $I_p$ resp. $I_{-1} \cap I_0$ vanishes at $t$.

(4) and (5) are obvious by definition of $\mathcal{F}$.

(6) By the choice of $k$, $d(k) = d$, the linear matrix has maximal rank $r-d$ at a general point, i.e. the matrix $K_0$ has rank $r-d$ at a general point. The rank drops if the rank of the graded matrices corresponding to $H^{-1}$ and $H^0$ drop. Because of (2) and (3) $S_\tau$ is defined by $I_{-1} \cap I_0$. Q.E.D.
To illustrate the filtration $\mathcal{F}'$, let us consider another example. Let $f = x_1^{5} + x_2^{12}$, then $\alpha = 3$, $\beta = 5$

\begin{align*}
0 \times (3, 5) &= (0, 0) & \lambda_1 \\
1 \times (3, 5) &= (3, 5) & \mu_1 \\
2 \times (3, 5) &= (1, 10) & \mu_2 \\
3 \times (3, 5) &= (4, 3) \\
4 \times (3, 5) &= (2, 8) & \mu_3 \\
5 \times (3, 5) &= (0, 1) & \lambda_2 \\
6 \times (3, 5) &= (3, 6) & \mu_4 \\
7 \times (3, 5) &= (1, 11) \\
8 \times (3, 5) &= (4, 4) \\
9 \times (3, 5) &= (2, 9) & \mu_5 \\
10 \times (3, 5) &= (0, 2) & \lambda_3 \\
11 \times (3, 5) &= (3, 7) & \mu_6 \\
12 \times (3, 5) &= (1, 0) & \lambda_4 \\
13 \times (3, 5) &= (4, 5) \\
14 \times (3, 5) &= (2, 10) & \mu_7 \\
15 \times (3, 5) &= (0, 3) & \lambda_5 \\
16 \times (3, 5) &= (3, 8) & \mu_8 \\
17 \times (3, 5) &= (1, 1) & \lambda_6 \\
18 \times (3, 5) &= (4, 6) \\
19 \times (3, 5) &= (2, 11) \\
20 \times (3, 5) &= (0, 4) & \lambda_7 \\
21 \times (3, 5) &= (3, 9) & \mu_9 \\
22 \times (3, 5) &= (1, 2) & \lambda_8 \\
23 \times (3, 5) &= (4, 7) \\
24 \times (3, 5) &= (2, 0) & \lambda_9 \\
25 \times (3, 5) &= (0, 5) & \lambda_{10} \\
26 \times (3, 5) &= (3, 10) & \mu_{10}
\end{align*}
$z = \{0, \ldots, 26\}$ $z_u = \{1, 2, 4, 6, 9, 11, 14, 16, 21, 26\}$

$z_x = \{0, 5, 10, 12, 15, 17, 20, 22, 24, 25\}$

$z_0^5 = \{3, 8, 13, 18, 23\}$

$z_0^{12} = \{7, 19\}$

d = 3, $k = 11$, $\n = 15$

$h^0 = \{12, 13, 14\}$, $h^{-1} = \{10, 11\}$, $h^{-2} = \{5, 6, 7, 8, 9\}$, $h^{-3} = \{0, 1, 2, 3, 4\}$

$\mathcal{F}^{-3}z = z$

$\mathcal{F}^{-2}z = \{k \in \mathbb{Z}, k \geq 5\}$

$\mathcal{F}^{-1}z = \{k \in \mathbb{Z}, k \geq 10\}$

$\mathcal{F}^0z = \{k \in \mathbb{Z}, k \geq 12\}$

$\mathcal{F}^1z = \{k \in \mathbb{Z}, k \geq 15\}$

$\mathcal{F}^2z = \{k \in \mathbb{Z}, k \geq 17\}$

$\mathcal{F}^3z = \{k \in \mathbb{Z}, k \geq 22\}$

$\mathcal{F}^4z = \emptyset$

the corresponding $gr_p M_1, gr_p M_2$ are generated by

<table>
<thead>
<tr>
<th>p</th>
<th>$gr_p M_1$</th>
<th>$gr_p M_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>$\lambda_1$</td>
<td>$\mu_1, \mu_2, \mu_3$</td>
</tr>
<tr>
<td>-2</td>
<td>$\lambda_2$</td>
<td>$\mu_4, \mu_5$</td>
</tr>
<tr>
<td>-1</td>
<td>$\lambda_3$</td>
<td>$\mu_6$</td>
</tr>
<tr>
<td>0</td>
<td>$\lambda_4$</td>
<td>$\mu_7$</td>
</tr>
<tr>
<td>1</td>
<td>$\lambda_5$</td>
<td>$\mu_8$</td>
</tr>
<tr>
<td>2</td>
<td>$\lambda_6, \lambda_7$</td>
<td>$\mu_9$</td>
</tr>
<tr>
<td>3</td>
<td>$\lambda_8, \lambda_9, \lambda_{10}$</td>
<td>$\mu_{10}$</td>
</tr>
</tbody>
</table>

This is not the $(x_1, x_2)$-adic filtration on $M_2$ resp. the induced one on $M_1$. 
The corresponding linear matrix is:

\[
\begin{array}{cccccccccc}
\mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 & \mu_{10} \\
\lambda_1 & t_1 & 2t_2 & 4t_3 & 6t_4 & 9t_5 & 11t_6 & 14t_7 & 16t_8 & 21t_9 & 26t_{10} \\
\lambda_2 & & t_1 & 4t_3 & 6t_4 & 9t_5 & 11t_6 & 16t_8 & 21t_9 & & \\
\lambda_3 & & & t_1 & 4t_3 & 6t_4 & 11t_6 & 16t_8 & & & \\
\lambda_4 & & & & t_1 & 4t_3 & 6t_4 & 9t_5 & 14t_7 & & \\
\lambda_5 & & & & & t_1 & 6t_4 & 11t_6 & & & \\
\lambda_6 & & & & & & 4t_3 & 9t_5 & & & \\
\lambda_7 & & & & & & & t_1 & 6t_4 & & \\
\lambda_8 & & & & & & & & 4t_3 & & \\
\lambda_9 & & & & & & & & & t_1 & & \\
\lambda_{10} & & & & & & & & & & t_1 \\
\end{array}
\]

\[I_{-1} = (2t_2 \frac{3t_2}{5t_1}), \quad I_0 = (t_1)\]

\[S: \quad t_1 (2t_2 \frac{3t_2}{5t_1}) \neq 0.\]

In the case of reducible plane curves one gets similar results (L-M-P): We can construct the moduli space of all plane curve singularities having the topological type of a quasi-homogeneous plane curve singularity, more precisely connected by a topological trivial family with a singularity defined by a non-degenerated quasi-homogeneous polynomial of degree \(d\) with respect to the weights \(w_1, w_2\). The following three cases will occur:

(i) \(k\) branches with semigroup \(\Gamma = \langle a_0, b_0 \rangle; (a_0, b_0) = 1\)

\[w_1 = d/ka_0, \quad w_2 = d/kb_0, \quad f = x_1^a x_2^b\]

(ii) \(k\) branches with semigroup \(\Gamma\) and one smooth branch:

\[w_1 = d/ka_0, \quad w_2 = (d-w_1)/kb_0, \quad f = x_1^a x_1 x_2^b\]
(iii) \( k \) branches with semigroup \( \Gamma \) and two smooth branches:

\[
  w_1 = \frac{(d-w_2)}{ka_0}, \quad w_2 = \frac{(d-w_1)}{kb_0}, \quad f = x_1^{a}x_2 + x_1^{b}.
\]

**Theorem (6.6).** Fix the quasi-homogeneous type \((w,d)\) and the Tjurina-number \(\tau\), then the coarse moduli space \(T(w,d),\tau\) of all plane curve singularities with that quasi-homogeneous topological type and Tjurina-number \(\tau\), exists in the category of algebraic spaces. For \(\tau = \tau_{\text{min}}\), the moduli space is a scheme (except, maybe, in the homogeneous case, i.e. when \(w_1 = w_2 = d\), if \(d\) is even).
We keep the notation of §5 and §6 and give an algorithm for computing the matrix $K_0$ defined by $E \lambda_i = \sum h_{ij} \mu_j$ in the case $(a_1, a_2) = 1$ (cf. §5).

To construct the monomial bases $\{\lambda_i\}$ of $M_1$ and $\{\mu_i\}$ of $M_2$, we apply the method above using a primitive vector (cf. proof of (6.2)).

Part (i) of the algorithm:

In this part of the algorithm we compute a string $S$ related to $k^*(\alpha, \beta)$:

$$S(k) = \begin{cases} 
l & \text{if } k \in \mathbb{Z}_l \\
\ast & \text{if } k \in \mathbb{Z}_0 \\
u & \text{if } k \in \mathbb{Z}_u 
\end{cases}$$

Compute $\alpha$ and $\beta$ such that $a_2 \alpha \equiv 1 \mod a_1$, $a_1 \beta \equiv 1 \mod a_2$; $j := 1$; $i := e := e' := \emptyset$; $S(1) := 'l'$; $l(1) := \emptyset$; $d := a_1 a_2$

FOR $k := 1$ TO $d - 2(a_1 + a_2)$ DO BEGIN

$e := e + \alpha \mod a_1$; $e' := e' + \beta \mod a_2$; $e(k) := e$; $e'(k) := e'$; $\{e(k), e'(k)\}$ is needed only in part (iii) of the algorithm

IF $e = a_1 - 1$ OR $e' = a_2 - 1$ THEN $S(k+1) := '\ast'$ ELSE

IF $(e a_2 + e' a_1) < d$ THEN BEGIN

$i := i + 1$; $l(i) := k$; $S(k+1) := 'l'$ END

{characterizes monomials of $M_1$, $l(i)$ is the degree of $\lambda_i$}

ELSE BEGIN $j := j + 1$; $u(j) := k$; $S(k+1) := 'u'$ END

{characterizes monomials of $M_2$, $u(j) + d$ is the degree of $\mu_j$}

END
Example: $f = x_1^5 + x_2^{14}$, $\alpha = 4$, $\beta = 3$

$$S = 'l*u*uu*u*u*l*u*ll*u*ll*l*u'$$

Notice that $S(u) = 'u'$ iff $S(N+1-u) = 'l'$, $N =$ length of the string $= a_1 a_2 - 2a_1 - 2a_2 + 1$ (duality), i.e. it is sufficient to compute only half the string.

Now a monomial $t_{g(1)} \ldots t_{g(s)}$ OCCURS in the element $h_{ij}$ of the matrix $K_0$ iff

(i) $\lambda_i \mu_{g(1)} \ldots \mu_{g(s)} = \mu_j$

(ii) $\lambda_i \mu_{g(1)} \ldots \mu_{g(t)}$ is not in the monomial base for $t = 1, \ldots, S-1$.

This is equivalent (in the language of our string resp. the method used in 6.2) to

(i') $\lambda(i)+u(g(1))+\ldots+u(g(s)) = u(j)$

(ii') $S(\lambda(i)+u(g(1))+\ldots+u(g(t)))$ FOR T $= 1, \ldots, S-1$.

Using (i') and (ii') we can compute the monomials of $h_{ij}$.

**Part (ii) of the algorithm:** Computing the monomials of $h_{ij}$.

PROCEDURE bracket $(r,s)$;
BEGIN $b := s-r$; $F := F + 'C'$;
REPEAT $g := \max i : u(i) < b$;
WHILE $g > \emptyset$ DO BEGIN $b := u(g) - 1$;
IF $u(g) = s-r$ THEN $F := F + 't' + g$
ELSE IF $S(l+r+u(g))$ FOR T THEN
BEGIN $F := F + 't' + g$; bracket $(r+u(g), s)$ END;
$g := \max i : u(i) < b$ END;
IF (last character of $F$) THEN (delete last two char)
UNTIL $b = \emptyset$;
Replace last character of $F$ by ')' +
END;
BEGIN $u(\emptyset) := \emptyset$; $F := ''$
bracket $(l(i), u(j))$;
Delete last character of $F$
END.
Example: \( F = x_1^{5} + x_2^{14} \), \( i = 2, j = 9 \)

\[ F = \left( t_7 + t_3 (t_4 (t_1) + t_1 (t_4 + t_3 (t_2) + t_2 (t_3 + t_1 (t_1))) ) \right) \]

to finish this part of the algorithm we have to "solve the brackets.

In our example we get:

\[ F = t_7 + t_3 t_4 t_1 + t_3 t_1 t_4 + t_3 t_1 t_2 + t_3 t_1 t_2 t_3 + t_3 t_1 t_2 t_1 t_1 \]

Now we have to compute the coefficients of the corresponding monomials. If \( c \) is the coefficient of the monomial \( t_{g(1)} \ldots t_{g(s)} \) occurring in \( F \), then \( c = c_1 \ldots c_s \)

and \( c_1 = \text{deg } g(l)/d \), \( c_1 \) is one of the exponents of \( g(i) \)

divided by \( a_1 \) or \( a_2 \) depending on whether \( \frac{\delta F}{\delta x_1} \) resp. \( \frac{\delta F}{\delta x_2} \) was

involved in the reduction modulo \( (\delta x_1, \delta x_2) \) in this step.

Part (iii) of the algorithm: Computing a coefficient

\( c := (-1)^s n(g(l))/d; \ n := l(i); \)

FOR \( k := 2 \) TO \( s \) DO BEGIN

\( e := e(u) + e(u(g(k-1))); \ e' := e'(u) + e'(u(g(k-1))); \)

\( u := n + g(k-1); \)

IF \( e > a_1 - 1 \) THEN \( c := c \cdot e(u(g(k)))/a_1; \)

IF \( e' > a_2 - 1 \) THEN \( c := c \cdot e'(u(g(k)))/a_2 \)

END

Example: \( f = x_1^4 + x_2^9 \), \( i = 2, j = 9, t_3 t_1 t_2 t_1 t_1 \)

\[ c = \frac{4^5 \cdot 6^2 \cdot 6^3}{70 \cdot 14^5 \cdot 14^5} = -\frac{108}{42875} \]

Part (iv) of the algorithm:

Finally we have to order the variables in the monomials, to order the monomials in \( F \) lexicographically and to identify monomials of the same type by adding their coefficients.
Example: $f = x_1^5 + x_2^{14}$

$h_2,g = -\frac{13}{70}t^7 - \frac{39}{1225}t^4 t_1^3 + \frac{99}{8575}t_2^2 t_1 - \frac{108}{42875}t_3^2 t_1^3$

Remark: The algorithm can be modified for more than two variables. We can also omit the restriction that $a_1$ and $a_2$ be relatively prime.
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