The Riemann problem for a single conservation law in two space dimensions

Raphael Høegh–Krohn†
Nils Henrik Risebro
Matematisk Institutt, Universitetet i Oslo, Blindern, Oslo 3 N-0316, Norway

Abstract

This paper considers the single hyperbolic conservation law in two space dimensions:

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}+\frac{\partial g(u)}{\partial y}=0.$$

The Riemann problem in several space dimensions is the initial value problem when the initial values are constant on a finite number of wedges meeting in a single point. In section one we present a numerical method for solving this problem if the functions f and g are piecewise linear and continous with a finite number of breakpoints. In section two we obtain stability estimates for the computed solution with respect to the initial values of f and g. We can then show existence and uniqueness in L_1 if the initial function can be approximated by polygonwise constant functions and has compact support, and if the flux funtions can be approximated by piecewise linear curves. In section four we show that if the solution is sufficiently smooth, it will satisfy an entropy condition. Some of these results were presented at the fourth European Symposium on Enhanced Oil Recovery [12].

1. Introduction

An existence theorem for weak solutions of

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}+\frac{\partial g(u)}{\partial y}=0 \qquad 1.1$$
$$u(0,x,y)=u_0(x,y)$$

was proved in 1966 by Conway and Smoller [1]. They used a finite difference scheme which was an adaptation of Lax's [2] scheme for proving existence in the one-dimensional case. In 1970 Krusjkov [3] gave an existence and uniqueness theorem via the "vanishing viscosity" method, using that solutions to viscosity equations obey a maximum principle. Wagner [4] constructed the entropy solution if $f=g$ and both f and g convex, and the inital data

† Deceased on January 24, 1988.

constant in each of the four quadrants. Lindquist [5],[6] also dealt with the case $f = g$, letting f have a limited number of inflection points and with more general inital data. Tung and Xu-Xi [7] solved the quadrant inital data problem if $\left(\frac{f''(u)}{g''(u)}\right)' \neq 0$, and f and g piecewise smooth. This two dimensional Riemann problem arises in applications, for instance in simulating petroleum reservoirs using a front tracking method, see [8] and the references therein.

Let $(x, y) \in R^2$ and consider (1.1). Assume f and g are almost everywhere differentiable and continuous functions from $R \to R$ and u_0 is bounded and measurable. By a solution to this equation is meant a weak solution in the sense that

$$\int\limits_{t>0}\int\int\limits_{R^2} u\frac{\partial\varphi}{\partial t} + f(u)\frac{\partial\varphi}{\partial x} + g(u)\frac{\partial\varphi}{\partial y}dxdydt + \int\int\limits_{R^2} u_0(x,y)\varphi(0,x,y)dxdy = 0 \qquad 1.2$$

for all continously differentiable test functions $\varphi = \varphi(t, x, y)$ with compact support.

We now assume that f and g are piecewise linear continuous functions. Using the results in [9] we can then solve the Riemann-problem consisting of two states only, seperated by a straight line. The solution to this will be a number of straight lines parallel to the original line and moving away from each other. This can now be used to solve the Riemann problem outside a circle about the origin, where no interaction of fronts takes place. Given the solution in this area, one can then work out a solution in the whole plane by considering the intersection points of such lines.

We then use the Riemann-solution to construct the solution to more general initial value problems. We approximate the initial value function by a polygonwise constant function. Then we solve the Riemann-problems which are independant until they interact, whereupon we again have an initial value function which is polygonwise constant and the process is repeated.

For this we can obtain stability estimates both with respect to the flux-functions f and g , and with respect to the initial data. These estimates, together with suitable sequences $\{(f_n, g_n)\}$ and $\{u_{0,n}\}$, then makes it possible to prove that the solutions computed by this method converges. If the limit function has sufficiently smooth discontinuities, if will satisfy an entropy inequality across such.

2. The Riemann-problem

In what follows let f and g be piecewise linear functions. First we examine the case where u_0 only takes two values, let these two be separated by a straight line; which we will name a front or a shock. One can now introduce coordinates perpendicular and parallel to the front. The solution to this problem will then consist of a number of fronts, each beeing a straight line parallel to the initial front. (See e.g.[9] and figure 1.) Assume now that f and g only have one breakepoint (at u_2) in between the two values of u_0 , say at u_1 and u_3 respectivly. Assume further that the solution does not involve splitting of the front. Now let $\sigma^h_{ij} = h'|\langle u_i, u_j\rangle$, and let $\vec{\sigma}_{ij} = (\sigma^f_{ij}, \sigma^g_{ij})$. Then the unsplit front will move in the direction of its own normal, with a speed given by

$$\vec{\sigma}_{13} \cdot \vec{n} = \left(\frac{(u_3 - u_2)\vec{\sigma}_{23} + (u_2 - u_1)\vec{\sigma}_{12}}{u_3 - u_1} \right) \cdot \vec{n}$$

where $\vec{n}$ is the unit normal to the front. If γ is a coordinate in the normal direction of the front and $h(u, \vec{n}) = (f(u), g(u)) \cdot \vec{n}$ the equation becomes

$$\frac{\partial u}{\partial t} + \frac{\partial h(u)}{\partial \gamma} = 0$$

which is the one dimensional equation treated in e.g. [9]. The solution of this problem involves a splitting of the front in case $h(u, \vec{n})$ is concave between u_1 and u_3 . Considering h as a function of the direction of the front we see that a splitting will occur when $\vec{\sigma}_{12} \cdot \vec{n} = \vec{\sigma}_{23} \cdot \vec{n}$ that is, the front is a parallel to the line between $\vec{\sigma}_{12}$ and $\vec{\sigma}_{23}$. If the initial front turns further in this direction the solution will consist of two separate fronts moving apart. Both of these are weak solutions since they are weak one-dimensional solutions in the direction normal to themselves.

Consider now the case where u_0 takes three values u_1, u_2, u_3 which are also the breakpoints of f and g . Far from the origin all fronts are one-dimensional problems. Assume first that the $(1,3)$ will split up. Then we again have two separate problems moving apart. If $(1,3)$ will not be splitting then the solution can be found by the following algorithm: Move front (a, b) to $\vec{\sigma}_{ab}$, then draw a line from the intersection of $(1,2)$ and $(2,3)$ to $\vec{\sigma}_{13}$. this line is a front which completes the weak solution to this problem, and the solution is self similar under scaling $(t, x, y) \to (ct, cx, cy), c > 0$, since if one applies the same algorithm starting from $t = t_1$ the "same" picture is obtained because all lines move linearly and normally to themselves, and therefore their intersctions will also move linearly. It is a weak solution to the initial value problem since it is a weak solution to the problem starting at t_1 because $u(t_1, x, y) \to u_o(x, y)$ in L_1as $t_1 \to 0$. Note that the added line has a permissible direction.

If f and g have several breakpoints in between u_1 and u_3 it may be that the $(1,3)$ front that extends from the intersection of the $(1,2)$ to the $(2,3)$ front has an unpermissible direction for such a front and should be split into several fronts. We will first show that we obtain a "good" solution if this front will split into two; a $(3,*)$ and a $(1,*)$ front. See figure 2. Here $u_1 < u_* < u_2$ if $u_2 < u_* < u_3$ the situation is similar. Since $(1,3)$ is unsplit the point $\vec{\sigma}_{13}$ must be in sector A, else $(1,3)$ would split. Since $(1,2)$ is unsplit and $\vec{\sigma}_{13}$ is on the line between $\vec{\sigma}_{3*}$ and $\vec{\sigma}_{1*}$, we have that $\vec{\sigma}_{1*}$ lies in sector B. Now $\vec{\sigma}_{12}$ is on the line between $\vec{\sigma}_{1*}$ and $\vec{\sigma}_{2*}$, $\vec{\sigma}_{2*}$ is now fixed on the intersection of this line and the line from $\vec{\sigma}_{23}$ through $\vec{\sigma}_{3*}$.

We find a triangle $P_1 P_2 P_3$ such that :

P_1 is on the intersection of γ and $\overrightarrow{P_3, \sigma_{3*}}$
P_2 is on the intersection of β and $\overrightarrow{P_1, \sigma_{1*}}$
P_3 is on the intersection of α and $\overrightarrow{P_2, \sigma_{2*}}$

See figure 3. Because of the locations of $\vec{\sigma}_{1*}$,$\vec{\sigma}_{2*}$,$\vec{\sigma}_{3*}$ relative to the lines α , γ and β this is possible in a unique way. The triangle will look like figure 4. If we now assume that the lines P_1P_2 and P_2P_3 are permissible directions for $(1,*)$ and $(2,*)$, then the following will be a solution by the same arguments used in the previous case. See figure 5 and the example in [10]. To find a solution to this initial value problem in general one will only have to find a finite number of points P_i since f and g only have a finite number of breakpoints. To do this one has a system of linear equations, which at least almost always has a solution. The case where no solution can be found can be thought of as if the point in question lies at infinity. This means that we have infinite "speed of information" , i.e. that $\frac{\partial f}{\partial \xi}$ or $\frac{\partial g}{\partial \xi}$ is infinite for some coordinate ξ, which cannot be the case. The correct solution to this Riemann-problem may look like figure 6.

Now consider the general situation where f and g are piecewice linear continous functions with a finite number of breakpoints. Let u_0 take n values, u_0 constant on wedges meeting at $(0,0)$. This is a generalization of the algorithm from the last example:

Far from the origin, treat each front as if it were the only one. It will split up into several half lines wich are moved to their respecive velocity points. Outside the area of interections, these fronts will be correct solutions. Now we construct the remaining fronts, a front reaches from a point towards its velocity point, extends until this or until the first intersection with a front that separates between one of the same u-values such that one can assign function values in a sector on one side of the intersection. A front has a permitted direction, and we do not construct unneccesary fronts, that is, if front (a,b) is permissible we do not split it further, this will ensure that the solution is self similar.

By the same arguments as in the last example this gives a weak solution which is self similar and satisfies the Oleinik entropy condition for the single conservation law across each front exept at the points where several u-values meet.

We will prove later that in L_1 this is the only solution to this Riemann problem. By this method one can solve the initial value problem when the function u_0 is constant on a finite number of polygons. In this case the problem is a finite number of independent Riemann problems which can be solved independently until they interact. We will show that the $\text{T.V.}_{(x,y)}$ norm of the solution is nonincreasing in t . If the initial function is of bounded total variation then we have for almost all x and y : $\text{T.V.}_x|u(t,x,y)| < \infty$ and $\text{T.V.}_y|u(t,x,y)| < \infty$ respectivly. This implies that we cannot have an infinite number of collisions in finite time. If so the collision times would accumulate for some time t' , and since our solutions can take only a finite number of values at any time, and since the resulting fronts from a collision move apart, this implies the existence of sets X and Y , at least one of which is of positive measure, and for $x \in X$ and $y \in Y$ the total variation with respect to that variable is infinite. Since $\text{T.V.}_{(x,y)} = \int \text{T.V.}_x dy + \int \text{T.V.}_y dx$ this is impossible.

If $f = k \cdot g$ then we have a stronger result. The solution will always stay constant on a finite number of intervals on each line $y = kx + c$, and on each such line the solution is constant in a finite number of regions for $t > 0$. If the initial function is constant on a finite number of polygons, we can construct the solution by pathing together such lines, this solution will then be constant on a finite number of regions in $R^2 \times R^+$.

3. Stability estimates

In this section we will need a theorem from [11]:

Theorem 3.1. *Let $u(t,x)$ and $v(t,x)$ be solutions of*

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

with the initial condition

$$u(0,x) = u_o(x) \qquad v(0,x) = v_0(x)$$

where u_0 and v_0 are bounded step functions with compact support and a finite number of discontinuities. Let f be piecewise linear with a finite number of breakpoints. Then

$$\int |u(t,x) - v(t,x)|dx \leq \int |u_0(x) - v_0(x)|dx$$

Proof: Although the proof is found in [11], we will include it here since the proof of the equivalent theorem for the two dimensional case is very similar.

Assume that u_0 and v_0 are constant in the intervals $I_j = \langle a_j, a_{j+1}\rangle$ for $j = 1, \ldots, N$ and $a_1 = -\infty$ and $a_N = \infty$ and $u|I_j = w_c$ and $v|I_j = w_{c'}$. We define the increasing sequence $\{s_i\}_1^{M+1}$ by $s_1 = 1$ and $s_{i+1} - s_i = |c - c'|$.

Let $H = 3s_{M+1}$. The sequence $\{u_{0,i}\}_{3s_i}^{3s_i+1}$ is defined by:

$$u_{0,n}(x) = \begin{cases} u_0, & \text{if } x \in I_j, j > i \\ v_0, & \text{if } x \in I_j, j < i \\ w_{s+\text{sign}(c-c')\frac{(n-3s_i+2)}{3}}, & \text{if } x \in \langle a_i, \frac{2a_i+a_{i+1}}{3}\rangle \\ w_{s+\text{sign}(c-c')\frac{(n-3s_i+1)}{3}}, & \text{if } x \in \langle \frac{2a_i+a_{i+1}}{3}, \frac{a_i+2a_{i+1}}{3}\rangle \\ w_{s+\text{sign}(c-c')\frac{(n-3s_i)}{3}}, & \text{if } x \in \langle \frac{a_i+2a_{i+1}}{3}, a_{i+1}\rangle \end{cases}$$

We also have

$$u_{o,3} = u_0$$
$$u_{0,H} = v_0$$

and

$$u_{0,3s_i} = \begin{cases} u_0, & \text{for } x \in I_j j \geq i \\ v_0, & \text{for } x \in I_j j \leq i-1 \end{cases}$$

$u_{0,i}$ and $u_{0,i+1}$ are identical exept on a third of an I_j interval where the difference between them is the minimum possible; $w_c - w_{c'}$. We also have that

$$|u_0 - v_0| = \sum_{i=3}^{H-1} |u_{0,i} - u_{0,i+1}|$$

Let $u_i(t,x)$ be the solution to the initial value problem with initial value $u_{0,i}$. Assume we are able to prove that

$$\int |u_i(t,x) - u_{i+1}(t,x)|dx \leq \int |u_{0,i}(x) - u_{0,i+1}(x)|dx \qquad 3.2$$

Then we obtain

$$\int |u-v|dx \leq \sum_{i=3}^{H-1} \int |u_i(t,x) - u_{i+1}(t,x)|dx$$

$$\leq \sum_{i=3}^{H-1} \int |u_{0,i}(x) - u_{0,i+1}(x)|dx = \int |u_0 - v_0|dx$$

Therefore it is sufficient to prove 3.2. The shock fronts are straight lines not parallel with the x–axis, and therefore $\int |u_i(t,x) - u_{i+1}(t,x)|dx$ is continuous and piecewise linear in t. To prove 3.2 it is sufficient to show that for any t the right derivative $\frac{\partial}{\partial t_+}(\int |u_i(t,x) - u_{i+1}(t,x)|dx) \leq 0$. Let $c = u_{0,i}$ and $d = u_{0,i+1}$. We have four cases. First two cases depending on whether c and d are equal on the middle third or on the left or right third of an interval. If the two functions are equal on the left this is eqivalent to equality on the right third if we substitute x with $-x$ and f with $-f$. If the two are different on the left third we divide this into four subcases depending upon the common value to the left of this interval. The subcases are: the value is larger than both the values of c and d, the value is smaller than both the values of c or d or the value is equal to one of the values. In each case the derivative is equal to zero. ∎

Using a similar technique we can prove a corresponding theorem for the two dimensional case:

Theorem 3.3. *Let $u(t,x,y)$ and $v(t,x,y)$ be solutions of 1.1 with the initial conditions*

$$u(0,x,y) = u_0(x,y)$$
$$v(0,x,y) = v_0(x,y)$$

where u_0 and v_0 are poygonwise constant step functions with compact support on a finite number of polygons. Let f and g be piecewise linear continuous with a finite number of breakpoints. Then we have

$$\int\int |u(t,x,y) - v(t,x,y)|dxdy \leq \int\int |u_0(x,y) - v_0(x,y)|dxdy.$$

Proof: We will prove that

$$\frac{\partial}{\partial t}\Big(\int\int |u(t,x,y) - v(t,x,y)|dxdy\Big) \leq 0$$

It is sufficient to show this for $t = 0$ We may assume that u_0 and v_0 are constant on the same polygons which are triangles that we number $\{T_i\}$. As in the one dimensional case we define a sequence $\{u_{0,i}\}$ and $\{u_i\}$ such that $u_{0,i} - u_{0,i+1}$ is the minimum possible . We divide each triangle where u_0 and v_0 are constant into three smaller triangles meeting at a point in the interior of the bigger triangle, $u_{0,i}$ and $u_{0,i+1}$ only differ on one of the smaller triangles where the difference is the minimum possible. To continue with the bookeeping we define an integer valued function $F(i)$ such that

$$u_{0,i}(x,y) = \begin{cases} v_0, & \text{if } (x,y) \in T_j \; j > F(i) \\ v_0, & \text{if } (x,y) \in T_j \; j < F(i) \end{cases}$$

Inside a triangle T_i, $u_{0,i}$ only has jumps which are minimal. We can find a constant M such that the speed of any triple point of u_i is less than M and that also M is a bound on $|u_i - u_{i+1}|$. Let $N(t)$ be the union of the four balls of radius Mt about the triple points where $u_{0,i}$ and $u_{0,i+1}$ differ. See figure 7. We have that

$$\begin{aligned} \iint_{R^2} |u_i - u_{i+1}| dx dy &= \iint_{R^2 \setminus N(t)} |u_i - u_{i+1}| dx dy \\ &\leq \iint_{R^2 \setminus N(t)} |u_i - u_{i+1}| dx dy + 4\pi(Mt)^2 \cdot M \end{aligned}$$

Therefore

$$\begin{aligned} &\Big(\iint_{R^2} |u_i - u_{i+1}| dx dy - \iint_{R^2} |u_{0,i} - u_{0,i+1}| dx dy\Big) \\ &\leq \iint_{R^2 \setminus N(t)} |u_i - u_{i+1}| dx dy - \iint_{R^2 \setminus N(t)} |u_{0,i} - u_{0,i+1}| dx dy \\ &\quad + Ct^2 - \iint_{N(t)} |u_{0,i} - u_{0,i+1}| dx dy \\ &= \iint_{R^2 \setminus N(t)} |u_i - u_{i+1}| - |u_{0,i} - u_{0,i+1}| dx dy + Ct^2 \end{aligned}$$

which implies

$$\begin{aligned} \frac{d}{dt}\Big|_{0+} \Big(\iint_{R^2} |u_i - u_{i+1}| dx dy\Big) &\leq \lim_{t \to 0} \frac{1}{t}\Big(\iint_{R^2 \setminus N(t)} |u_i - u_{i+1}| - |u_{0,i} - u_{0,i+1}| dx dy + Ct^2\Big) \\ &= \frac{d}{dt}\Big|_{0+} \Big(\iint_{R^2 \setminus N(t)} |u_i - u_{i+1}| dx dy\Big) + 0 \\ &= 0 \end{aligned}$$

What regards the difference between $u_{0,i}$ and $u_{0,i+1}$ we have three cases, see figure 8. And since we are interested in $R^2 \setminus N(t)$ we can ignore what originates in $N(t)$, if this is ignored the difference in each of the three cases is a triangle moving with fixed speed, and the derivative in question is equal to zero. We therefore have that

$$\int\int_{R^2} |u_i(t,x,y) - u_{i+1}(t,x,y)|dxdy \le \int\int_{R^2} |u_{0,i}(x,y) - u_{0,i+1}(x,y)|dxdy$$

and we can use the triangle inequality and sum to prove the theorem as in the one dimensional case. ■

To prove the next inequality we need two lemmas from [11]:

Lemma 3.4. *Let $u(t,x)$ and $v(t,x)$ be solutions of*

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$
$$\frac{\partial u}{\partial t} + \frac{\partial g(u)}{\partial x} = 0$$

with the initial condition

$$u(0,x) = v(0,x) = \begin{cases} c_1 & \text{if } x \ge 0 \\ c_2 & \text{if } x < 0 \end{cases}$$

with f and g piecewise linear continuous and either

$$c_1 < c_2 \ f \text{ and } g \text{ convex}$$
$$c_1 > c_2 \ f \text{ and } g \text{ concave}$$

Then

$$\frac{\partial}{\partial t}\Big(\int |u(t,x) - v(t,x)|dx\Big) \le \int_{\min(c_1,c_2)}^{\max(c_1,c_2)} |f'(u) - g'(u)|du.$$

Let now f_{cab} denote the convex envelope of $f|\langle a,b\rangle$

Lemma 3.5. *Let f and g be piecewise linear on $\langle a,b\rangle$. Then*

$$\int_a^b |f'_{cab}(u) - g'_{cab}(u)|du \le \int_a^b |f'(u) - g'(u)|du$$

A corresponding lemma holds for the concave envelope.

These two lemmas together give for the Riemann problem of lemma 3.4 that

$$\int |u(t,x) - v(t,x)|dx \le t \cdot \int_{\min(c_1,c_2)}^{\max(c_1,c_2)} |f'(u) - g'(u)|du \qquad 3.6$$

Theorem 3.7. *Let $u(t,x,y)$ and $v(t,x,y)$ be solutions of*

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{f}(u) = 0$$
$$\frac{\partial v}{\partial t} + \nabla \cdot \vec{g}(v) = 0$$
$$u(0,x,y) = v(0,x,y) = u_0(x,y)$$

Where $\vec{f} = (f_x, f_y)$ and $\vec{g} = (g_x, g_y)$ are piecewise linear continuous , u_0 is a bounded polygonwise constant step function with compact support. Then

$$\frac{\partial}{\partial t}\Big(\int\int |u(t,x,y) - v(t,x,y)|dxdy\Big) \le \text{T.V.}_{\cdot(x,y)}|\vec{f}(\tilde{u}(x,y)) - \vec{g}(\tilde{u}(x,y))|$$

where the notation on the left will be explained in the proof.

Proof: If $f(x,y)$ is a function let $h_y(x) = f(x,y)$; x fixed. Similarely for $h_x(y)$. If then $\int \text{T.V.}_{\cdot x}|h_y(x)|dy + \int \text{T.V.}_{\cdot y}|h_x(y)|dx < \infty$, we call this number $\text{T.V.}_{\cdot(x,y)}|f(x,y)|$. We have that this is equivalent to $\int\int \|\text{grad} f\| dxdy$, where $\text{grad} f$ is to be interpreted as a measure, this is then the total variation of this measure. If $\vec{f}$ is a vector $\text{T.V.}(\vec{f})$ is the sum of the total variation of each component function.

It is sufficient to prove this inequality at $t = 0$, since for sufficiently small time u and v are self similar about the points where several values of u_0 meet, and for such small t the integral is linear in t. We number the fronts of u_0 (index j), and each front divides between c_{jk} $k = 1,2$. Let the length of that front be L_j. As in the proof of the last theorem we can find a constant M such that the speed of any triple point of u or v is less then M, and M is also a bound on $|v|$ and $|u|$. Now consider front j. Divide the area around that front in three; two sectors of radius Mt_0 about the endpoints of the front, and a strip on both of the remaining parts of the front. For $t < t_0$ we may use 3.6 integrating across the front, and then integrate along it. Let α_j be the total of the abovementioned area, and let

$N_j(t_0)$ be the parts around the endpoints. Then we have

$$\begin{aligned}
\iint_{\alpha_j} |u(t,x,y)-v(t,x,y)|dxdy &= \iint_{\alpha_j\setminus N(t_0)} |u(t,x,y)-v(t,x,y)|dxdy \\
&\quad - \iint_{N(t_0)} |u(t,x,y)-v(t,x,y)|dxdy \\
&\leq \iint_{\alpha_j\setminus N(t_0)} |u(t,x,y)-v(t,x,y)|dxdy \\
&\qquad + M\pi(Mt_0)^2 2\frac{1}{2} \\
&\leq tL_j \int_{\min c_{jk}}^{\max c_{jk}} |\vec{f}\cdot\vec{n}'(u) - \vec{g}\cdot\vec{n}'(u)|du + Kt_0^2 \\
&\quad ; \quad \vec{n} \text{ is unit normal} \\
&\leq tL \int_{\min c_{jk}}^{\max c_{jk}} |\vec{f}'(u) - \vec{g}'(u)|du + Kt_0^2 \\
&\quad ; \quad L = \max\{L_j\}
\end{aligned}$$

which implies

$$\begin{aligned}
\iint_{\alpha_j} |u(t,x,y)-v(t,x,y)|dxdy - \iint_{\alpha_j} |u(0,x,y)-v(0,x,y)|dxdy \\
\leq tL \int_{\min c_{jk}}^{\max c_{jk}} |\vec{f}'(u) - \vec{g}'(u)|du + Kt_0^2
\end{aligned}$$

We have that $t \leq t_0$, but when $t \to 0$, we can also let t_0 become small. So

$$\begin{aligned}
\lim_{t\to 0}\Big(\frac{1}{t}\Big(\iint |u(t,x,y)-v(t,x,y)|dxdy - 0\Big)\Big) \\
&\leq \lim_{t\to 0}\Big(\frac{1}{t}tL\Big(\int_{\min c_{jk}}^{\max c_{jk}} |\vec{f}'(u) - \vec{g}'(u)|du + Kt_0^2\Big)\Big) \\
&= L\cdot \int_{\min c_{jk}}^{\max c_{jk}} |\vec{f}'(u) - \vec{g}'(u)|du
\end{aligned}$$

since the last term can be made arbitrarely small. Outside $\alpha = \cup\alpha_i$ we have that $u = v$

for sufficientley small time. For derivatives taken at $t = 0$ we have

$$\frac{d}{dt}\int\int_{\alpha} |u(t,x,y) - v(t,x,y)|dxdy = \frac{d}{dt}\sum_{j=1}^{n}\Big(\int\int_{\alpha_j} |u(t,x,y) - v(t,x,y)|dxdy\Big)$$
$$\leq \sum_{j=1}^{n} L_j \cdot \int_{\min c_{jk}}^{\max c_{jk}} |\vec{f}'(u) - \vec{g}'(u)|du$$

We can now introduce coordinates (η_j, γ_j) along and perpendicular to the front respectivly, defined in a small neigbourhood of the front, let $\tilde{u}(\eta,\gamma)$ be equal to u_0 outside of this neighbourhood but $\tilde{u}$ is continuous and $\frac{\partial \tilde{u}}{\partial \eta} = 0$. See figure 9. Then we can change variables to get

$$L_j \cdot \int_{\min c_{jk}}^{\max c_{jk}} |\vec{f}'(u) - \vec{g}'(u)|du = \int d\eta_j \int |\vec{f}'(\tilde{u}) - \vec{g}'(\tilde{u})|\frac{\partial \tilde{u}}{\partial \gamma_j}d\gamma_j$$
$$\leq \sum_{i}^{2}\int\int \|\text{grad}(f_i(\tilde{u}(x,y)) - g_i(\tilde{u}(x,y)))\|d\eta_j d\gamma_j$$

If we sum this over all fronts

$$\frac{\partial}{\partial t}\Big(\int\int |u(t,x,y) - v(t,x,y)|dxdy\Big) \leq \sum_{i}^{2}\int\int \|\text{grad}\Big(f_i(\tilde{u}(x,y)) - g_i(\tilde{u}(x,y))\Big)\|dxdy$$
$$= \text{T.V.}_{(x,y)}|\vec{f}(\tilde{u}(x,y)) - \vec{g}(\tilde{u}(x,y))|$$

To prove this inequality for any t we use the triangle inequality. Let $TV(u,t)$ denote the right side of the inequality in the theorem. Let t_1 be the time of the first collision between multiple points. Let $w(t,x,y)$ be defined for $t_1 < t < t_2$ which is the time of the second such collision, by

$$\frac{\partial w}{\partial t} + \nabla \cdot \vec{f}(w) = 0$$
$$w(t_1,x,y) = v(t_1,x,y)$$

Then we can use theorem 3.3 and what we have just shown to obtain for $t_1 < t < t_2$:

$$\int\int |u - v|dxdy \leq \int\int |u - w|dxdy + \int\int |w - v|dxdy$$
$$\leq \int\int |u(t_1,x,y) - w(t_1,x,y)|dxdy + (t - t_1)TV(u,t_1)$$
$$\leq t_1 TV(u,0) + (t - t_1)TV(u,0)$$

since $TV(u.t_1)$ is smaller or equal to $TV(u,0)$ because $TV(\tilde{u},t)$ is smaller or equal to $TV(\tilde{u},0)$ and the two functions take values in the same range. ∎

4. Convergence

Theorem 4.1. *Let $u_0(x, y)$ be a measurable bounded function $R^2 \rightarrow R$ with compact support. Let $\vec{f}(s)$ be a function $R \rightarrow R^2$ which is continuous and smooth almost everywhere. Then there exists a unique function $u(t, x, y)$ which for each $t \leq T$ is in $L_1(R^2)$ and in the weak sense satisfies:*

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{f}(u) = 0$$
$$u(0, x, y) = u_0(x, y)$$

and that for each polygonwise constant bounded $v_0(x, y)$, for all piecewise linear continuous $\vec{g}(s)$ we have

$$\int\int |u(t, x, y) - v(t, x, y)| dx dy \leq$$
$$\int\int |u_0(x, y) - v_0(x, y)| dx dy + t \mathrm{T.V.}_{\cdot(x,y)} |\vec{f}(\tilde{u}(x, y)) - \vec{g}(\tilde{u}(x, y))|$$

where $\tilde{u}$ is as in the last theorem, and the function $v = v(t, x, y)$ is the entropy weak solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \vec{g}(v) = 0$$
$$v(0, x, y) = v_0(x, y).$$

Proof: Define $u_{0,n}(x, y)$ to be a sequence of polygonwise constant step functions appproximating u_0 in L_1. Let N_n be the number of fronts of $u_{0,n}(x, y)$, and let $C_n = \max_{i \leq n}\{N_i\}$. We have that $\vec{f} = (f_1, f_2)$ and that f_i' is in$L_1(-M, M)$ since they are bounded and defined almost everywhere. We choose a sequence of step functions approximating $\vec{f}'$ in $L_1(-M, M)$: $\{\vec{g}_n\}_1^\infty$ such that:

$$\|g_{i,n} - g_{i,m}\|_{L_1} \leq \frac{1}{mC_m} \qquad 4.2$$

$$\|f_i' - g_{i,n}\|_{L_1} \leq \frac{1}{nC_n} \qquad 4.3$$

Since we have that step functions are dense in L_1, we can find such a sequence as long as $\{C_i\}$ is independent of $\{\vec{g}_n\}$.

We now define $\vec{f}_n(s)$ by

$$\vec{f}_n(s) = \vec{f}(-M) + \int_{-M}^{s} \vec{g}_n(t) dt$$

Here M is a bound on u_0 . We define $u_n(t,x,y)$ to be the solution of

$$\frac{\partial u_n}{\partial t} + \nabla \cdot \vec{f_n}(u_n) = 0$$
$$u_n(0,x,y) = u_{0,n}(x,y).$$

Let $m > n$, $u_{0,m}$ can be a refinement of $u_{0,n}(x,y)$. We have

$$\int\int |u_n(t,x,y) - u_m(t,x,y)|dxdy \leq$$
$$\int\int |u_{0,n}(x,y) - u_0, m(x,y)|dxdy + t\text{T.V.}_{\cdot(x,y)}|\vec{f_n}(\tilde{u}(x,y)) - \vec{f_m}(\tilde{u}(x,y))|$$

where $\tilde{u} = \tilde{u}(x,y)$ is constructed from $u_{0,m}$.

$$\text{T.V.}_{\cdot(x,y)}|\vec{f_n}(\tilde{u}(x,y)) - \vec{f_m}(\tilde{u}(x,y))| = \sum_i^2 \Big(\text{T.V.}_{\cdot(x,y)}|f_{i,n}(\tilde{u}(x,y)) - f_{i,m}(\tilde{u}(x,y))|\Big)$$

$$\text{T.V.}_{\cdot(x,y)}|f_{i,n}(\tilde{u}(x,y)) - f_{i,m}(\tilde{u}(x,y))| =$$
$$\int \text{T.V.}_{\cdot x}|f_{i,n}(\tilde{u}(x,y)) - f_{i,m}(\tilde{u}(x,y))|dy + \int \text{T.V.}_{\cdot y}|f_{i,n}(\tilde{u}(x,y)) - f_{i,m}(\tilde{u}(x,y))|dx$$

and

$$\begin{aligned}\text{T.V.}_{\cdot x}|f_{i,n}(\tilde{u}(x,y)) - f_{i,m}(\tilde{u}(x,y))| &\leq N_m \int_{-M}^{M} |f'_{i,n}(s) - f'_{i,m}(s)|ds \\ &= N_m \|g_{i,n} - g_{i,m}\|_{L_1} \\ &\leq \frac{1}{n}\end{aligned}$$

Therefore

$$\text{T.V.}_{\cdot(x,y)}|\vec{f_n}(\tilde{u}(x,y)) - \vec{f_m}(\tilde{u}(x,y))| \leq \frac{4 \cdot \text{diameter of support } (u_0)}{n}$$

We have that $u_{0,n}(x,y)$ is Cauchy in $L_1(R^2)$. Then we obtain

$$\int_0^T \int\int |u_n(t,x,y) - u_m(t,x,y)|dxdydt \leq$$
$$T \cdot \int\int |u_{0,n}(x,y) - u_{0,m}(x,y)|dxdy + \frac{1}{2}T^2 \Big(\text{T.V.}_{\cdot(x,y)}|\vec{f_n}(\tilde{u}(x,y)) - \vec{f_m}(\tilde{u}(x,y))|\Big)$$

which goes to zero as $n, m \to \infty$. u_n is Cauchy in $L_1([0:T] \times R^2)$ and we can define $u(t, x, y)$ as

$$u(t,x,y) = \lim_n \big(u_n(t,x,y)\big).$$

By construction we have that all u_n are weak solutions. Therefore for all appropriate test functions $\phi(t, x, y)$:

$$\Big|\int_0^T \int\int (u - u_n)\phi_t + (\vec{f}(u) - \vec{f}_n(u_n))\nabla \cdot \phi dxdydt + \int\int_{t=0} (u_0 - u_{0,n})\phi(0,x,y)dxdy\Big|$$

$$\leq \int_0^T \int\int |(u-u_n)||\phi_t|dxdydt +$$

$$\int_0^T \int\int |\vec{f}(u) - \vec{f}_n(u_n)||\nabla \cdot \phi|dxdydt + \int\int_{t=0} |u_0 - u_{0,n}||\phi(0,x,y)|dxdy$$

By construction the first and last term become small as $n \to \infty$. Since $|\nabla \cdot \phi|$ is bounded, for the middle term it suffices to show that: $\lim_{n\to\infty} \int_0^T \int\int |\vec{f}(u) - \vec{f}_n(u_n)|dxdydt = 0$. We have that

$$|\vec{f}(u) - \vec{f}_n(u)| \leq \|\vec{f'} - \vec{g_n}\|_{L_1} \to 0 \text{ as } n \to \infty$$

that is $f_n \to f$ uniformly. f is also continuous on $[-M, M]$ and therefore uniformly continuous there.

$$\int_0^T \int\int |\vec{f}(u) - \vec{f}_n(u_n)|dxdydt \leq$$

$$\int_0^T \int\int |\vec{f}(u) - \vec{f}(u_n)|dxdydt + \int_0^T \int\int |\vec{f}(u_n) - \vec{f}_n(u_n)|dxdydt$$

The first of these terms will tend to 0 because $\vec{f}(s)$ is uniformly continuous , the second because $f_n \to f$ uniformly. Thus $u = \lim u_n$ is a weak solution.

Now let $v(t, x, y)$ be the v-function in the theorem.

$$\int\int |u(t,x,y) - v(t,x,y)|dxdy \leq \int\int |u - u_i|dxdy + \int\int |u_i - v|dxdy$$

$$\leq \int\int |u - u_i|dxdy + \int\int |v_0 - u_0|dxdy + \int\int |u_0 - u_{0,i}|dxdy$$

$$+t \cdot \text{T.V.}_{\cdot(x,y)}|\vec{f}(\tilde{u}(x,y)) - \vec{g}(\tilde{u}(x,y))| + t \cdot \text{T.V.}_{\cdot(x,y)}|\vec{f}_i(\tilde{u}(x,y)) - \vec{f}(\tilde{u}(x,y))|$$

Here $\tilde{u}$ is constructed from $u_{0,i}$. To prove the inquality in the theorem we must show that the first, third and last terms tend to zero as $i \to \infty$. The first two do so by construction.

Let $\{s_j\}_1^J$ be a partition approximating the T.V. norm such that

$$\begin{aligned}\text{T.V.}_{\cdot s}|f_i - f_{n,i}| &\leq \sum_1^J \big|f_i(s_j) - f_{n,i}(s_j) - \big(f_i(s_{j+1}) - f_{n,i}(s_{j+1})\big)\big| + \epsilon \\ &= \sum_1^J \int_{s_j}^{s_{j+1}} |f_i'(s) - f_{n,i}'(s)|ds + \epsilon \\ &\leq \int_{-M}^{M} |f_i'(s) - f_{n,i}'(s)|ds + \epsilon \\ &;\ \text{for } i = 1,2\end{aligned}$$

this holds for any $\epsilon > 0$, which implies

$$\text{T.V.}_{\cdot s}|f_i - f_{n,i}| \leq \int_{-M}^{M} |f_i'(s) - f_{n,i}'(s)|ds.$$

We can now repeat each step in the argument where we showed that $\text{T.V.}_{\cdot(x,y)}|\vec{f_n}(\tilde{u}(x,y)) - \vec{f_m}(\tilde{u}(x,y))| \to 0$ to show that $\text{T.V.}_{\cdot(x,y)}|\vec{f}(\tilde{u}(x,y)) - \vec{f_i}(\tilde{u}(x,y))| \to 0$ using condition 4.3.

Let $w(t,x,y)$ be another function which satisfies the inequality in the theorem.

$$\int\int |u-w|dxdy \leq \int\int |u-u_i|dxdy + \int\int |u_i - w|dxdy$$
$$\leq \int\int |u-u_i|dxdy + \int\int |u_{0,i} - w_0|dxdy + \text{T.V.}_{\cdot(x,y)}|\vec{f}(\tilde{u}(x,y)) - \vec{f_i}(\tilde{u}(x,y))|$$

The first two terms tend to zero since $w_0 = u_0$, and in the existence part of this theorem we showed that the last term tend to zero. Therefore $u = w$ almost everywhere. ∎

We remark that the integral inequality in the theorem remains valid also if the integral at $t = 0$ is taken over e.g. a rectangle, the left hand integral should then be taken over the same region magnified by a factor M; a bound on $|\vec{f}'|$.

5. The entropy condition

Theorem 5.1. *Let $u(t,x,y)$ be the weak solution constructed in the last theorem. Assume $u(t,x,y)$ has disontinuities which are piecewise smooth surfaces in (t,x,y). Then $u(t,x,y)$satisfies the Oleinik entropy condition everywhere where the limits u_+ and u_- as (t,x,y) approaches a discontinuity from each side is defined.*

Proof: Let p be a point on the discontinuity surface where the entropy condition is violated. We will prove that if $u(t,x,y)$is continuous on each side of this surface this implies a contradiction. We may assume that p is the origin, and that the surface is tangential to the x-axis and normal to the y-axis at p. Let

$$u_+ = \lim_{x \to 0, y \to 0_+} u(0, x, y)$$
$$u_- = \lim_{x \to 0, y \to 0_-} u(0, x, y)$$

We may assume $u_- < u_+$. Since the entropy condition is violated at p, for a $z \in [u_-, u_+]$ we have that: ($\vec{n}$ is unit normal to surface)

$$\frac{\vec{n} \cdot \vec{f}(z) - \vec{n} \cdot \vec{f}(u_-)}{z - u_-} = \frac{\vec{n} \cdot \vec{f}(u_+) - \vec{n} \cdot \vec{f}(u_-)}{u_+ - u_-} - B = s - B$$

for a $B > 0$.

We can now find a function $\vec{h}(s)$ which is linear in $\langle u_-, u_+ \rangle$ and $\vec{h}(u_-) = \vec{f}(u_-)$ and $\vec{n} \cdot \vec{h}'(u_-) = s$ and

$$-\left(\vec{f}(z) \cdot n - \vec{h}(z) \cdot n\right) > C > 0$$

Sice u is continuous on each side of the discontinuity we can approximate u with a function v defined by:

$$v(t, x, y) = \begin{cases} u_- & \text{if } y < st \\ u_+ & \text{if } y > st \end{cases}$$

Let $D = [-\delta, \delta]^2$. Then we have

$$\iint_D |u(t, x, y) - v(t, x, y)| dx dy \leq O(\delta^2) + O(t^2)$$

since we can choose δ such that $|u - v| \leq \delta$ on D by the continuity of u on each side of the discontinuity surface at p.

Let now w be the entropy solution to the problem

$$\frac{\partial w}{\partial t} + \nabla \cdot \vec{g}(w) = 0$$
$$w(0, x, y) = v(0, x, y)$$

Then

$$\iint_D |w(t, x, y) - u(t, x, y)| dx dy \leq$$

$$\iint_D |w(0, x, y) - u(0, x, y)| dx dy + t \cdot \delta \text{T.V.}_{u \in [u_-, u_+]} |\vec{f}(u) - \vec{g}(u)|$$

But v is also the entropy solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \vec{h}(v) = 0$$
$$v(0,x,y) = v(0,x,y)$$

We then have that:

$$\begin{aligned}
\int\int_D |w(t,x,y)-v(t,x,y)|dxdy &= \delta t \int_{u_-}^{u_+} |(\vec{g}(t)\cdot\vec{n})_c' - (\vec{h}(t)\cdot\vec{n})'|dt \\
&\geq O(\delta)t|\vec{h}(u_+)\cdot\vec{n} - \vec{h}(u_-)\cdot\vec{n} - (\vec{g}(u_+)\cdot\vec{n} - \vec{g}(u_-)\cdot\vec{n})| \\
&\geq O(\delta)t(u_+ - u_-)\left|\frac{\vec{h}(u_+)\cdot\vec{n} - \vec{h}(u_-)\cdot\vec{n}}{(u_+ - u_-)} - \frac{\vec{g}(u_+)\cdot\vec{n} - \vec{g}(u_-)\cdot\vec{n}}{(u_+ - u_-)}\right| \\
&\geq O(\delta)t(u_+ - u_-)\left|\frac{\vec{h}(z)\cdot\vec{n} - \vec{h}(u_-)\cdot\vec{n}}{(z - u_-)} - \frac{\vec{g}(z)\cdot\vec{n} - \vec{g}(u_-)\cdot\vec{n}}{(z - u_-)}\right| \\
&= O(\delta)t\frac{u_+ - u_-}{z - u_-}(\vec{h}(z)\cdot\vec{n} - \vec{g}(z)\cdot\vec{n}) \qquad \text{if } \vec{h}(u_-) = \vec{g}(u_-) \\
&= O(\delta)t\frac{u_+ - u_-}{z - u_-}\Big(\vec{h}(z)\cdot\vec{n} - \vec{f}(z)\cdot\vec{n} - (\vec{g}(z)\cdot\vec{n} - \vec{f}(z)\cdot\vec{n})\Big) \\
&\geq O(\delta)t\frac{u_+ - u_-}{z - u_-}\Big(C + (\vec{f}(z)\cdot\vec{n} - \vec{g}(z)\cdot\vec{n})\Big)
\end{aligned}$$

We also have that

$$\int\int |v - w|dxdy \leq \int\int |v - u|dxdy + \int\int |u - w|dxdy.$$

We choose $\vec{g}(s)$ u such that $\vec{g}(z) = \vec{f}(z)$ and T.V.$|\vec{f} - \vec{g}| < \epsilon$. Then this less than

$$\epsilon O(\delta^2) + \epsilon O(\delta)tO(t^2),$$

but also greater than

$$tO(\delta)K$$

for some $K > 0$, which is a contradiction. ∎

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References

[1] C. Conway and J. Smoller *"Global solutions of the Cauchy problem for quasi-linear first order equations in several space dimensions."* Comm. Pure and Appl. Math. 6 (1966) pp. 95–105.

[2] P. Lax, *"Weak solutions of nonlinear hyperbolic equations and their numerical computation"* Comm. Pure Appl. Math. 7(1954), pp.159–193.

[3] N.Krusjkov. *"Quasi-linear equations of the first order"*. Matem. Sbornik 2 (1970) pp. 228–225.

[4] D. Wagner. *"The Riemann problem in two space dimensions for a single conservation law"* SIAM J. Math Anal. 14,3,(1983)

[5] B. Lindquist *"The scalar Riemann problem in two spatial dimensions: piecewise smoothness and its breake-down"*, SIAM J. Math. Anal. 17, 5 (1986) pp.1178–1197.

[6] B. Lindquist *"Construction of solutions for two dimensional Riemann-problems"*, Comp. and Maths with Appls. Vol. 12(1986), pp. 615–630.

[7] C. Tung and X. Yu-Xi *"Two dimensional Riemann problems for a single conservation law"*, Research mem. no. 20, Acad. Sin. Oct. 1985.

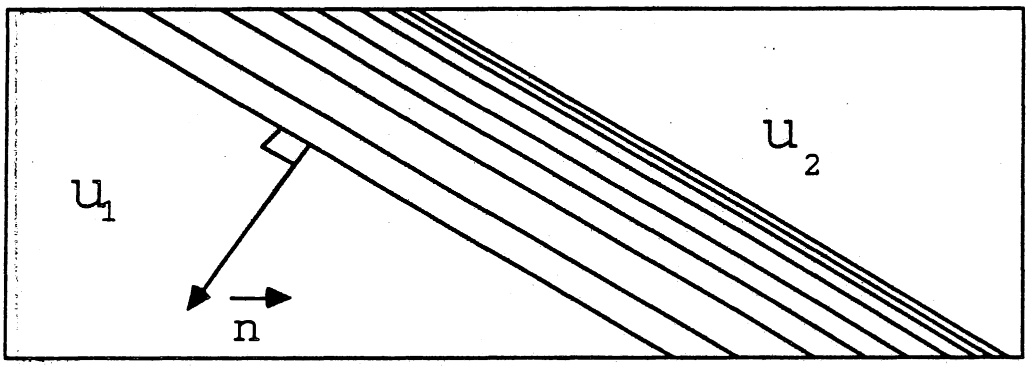
[8] P. Daripa, J. Glimm, B. Lindquist, M. Maesumi, O. McBryan *"On the simulation of heterogeneous petroleum reservoirs"*, New York Univ. preprint,April 1987

[9] H. Holden, L. Holden and R. Høgh–Krohn. *"A numerical method for first order nonlinear scalar hyperbolic conservation laws in one space dimension."* Univ. of Oslo, preprint series 12, Oct., 1986.

[10] J. Guckenheimer *"Shocks and rarefactions in two space dimensions"*, Arch. Rational Mech. Anal. , 59(1975), pp. 281–291.

[11] L. Holden. Private communication.

[12] N.H. Risebro *"The Riemann problem in several space dimensions; a numerical method"* Proc. of the 4^{th}. Eur. Symp. on Enhanced Oil recov.(Hamburg 1987) pp. 735–743.



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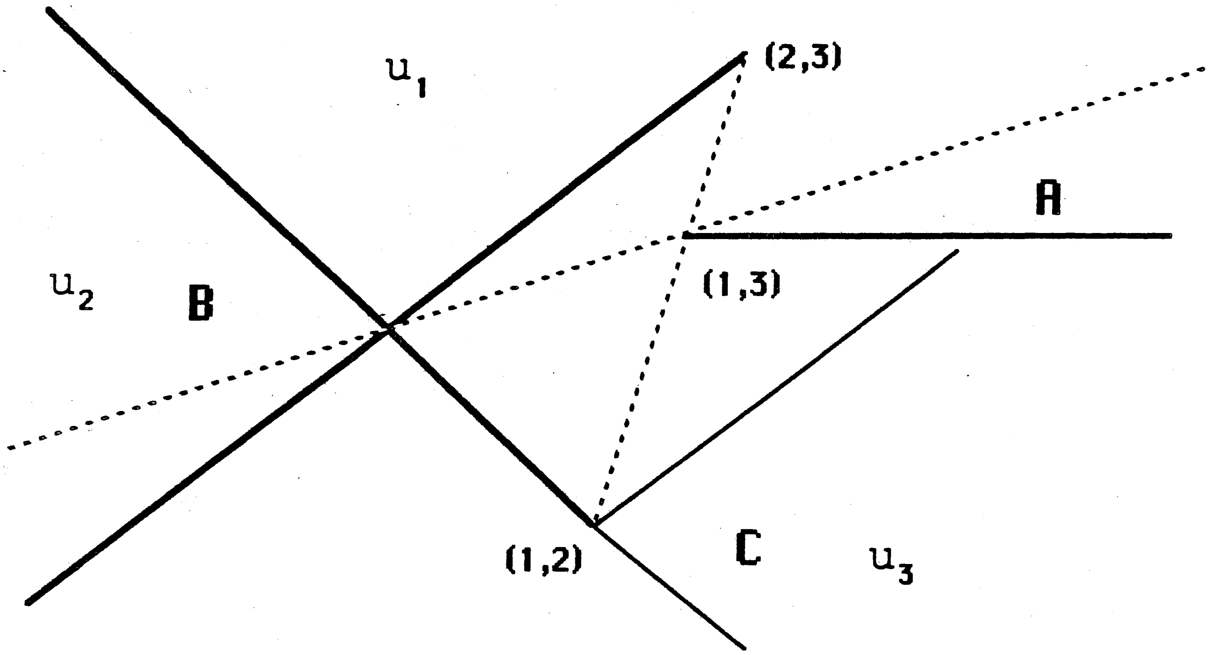

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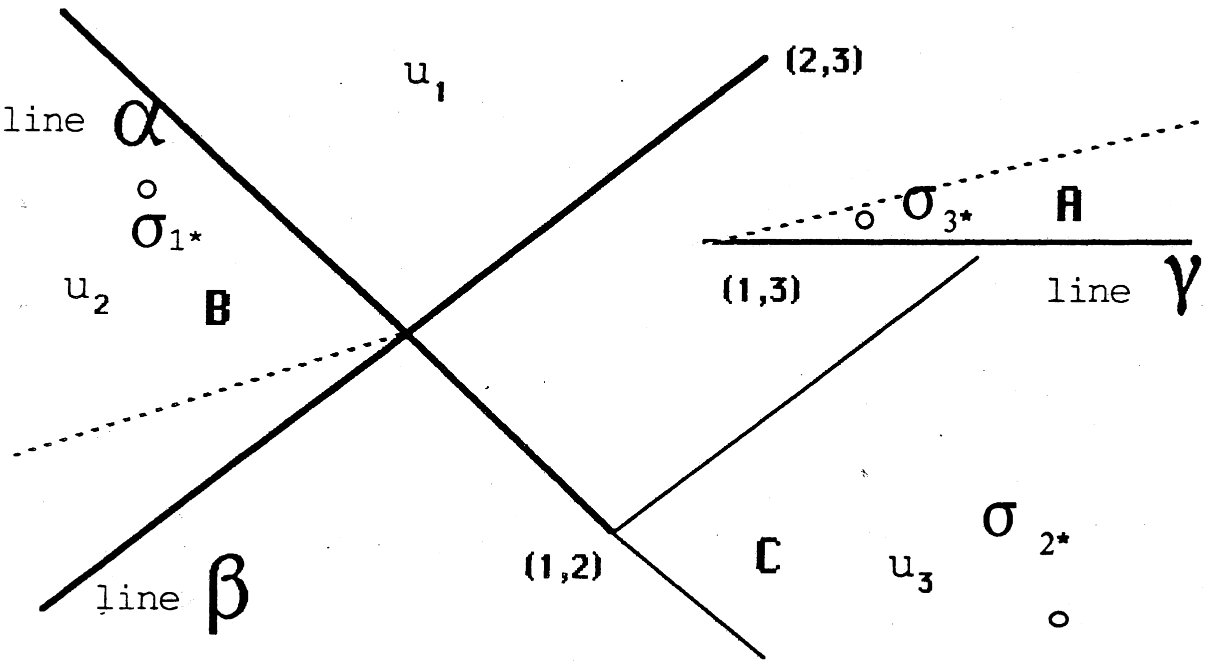

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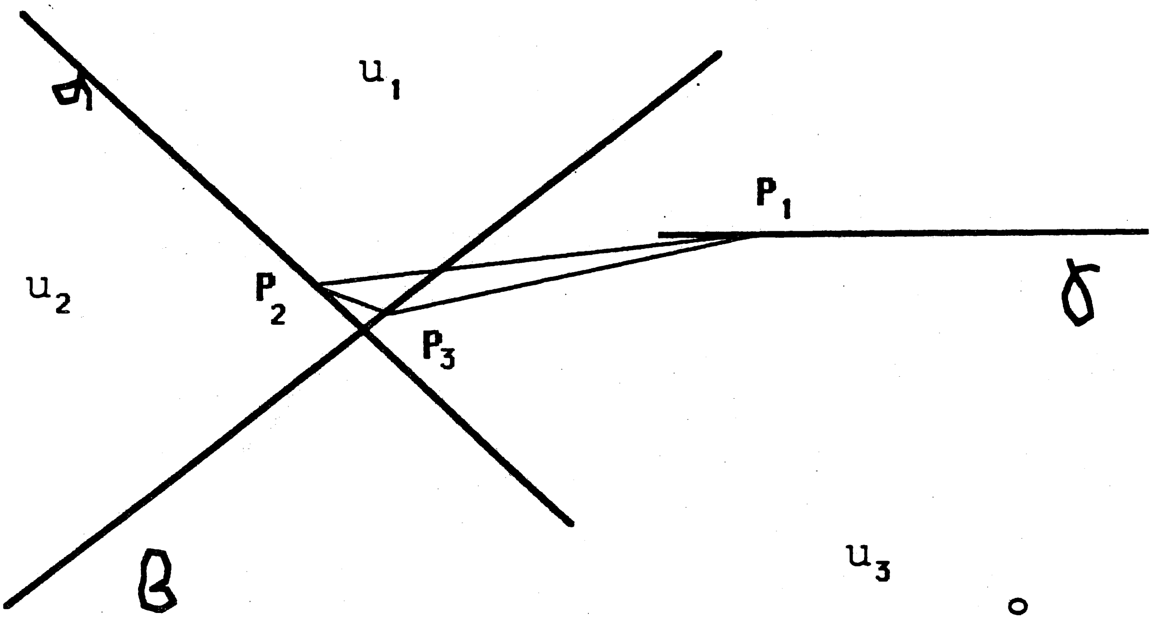

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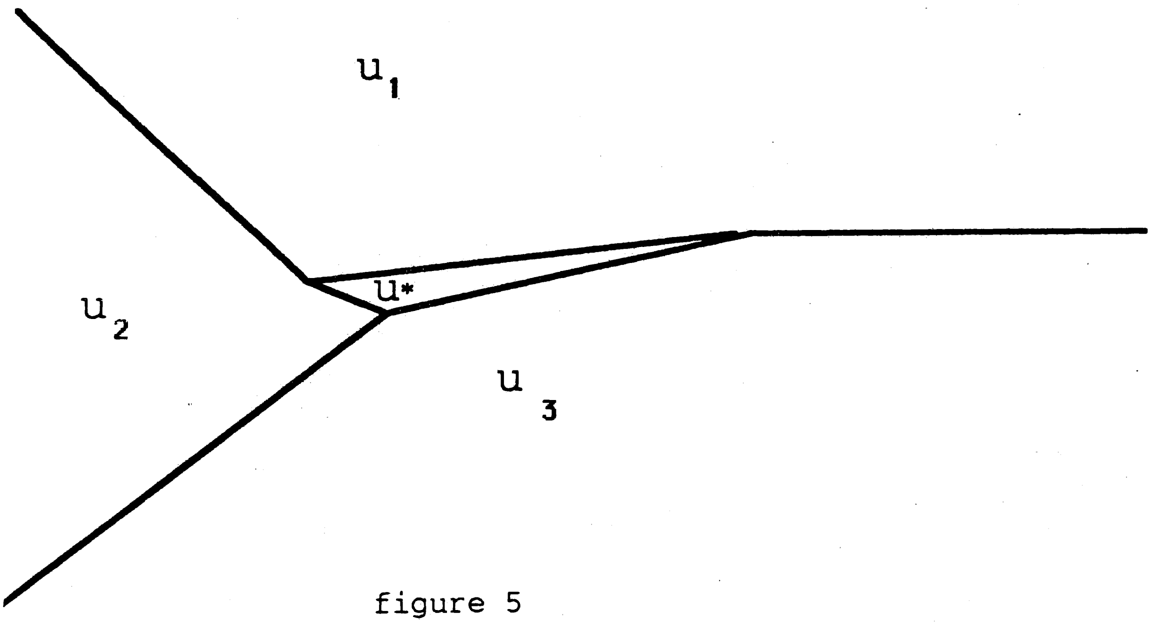
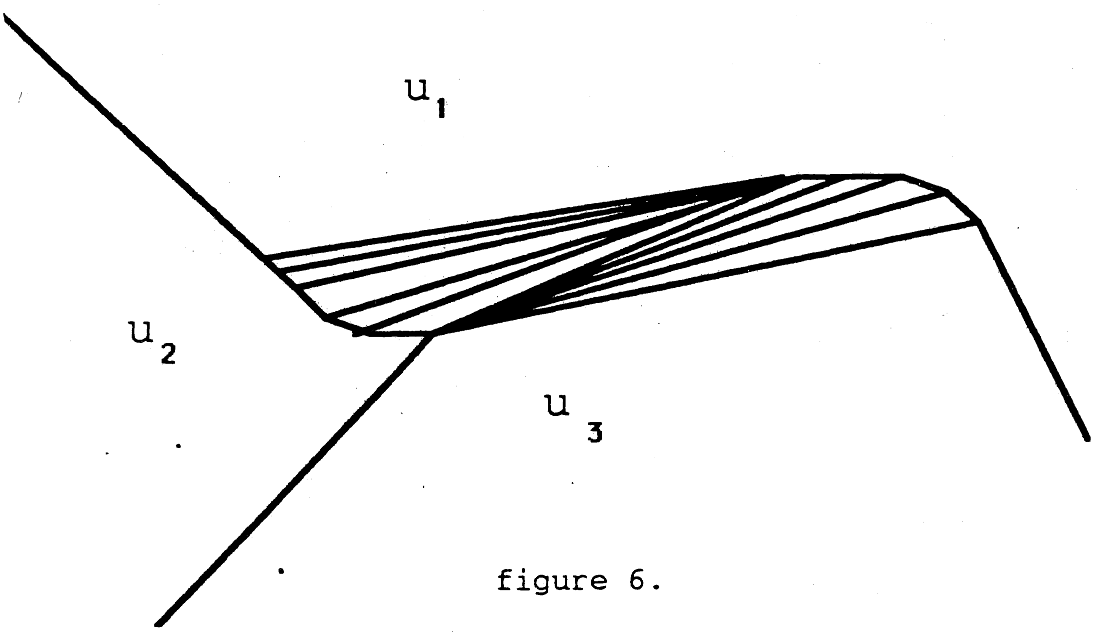


figure 5



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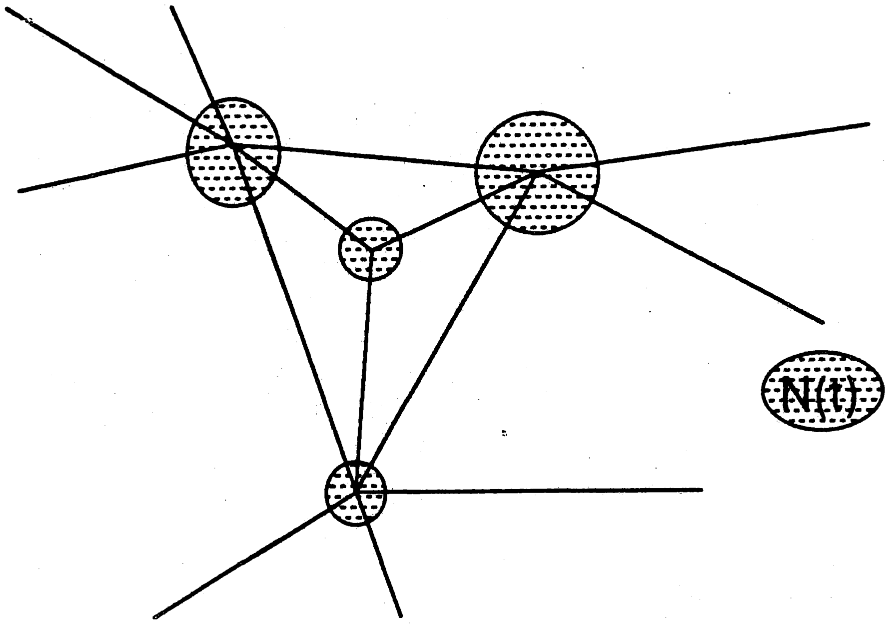

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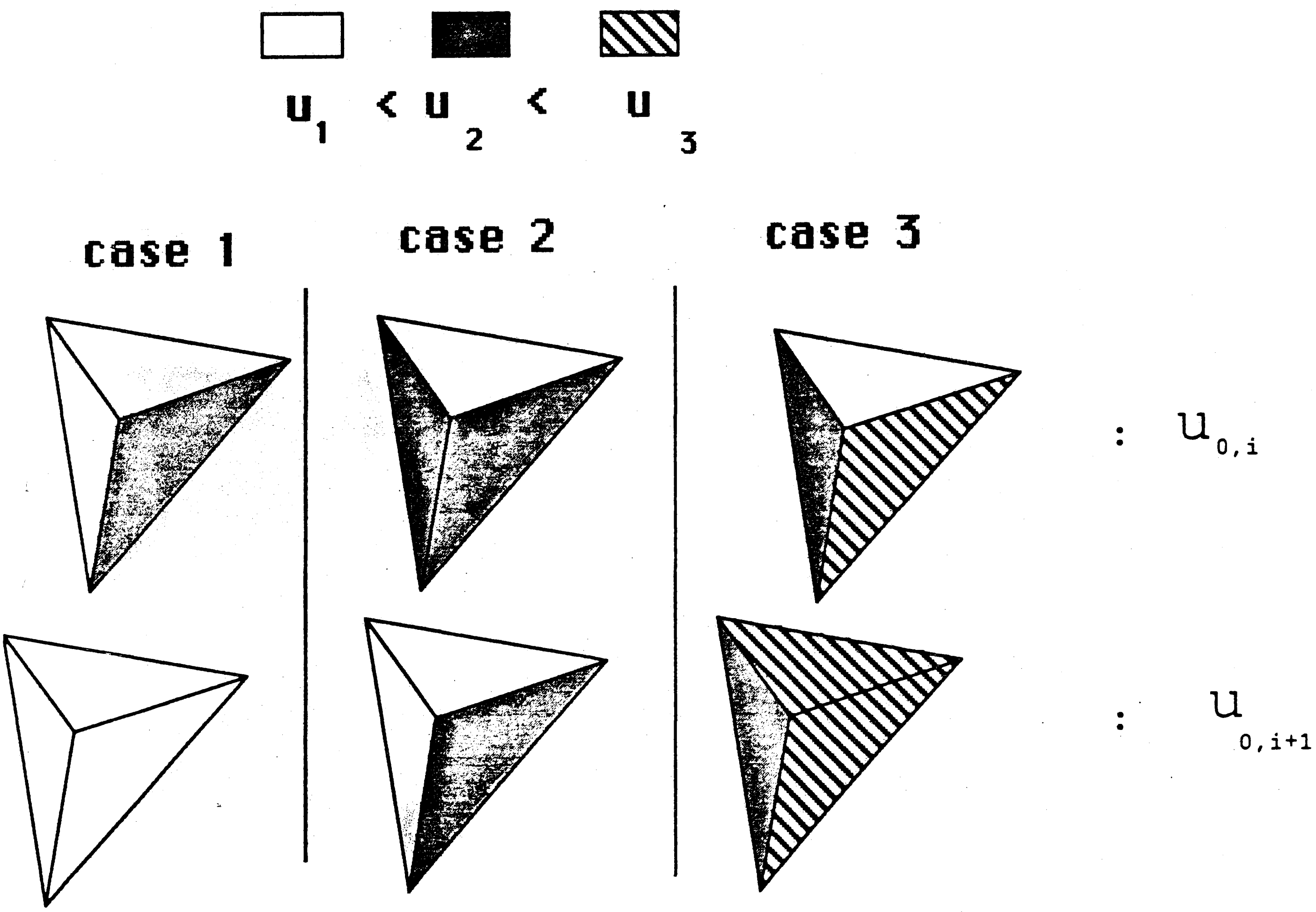

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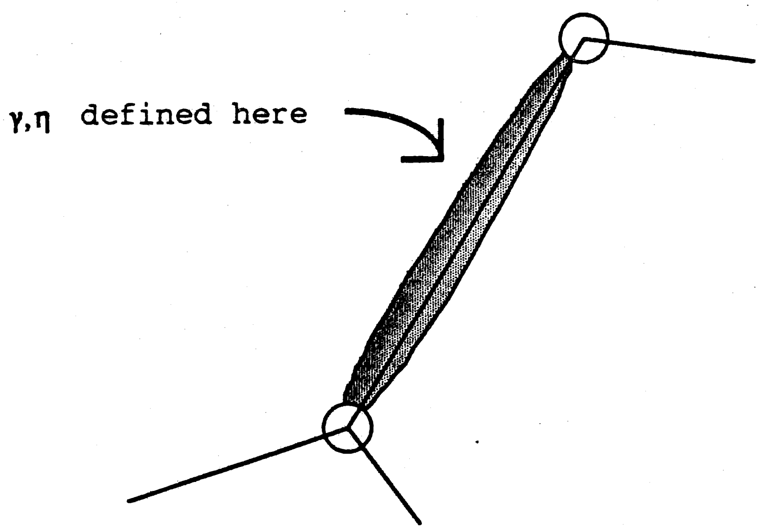

figure 9.