A Characterization of Balanced Rational Normal Scrolls in Terms of their Osculating Spaces

by
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1. Introduction

Consider a smooth, complex projective surface $X \subset \mathbf{P}^N$, $N \geq 5$, and assume X is not contained in a hyperplane. Recall ([2], [4]) that the m-th order osculating space to X at a point x is the linear subspace $\mathrm{Osc}_X^m(x)$ of $\mathbf{P}^N$ determined by the partial derivatives of order $\leq m$ of the coordinate functions, with respect to a system of local parameters for X at x, and evaluated at x. At a general point $x \in X$, one expects the m-th order osculating space to X to have dimension $\binom{m+2}{2} - 1$, if m is such that this number is not greater than N. Points where the dimension of the osculating space is smaller than expected, are called points of hyperosculation — these are "flat" points of X. Certain surfaces are such that all points are points of hyperosculation, in this sense. For example, if X is a ruled surface, then

$$(L) \qquad \dim \mathrm{Osc}_X^2(x) \leq 4 \quad \text{for all points } x \in X$$

and hence dim $\mathrm{Osc}_X^m(x) \leq 2m$, for all m. Not all surfaces satisfying the condition (L) are ruled: Togliatti ([10]) gives as an example a special projection to $\mathbf{P}^5$ of the Del Pezzo surface of degree 6 in $\mathbf{P}^6$. (This seems to be the only known example of a non-ruled smooth surface satisfying (L).)

If a surface satisfies (L), one can show that, locally around a point, the coordinate functions of the surface satisfy a linear partial differential equation of order 2, or a Laplace equation — a term used classically by projective differential geometers (see [6], [10], [3], [7]).

Here we study those surfaces that satisfy condition (L) and do not exhibit further hyperosculating behavior. Our aim is to give a characterization of rational normal scrolls (of dimension 2) similar to the one given for the Veronese embeddings of projective space given in [2].

Definition. A surface $X \subset \mathbf{P}^N$, $N \geq 5$, satisfies condition (L') if X is smooth, X is not contained in a hyperplane, X satisfies condition (L), and in addition dim $\mathrm{Osc}_X^n(x) = 2n$, for all $x \in X$, where $n = [\frac{N-1}{2}]$.

Then we have the following conjecture.

Conjecture. *Let $X \subset \mathbf{P}^N$ be a smooth, projective surface that satisfies (L').*

(i) If $N = 2n+1$ is odd, then X is a balanced rational normal scroll, of degree $2n$ (i.e., of type (n,n)).

(i) If $N = 2n+2$ is even, then X is a semi-balanced rational normal scroll, of degree $2n+1$ (i.e., of type $(n,n+1)$).

Recall ([5]) that a rational normal scroll X of type (d_1, d_2) is defined as the image of a $\mathbf{P}^1$-bundle:

$$\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(d_1) \oplus \mathcal{O}_{\mathbf{P}^1}(d_2)) \to \mathbf{P}(H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d_1)) \oplus H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d_2))) \cong \mathbf{P}^{d_1+d_2+1}.$$

We call X *balanced* if $d_1 = d_2$, and *semi-balanced* if $|d_1 - d_2| = 1$. A balanced scroll is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, and a semi-balanced scroll is isomorphic to the surface $\mathbf{F}_1$ obtained from $\mathbf{P}^2$ by blowing up a point. Note that both $\mathbf{P}^1 \times \mathbf{P}^1$ and $\mathbf{F}_1$ have Chern numbers $c_2 = 4$ and $c_1^2 = 8$.

The only rational normal scrolls satisfying (L') are the balanced and semi-balanced ones; moreover, these are the only ones with the property that the strict dual variety $(X^*)^*$ of the strict dual variety X^* of X is equal to X; the balanced ones are the only ones such that X and X^* are isomorphic ([5]).

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Case (i) of the above Conjecture was formulated by the second named author and proved by him in the case $n = 2$, under some additional assumptions ([9]). The purpose of the present paper is to give a proof in case (i) that works for $n \leq 4$. In particular, by taking $n = 2$, this proves the conjecture in ([7]).

We begin the next section by recalling the definition of osculating spaces and strict dual variety. Then we prove several lemmas and a proposition, that are valid for any surface $X \subset \mathbf{P}^{2n+1}$ satisfying $(L^!)$. We then prove our main result by showing how to deduce case (i) of the Conjecture from the Proposition, under the additional assumption $n \leq 4$.

In the last section we discuss briefly how one could approach case (ii) of the Conjecture. We also speculate on the possibillity of proving directly that a surface in $\mathbf{P}^N$, satisfying (L) and with $N \geq 6$, must be linearly normal and ruled. Needless to say, we hope to return to a complete proof of the Conjecture in a future work.

2. Surfaces in $\mathbf{P}^{2n+1}$

Let V be a complex vector space of dimension $2n+2$, $n \geq 2$, and $X \subset \mathbf{P}(V) \cong \mathbf{P}^{2n+1}$ a smooth surface. Let $\mathcal{P}^m_X(1)$ denote the sheaf of principal parts of order m of the line bundle $\mathcal{O}_X(1) \cong \mathcal{O}_{\mathbf{P}(V)}(1)|_X$, for $m \geq 0$. Recall ([4]) that there are homomorphisms

$$a^m : V_X \to \mathcal{P}^m_X(1),$$

such that $\mathrm{Im}(a^m(x))$ defines the m-th order osculating space to X at x, i.e.,

$$\mathrm{Osc}^m_X(x) = \mathbf{P}(\mathrm{Im}(a^m(x))) \subset \mathbf{P}(V).$$

The sheaf $\mathcal{P}^m_X(1)$ is locally free, with rank $\binom{m+2}{2}$, and there are natural exact sequences (compatible with the maps a^m)

$$0 \to S^m\Omega^1_X \otimes \mathcal{O}_X(1) \to \mathcal{P}^m_X(1) \to \mathcal{P}^{m-1}_X(1) \to 0.$$

Set $K_m = \mathrm{Ker}(a^m)$, $P_m = \mathrm{Im}(a^m)$, and $Q_m = \mathrm{Coker}(a^m)$. Then Q_m is locally free at x if and only if $\mathrm{Im}(a^m(x)) = P_m(x)$. If P_m is locally free, we call P_m the m-th order osculating bundle of X (note that P_m may be locally free even if Q_m is not). Since X is a smooth surface, K_m is always locally free.

Denote by $s(m)$ the integer such that a^m has generic rank $s(m)+1$. Then $\mathrm{Osc}^m_X(x)$ has dimension $s(m)$ for almost all points $x \in X$. If $x \in X$ is such that $\dim \mathrm{Osc}^m_X(x) \leq s(m)-1$, we call x a point of hyperosculation (of order $\leq m$).

Let $\bar{m}$ denote the largest integer such that $s(\bar{m}) < 2n+1$, $s(m) < s(\bar{m})$ for all $m < \bar{m}$, and such that $P_{\bar{m}}$ is not the trivial bundle. Define the strict dual variety of X, as in ([5]), $X^* \subset \mathbf{P}(V^\vee)$, to be the closure of the set $\{H \subset \mathbf{P}(\mathrm{V});\ H \text{ hyperplane},\ H \supseteq \mathrm{Osc}^{\bar{m}}_X(x), \text{ for some } x \text{ s.t. } \dim \mathrm{Osc}^{\bar{m}}_X(x) = s(\bar{m})\}$.

Assume now that the surface X satisfies condition (L). The maps a^m are, locally at each point x, Taylor series expansions of the coordinate functions of X, of order $\leq m$. The relation that makes a^2 drop rank can be interpreted, around each point of X, as a second order linear partial differential equation of the coordinate functions of X (in terms of local parameters of X at the point). This is the Laplace equation referred to in the introduction. When we differentiate further, this relation gives more relations — so the generic rank of a^m is $\leq 2m+1$, for all m. Moreover, if $\mathrm{rank}(a^m(x)) < 2m+1$ for some x, then $\mathrm{rank}(a^{m'}(x)) < 2m'+1$ for all $m' \geq m$. Hence, if X satisfies $(L^!)$, then $\mathrm{rank}(a^m(x)) = 2m+1$ for all $x \in X$ and all m, $0 \leq m \leq n$. In particular, the sheaves Q_m are locally free, with rank $\binom{m}{2}$, and the osculating bundles P_m, with rank $2m+1$, represent the osculating spaces at all points of X, for $m \leq n$.

Suppose X satisfies $(L^!)$. Then we have $\bar{m} = n$, $K_n = \mathrm{Ker}(a^n)$ is locally free with rank 1, and the dual homomorphism $V^\vee_X \to K^\vee_n$ is surjective. Denote by

$$\pi : X \to \mathbf{P}(V^\vee) \cong (\mathbf{P}^{2n+1})^\vee$$

the morphism defined by this 1-quotient. Then we have $\pi(X) = X^*$, and we call π the strict dual morphism.

Lemma 1. *Assume X satisfies $(L^!)$. Then the strict dual morphism $\pi : X \longrightarrow X^*$ is finite; in particular, the strict dual variety X^* is a surface.*

Proof: Since $X \not\subset$ hyperplane, X^* cannot be a point. Assume $C \subset X$ is an integral curve such that $\pi(C) = y$ is a point. If H_y denotes the hyperplane corresponding to y, then $H_y \cap X$ contains $(n+1)C$. Since X has no points of hyperosculation, no points of the curve $H_y \cap X$ can have multiplicity greater than $n+1$, hence C must be smooth, and $H_y \cap X$ can have no other components (since it is connected). But this gives a contradiction: the curve C has self-intersection ≤ 0 since it is contracted, whereas $H_y^2 \cap X$ has positive degree (the degree of X). □

Lemma 2. *Assume X satisfies $(L^!)$.Then, for $m \geq 2$, there exist exact sequences*

$$0 \longrightarrow S^{m-2}\Omega^1_X \otimes Q \longrightarrow Q_m \xrightarrow{\beta} Q_{m-1} \longrightarrow 0.$$

Proof: The surjection β is induced by the natural surjection $\mathcal{P}^m_X(1) \rightarrow \mathcal{P}^{m-1}_X(1)$, because of the commutative diagram

$$\begin{array}{ccccccc} V_X & \xrightarrow{a^m} & \mathcal{P}^m_X(1) & \longrightarrow & Q_m & \longrightarrow & 0 \\ \| & & \downarrow & & \beta\downarrow & & \\ V_X & \xrightarrow{a^{m-1}} & \mathcal{P}^{m-1}_X(1) & \longrightarrow & Q_{m-1} & \longrightarrow & 0 \end{array}$$

Now set $A_m = \mathrm{Ker}(P_m \rightarrow P_{m-1})$ and $B_m = \mathrm{Ker}(\beta)$ — then A_m is locally free with rank 2, and B_m is locally free with rank $m-1$ — and consider the diagrams of locally free sheaves

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_m & \longrightarrow & P_m & \longrightarrow & P_{m-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S^m\Omega^1_X \otimes \mathcal{O}_X(1) & \longrightarrow & \mathcal{P}^m_X(1) & \longrightarrow & \mathcal{P}^{m-1}_X(1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_m & \longrightarrow & Q_m & \xrightarrow{\beta} & Q_{m-1} & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Note that $Q_1 = 0$, so $B_2 \cong Q_2$. Set $Q = Q_2$, and observe that Q is invertible. Then we may consider $Q^\vee \rightarrow S^2T_X \otimes \mathcal{O}_X(-1)$ as a sub-bundle, where $T_X = (\Omega^1_X)^\vee$ denotes the tangent bundle of X. Consider the map $Q^\vee_m \rightarrow S^mT_X \otimes \mathcal{O}_X(-1)$, whose image is $B^\vee_m$, and the composed map

$$\gamma : S^{m-2}T_X \otimes Q^\vee \rightarrow S^{m-2}T_X \otimes S^2T_X \otimes \mathcal{O}_X(-1) \rightarrow S^mT_X \otimes \mathcal{O}_X(-1).$$

It is enough to show $\gamma(S^{m-2}T_X \otimes Q^\vee) = B^\vee_m$, since this gives a surjection $S^{m-2}T_X \otimes Q^\vee \rightarrow B^\vee_m$, which must be an isomorphism, since the bundles have the same rank. This verification can now be done locally: Suppose (u, v) are local parameters of X at x. Then, around x, $Q^\vee \rightarrow S^2T_X \otimes \mathcal{O}_X(-1)$ is determined by an element

$$q = a\frac{\partial^2}{\partial u^2} + b\frac{\partial^2}{\partial u \partial v} + c\frac{\partial^2}{\partial v^2}.$$

Then $B^\vee_m \subset S^mT_X \otimes \mathcal{O}_X(-1)$ is spanned, locally around x, by the elements

$$q_i = a\frac{\partial^m}{\partial u^{i+1}\partial v^{m-i-1}} + b\frac{\partial^m}{\partial u^i \partial v^{m-i}} + c\frac{\partial^m}{\partial u^{i-1}\partial v^{m-i+1}},$$

for $i = 1,\ldots,m-1$. But the image of γ is spanned by

$$\gamma(\frac{\partial^{m-2}}{\partial u^{i-1}\partial v^{m-i-1}} \otimes q) = q_i. \quad \square$$

This lemma implies that Q_m resembles $\mathcal{P}_X^{m-2}(Q)$ — in particular, the Chern classes of these two bundles are equal.

Lemma 3. *Assume X satisfies ($L^!$). Then the strict dual surface X^* is smooth, and there exists an exact sequence*

$$0 \longrightarrow \pi^*(\Omega^1_{X^*} \otimes \mathcal{O}_{X^*}(1))^\vee \longrightarrow S^n\Omega^1_X \otimes \mathcal{O}_X(1) \longrightarrow S^{n-2}\Omega^1_X \otimes Q \longrightarrow 0.$$

Proof: Let $\mathcal{P}^m_{X^*}(1)$ denote the sheaf of principal parts of order m of $\mathcal{O}_{X^*}(1) = \mathcal{O}_{\mathbf{P}(V^\vee)}(1)|_{X^*}$, and let

$$a^m_* : V^\vee_{X^*} \to \mathcal{P}^m_{X^*}(1)$$

denote the natural homomorphisms. As in ([4]) one observes that the composed map

$$\pi^*(a^1_*) \circ (a^{n-1})^\vee : (\mathcal{P}^{n-1}_X(1))^\vee \to V^\vee_X \to \pi^*(\mathcal{P}^1_{X^*}(1)$$

is 0. Hence there is an induced surjection

$$K^\vee_{n-1} \to \pi^*(\mathcal{P}^1_{X^*}(1).$$

Since $\mathcal{P}^1_{X^*}(1)$ has generic rank 3, and K_{n-1} is locally free with rank 3, this surjection must be an isomorphism. Therefore $\pi^*(\mathcal{P}^1_{X^*}(1))$ is locally free with rank 3, hence $\pi^*\Omega^1_{X^*}$ is locally free with rank 2. Since the strict dual map π is a surjective morphism between integral schemes, and $\Omega^1_{X^*}$ has generic rank 2, it follows — from the functoriality of Fitting ideals — that $\Omega^1_{X^*}$ is locally free, with rank 2. Hence X^* is smooth, and $K^\vee_{n-1} \cong \pi^*(\mathcal{P}^1_{X^*}(1))$.

In order too establish the exact sequence, it suffices to observe that A_n is isomorphic to Coker($K^\vee_n \to K^\vee_{n-1}$), which again is isomorphic to Ker($\pi^*(\mathcal{P}^1_{X^*}(1)) \to \pi^*\mathcal{O}_{X^*}(1)) = \pi^*\Omega^1_{X^*} \otimes \mathcal{O}_X(1)$, and then apply the diagram of Lemma 2 with $m = n$. □

Lemma 4. *Assume X satisfies ($L^!$). Then the strict dual morphism $\pi : X \to X^*$ is finite and étale, and the first Chern classes of the line bundles $\mathcal{O}_X(1)$ and $\pi^*(\mathcal{O}_{X^*}(1))$ are equal in $A^1X \otimes \mathbf{Q}$.*

Proof: Since we know already that X and X^* are smooth and π is finite, π is flat. In order to prove the first part, it therefore suffices to prove that π is unramified.

Let $y \in X^*$, let H_y denote the corresponding hyperplane, and consider the flat family of curves on X (over X^*)

$$\mathcal{H} = \{H_y \cap X | y \in X^*\}.$$

Since X satisfies ($L^!$), each point $x \in \pi^{-1}(y)$ is a point of multiplicity exactly $n+1$ on the curve $H_y \cap X$ — and all other points on this curve have multiplicity $\leq n$. Set $e = \deg(\pi)$. For most y, $\pi^{-1}(y)$ consists of e distinct points — since no member of $\mathcal{H}$ has points of multiplicity $> n+1$, no member can have fewer than e points of multiplicity $n+1$. Hence $\# \pi^{-1}(y) = e$ for all $y \in X^*$, and so π has no points of ramification.

From this it follows that $\pi^*\Omega^1_{X^*} \cong \Omega^1_X$. Set $K = c_1(\Omega^1_X)$, $H = c_1(\mathcal{O}_X(1))$, and $H^* = c_1(\pi^*\mathcal{O}_{X^*}(1))$.

To prove the second part, we shall compute two expressions for $c_1(Q)$ in A^1X. First we note that

$$\pi^*\mathcal{O}_{X^*}(1) \cong K_n^{-1} \cong \overset{max}{\wedge} P_n,$$

hence

$$H^* = -c_1(K_n) = c_1(P_n).$$

From the various exact sequences established earlier, we obtain the following equalities in A^1X:

$$c_1(Q_n) = c_1(\mathcal{P}^n_X(1)) - c_1(P_n) = \binom{n+2}{3}K + \binom{n+2}{2}H - H^*.$$

From Lemma 2 we obtain

$$c_1(Q_n) = c_1(\mathcal{P}^{n-2}_X(Q)) = \binom{n}{3}K + \binom{n}{2}c_1(Q).$$

This gives

$$\binom{n}{2} c_1(Q) = n^2 K + \binom{n+2}{2} H - H^*.$$

The sequence of Lemma 3 yields

$$(n-1)c_1(Q) = c_1(S^n\Omega^1_X \otimes \mathcal{O}_X(1)) - c_1(S^{n-2}\Omega^1_X) + K + 2H^*.$$

Carrying out the computations and eliminating $c_1(Q)$, we obtain the equality in A^1X,

$$2(n+1)H = 2(n+1)H^*. \quad \square$$

Now let c_2 denote the degree of the second Chern class $c_2(T_X)$ of X. For A, $B \in A^1X$, we let $A \cdot B \in \mathbf{Z}$ denote the intersection number. Note that $H^2 = H \cdot H$ is the degree of X.

Proposition. *Assume the surface X satisfies condition ($L^!$). Then the following formulas hold:*
(1) $(n-1)^2c_2 + (nK+2H)\cdot(K+2H) = 0$
(2) $(n^2+2n+3)c_2 + nK^2 + 2(n+3)K\cdot H = 0.$
Or, equivalently,
(1′) $(2n+1)c_2 + 2K\cdot H - 2H^2 = 0$
(2′) $n(2n+1)K^2 + 2n(n+5)K\cdot H + 2(n^2+2n+3)H^2 = 0.$

Proof: Since H^* and H are numerically equivalent, we may identify them in the computations. By Lemma 4, we may replace $\pi^*\Omega^1_{X^*}$ by Ω^1_X in the exact sequence of Lemma 3. This sequence gives the equality in $A^1X \otimes \mathbf{Q}$,

$$(n-1)c_1(Q) = 2nK + (n+3)H.$$

Taking this equality into account, we then obtain (1) by computing $c_2(\Omega^1_X)$ from the same exact sequence.

The formula (2) is obtained by setting equal the two expressions for $c_2(P_n)$ obtained from the two exact sequences

$$0 \longrightarrow \pi^*\mathcal{O}_{X^*}(-1) \longrightarrow V_X \longrightarrow P_n \longrightarrow 0$$

and

$$0 \longrightarrow P_n \longrightarrow \mathcal{P}^n_X(1) \longrightarrow Q_n \longrightarrow 0,$$

using the fact

$$c(Q_n) = c(\mathcal{P}^{n-2}_X(Q))$$

proved in Lemma 2. $\square$

Corollary. *If the surface X satisfies ($L^!$), then X is birationally equivalent to a ruled surface, and $X \not\cong \mathbf{P}^2$.*

Proof: We use only formula (1). Suppose X is not birationally ruled, and let $X \to S$ be a minimal model of X. Then $c_2 = c_2(S) + b$ and $K^2 = c_1(S)^2 - b$, for some integer $b \geq 0$. Since S is minimal and not ruled, we have $c_2(S)$, $c_1(S)^2 \geq 0$. Since X is not birationally ruled and H is ample, we have $K \cdot H \geq 0$. From (1) we obtain

$$(n-1)^2c_2(S) + nc_1(S)^2 + 2(n+1)K\cdot H + 4H^2 = -(n^2-3n+1)b.$$

For $n \geq 3$ this gives a contradiction, since $n^2 - 3n + 1 > 0$ in that case.

For $n = 2$, (1) becomes

$$c_2 + 2(K+H)^2 + H^2 = 0.$$

Hence we must have $(K+H)^2 < 0$, since $c_2 \geq 0$ and $H^2 > 0$. Now we apply a theorem of Sommese and Van de Ven ([8], [11]), which says that $K + H$ is generated by global sections (and hence that $(K+H)^2 \geq 0$) unless X is a plane, the Veronese surface, or is ruled (by lines). This gives the desired contradiction.

The last assertion holds because (1) implies that c_2 and K^2 cannot both be odd. □

We now state our main result and show how to deduce it from the formulas of the Proposition.

Theorem. *Assume the surface $X \subset \mathbf{P}^{2n+1}$ satisfies condition $(L^!)$. If $n \leq 4$, then X is a balanced rational normal scroll, of degree $2n$ and type (n,n).*

Proof: Let

$$\tau = \frac{1}{3}(K^2 - 2c_2)$$

denote the Hirzebruch index of X. Since X is birationally ruled and $X \not\cong \mathbf{P}^2$, we have $\tau \leq 0$, and $\tau = 0$ holds if and only if X is ruled (not necessarily by lines in $\mathbf{P}^{2n+1}$).

We claim that in order to prove the Theorem for *any* n (and hence case (i) of the Conjecture), it suffices to prove $\tau = 0$. For suppose this holds. Then (1′) and (2′) imply

$$nK \cdot H + (n+1)H^2 = 0.$$

Hence

$$n(2g-2) + H^2 = 0,$$

where g denotes the sectional genus of X, i.e., $2g-2 = K \cdot H + H^2$. Since X is not contained in a hyperplane, $H^2 \geq 2n$ holds; hence we get $g = 0$ and $H^2 = 2n$. But then it is well known (see e.g.[1]) that X is a rational normal scroll (since $X \not\cong \mathbf{P}^2$, the exceptional case of the Veronese surface is excluded). It follows from ([5]) that the only such scrolls that satisfy condition $(L^!)$, are the balanced ones.

To finish the proof of the Theorem, it remains to show that if $n \leq 4$, then $\tau = 0$ holds.

From the formulas (1′) and (2′) we obtain the following expression for the index τ:

$$3n(2n+1)\tau = -2(n+3)\big(nK \cdot H + (n+1)H^2\big).$$

This gives

$$(K+H)^2 = -\frac{3(n-4)}{n+3}\tau - \frac{n-2}{n}H^2.$$

If $n = 3$ or $n = 4$, then this implies, since $\tau \leq 0$,

$$(K+H)^2 \leq -\frac{n-2}{n}H^2 < 0,$$

and we conclude by the theorem of Sommese–Van de Ven [loc.cit.] that X is ruled (even by lines), hence that $\tau = 0$, and we are done.

If $n = 2$, we obtain

$$(K+H)^2 = \frac{6}{5}\tau.$$

Now if $\tau < 0$, then X is not ruled, so it cannot be a scroll — this gives a contradiction, again by the theorem of Sommese–Van de Ven. Hence we must have $\tau = 0$ also in this case. □

Clearly, one would like to prove directly that $\tau = 0$ holds (for all n), i.e., that X is ruled. Geometrically, it seems likely that the presence of reducible rulings would give points of hyperosculation on X, namely the points of intersection of the irreducible components of these rulings, and thus contradict the fact that X satisfies $(L^!)$. Unfortunately, we have not been able to make this idea work.

Remark. Instead of using the numerical invariants c_2, K^2, and $K \cdot H$, one could use τ, g, and q, where τ is the Hirzebruch index of X, g is the sectional genus, and $q = h^1(X, \mathcal{O}_X(1))$ is the irregularity. Since X is birationally ruled, we have $K^2 = 8(1-q) + \tau$ and $c_2 = 4(1-q) - \tau$. One has $g \geq q \geq 0$, and one easily sees that the formulas (1) and (2) imply that c_2 and K^2, hence also τ, are divisible by 4, and that H^2 is divisible by $2n$. Set $e = \frac{H^2}{2n}$. We get

$$e - 1 + q = -\frac{n+6}{4(n+3)}\tau \tag{1''}$$

and

$$g - q = -\frac{5n-3}{4(n+3)}\tau. \tag{2''}$$

This shows (again)

$$\tau = 0 \Longleftrightarrow e = 1 \quad \text{and} \quad g = q = 0.$$

In fact, any one of the conditions $e = 1$, $g = q$, or $g = 0$ implies $\tau = 0$. We have already observed this for $e = 1$. If $g = q$, then $\tau = 0$. If $g = 0$, then $q = 0$, and hence $\tau = 0$. Moreover, it is also true that in order to prove the Theorem for $n \geq 3$, it suffices to show that $q = 0$ holds, i.e., that X is a rational surface. For suppose $q = 0$ holds. Compute $(K + H)^2$ using this and $(1'')$ to get

$$(K+H)^2 = -\frac{2}{n+6}((n^2 - 2n + 12)e + 6(n-4)).$$

This shows that $(K + H)^2 < 0$ if $n \geq 3$. Hence $K + H$ cannot be generated by its global sections, and we conclude once again by the theorem of Sommese – Van de Ven.

3. Some remarks

Suppose the surface $X \subset \mathbf{P}(V) \cong \mathbf{P}^{2n+2}$ satisfies the condition $(L^!)$. Then the strict dual variety $X^* \subset \mathbf{P}(V^\vee)$ is of dimension ≤ 3, since it is the image of a $\mathbf{P}^1$-bundle on X; the strict dual morphism is in this case

$$\pi : \mathbf{P}(K_n^\vee) \longrightarrow \mathbf{P}(V^\vee),$$

defined by the 2-quotient $V_X^\vee \to K_n^\vee$, where $K_n = \mathrm{Ker}(a^n)$, and $X^* = \pi(\mathbf{P}(K_n^\vee))$.

If X is a semi-balanced rational normal scroll, then $\dim X^* = 3$ holds ([5]), so one wants to show this holds whenever X satisfies $(L^!)$. Granted this, one should proceed by trying to obtain formulas similar to those of the Proposition, hoping that the fact that π is no longer defined on X, but on $\mathbf{P}(K_n^\vee)$, does not complicate matters too much.

One could also ask whether there are any surfaces $X \subset \mathbf{P}^{2n+2}$ satisfying $(L^!)$ *and* such that $\mathrm{Osc}_X^{n+1}(x) = \mathbf{P}^{2n+2}$ for all points $x \in X$. But if the Conjecture is true, no such surface exist, since the only possibility — a semi-balanced scroll — satisfies $\dim \mathrm{Osc}_X^{n+1}(x) = 2n + 1$ for all points x on a rational normal curve of degree n on X ([5]).

Another approach to the Conjecture (in both cases) would be to establish more directly that a surface satisfying $(L^!)$ must be linearly normal and ruled. Togliatti ([10]) gives an example of a surface in $X \subset \mathbf{P}^5$ satisfying (L) and such that the Laplace equation satisfied by X, at a general point, is of hyperbolic type, hence X is not ruled. This surface is obtained by projecting the Del Pezzo surface of degree 6 in $\mathbf{P}^6$ from a point common to all the 2nd order osculating spaces to X (such a point exists!), so it is not linearly normal. It would be interesting to know whether this is in fact the only example of a smooth surface satisfying (L) which is neither ruled nor linearly normal — this would give a proof of our Conjecture.

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