A NUMERICAL METHOD FOR A SYSTEM OF EQUATIONS MODELLING ONE-DIMENSIONAL THREE-PHASE FLOW IN A POROUS MEDIUM.

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Summary

A numerical method for Riemann problems for a class of equations (system of conservation laws) is presented. Stability and convergence in a case of three-phase flow in porous media is shown and the application to general Cauchy problems is discussed.

Introduction

Based upon ideas presented by Dafermos [1], a numerical method for one-dimensional, scalar conservation laws :

$$u_t + f(u)_x = 0 \qquad \text{with} \qquad u(x,0) = u_0(x) \tag{1}$$

was developed by H.Holden, L.Holden and Høegh-Krohn [2]. The algorithm is tracing envelopes of the flow-function f. By approximating f by a piecewise linear function and $u_0(x)$ by a piecewise constant function one obtains a solution consisting of shocks only, finitely many at any time and a finite number of shock collisions as $t \to \infty$. Hence, this method is different from the usual methods of finite differences, first presented by Lax [3]. If f actually is piecewise linear and u_0 is piecewise constant, the solution is exact, else one has good error estimates (e.g. Lucier [4]). Existence and uniqueness for (1) are well known (e.g. Oleinik [5]). For a 2x2 system with some restrictions of f, Isaacson and Temple [6] have shown uniqueness of a weak solution for a global problem. As an approach to a system of conservation laws L.Holden and Høegh-Krohn have studied Riemann problems for a specific class of equations :

$$u_{i,t} + f_i(u_1,..u_i)_x = 0 \qquad i = 1,\dots,N$$
$$u_0(x) = u(x,0) = \begin{cases} u_- & \text{if } x < 0 \\ u_+ & \text{if } x > 0. \end{cases} \tag{2}$$

The results are presented in a preprint [7] in which their proofs suggest a numerical method for the problem (2).

The numerical method

The algorithm works inductively, so assume that

$$u_t + f(u)_x = 0 \ , \quad u(x,0) = \begin{cases} u_- & \text{if } x < 0 \\ u_+ & \text{if } x > 0 \end{cases} \tag{3}$$

is solved. u may be either a vector or a scalar. Different kinds of assumptions on f may be made, but we will not discuss that, just assume that f is continuous and piecewise linear. Let $u_1 = u_-$, u_2, u_3, . . , $u_n = u_+$ be the constant states that make up the solution and s_1, s_2, . . . , s_{n-1} the corresponding shock speeds.

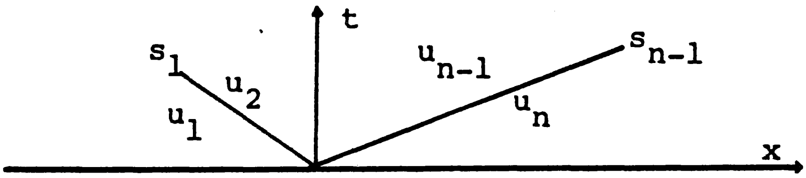


Fig. 1 The solution of : $u_t + f(u)_x = 0$.

Consider the next equation :

$$v_t + g(u,v)_x = 0 \quad , \quad v(x,0) = \begin{cases} v_- & \text{if } x < 0 \\ v_+ & \text{if } x > 0 \end{cases} \tag{4}$$

v is a scalar variable. We know that passing from $g(u_i, \ )$ to $g(u_{i+1}, \ )$, that is, passing from an area in the x-t plane of $u = u_i$ to an area of $u = u_{i+1}$, we have a shock of speed s_i. On the other hand, within each area u is constant, so there we have a scalar problem with a restriction of the permitted shock speeds : $s_{i-1} < s < s_i$. Hence, the sequence of u-values induces a sequence of g-functions to be considered; let g_i denote $g(u_i, \ )$. To help us explicitely constructing the solution we define two kinds of sets: $H_{i,in}$ is the set of v-values where we may land after having made a jump from g_{i+1} to g_i, and $H_{i+1,out}$ is the set of points from where this jump may originate. Hence, $H_{i,in}$ and $H_{i+1,out}$ are the permitted values to the left and to the right (respectively) of the u_i/u_{i+1} shock. We start out on the function g_1 at the point $v=v_-$. We then find the set of points on g_1 from where we may jump to $g_1(v_-)$ with speed less than or equal to s_1. By a jump we mean to find a path along the upper/lower convex envelope [2]. These are the points that we may invoke where $u=u_1$, and so make up $H_{1,in}$. Next we consider g_2 and find the points upon it from where we may jump with speed s_1 and land on g_1 at a point of $H_{1,in}$. This points make up $H_{2,out}$. $H_{2,out}$ is contained in $H_{2,in}$ (if we may jump from a point, we may of course come there first). In addition we know that we may move along g_2 with speed between s_1 and s_2. Therefore we have to include in $H_{2,in}$ the points of g_2 from where we may jump to $H_{2,out}$ with such speeds. From $H_{2,in}$ we now repeat the process for g_3 as we did with g_2 from $H_{1,in}$. The process is repeated until $H_{n,out}$ is constructed. We are now prepared to trace the solution, and start out in the point $g_n(v_+)$. If this point is in $H_{n,out}$ we jump across to g_{n-1}. Otherwise we first have to jump along g_n with decreasing speeds larger than s_{n-1} until we reach a point of $H_{n,out}$. Then pass to g_{n-1} (where we know we land in $H_{n-1,in}$), from where we may have to jump into $H_{n-1,out}$ before passing to g_{n-2} etc. In this way we construct our solution path all way down to $g_1(v_-)$. As shown by L.Holden and Høegh-Krohn [7], the solution exists, but is not generally unique. However, the set of initial values where we do not have uniqueness is finite, and in the case of two equations there is uniqueness. That will be the kind of system we will examine closer, a system of equations modelling a case of three-phase flow in a porous medium.

Flow equations

We write the equations :

$$\begin{aligned} u_t + f(u)_x &= 0 \\ v_t + g(u,v)_x &= 0 \end{aligned} \qquad (5)$$

and the initial states (to the left and right of x=0 respectively) (u_-, v_-) and (u_+, v_+). Interpretated physically u denotes gas- and v is oil-saturation. The saturation of water $w = 1 - u - v$. The equations describe a system where gas-flow is independent of whether it takes place in oil or water environment, whereas the oil-flow is sensible to the amount of both water and gas present. We have approximated f with a piecewise linear function. (Usually f is determined experimentally, and so it is piecewise linear in most applications.) We will denote the approximation f. Furthermore we assume that both f( ) and g(u,) are strictly increasing, continuous functions with at most one point of inflection. We also assume $f(0) = 0$ and $g(u,0) = 0$ (no substance gives no flow) and that $g_u < 0$ (the more gas present, the less relative amount of the flow is oil flowing). The physical situation implies that g is not defined for negative arguments nor for arguments so that $u+v > 1$.

Existence

We first state that the solution exists, that is, we can always find the H-sets and the solution always remains within the phase-space $0 \le u + v \le 1$.

<u>Lemma 1.</u>
The slope of the line connecting two g-functions at their endpoints $g_i(1-u_i)$ and $g_{i+1}(1-u_{i+1})$ equals s_i, the u_i/u_{i+1} shock-speed.
<u>Proof:</u>
The slope of the line is

$$s = \frac{g(u_{i+1}, 1-u_{i+1}) - g(u_i, 1-u_i)}{(1-u_{i+1}) - (1-u_i)}$$

Now, if h is the fractional flow function of water, we always have $f(u) + g(u,v) + h(u,v,w) = 1$ (all that flows is u, v and w). If $v = 1 - u$, $w = 0$, and so $h = 0$. Hence $g(u, 1-u) = 1 - f(u)$, and :

$$s = \frac{(1 - f(u_{i+1})) - (1 - f(u_i))}{(1 - u_{i+1}) - (1 - u_i)} = \frac{f(u_{i+1}) - f(u_i)}{u_{i+1} - u_i} = s_i$$

This lemma guarantees we do not pass out of the phase-space at u+v=1. It also implies that if $v=1-u_i \in H_i$ (in or out), then $1-u_j \in$

H_j for all j. The same is true for v=0, since all g-functions coincide here. To show that the H-sets are non-empty, observe that $v_- \in H_{1,in}$ (by construction). If $H_{1,in} = \{v_-\}$ then (by Lemma 1) the line through v_- with slope s_1 will cut g_2. This cutting point will be in $H_{2,out}$. Then, by induction no H is empty. On the other hand, if $H_{1,in}$ consists of more than one point, at least one of the two points 0 and $1-u_1$ is in it, and again no H is empty. Hence :

<u>Theorem 1.</u>
The solution of the Riemann problem (5) exists and is well-defined inside the phase-space $0 \leq u + v \leq 1$.

The H-sets

For the S-shaped functions that we will be interested in, there are basically three kinds of H-sets.

1) A cutting H-set. A H-set including one point of g where g_v is greater than s (fig.3a).
2) An upper-touching H . A H-set including all $v > v'$ where $g(u,v')_v = s$ (fig.3b).
3) A lower-touching H . As for 2), but for v smaller than some v' (fig.3c).

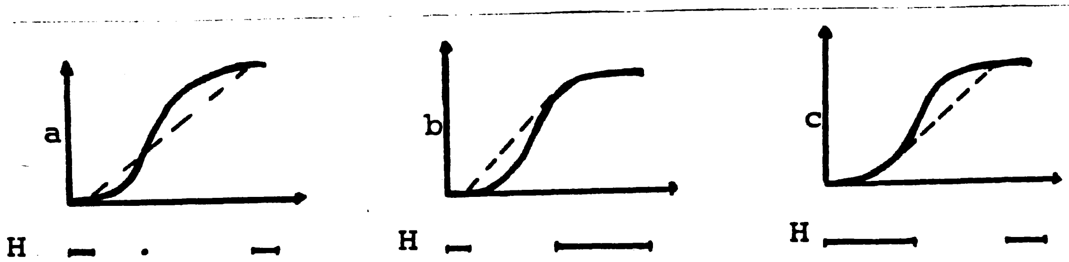


Fig. 2 The three kinds of H-sets.

If $H_i = (0,1-u_i)$ we may name it both upper- and lower-touching. (These names are motivated by the properties of the so called h-functions of [7], our H-sets are the intervals where h = g.) We define the points that determine the H-s in the following way:
A H-set consists of at most three parts, two intervals and possibly one single point, for each index "i,xx" we call the right point of the left part $v_{i,xx,l}$, the middle point (cutting point) $v_{i,xx,c}$ and the lefthand point of the right part $v_{i,xx,r}$. The v_l or v_r ("i,xx" is omitted when no confusion is possible) is called the touching point if H is touching and $g_v(v_l)$ or $g_v(v_r)$ equals the corresponding s. Denoting the u-solution sequence $u_1 = u_-$, u_2, u_3, . . , $u_n = u_+$, we order the H-sets :

$$H_{1,in},\ H_{2,out},\ H_{2,in},\ H_{3,out},\ \ldots,\ H_{n,out}.$$

With respect to this order we have the following useful property :

<u>Lemma 2.</u>
Except for the first H-set we may divide the sequence into two parts (possibly one is empty). The first part consists of cutting, the latter part consists of only upper or lower-touching sets.

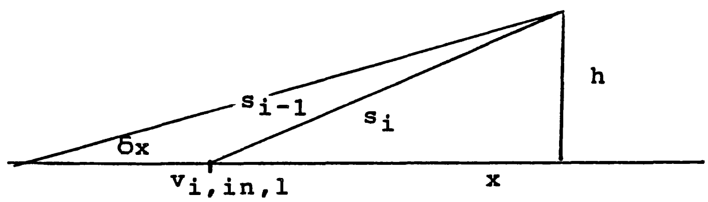
Proof :
Assume that $u_1 < u_2 < . . . < u_n$. Then $g_i > g_{i+1}$. Assume that H_i is upper-toucing. We will prove that the next H is upper touching. If H_i is an "out" set, we construct $H_{i,in}$ by adding the interval (v', $v_{i,out,r}$) where $g_i(v')_v = s_i$. (Or $v' = 0$ if the slope of g_i is always smaller.) In addition we include the interval ($v_{i,out,l}$, v'') where v'' is the point where the line through v' with slope s_i cuts g_i. Then, $H_{i,in}$ is upper touching. If H_i is an "in" set, $H_{i+1,out}$ is constructed by tracing g_{i+1} from the right until $g_{i+1,v} = s_i$ (touching point), then including the part to the left of the point where the line through this point with slope s_i cuts g_{i+1}. So $H_{i+1,out}$ will be upper touching also in this case. A lower touching or a cutting H may induce a cutting or an upper touching H in the next step. The case of a decreasing u-sequence is treated symmetrically, upper should be substituted with lower, right with left and vice versa. #

By simple use of Lemma 2 we find the following monotonity property of the solution (see Gimse [8]).

Theorem 2 :
There is a value s_0, so that both : $v(s)$, for $s < s_0$ and $v(s)$, for $s > s_0$ are monotone functions. #

The following lemma determines how H_{in} is related to H_{out} :

Lemma 3:
If the points are defined :
$v_{i,in,l} > v_{i,out,l}$ and $v_{i,in,r} < v_{i,out,r}$.
Proof:
Since we have a finite number of u-shocks, there is only a finite number of shock speeds to consider. Hence, there is some minimum difference $\delta s_{min} = \min(s_i - s_{i-1}) > 0$. Consider the following figure



Here $s_i = h/x$ and $s_{i-1} = h/(x+\delta x)$ (the H-sets are assumed cutting, if one (or both) are touching , the upper line is higher above, and so δx will be even greater). Thereby $\delta s_{min} \le$ ($h\,\delta x / x(x+\delta x)$) $\le$ ($h\,\delta x / x^2$) , which gives: $\delta x \ge \delta s_{min}\, x^2/h$. Since g_i is increasing, $\delta x \le v_{i,in,l} - v_{i,out,l}$. The argument for the upper part is similar, if H_i is upper touching, $v_{i,in,r}$ is the point where $g_v = s_i$ and where $g_v = s_{i-1}$, $v_{i,out,r}$ is greater . #

Jumps between H-sets.

Before investigating continuity and stability properties we make the following observations concerning jumps between the H-sets. (We assume that the u-sequence is increasing, the case of a decreasing sequence is treated symmetrically.)

1) The points of the right part of $H_{i,out}$ is mapped continuously onto the right part of $H_{i-1,in}$ (say $(v'', 1-u_{i-1})$).
2) Either : The middle point of $H_{i,out}$ is mapped into the middle point of $H_{i-1,in}$.
   Or : There is an interval $(v', v_{i,out,l})$ that is mapped continously onto the interval $(v_{i-1,in,r}, v'')$
3) The rest of $H_{i,out}$ is mapped continuously onto $(0, v_{i,in,l})$.
4) The rightmost part of the left and the leftmost part of the right part of H_{iin} is mapped into the middle or touching point of $H_{i,out}$.

Continuity with respect to v_+.

Consider two values v_+ and v_+'. We will demonstrate that if v_+' is close to v_+, the solution paths are close (L_1-close). Assume the last H_{out} is upper-touching. (The case of H_{out} lower-touching (decreasing u-sequence) is treated symmetrically.) If both v_+ and v_+' are outside H_{out}, we have to jump into the touching point, and so their paths coincide from there. Next, if v_+ is on H_{out}, but v_+' is not, they move closer if v_+ is in the upper part, while they are separated if v_+ is in the lower part. In the latter case, v_+' passes to the touching point. This situation is however equivivialent to the case of v_+' in the rightmost point of the lower part of H_{out}. The only difference in the solution path is the jump up to the touching point and down. Observe that the speed of these two jumps are close, and that the difference tends to zero as $v_+' \to v_+$. It remains to consider the case when both v_+ and v_+' is on H_{out} (or have come there by jumping as above). If the two points pass to the same part of the next H_{in}, (and if g_{vv} is not zero in some interval), it is obvious that the mapping is continuous with respect to the distance between the v-values (the observations above). Then assume the two points do not pass to the same part of H_{in}. The leftmost point then end in the lower part of H_{in}, while the right point goes to the upper part. However, by Lemma 2, the leftmost point, if it was sufficiently close to the other, cannot be on the next H_{out}. (If H_{out} is cutting, nor can the right point.) Thus, the leftmost point have to pass to the upper part (with speed between the incoming and the outgoing) and so will come closer to the right point. Finally, assume that we start out on a cutting H_{out}, If both v_+' and v_+ are on H_{out}, we pass continuously over as above. If none of them are, both jump into the cutting point, from where the paths are identical. If one is and the other is not part of H_{out}, the latter jumps into the cutting point, from where it continues back to v_-, while the other passes over, but (by Lemma 3) it will not land in a point of the next H_{out}. Therefore, in the next step this point also passes into the secuence of cutting points. Hence, the solutions, as curves in phase-space, are close in the two cases ; Single shocks are not necessarily stable, but rarefaction waves are.

Remark : If g is approximated by piecewise linear functions, one may have an interval of slope equal to some s_i. Thereby the jumps between the g's do not map the distance from v' continuously in a small neighbourhood of v'. However, as the approximation is done

finer, the discontinuities tend to zero.
In fig.3 we have illustrated the construction of the solution in a simple case of three different u-values ($u_1 < u_2 < u_3$).

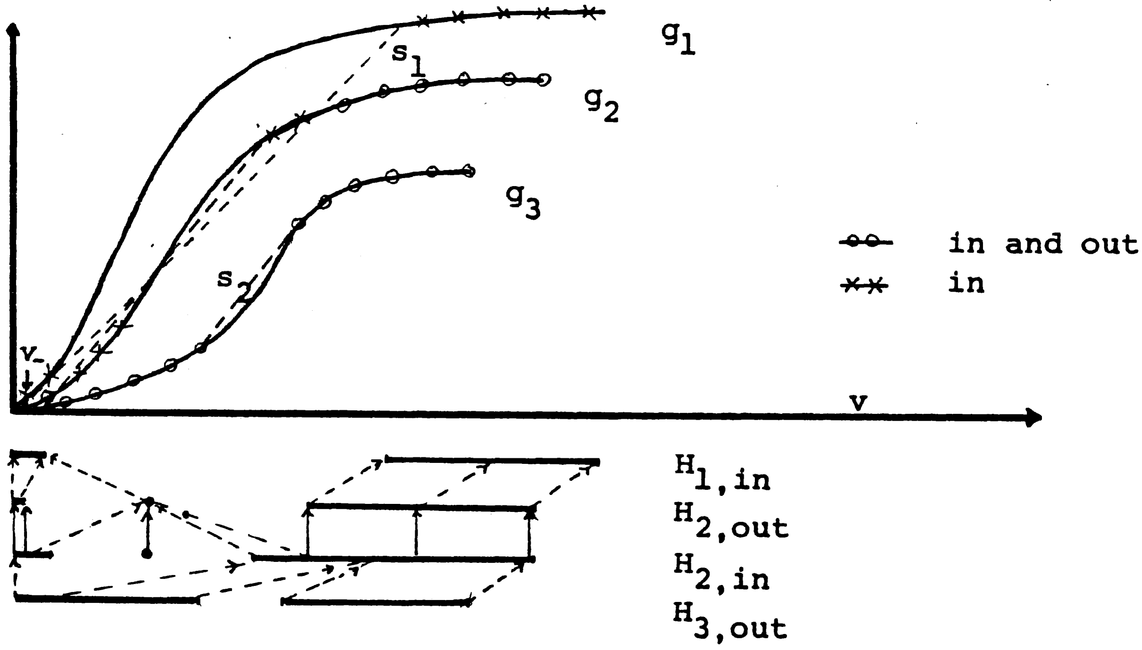


Fig. 3 An example of H-sets and solution paths.

Continuity with respect to v_-.

For any value v_-' in a small neighbourhood of v_-, let H' be the H-sets constructed from v_-'.

Lemma 4 :
If H_j' and H_j are both touching, then $H_i = H_i'$ for all $i \geq j$.
Proof:
The slope at the touching point is the same (independent of v_-), hence the touching points are equal, and so the entire sets.#

We turn to the case of two cutting H-sets. We have :

Lemma 5 :
The perpendicular distance between the two parallell lines through the cutting points is less than the distance $|v_- - v_-'|$.
Proof:
The lemma trivially holds for the first pair of H-sets. Assume it is valid for index j. Since g(u,) is increasing, the upper line intersects with $g(u_j,)$ above and to the right of the intersection of the lower line. The algorithm tells us to tilt the lines a bit more ($s_{j-1} < s_j$), and so the perpendicular distance shrinks. #

Observe that this lemma is also valid if one H_j is touching, while the other is still cutting. Then assume H_j' is touching while H_j is cutting. Since $g_u < 0$, we know that the perpendicular distance between the lines from the point of g_j with slope s_j to the similar point of g_{j+1}, δ, is greater than 0. If $|v_- - v_-'| < δ$ then the perpendicular distance between the touching line of H' and the cutting line of H is also < δ, (by Lemma 5), and so the next H cannot be cutting. Hence,

Lemma 6 :
If H_j' is touching and H_j is cutting, then $H_i' = H_i$ for $i > j$ provided v_-' is sufficiently close to v_-. #
We are now prepared to trace the solutions. As long as we have touching functions there are no problems, the first step where we have to differ, is when reaching a point on some H' but not on the corresponding H (or vice versa). In the latter case we proceed to the cutting point, while in the first case we jump across. However, by Lemma 3, we will have to enter the sequence of cutting points in the next step. The points of these sequences are close (Lemma 5), and so are the solutions.

Continuity and stability

In the proceeding sections we have proved stability with respect to the initial values v_- and v_+ independently. It is easy to see it is not necessary for one of the initial v-values to be fixed : The solution with initial values (v_-,v_+) is close both to the solution of (v_-,v_+') and of (v_-',v_+). Hence, (v_-',v_+') which is close to any of the two, is close to the solution of (v_-,v_+).
Finally it remains to discuss stability with respect to u_- and u_+. Consider again an increasing sequence of u-values, (the opposite is treaded symmetrically,) and observe that the solution of $u_t + f(u)_x = 0$, will consist of at most one rarefaction wave (an approximated rarefaction wave) and one shock. (This is due to the shape of f.) Assume it starts out with a rarefaction wave. Then, by taking some u_-' close to u_-, we introduce or lose one (or a few) u-value(s). The remaining u-values of the approximated wave are the same. (Assume $u_-' > u-$, else, rename.) If $H_{1,in}$ is upper touching, so is all H'. If $H_{1,in}$ is lower-touching or cutting we know, by the continuity of g, that the touching/cutting point of the first H' is close to the corresponding point of H. Hence, the situation will be similar to the problem of variation of v_-. On the other hand, if u_+ is varied slightly, the last g-function will be slightly different (again by continuity of g), and so will H_{out}. (If there is no distinct shock at u_+, one (or a few) g(s) may be added or subtracted at the end of the sequence.) Thus the jump from the last function will be slightly altered only. We have:

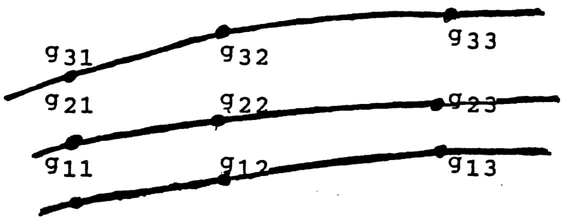
Theorem 3 :
The essential structures of the solution of the initial value problem is stable with respect to variation of the initial values. (By essential structures we mean rarefaction waves, approximated rarefaction waves or major discontinuities. The structure of single peaks are not necessarily stable.) #
Corollary .
The solution (as a curve in phase-space) depends L_1-continuously upon the initial data. #
This Corollary is weaker, since approximated rarefaction waves consist of single points in phase-space.

The approximation

Finally we investigate the correspondance between the exact solution, where f is not approximated, and the numerical solution where it is. If the not-approximated f gives a discontinuity in u, so does the approximated, so assume that u is continuous. If u is constant, we solve a scalar problem exactly (g is not approximated), so it remains to consider a not-constant u. Then, by carrying out the differentiation of $v_t + g(u,v)_x = 0$ and rearranging (e.g. [7]) :

$$s = g_u \frac{u_s}{v_s} + g_v$$

The algorithm for the approximated case gives us a sequence of g-functions ($g_{ij} = g(u_i, v_j)$) :



We make a difference approximation to s at g_{22} by setting :

$$g_u = \frac{g_{32} - g_{12}}{u_3 - u_1} \qquad \text{and} \qquad g_v = \frac{g_{23} - g_{21}}{v_3 - v_1}$$

and $$u_s = \frac{u_3 - u_1}{\delta s} \qquad v_s = \frac{v_3 - v_1}{\delta s}$$

Then put these approximations into the expression for s:

$$s \approx (g_{32} - g_{12} + g_{23} - g_{21})/(v_3 - v_1)$$

Now, by using the same approximation for g_u and g_v in a first order Taylor's formula (expanded for the point g_{22}) we find :

$$g_{ij} \approx g_{22} + [(g_{32}-g_{12})/(u_3-u_1)]\cdot(u_i-u_2) + [(g_{23}-g_{21})/(v_3-v_1)]\cdot(v_j-v_2)$$

We solve for $s_1 = (g_{22}-g_{11})/(v_2-v_1)$ and $s_2 = (g_{33}-g_{22})/(v_3-v_2)$:

$$s_1 \approx [(g_{32} - g_{12})\cdot(v_3 - v_1)/(2(v_2 - v_1)) + g_{23} - g_{21}]$$

$$s_2 \approx [(g_{32} - g_{12})\cdot(v_3 - v_1)/(2(v_3 - v_2)) + g_{23} - g_{21}]$$

Where we have assumed : $u_{i+1} - u_i = \delta u$ for all i (uniform approximation).
Hence : $s_1 \leq s \leq s_2$, or $s_1 \geq s \geq s_2$, and $s_1 \to s_2$ when $\delta u \to 0$.
Furthermore, we know that a difference approximation will converge, so our approximated s will tend to the exact s value. Thus,

<u>Theorem 4 :</u>
The solution when f is approximated by a piecewise linear function will converge to the exact solution as the approximation converges.

Applications

We close this paper with some remarks on applications. In general Cauchy problems we may approximate the initial value function by a piecewise constant function to obtain a finite number of Riemann problems. These may be solved as above. We may apply the results of [2] to the (scalar) gas-flow equation. They proved that when a finite number of Riemann problems are considered, there is only a finite number of shock collisions, and so the method solves the problem in a finite number of steps. In our case, in a bounded spatial area, we will find a single, constant u-value after some time (since all speeds are greater that zero, all shocks will move out of our area of interest). Then our second equation is scaler, and we may apply [2]'s argument once more. In [8] some examples of such problems are shown, also with some comparison to upwind schemes. Also note that our model, when applicable, gives no problems with elliptic regions (e.g. Bell, Trangenstein and Shubin [9] nor unbounded variation (e.g.Temple [10]). The second important application of the ideas presented here, is the problem of discontinuities (e.g. in geological datas in oil reservoir simulation). Such differences give rise to one flow function in one region and a different flow function in the neighbouring region. By assuming a shock of zero speed at the discontinuity we may solve the problem by making a jump from the one to the other as we do between different g-functions. Alternatively we may add an equation with a zero flow function : $u_{0,t} = 0$, with appropriate initial conditions.

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