$\mathbb{C}$-valued stochastic integrals in the plane.

by

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Abstract

In this paper we are going to discuss various stochastic integrals over a 2-parameter Wiener process. Our main interest is the relationship between Brownian motion and analytic functions, and we want to demonstrate how complex notation may be used to study these objects.

Introduction

A two parameter Wiener process admits a theory of stochastic line and surface integrals. When the stochastic line-integrals are defined, it is natural to ask what processes have line-integrals independent of the particular path joining the end points.

In the case of a real valued Wiener process B_{st}, Cairoli & Walsh [1], proved the following.

Theorem (Cairoli & Walsh 1974)

The line integral $\int_\Gamma \phi \partial B$ is independent of the path joining the end points if and only if $\phi_{st} = \sum_{n=0}^{\infty} a_n H_n(B_{st}, s\cdot t)$ with $\sum_{n=0}^{\infty} a_n^2 \frac{t^n}{n!} < \infty \ \forall t$, where $H_n(x,t)$ is the n-th Hermite polynomial.

I think the proof of this is very fascinating . Cairoli & Walsh introduced a whole new theory of stochastic calculus. They proved a stochastic version of Green's theorem connecting line integrals with surface integrals, and used a martingale representation theorem together with a theory of quadratic variation to prove their result.

At the time the connection between Brownian motion and analytic functions was already very apparent, and Yor [2] observed a complex version of the theorem.

Theorem (Yor 1977)

When W_{st} is a $\mathbb{C}$-valued 2-parameter Wiener process, the line integral $\int_\Gamma \Phi \partial W$ is independent of the path if and only if

$$\Phi_{st} = \sum_{n=0}^{\infty} a_n (W_{st})^n \quad \text{with} \quad \sum_{n=0}^{\infty} |a_n|^2 n! t^n < \infty \ \forall t.$$

To prove this theorem Yor studied the integrals as real objects and managed to match the real and imaginary part to prove the theorem. The proof of this was efficient and fair enough, but I believe it has some interest to see how this theorem can be proved from a purely complex point of view. In this paper I will explain how to build up the complex objects, and we will see that the proof in the Cairoli & Walsh paper can be carried out directly in this setting.

Acknowledgement

I wish to thank Eugene Wong and John Walsh for private communication on this work.

Some basic definitions and notation

Let B_{1z} and B_{2z} denote two independent real-valued 2-parameter Wiener processes on a probability-space $(\Omega,\mathfrak{F},P)$ and put

$$W_z = B_{1z} + iB_{2z} \quad \text{i-imaginary unit, } z = (s,t) \in \mathbb{R}_+^2$$

We have the order relations

$$(s,t) < (s',t') \text{ iff } s \leq s' \quad t \leq t'$$

$$(s,t) << (s',t') \text{ iff } s < s' \quad t < t'$$

$$(s,t) \wedge (s',t') \text{ iff } s \leq s' \quad t \geq t'$$

$$(s,t) \hat{\wedge} (s',t') \text{ iff } s < s' \quad t > t'$$

We let $\mathfrak{F}_z$ denote the σ-algebra generated by $\{W_{z'}, z' < z\}$ and we also have the σ-algebras

$$\mathfrak{F}_z^1 = \mathfrak{F}_{s\infty} = \bigvee_v {}_{sv}$$

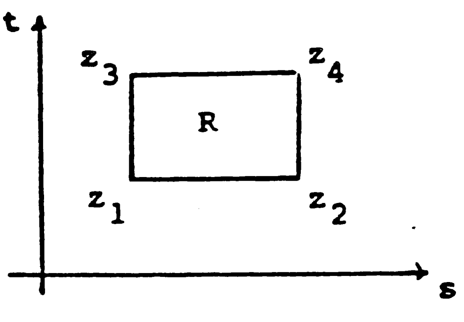
$$\mathfrak{F}_z^2 = \mathfrak{F}_{\infty t} = \bigvee_v {}_{vt}$$

We say that a stochastic process X_z is a martingale if

$$E[X_{z'} | \mathfrak{F}_z] = X_z \quad \text{whenever } z < z'$$

For a rectangle R with corners $z_1 z_2 z_3$ and z_4 as below, we define

$\Delta_R \Phi = \Phi_{z_4} - \Phi_{z_3} - \Phi_{z_2} + \Phi_{z_1}$ where Φ is a 2-parameter process.



ΔR = Area of R and R_z denotes the set $\{z' \in \mathbb{R}^2_+ \; z' < z\}$. For each $z << z'$ let $(z,z']$ denote the rectangle $(s,s'] \times (t,t']$. We say that an adapted integrable process X_{st} is

a weak martingale iff $E[\Delta_{(z,z']} X | \mathfrak{F}_z] = 0 \; \forall z << z'$

an i-martingale iff $E[\Delta_{(z,z']} X | \mathfrak{F}^i_z| = 0 \; \forall \; z << z' \quad i = 1,2$

It is convenient to observe that a martingale is both a 1- and a 2-martingale , see Cairoli & Walsh [1] p. 115.

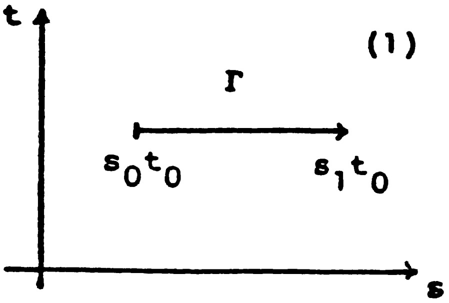
We call a $\mathbb{C}$-valued process increasing iff both componønts are increasing. The joint quadratic variation $\langle x,y \rangle_{st}$ is any difference of increasing processes s.t. $x_{st}\bar{y}_{st} - \langle x,y \rangle_{st}$ is at least a weak martingale. We also write $\langle x,x \rangle_{st} = \langle x \rangle_{st}$. A process Φ_{st} is said to be adapted measurable whenever

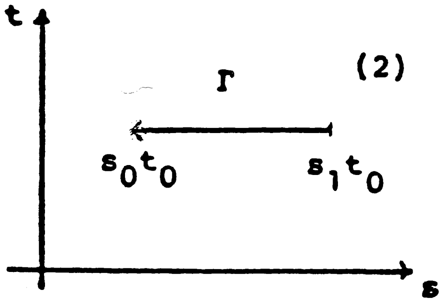
(i) Φ_{st} is $\mathfrak{F}_{st}$ - measurable

(ii) $(s,t,\omega) \to \Phi_{st}(\omega)$ is $\mathfrak{F} \times \mathcal{B}$ measurable, where $\mathcal{B}$ is the class of Borel sets on $\mathbb{R}^2_+$.

Part 1 - Line integrals

Fixing one parameter in a two parameter process gives a one parameter process. If the process is adapted and reasonably nice, stochastic integrals along straight line segments parallel to the coordinate-axes can be defined in the obvious way.





In case (1) we define

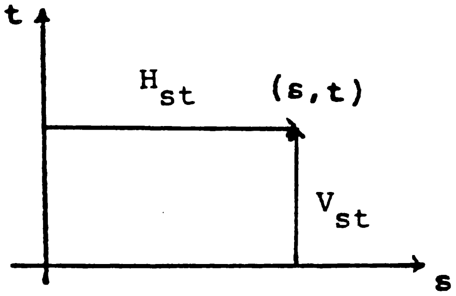
$$\int_\Gamma \Phi \partial W = \int_{s_0}^{s_1} \Phi_{ut_0} d_u W_{ut_0}$$

where $d_u W_{ut_0}$ means integration w.r.t the Brownian motion W_{st_0}.
This is well defined as an Ito-integral if Φ_{st_0} is adapted measurable with $E|\Phi_{st_0}|^2$ bounded on compact sets.

In case (2) we define

$$\int_\Gamma \Phi \partial W = -\int_{-\Gamma} \Phi \partial W$$

The integrals on vertical paths are defined similarly, and the integral extends immediately to rectangular paths by linearity. It is not hard to see that for a large class of processes, the integrals can be defined along any sufficiently smooth path by approximating the path with rectangular paths. This, however, will be of little importance to us, and we will choose to ignore it, at least for the time being. Our main interest will be with the paths below, which we denote by V_{st} and H_{st}.



Definition

We call a process Φ_{st} weakly holomorphic if there exists an adapted measurable process Φ'_{st} with $E|\Phi'_{st}|^2$ bounded on compact sets in $\mathbb{R}^2_+$ and s.t.

$$\Phi_{st} = \Phi_0 + \int_{V_{st}} \Phi' \partial W = \Phi_0 + \int_{H_{st}} \Phi' \partial W \quad \text{for each} \quad (s,t) \in \mathbb{R}^2_+$$

We call Φ'_{st} a derivative of Φ_{st}, and write $\Phi_{st} \in H$.

It follows by linearity that $\Phi_{st} = \Phi_0 + \int_\Gamma \Phi' \partial W$ where Γ is any rectangular path joining (s,t) to $(0,0)$. If Φ_{st} has derivatives up to order n, we say that $\Phi_{st} \in H^n$ i.e. $\Phi_{st} \in H^n$ iff

$$\Phi_{st} = \Phi^0_0 + \int_\Gamma \Phi^1 \partial W$$

$$\Phi^1{}_{st} = \Phi^1_0 + \int_\Gamma \Phi^2 \partial W$$

$$\vdots$$

$$\Phi^{n-1}_{st} = \Phi^{n-1}_0 + \int_\Gamma \Phi^n \partial W$$

Before we go on to study the holomorphic processes, we observe that the usual Ito-formula and Ito-isometry applies to each line-segment. i.e.

$$\begin{aligned} f(Z_{st}) = f(Z_{0t}) &+ \int_0^s \frac{\partial f}{\partial z}(Z_{ut}) d_u Z_{ut} \\ &+ \int_0^s \frac{\partial f}{\partial \bar{z}}(Z_{ut}) d_u \bar{Z}_{ut} \\ &+ \int_0^s \frac{1}{2} \frac{\partial^2 f}{\partial z^2}(Z_{ut}) d_u Z_{ut} d_u Z_{ut} \\ &+ \int_0^s \frac{\partial^2 f}{\partial z \partial \bar{z}}(Z_{ut}) d_u Z_{ut} d_u \bar{Z}_{ut} \\ &+ \int_0^s \frac{1}{2} \frac{\partial^2 f}{\partial \bar{z}^2}(Z_{ut}) d\bar{Z}_{ut} d\bar{Z}_{ut} \end{aligned}$$

where we have the formal relations

$$d_u W_{ut} d_u W_{ut} = d_u \bar{W}_{ut} d_u \bar{W}_{ut} = 0 \qquad d_u W_{ut} d_u \bar{W}_{ut} = 2t du$$

The Ito-isometry applies in the same way, so

$$E\left|\int_0^s \Phi_{ut} d_u W_{ut}\right|^2 = 2t \int_0^s E|\Phi_{ut}|^2 du$$

We first note some easy consequences of the Ito-formula.

Proposition 1.1.

If $\Phi_{st} = (W_{st})^n$ then $\Phi_{st} \in H^\infty$ and

$$\Phi_{st} = \int_\Gamma nW^{n-1} \partial W$$

$$\Phi'_{st} = \int_\Gamma n(n-1)W^{n-2} \partial W$$

$$\vdots$$

$$\Phi^{n-1}_{st} = \int_\Gamma n! \, \partial W$$

$$\Phi^{n}_{st} = n!$$

$$\Phi^{n+1}_{st} = 0$$

Proof

By the Ito-formula on $f(z) = z^n$, t fixed

$$f(W_{st}) = \int_0^s nW^{n-1}_{ut} d_u W_{ut} = \int_{H_{st}} nW^{n-1} \partial W$$

Since $d_u W_{ut} d_u W_{ut} = 0$. The same relation with s fixed gives

$$f(W_{st}) = \int_0^t nW^{n-1}_{su} d_u W_{su} = \int_{V_{st}} nW^{n-1} \partial W$$

so the derivative of W^n_{st} is nW^{n-1}_{st}

□

Lemma 1.2

$$E[W_{st}^n \bar{W}_{st}^m] = \delta_{nm} \cdot n! \cdot (2st)^n$$

Proof

By Ito's formula on $f(z) = z^n \bar{z}^m$, t fixed

$$f(W_{st}) = \int_0^s n W_{ut}^{n-1} \bar{W}_{ut} d_u W_{ut} + \int_0^s W_{ut}^n \cdot m \bar{W}_{ut}^{m-1} d_u \bar{W}_{ut}$$

$$+ 2t \int_0^s n W_{ut}^{n-1} m \bar{W}_{ut}^{m-1} du$$

so

$$E[W_{st}^n \bar{W}_{st}^m] = 2t \cdot m \cdot n \int_0^s E[W_{ut}^{n-1} \bar{W}_{ut}^{m-1}] du$$

and the lemma follows by iteration.

□

Proposition 1.1 may be extended to $f(z) = \sum_{n=0}^{\infty} a_n z^n$ whenever $f'(W_{st})$ is adapted measurable with $E|f'(W_{st})|^2$ bounded on compact sets. Since the first statement is trivial, we turn to the L^2-norm. By lemma 1.2 it is easy to see that

$$E|\sum_{n=M}^{N} a_n W_{st}^n|^2 = \sum_{n=M}^{N} |a_n|^2 E|W_{st}|^{2n} = \sum_{n=M}^{N} |a_n|^2 \cdot n!(2st)^n$$

If $\sum_{n=0}^{\infty} |a_n|^2 \cdot n! t^n < \infty \forall t$, it will follow immediately that $\sum_{n=0}^{\infty} a_n W_{st}^n \to f(W_{s_t})$ in L^2. The same applies to all the derivatives of f and we have the following

Theorem 1.3 (Yor)

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\sum_{n=0}^{\infty} |a_n|^2 \cdot n! t^n < \infty \forall t$

Then

$$f(W_{st}) = f(0) + \int_0^{(s,t)} f'(W)\,\partial W$$

where the line integral is independent of the path Γ joining the origin with (s,t). Since the same applies to all the derivatives, $f(W_{st}) \in H^{\infty}$.

We will now see how to prove the converse when the process $\Phi_{st} \in H^{\infty}$. The proof is a complex version of the proof presented in Cairoli & Walsh [1]. See also Nguyen [3] for local versions of these theorems.

Proposition 1.4

If Φ_{st} and $\psi_{st} \in H^{n+1}$ then

$$E[\Phi_{st}\bar{\psi}_{st}] = \sum_{j=0}^{n} \Phi_0^j \bar{\psi}_0^j \cdot \frac{(2st)^j}{j!} + (2t)^{n+1} \int_0^s \int_0^{s_n} \cdots \int_0^{s_1} E[\Phi_{ut}^{n+1} \bar{\psi}_{ut}^{n+1}]\,du\,d\vec{s}$$

Proof

By Ito's formula on $f(z,w) = z\bar{w}$ we get

$$E[\Phi_{st}\bar{\psi}_{st}] = \Phi_0^0 \bar{\psi}_0^0 + 2t \int_0^s E[\Phi_{ut}^1 \bar{\psi}_{ut}^1]\,du$$

as in lemma 1.2. Proposition 1.4 then follows by repeated use of this relation.

□

Lemma 1.5

If $\Phi_{st} \in H^{\infty}$ then

$$\lim_{n\to\infty} \frac{(2st)^n}{n!} E|\Phi_{st}^{n+1}|^2 = 0$$

Proof

Define

$$g_n(s,t) = E|\Phi^n_{st}|^2 \quad n = 0,1,2.....$$

By proposition 1.4 we have

$$g_n(s,t) = |\Phi^n_0|^2 + 2t\int_0^s g_{n+1}(u,t)du$$

Put $f_n(x) = g_n(x,1)$, then

$$f'_n(x) = \frac{\partial g_n}{\partial s}(x,1) = 2g_{n+1}(x,1) = 2f_{n+1}(x)$$

It follows that

$$f^{(n)}(x) = 2^n f_n(x)$$

By Taylors formula

$$f(x) = \sum_{j=0}^{N} f^{(n)}(x_0)\frac{(x-x_0)^n}{n!} + \frac{f^{(N+1)}(\theta)}{(N+1)!}(x-x_0)^{N+1}$$

Since

$$f^{N+1}(\theta) = 2^{N+1}f_{N+1}(\theta) = 2^{N+1}g_{N+1}(\theta,1) = 2^{N+1}E|\Phi^{N+1}_{\theta 1}|^2 \geq 0$$

we have

$$f(2x_0) \geq \sum_{j=0}^{N} f^{(n)}(x_0)\frac{x_0^n}{n!} \quad \text{for all } N, \text{ all } x_0 \geq 0$$

Now $f(2x_0) = g_0(2x_0,1) = E|\Phi_{2x_0 1}|^2 < \infty$. Then the series converges

for all $x_0 \geq 0$ and $\lim_{n\to\infty} f^{(n)}(x_0)\frac{x_0^n}{n!} = 0 \quad \forall\, x_0 \geq 0$

It is also true that

$$\lim_{n\to\infty} f_n(x_0)\frac{(2x_0)^n}{n!} = 0 \quad \forall \ x_0 > 0$$

It remains to observe that $E|\Phi^n_{st}|^2 = f_n(s\cdot t)$.

Look at

$$g_n(s,t) = g_{no} + 2t\int_0^s g_n(u,t)\,du$$

and the symmetric relation

$$g_n(s,t) = g_{n0} + \int_0^t g_{n+1}(s,u)du$$

From these one easily sees that

$$s\frac{\partial g_n}{\partial s} = t\frac{\partial g_n}{\partial t}$$

But then

$$E|\Phi^n_{st}|^2 = g_n(s,t) = g_n(st,1) = f_n(s\cdot t)$$

since g_n only depends on (s,t) by their product.

<u>Corollary 1.6</u>

If $\Phi_{st} \in H^\infty$ then

$$E|\Phi_{st}|^2 = \sum_{j=0}^{\infty} \frac{(2st)^j}{j!}|\Phi_0^j|^2$$

<u>Proof</u>

By proposition 1.4

$$E|\Phi_{st}|^2 = \sum_{j=0}^{n} \frac{(2st)^j}{j!}|\Phi_0^j|^2 + (2t)^{n+1}\int_0^s\int_0^{s}{}^{n}\text{---}\int_0^{s}{}^{1} E|\Phi_{ut}^{n+1}|^2 du d\vec{s}$$

Since $\Phi_{st} \in H^{\infty}$, Φ_{st}^{n+1} is a martingale, and $E|\Phi_{ut}^{n+1}|^2$ is bounded by $E|\Phi_{st}^{n+1}|^2$. Then the integral term is bounded by $\frac{(2st)^{n+1}}{(n+1)!}E|\Phi_{st}^{n+1}|^2$ and this goes to zero by lemma 1.5

Proposition 1.7

If $\Phi_{st} \in H^{n+1}$ then

$$E[\Phi_{st}\bar{W}_{st}^{n}] = \Phi_0^n \cdot (2st)^n$$

Proof

Put $\psi_{st} = W_{st}^n$ proposition 1.4. By proposition 1.1 all terms except $\Phi_0^n \bar{\psi}_0^n \cdot \frac{(2st)^n}{n!} = \Phi_0^n \cdot (2st)^n$ vanish.

Corollary 1.8

If $\Phi_{st} \in H^{\infty}$ with $E[\Phi_{st}\bar{W}_{st}^{n}] = 0$ for all n, then $\Phi_{st} \equiv 0$.

Proof

By corollary 1.6 and proposition 1.7

$$E|\Phi_{st}|^2 = \sum_{j=0}^{\infty} \frac{(2st)^j}{j!}|\Phi_0^j|^2 = 0$$

Theorem 1.10 (Yor)

If $\Phi_{st} \in H^{\infty}$ then $\Phi_{st} = f(W_{st})$ where $f(z) = \sum_{n=0}^{\infty} \frac{\Phi_0^n}{n!} z^n$ satisfies the conditions in theorem 1.3.

Proof

Put $f(z) = \sum_{n=0}^{\infty} \frac{\Phi_0^n}{n!} z^n$ i.e $a_n = \frac{\Phi_0^n}{n!}$

By corollary 1.6

$$\sum_{n=0}^{\infty} |a_n|^2 n! t^n = \sum_{n=0}^{\infty} \left|\frac{\Phi_0^n}{n!}\right|^2 n! t^n = \sum_{n=0}^{\infty} \frac{(2t \cdot \frac{1}{2})^n}{n!} |\Phi_0^n|^2 = E|\Phi_{\frac{1}{2},t}|^2 < \infty$$

Then

$\psi_{st} = f(W_{st})$ satisfies the conditions in theorem 1.3. We have $\psi_{st} \in H^{\infty}$ and $\psi_0^n = \Phi_0^n$ for each n. Since $\Phi_{st} - \psi_{st} \in H^{\infty}$ and

$$E[(\Phi_{st} - \psi_{st})\bar{W}_{st}{}^n] = \Phi_0^n \cdot (2st)^n - \psi_0^n \cdot (2st)^n = 0$$

for each n, $\Phi_{st} - \psi_{st} \equiv 0$ by corollary 1.8

Now it only remains to prove that all weakly holomorphic processes are in fact H^{∞}. To prove this we turn our attention to various stochastic integrals.

Part 2 - Stochastic integrals and Green's formula.

We want to define the surface integral $\int_{R_{st}} \Phi dW$. To define this integral on simple functions, partition R_{st} into rectangles R_{ij} with lower left corners z_{ij} and let the values Φ_{ij} on these rectangles be $\mathfrak{F}_{z_{ij}}$-measurable.

Then

$$E\left|\sum_{i,j} \Phi_{ij} \Delta_{ij} W\right|^2 = \sum_{i,j} E|\Phi_{ij}|^2 E|\Delta_{ij} W|^2 = \sum_{i,j} E|\Phi_{ij}|^2 \cdot 2\,\Delta R_{ij}$$

Once you have this isometry, the integral can be extended to adapted measurable Φ_{st} with $E|\Phi_{st}|^2$ bounded on compact sets. This is exactly as in the theory of oneparameter Ito-integrals. If we multiply two such integrals, however, we end up with something new. i.e. let

$$X_{st} = \int_{R_{st}} \Phi dW \qquad Y_{st} = \int_{R_{st}} \psi dW$$

and look at

$$\sum_{i,j} \Phi_{ij} \Delta_{ij} W \cdot \sum_{k,l} \psi_{kl} \Delta_{kl} W$$

$$= \sum_{i,j} \Big(\sum_{k,l}^{i-1\ j-1} \psi_{kl} \Delta_{kl} W \cdot \Phi_{ij} \Big) \Delta_{ij} W \approx \int_{R_{st}} Y\Phi dW \qquad (1)$$

$$+ \sum_{k,l} \Big(\sum_{i,j}^{k-1\ l-1} \Phi_{ij} \Delta_{ij} W \cdot \psi_{ij} \Big) \Delta_{kl} W \approx \int_{R_{st}} X\psi dW \qquad (2)$$

$$+ \sum_{j,k}^{k-1} \sum_{i,l}^{j-1} \Phi_{ij} \psi_{kl} \Delta_{ij} W \Delta_{kl} W \qquad (3)$$

$$+ \sum_{i,l}^{i-1} \sum_{j,k}^{l-1} \Phi_{ij} \psi_{kl} \Delta_{ij} W \Delta_{kl} W \qquad (4)$$

+ remaing terms

In this particular case the L^2-norm of the remaining terms can be made uniformly small by choosing the partition fine enough. The terms (1) and (2) can be accounted for as ordinary surface integrals. We cannot, however, include any more terms in these sums as long as we only want to integrate adapted processes. The terms (3) and (4) represents roughly one half on the terms, so they cannot be ignored. At first sight these terms look pretty hopeless, but the particular positioning of the indices turn out to be very convenient. We actually have the following isometry.

If α_{ijkl} is $\mathcal{F}_{(i,j)\vee(k,l)}$ - measurable, then

$$E\Big|\sum_{\substack{i,j,k,l\\(i,j)\hat{\wedge}(k,l)}} \alpha_{ijkl}\Delta_{ij}W\Delta_{kl}W\Big|^2 = \sum_{\substack{i,j,k,l\\(i,j)\hat{\wedge}(k,l)}} E|\alpha_{ijkl}|^2 2\Delta R_{ij} 2\Delta R_{kl}$$

The expression $\sum_{\substack{i,j,k,l\\(i,j)\hat{\wedge}(k,l)}} \alpha_{ijkl}\Delta_{ij}W\Delta_{kl}W$ defines the integral

$\int_{R_{st}\times R_{st}} \alpha(\xi,\eta)dW_\xi dW_\eta$ on simple four-parameter processes. The isometry above then makes it possible to extend this integral to any four-parameter process $\alpha(\xi,\eta)$ with

(i) $\alpha(\xi,\eta)$ $\mathcal{F}_{\xi V\eta}$ - measurable

(ii) $\alpha(\xi,\eta) = 0$ unless $\xi\hat{\wedge}\eta$

(iii) $E|\alpha(\xi,\eta)|^2$ bounded on compact sets in $\mathbb{R}_+^4$

To account for the term (3) just observe that

$$\sum_{j,k}\sum_{i,j}^{k-1\;\;j-1} = \sum_{\substack{i,j,k,l\\(i,j)\hat{\wedge}(k,l)}}$$ so that this will

approximate the term $\int_{R_{st}\times R_{st}} \Phi_\xi\phi_\eta dW_\xi dW_\eta$.

The term (4) is accounted for in same way.

The same procedure can be used to define the integrals $\int_{R_{st}\times R_{st}} \alpha dW d\bar{W}$, $\int_{R_{st}\times R_{st}} \alpha d\bar{W} d\bar{W}$ and so on. These are the integrals we will be working with.

L^2-martingales

In their paper [4], Wong & Zakai proved that every L^2-martingale w.r.t. a 2-parameter Wiener process B_{st}, can be represented as a sum of two stochastic integrals $\int_{R_{st}} \Phi dB + \int_{R_{st}\times R_{st}} \psi dBdB$. This also applies when the σ-algebras are generated by several independent two-parameter Wiener processes. In our case each W_{st}-martingale can be written on the form

$$\sum_{i=1}^{2} \int_{R_{st}} \Phi_i dB_i + \sum_{i,j}^{2} \int_{R_{st}\times R_{st}} \psi_{ij} dB_i dB_j$$

If you split the matrices involved into $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts, it is easy to see that you have the following representation.

<u>Theorem 2.1 (Wong & Zakai)</u>

If X_{st} is a $\mathbb{C}$-valued L^2-martingale, then

$$X_{st} = X_0 + \int_{R_{st}} \Phi dW + \int_{R_{st}} \psi d\bar{W} + \int_{R_{st}\times R_{st}} \alpha dWdW + \int_{R_{st}\times R_{st}} \beta dWd\bar{W} + \int_{R_{st}\times R_{st}} \gamma d\bar{W}dW + \int_{R_{st}\times R_{st}} \delta d\bar{W}d\bar{W}$$

It turns out that the terms in this representation are actually orthogonal in L^2. More exactly we have.

<u>Proposition 2.2</u>

Let

$$I_1 = \int_{R_{z_0}} \Phi dW \quad I_2 = \int_{R_{z_0}} \psi d\bar{W} \quad I_3 = \int_{R_{z_0}\times R_{z_0}} \alpha dWdW \quad I_4 = \int_{R_{z_o}\times R_{z_0}} \beta dWd\bar{W}$$

$$I_5 = \int_{R_{z_0}\times R_{z_0}} \gamma d\bar{W}dW \quad I_6 = \int_{R_{z_0}\times R_{z_o}} \delta d\bar{W}d\bar{W}$$

Then

$$E[I_i\bar{I}_j] = 0 \quad \text{unless} \quad i = j$$

Also if $X_{st} = X_0 + \int_{\Gamma} X' \partial W \in H$ we get $E[X_{z_0}\bar{I}_j] = 0$ if $j = 2,4,5,6$

Proof

Partition R_{z_0} into rectangles Δ_{ij} and assume to begin with that all integrands are constant on Δ_{ij}. We first look at

$$E[I_1\bar{I}_2] = \sum_{i,j,k,l} E[\Phi_{ij}\overline{\varphi_{kl}}\Delta_{ij}W\Delta_{kl}W]$$

Here either $\Delta_{ij}W$, $\Delta_{kl}W$ or both are independent from the rest. Since

$$E[\Delta_{ij}W] = E[\Delta_{ij}W\Delta_{kl}W] = 0 \text{ all terms vanish.}$$

Case two

$$E[I_1\bar{I}_3] = \sum_{i,j} \sum_{\substack{k,l,m,n \\ (k,l)\hat{\wedge}(n,n)}} E[\Phi_{ij}\bar{\alpha}_{klmn}\Delta_{ij}\bar{W}\Delta_{mn}\bar{W}]$$

Here either $\Delta_{ij}W_1$, $\Delta_{kl}\bar{W}_1$, $\Delta_{mn}\bar{W}$ or the pairs $(\Delta_{ij}W\Delta_{kl}\bar{W})$ $(\Delta_{ij}W\Delta_{mn}\bar{W})$ are independent from the rest. The first three cases are trivial, so let us consider the remaining two. When $(k,l)\hat{\wedge}(m,n)$ Φ_{kl} is $\mathfrak{F}^1_{mn}$- measurable and as such independent of $\Delta_{mn}\bar{W}$. When $\Delta_{ij}W\Delta_{kl}\bar{W}$ are dependent, but independent from the rest $(i,j)=(k,l)$ and

$$E[\phi_{ij}\bar{\alpha}_{klmn}\Delta_{ij}W\Delta_{kl}\bar{W}\Delta_{mn}\bar{W}] = E[\Phi_{kl}\bar{\alpha}_{klmn}]E[\Delta_{kl}W\Delta_{kl}\bar{W}]E[\Delta_{mn}\bar{W}] = 0$$

The first part of the proposition is proved along the same lines and are left to the reader. As for the second part $E[X_{z_0}\bar{I}_2] = 0$ and $E[X_{z_0}\bar{I}_6] = 0$ are easy since there are no non-conjugate terms. The two remaining terms require a bit more carefulness. Look at

$$E[X_{z_0} \cdot \sum_{\substack{i,j,k,l \\ (i,j)\hat{\wedge}(k,l)}} \bar{\beta}_{ijkl}\Delta_{ij}\bar{W}\Delta_{kl}W]$$

Write

$$X_{z_0} = X_0 + \int_0^s X'_{ut_0} d_u W_{ut_0} = X_0 + \sum_m X'_{s_m t_0} (W_{s_{m+1} t_0} - W_{s_m t_0})$$

where $s_1 \ldots s_N$ are the lower s-coordinates of the partition Δ_{ij}. The case with X_0 is trivial. Consider

$$\sum_{\substack{i,j,k,l \\ (i,j)\hat{\wedge}(k,l)}} \sum_m [X'_{s_m t_0} \bar{\beta}_{ijkl} (W_{s_{m+1} t_0} - W_{s_m t_0}) \Delta_{ij} \bar{W} \Delta_{kl} W]$$

The nontrivial case occurs when $(W_{s_{m+1} t_0} - W_{s_m t_0})$ and $\Delta_{ij}\bar{W}$ are dependent, but independent from the rest. This only happens when the rectangle R_{ij} have upper and lower s-coordinates s_{m+1} and s_m. Since $i<k$ $X'_{s_m t_0}$ is $\mathfrak{F}^2_{kl}$-measurable so it is independent from $\Delta_{kl}W$. Then $\Delta_{kl}W$ can also be split out, and we are through.
In general $X_{s_m t_0}$ will not be independent from $\Delta_{kl}W$. To prove $E[X_{z_0} \bar{I}_5] = 0$ you have to use $X_{z_0} = X_0 + \int_0^{t_0} X'_{s_0 u} d_u W_{s_0 u}$. Then the proof can be carried out along the same lines. The above also explains why you cannot expect to get $E[X_{z_0} \bar{I}_3] = 0$. In this case you would end up with terms of the form $E[X'_{s_m t_0} \bar{\alpha}_{ijkl} \Delta_{kl} \bar{W}]$ and this may not vanish because $X'_{s_m t_0}$ and $\Delta_{kl}\bar{W}$ may be dependent. E.g. $X' = W$

<u>Corollary 2.3</u>

If $\Phi_{st} \in H$ then

$$\Phi_{st} = \Phi_0 + \int_{R_{st}} \phi dW + \int_{R_{st} \times R_{st}} \alpha dW dW$$

Proof

When $\Phi_{st} \in H$, Φ_{st} is clearly an L^2-martingale so by theorem 2.1

$$\Phi_{st} = \Phi_0 + I_1 + I_2 + I_3 + I_4 + I_5 + I_6$$

By proposition 2.2

$$0 = \|\Phi_{st} - \Phi_0 - I_1 - I_2 - I_3 - I_4 - I_5 - I_6\|^2$$

$$= \|\Phi_{st} - \Phi_0 - I_1 - I_3\|^2 + \|I_2\|^2 + \|I_4\|^2 + \|I_5\|^2 + \|I_6\|^2$$

so all the conjugate terms vanish.

The process J and Green's formula.

We define the process J_z by the relation

$$J_z = \int_{R_z \times R_z} \psi dW dW \quad \text{where} \quad \psi(\xi,\eta) = \begin{cases} 1 & \text{if } \xi \hat{\wedge} \eta \\ 0 & \text{otherwise} \end{cases}$$

If you approximate ψ by simple processes and calculate the conditional expectations, if is easy to see that J_z is a martingale with quadratic variation

$$\langle J \rangle_{st} = \int_{R_{st} \times R_{st}} 4|\psi|^2 d\xi d\eta$$

The process J_z gives a connection between surface integrals and line integrals. We first observe the following

Proposition 2.4

$$J_{st} = \int_{H_{st}} W \partial W - \int_{R_{st}} W dW$$

$$J_{st} = \int_{V_{st}} W \partial W - \int_{R_{st}} W dW$$

Proof

Partition $R_{s_0t_0}$ into rectangles Δ_{ij} $i,j < n$. Then

$$J_{s_0t_0} \approx \sum_{\substack{i,j,k,l \\ (i,j)\hat{\wedge}k,l}} \Delta_{ij}W\Delta_{kl}W$$

$$= \sum_{j,k} \sum_{i}^{k-1} \sum_{l}^{j-1} \Delta_{ij}W\Delta_{kl}W$$

$$= \sum_{j,k} (W_{kj+1}-W_{kj}) \sum_{l}^{j-1} \Delta_{kl}W$$

$$= \sum_{j,k} (W_{kj+1}-W_{kj})(-\Delta_{kj}W) + \sum_{j,k} (W_{kj+1}-W_{kj}) \sum_{l}^{j} \Delta_{kl}W$$

$$= - \sum_{j,k} (W_{kj+1}-W_{kj})\Delta_{kj}W$$

$$+ \sum_{j,k} W_{kj+1} \sum_{l}^{j} \Delta_{kl}W - W_{kj} \sum_{l}^{j-1} \Delta_{kl}W$$

$$- \sum_{j,k} W_{kj} \left(\sum_{l}^{j} \Delta_{kl}W - \sum_{l}^{j-1} \Delta_{kl}W\right)$$

The second sum telescopes in j and l and we get

$$= - \sum_{j,k} (W_{kj+1}-W_{kj})\Delta_{kj}W$$

$$+ \sum_{k} W_{kn}(W_{k+1n}-W_{kn})$$

$$- \sum_{j,k} W_{kj}\Delta_{kj}W$$

$$\approx 0 + \int_{H_{s_0t_0}} W\partial W - \int_{R_{s_0t_0}} WdW$$

since the first term obviously can be made uniformly small. The proof of the second relation is similar.

A complex version of Greens formula follows almost immediately from proposition 2.4. First you need to observe that it is possible to integrate against J_{st}. The definition on simple processes is of course $\sum_{i,j} \Phi_{ij} \Delta_{ij} J$ and there is an isometry also in this case. The class of integrable processes against a general L^2-martingale M_{st} depends on the quadratic variation $\langle M \rangle_{st}$. In the case of J_{st}, however, the quadratic variation is so small that is suffices to have $E[\Phi_{st}]^2$ bounded on compact sets.

Once it is meaningful to integrate against J_{st} we can state an integration formula.

Greens formula 2.5

Assume that Φ'_{st} is adapted measurable with $E|\Phi'_{st}|^2$ bounded on compact sets. When $\Phi_{st} = \Phi_0 + \int_{V_{st}} \Phi' \partial W$, then for any rectangle A

$$\int_{\partial A} \Phi \partial_s W = - \int_A \Phi dW - \int_A \Phi' dJ$$

The integration is counter-clockwise and integration ∂_s means that the vertical segments are ignored. With the same conditions on Φ', the symmetric relation along the vertical segments is that if

$$\Phi_{st} = \Phi_0 + \int_{H_{st}} \Phi' \partial W \quad \text{then}$$

$$\int_{\partial A} \Phi \partial_t W = \int_A \Phi dW + \int_A \Phi' dJ$$

Proof

By a standard argument you can reduce to the case where Φ' is constant on A. Then the formula follows from proposition 2.4. For details see Cairoli & Walsh [1] p. 151

This Green's formula has two important corollaries.

<u>Corollary 2.6</u>

If $\phi \in H$ then ϕ has a primitive Φ s.t

$$\Phi = \Phi_0 + \int_\Gamma \phi \partial W$$

<u>Proof</u>

By 2.5.

$$\int_{H_{st}} \phi \partial W = \int_{R_{st}} \phi dW + \int_{R_{st}} \phi dJ = \int_{V_{st}} \phi \partial W$$

<u>Corollary 2.7</u>

If $\Phi \in H^2$ then

$$\Phi_{st} = \Phi_0 + \int_{R_{st}} \Phi' dW + \int_{R_{st}} \Phi'' dJ$$

<u>Proof</u>

Immediate from 2.5 since $\Phi_{st} = \Phi_0 + \int_{\partial R_{st}} \Phi' \partial_t W$

Corollary 2.7 is the basic idea to prove that all processes $\Phi \in H$ are in fact H^∞. From corollary 2.3 we know that

$$\Phi_{st} = \Phi_0 + \int_{R_{st}} \psi dW + \int_{R_{st} \times R_{st}} \alpha dW dW$$

One can hope to prove that $\psi = \Phi'$ and that α represents Φ'' in some sence. To pursue this further, we need to be able to translate an integral $\int_{R_{st}} \phi dJ$ to the form $\int_{R_{st} \times R_{st}} \alpha dW dW$.

For this you have the following.

Proposition 2.8.

When ϕ_{st} is adapted measurable with $E|\phi_{st}|^2$ bounded on compact sets, and

$$\alpha(r,s,t,u) = \begin{cases} \phi_{st} & \text{when } (r,s)\hat{\wedge}(t,u) \\ 0 & \text{otherwise} \end{cases}$$

then

$$\int_{R_{st}} \phi dJ = \int_{R_{st}\times R_{st}} \alpha dW dW$$

Proof

Partition $R_{s_0 t_0}$ into rectangles Δ_{ij} $i,j < n$ and replace J_{st} by its approximertion J^n_{st} given by $J^n_{st} = E[\sum_{\substack{i,j,k,l \\ (i,j)\hat{\wedge}(k,l)}} \Delta_{ij}W\Delta_{kl}W | \mathfrak{F}_{st}]$

Then the J^n-s are martingales and $J^n \to J$ uniformly in L^2. When n is fixed according to the partition we have

$\Delta_{ij}J^n = \sum_{k,l}^{i-1\ j-1} \Delta_{kj}W\Delta_{il}W$ and we get

$$\begin{aligned}
\int_{R_{st}} \phi dJ &\approx \sum_{i,j} \phi_{ij}\Delta_{ij}J^n \\
&= \sum_{i,j} \sum_{k,l}^{i-1\ j-1} \phi_{ij}\Delta_{kj}W\Delta_{il}W \\
&= \sum_{i,j} \sum_{k,l}^{i-1\ j-1} \alpha_{kjil}\Delta_{kj}W\Delta_{il}W \\
&= \sum_{k,l} \sum_{i,l}^{k-1\ j-1} \alpha_{ijkl}\Delta_{ij}W\Delta_{kl}W \\
&= \sum_{\substack{i,j,k,l \\ (i,j)\hat{\wedge}(k,l)}} \alpha_{ijkl}\Delta_{ij}W\Delta_{kl}W \\
&\approx \int_{R_{st}\times R_{st}} \alpha dW dW
\end{aligned}$$

Part 3 Weakly holomorphic processes are H^∞

When the processes $X_{st} = \int_{R_{st}} \Phi dW$ and $Y_{st} = \int_{R_{st}\times R_{st}} \psi dWdW$ are restricted to the straight line-segments V_{st} and H_{st}, they become 1-parameter martingales w.r.t. the σ-algebras $\mathcal{F}_{\infty t}$ and $\mathcal{F}_{s\infty}$. As such they have unique quadratic varitions along these line segments. We denote the variations by $\langle x^V\rangle$, $\langle x^H\rangle$ and so on. It is fairly straigtforward to generalize the quadratic variations from the real case, see Cairoli & Walsh [1] p. 158. The result is the following (we omit the proof).

Proposition 3.1

Let $X_{st} = \int_{R_{st}} \Phi dW$ $\hat{X}_{st} = \int_{R_{st}} \hat{\Phi} dW$ $Y_{st} = \int_{R_{st}\times R_{st}} \psi dWdW$ $\hat{Y}_{st} = \int_{R_{st}\times R_{st}} \hat{\psi} dWdW$

Then

$$\langle X^H,\hat{X}^H\rangle_{st} = \int_{R_{st}} 2\phi\bar{\hat{\phi}}d\xi \qquad \langle X^H\rangle_{st} = \int_{R_{st}} 2|\phi^2|d\xi$$

$$\langle X^H,Y^H\rangle_{st} = \int_{R_{st}} 2\phi(\xi) \int_{R_{\infty t}} \bar{\psi}(\eta,\xi)d\bar{W}_\eta d\xi$$

$$\langle Y^H,\hat{Y}^H\rangle_{st} = \int_{R_{st}} 2\int_{R_{\infty t}} \psi(\eta,\xi)dW_\eta \int_{R_{\infty t}} \bar{\hat{\psi}}(\eta',\xi)d\bar{W}_{\eta'}d\xi$$

$$\langle Y^H\rangle_{st} = \int_{R_{st}} 2\left|\int_{R_{\infty t}} \psi(\eta,\xi)dW_\eta\right|^2 d\xi$$

When $\phi_{st} \in H$ we have

$$\int_{H_{st}} \phi'\partial W = \int_{R_{st}} \psi dW + \int_{R_{st}\times R_{st}} \alpha dWdW = \int_{V_{st}} \phi'\partial W$$

Cairoli & Walsh observed that an equality of the above type cannot hold unless ψ and α are intimately related. More exactly we have

Proposition 3.2

If $\Phi_{st} \in H$ then for (s,t) outside a negligible set $G \subset \mathbb{R}_+^2$ the functions

$\tau \to \alpha(\sigma,t,s,\tau)$ is a.s. essentially constant in $[0,t]$ for a.e. σ

$\sigma \to \alpha(\sigma,t,s,\tau)$ is a.s. essentially constant in $[0,s]$ for a.e. τ

Moreover for (s,τ) and (s,τ') outside a negligible set F and s.t. $\tau < \tau' < t$ we have

$$\psi_{s\tau} - \psi_{s\tau'} = \int_{R_{\infty t}} \alpha(u,v,s,\tau') - \alpha(u,v,s,\tau)dW_{uv}$$

and for (σ,t) , $(\sigma' t)$ outside a negligible set

$$\psi_{\sigma t} - \psi_{\sigma' t} = \int_{R_{\infty t}} \alpha(\sigma,t,u,v) - \alpha(\sigma',t,u,v)dW_{uv}$$

Proof

Since

$$\Phi_{st_0} = \int_{H_{st_0}} \Phi \partial W = \int_0^s \Phi'_{ut_0} d_u W_{ut_0}$$

we get from Ito's formula that

$$<\Phi^H>_{st_0} = \int_0^s 2t_0 |\Phi'_{ut_0}|^2 du$$

and

$$<\Phi^H, W^H>_{st_0} = \int_0^s 2t_0 \Phi'_{ut_0} du$$

By the first relation we see that

$$\Phi'_{st_0} = \frac{1}{2t_0} \frac{\partial}{0s} <\Phi^H, W^H>_{st_0} \quad \text{for a.e.s}$$

When we insert this in the first relation, we have

(*) $$\langle\Phi^H\rangle_{st_0} = \int_0^s \frac{1}{2t_0}\left|\frac{\partial}{\partial u}\langle\Phi^H,W^H\rangle_{ut_0}\right|^2 du$$

Since

$$\Phi_{st} = \int_{R_{st}} \psi dW + \int_{R_{st}} \alpha dW dW \quad \text{and} \quad W_{st} = \int_{R_{st}} 1 dW$$

we may also compute the quadratic variations from proposition 3.1 i.e.

$$\langle\Phi^H,W^H\rangle_{st_0} = \int_0^s\int_0^{t_0} 2\psi_{uv} + \int_{R_{\infty t_0}} 2\alpha(\eta,u,v)dW_\eta dv du$$

Then for a.e.s

$$\frac{\partial}{\partial s}\langle\Phi^H,W^H\rangle_{st_0} = 2\int_0^{t_0}\psi_{sv} + \int_{R_{\infty t_0}} \alpha(\eta,u,v)dW_\eta dv$$

The same argument also gives

$$\langle\Phi^H\rangle_{st_0} = 2\int_{R_{st_0}} \left|\psi_{uv} + \int_{R_{\infty t_0}} \alpha(\eta,u,v)dW_\eta\right|^2 du dv$$

When we insert these expressions in (*) we get

$$\int_0^s\int_0^{t_0} 2\left|\psi_{uv} + \int_{R_{\infty t_0}} \alpha(\eta,u,v)dW_\eta\right|^2 dv du =$$

$$\int_0^s \frac{2}{t_0}\left|\int_0^{t_0}\psi_{uv} + \int_{R_{\infty t_0}} \alpha(\eta,u,v)dW_\eta dv\right|^2 du$$

So for a.e.s we must have

$$\int_0^{t_0}\left|\psi_{sv} + \int_{R_{\infty t_0}} \alpha(\eta,s,v)dW_\eta\right|^2 dv = \frac{1}{t_0}\left|\int_0^{t_0}\psi_{sv} + \int_{R_{\infty t_0}} \alpha(\eta,s,v)dW_\eta dv\right|^2$$

But by the Cauchy-Schwarz inequality it is easy to see that this can only happen whenever the integral is a.s. constant in v! We get that for a.e.s there is a random variable $\rho(s)$ s.t. for a.e $v < t$

$$\rho(s) = \psi_{sv} + \int_{R_{\infty t_0}} \alpha(\eta, s, v) dW_\eta$$

We can also choose ρ measurably by averageing. Then outside a negligible set F with $\tau < \tau' < t_0$

$$\psi_{s\tau} - \psi_{s\tau'} = \int_{R_{\infty t_0}} \alpha(\eta, s, \tau') - \alpha(\eta, s, \tau) dW_\eta$$

Since the left-hand side is $\mathcal{F}_{\infty\tau'}$-measurable it is easy to see that we must have $\alpha(\eta, s, \tau') = \alpha(\eta, s, \tau)$ for a,e, $\eta \in R_{\infty t_0} - R_{\infty\tau'}$. When this is applied to all possible pairs (τ, τ') we get a process

$\Psi(u,v,s)$ s.t

$\Psi(u,v,s) = \alpha(u,v,s,\tau)$ for a.e. $\tau < v$.

The second pair of relations is proved along the same lines.

□

Let us for a moment forget about the negligible sets in proposition 3.2, and let us see what we would have if the relations were true everywhere.

When $\Phi \in H$ the functions

$\tau \to \alpha(\sigma, t, s, \tau)$

$\sigma \to \alpha(\sigma, t, s, \tau)$

are essentially constant and α will essentially satisfy the conditions in proposition 2.8. When we define $\Psi(s,t) = \alpha(0,t,s,0)$ we get

$$\int_{R_{st}\times R_{st}} \alpha dW dW = \int_{R_{st}} \Psi dJ$$

We then have

$$\Phi_{st} = \Phi_0 + \int_{R_{st}} \phi dW + \int_{R_{st}} \Psi dJ$$

Essentially we also have

$$\phi_{st'} - \phi_{st} = \int_{R_{\infty t'}} \alpha(\eta,s,t) - \alpha(\eta,s,t')dW_\eta \quad \text{if} \quad t < t'$$

$$= \int_{R_{st'}-R_{st}} \alpha(u,v,s,t)dW_{uv}$$

since everything else vanish outside this set. But we also know that α essentially doesn't depend to u. i.e. $\alpha(u,v,s,t) = \Psi(s,v)$. Then we get

(1) $$\phi_{st'} - \phi_{st} = \int_{R_{st'}-R_{st}} \Psi(s,v)dW_{uv} = \int_{V_{st'}} \Psi \partial W - \int_{V_{st}} \Psi \partial W$$

By the same way of reasoning we also get

(2) $$\phi_{s't} - \phi_{st} = \int_{H_{s't}} \Psi \partial W - \int_{H_{st}} \Psi \partial W$$

If (1) and (2) were true everywhere it would follow that

$$\phi_{st} = \phi_0 + \int_\Gamma \Psi \partial W \quad \text{i.e} \quad \Psi = \phi'$$

By Green's formula 2.5 we would also have

$$\int_{H_{st}} \phi \partial W = \int_{R_{st}} \phi dW + \int_{R_{st}} \phi dJ = \Phi_{st} - \Phi_0$$

$$\int_{V_{st}} \partial W = \int_{R_{st}} \phi dW + \int_{R_{st}} \Psi dJ = \Phi_{st} - \Phi_0$$

This would prove that $\phi = \Phi'$ and that $\Psi = \Phi''$. The process Φ

then has a second derivative Φ'', and by iteration $\Phi \in H^{\infty}$.

In general we need to correct the processes on sets of measure zero, and to extend the equalities using martingale properties and conditional expectation. The details are the same as in Cairoli & Walsh [1], see p. 174, 178, 179. Since these arguments are technical, and have little to do with the complex aspect of this theory, they are left to the reader.

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