C-valued stochastic integrals in the plane.
by
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Abstract

In this paper we are going to discuss various stochastic integrals over a 2-parameter Wiener process. Our main interest is the relationship between Brownian motion and analytic functions, and we want to demonstrate how complex notation may be used to study these objects.
Introduction

A two parameter Wiener process admits a theory of stochastic line and surface integrals. When the stochastic line-integrals are defined, it is natural to ask what processes have line-integrals independent of the particular path joining the end points.

In the case of a real valued Wiener process $B$, Cairoli & Walsh [1], proved the following.

Theorem (Cairoli & Walsh 1974)
The line integral $\int_{\Gamma} \phi dB$ is independent of the path joining the end points if and only if $\phi_{st} = \sum_{n=0}^{\infty} a_n H_n(B_{st}, s, t)$ with $\sum_{n=0}^{\infty} |a_n|^2 n! t^n < \infty \forall t$,

where $H_n(x,t)$ is the n-th Hermite polynomial.

I think the proof of this is very fascinating. Cairoli & Walsh introduced a whole new theory of stochastic calculus. They proved a stochastic version of Green's theorem connecting line integrals with surface integrals, and used a martingale representation theorem together with a theory of quadratic variation to prove their result.

At the time the connection between Brownian motion and analytic functions was already very apparent, and Yor [2] observed a complex version of the theorem.

Theorem (Yor 1977)

When $W_{st}$ is a $\mathbb{C}$-valued 2-parameter Wiener process, the line integral $\int_{\Gamma} \phi dW$ is independent of the path if and only if $\phi_{st} = \sum_{n=0}^{\infty} a_n (W_{st})^n$ with $\sum_{n=0}^{\infty} |a_n|^2 n! t^n < \infty \forall t$. 
To prove this theorem Yor studied the integrals as real objects and managed to match the real and imaginary part to prove the theorem. The proof of this was efficient and fair enough, but I believe it has some interest to see how this theorem can be proved from a purely complex point of view. In this paper I will explain how to build up the complex objects, and we will see that the proof in the Cairoli & Walsh paper can be carried out directly in this setting.

Acknowledgement

I wish to thank Eugene Wong and John Walsh for private communication on this work.
Some basic definitions and notation

Let $B_{1z}$ and $B_{2z}$ denote two independent real-valued 2-parameter Wiener processes on a probability-space $(\Omega,\mathcal{F},\mathbb{P})$ and put

$$W_z = B_{1z} + iB_{2z} \quad \text{i-imaginary unit, } z = (s,t) \in \mathbb{R}^2$$

We have the order relations

$$(s,t) < (s',t') \text{ iff } s < s' \quad t < t'$$
$$(s,t) \ll (s',t') \text{ iff } s < s' \quad t < t'$$
$$(s,t) \wedge (s',t') \text{ iff } s < s' \quad t > t'$$
$$(s,t) \wedge (s',t') \text{ iff } s < s' \quad t > t'$$

We let $\mathcal{F}_z$ denote the $\sigma$-algebra generated by $\{W_z, z' < z\}$ and we also have the $\sigma$-algebras

$$\mathcal{F}_z^1 = \mathcal{F}_{s=} = \bigvee_v s_v$$
$$\mathcal{F}_z^2 = \mathcal{F}_{t=} = \bigvee_v t_v$$

We say that a stochastic process $X_z$ is a martingale if

$$E[X_z, |\mathcal{F}_z] = X_z \text{ whenever } z < z'$$

For a rectangle $R$ with corners $z_1, z_2, z_3, z_4$ as below, we define

$$\Delta_R \phi = \phi_{z_4} - \phi_{z_3} - \phi_{z_2} + \phi_{z_1} \text{ where } \phi \text{ is a 2-parameter process.}$$

![Rectangle Diagram](image)
ΔR = Area of \( R \) and \( R_z \) denotes the set \( \{ z' \in \mathbb{R}^2 \mid z' < z \} \). For each \( z << z' \) let \( [z, z'] \) denote the rectangle \( (s, s') \times (t, t') \). We say that an adapted integrable process \( X_{st} \) is

a weak martingale iff \( E[\Delta(z, z')X | \mathcal{F}_z] = 0 \) for all \( z << z' \)

an i-martingale iff \( E[\Delta(z, z')X | \mathcal{F}_z^i] = 0 \) for all \( z << z' \), \( i = 1,2 \)

It is convenient to observe that a martingale is both a 1- and a 2-martingale, see Cairoli & Walsh [1] p. 115.

We call a \( \mathbb{C} \)-valued process increasing iff both components are increasing. The joint quadratic variation \( \langle x, y \rangle_{st} \) is any difference of increasing processes s.t. \( x_{st} - y_{st} \) is at least a weak martingale. We also write \( \langle x, x \rangle_{st} = \langle x \rangle_{st} \). A process \( \Phi_{st} \) is said to be adapted measurable whenever

(i) \( \Phi_{st} \) is \( \mathcal{F}_{st} \) - measurable

(ii) \( (s, t, \omega) \rightarrow \Phi_{st}(\omega) \) is \( \mathcal{F} \times \mathcal{B} \) measurable, where \( \mathcal{B} \) is the class of Borel sets on \( \mathbb{R}^2_+ \).
Part 1 - Line integrals

Fixing one parameter in a two parameter process gives a one parameter process. If the process is adapted and reasonably nice, stochastic integrals along straight line segments parallel to the coordinate-axes can be defined in the obvious way.

\[
\begin{align*}
\int_{\Gamma} \phi \circ \omega &= \int_{s_0}^{s_1} \phi \circ u \circ t_0 \, dW \circ t_0 \\
\end{align*}
\]

In case (1) we define

\[
\int_{\Gamma} \phi \circ \omega = \int_{s_0}^{s_1} \phi \circ u \circ t_0 \, dW \circ t_0
\]

where \( dW \circ t_0 \) means integration w.r.t the Brownian motion \( W_{st_0} \). This is well defined as an Ito-integral if \( \phi_{st_0} \) is adapted measurable with \( E|\phi_{st_0}|^2 \) bounded on compact sets.

In case (2) we define

\[
\int_{\Gamma} \phi \circ \omega = -\int_{-\Gamma} \phi \circ \omega
\]

The integrals on vertical paths are defined similarly, and the integral extends immediately to rectangular paths by linearity. It is not hard to see that for a large class of processes, the integrals can be defined along any sufficiently smooth path by approximating the path with rectangular paths. This, however, will be of little importance to us, and we will choose to ignore it, at least for the time being. Our main interest will be with the paths below, which we denote by \( V_{st} \) and \( H_{st} \).
Definition

We call a process $\Phi_{st}$ weakly holomorphic if there exists an adapted measurable process $\Phi'_st$ with $E|\Phi'_st|^2$ bounded on compact sets in $\mathbb{R}^2_+$ and s.t.

$$\Phi_{st} = \Phi_0 + \int \Phi'_st \, dt = \Phi_0 + \int \Phi'_st \, dt \quad \text{for each } (s,t) \in \mathbb{R}^2_+$$

We call $\Phi'_st$ a derivative of $\Phi_{st}$, and write $\Phi_{st} \in H$.

It follows by linearity that $\Phi_{st} = \Phi_0 + \int \Phi'_st \, dt$ where $\Gamma$ is any rectangular path joining $(s,t)$ to $(0,0)$. If $\Phi_{st}$ has derivatives up to order $n$, we say that $\Phi_{st} \in H^n$ i.e. $\Phi_{st} \in H^n$ iff

$$\Phi_{st} = \Phi_0 + \int \Phi'_st \, dt$$

$$\Phi'_st = \Phi'_0 + \int \Phi''_st \, dt$$

$$\vdots$$

$$\Phi^{n-1}_{st} = \Phi^{n-1}_0 + \int \Phi^n_{st} \, dt$$

Before we go on to study the holomorphic processes, we observe that the usual Ito-formula and Ito-isometry applies to each line-segment. i.e.

$$f(Z_{st}) = f(Z_{st}) + \int_{0t}^{st} \frac{\partial f}{\partial Z_{ut}} (Z_{ut}) dZ_{ut} + \int_{0t}^{st} \frac{\partial^2 f}{\partial Z_{ut}^2} (Z_{ut}) dZ_{ut}^2$$

$$+ \int_{0t}^{st} \frac{1}{2} \frac{\partial^2 f}{\partial Z_{ut} \partial Z_{ut}^2} (Z_{ut}) dZ_{ut} dZ_{ut}$$

$$+ \int_{0t}^{st} \frac{1}{2} \frac{\partial^2 f}{\partial Z_{ut} \partial Z_{ut}^2} (Z_{ut}) dZ_{ut} dZ_{ut}$$

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$$+ \int_{0t}^{st} \frac{1}{2} \frac{\partial^2 f}{\partial Z_{ut} \partial Z_{ut}^2} (Z_{ut}) dZ_{ut} dZ_{ut}$$
where we have the formal relations

\[ dW_t \, dw = 0 \]

The Ito-isometry applies in the same way, so

\[ E\left| \int_0^s \phi \, dW_t \right|^2 = 2t \int_0^s E\left| \phi \right|^2 \, du \]

We first note some easy consequences of the Ito-formula.

**Proposition 1.1.**

If \( \phi_{st} = (W_{st})^n \) then \( \phi_{st} \in H^\infty \) and

\[
\begin{align*}
\phi_{st} &= \int nW^{n-1} \, dW \\
\phi'_{st} &= \int n(n-1)W^{n-2} \, dW \\
\vdots \\
\phi^{n-1}_{st} &= \int n! \, dW \\
\phi^n_{st} &= n! \\
\phi^1_{st} &= 0
\end{align*}
\]

**Proof**

By the Ito-formula on \( f(z) = z^n \), \( t \) fixed

\[ f(W_{st}) = \int nW^{n-1} \, dW = \int nW^{n-1} \, dW_{st} \]

Since \( dW_t \, dw = 0 \). The same relation with \( s \) fixed gives

\[ f(W_{st}) = \int nW^{n-1} \, dW = \int nW^{n-1} \, dW_{st} \]

so the derivative of \( W^n_{st} \) is \( nW^{n-1}_{st} \).
Lemma 1.2

\[ E[W_{st}^n - W_{st}^m] = \delta_{nm} \cdot n! \cdot (2st)^n \]

Proof

By Ito's formula on \( f(z) = z^{n-m} \), \( t \) fixed

\[
f(W_{st}) = \int_0^s nW_{ut}^{n-1} \, dW_{ut} + \int_0^s W_{ut} \cdot mW_{ut}^{m-1} \, dW_{ut} + 2t \int_0^s nW_{ut}^{n-1} \, dW_{ut}
\]

so

\[ E[W_{st}^n - W_{st}^m] = 2t \cdot m \cdot n \int_0^s E[W_{ut}^{n-1}W_{ut}^{m-1}] \, du \]

and the lemma follows by iteration.

Proposition 1.1 may be extended to \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) whenever \( f'(W_{st}) \) is adapted measurable with \( E|f'(W_{st})|^2 \) bounded on compact sets. Since the first statement is trivial, we turn to the \( L^2 \)-norm. By lemma 1.2 it is easy to see that

\[ E \left[ \sum_{n=M}^{N} a_n W_{st}^n \right]^2 = \sum_{n=M}^{N} |a_n|^2 \cdot E[W_{st}^n]^2 \cdot n! \cdot (2st)^n \]

If \( \sum_{n=0}^{\infty} |a_n|^2 \cdot n! t^n < \infty \), it will follow immediately that

\[ \sum_{n=0}^{\infty} a_n W_{st}^n \to f(W_{st}) \] in \( L^2 \). The same applies to all the derivatives of \( f \) and we have the following.

Theorem 1.3 (Yor)

If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) with \( \sum_{n=0}^{\infty} |a_n|^2 \cdot n! t^n < \infty \)
Then
\[ f(W_{st}) = f(0) + \int_0^{(s,t)} f'(W) dW \]
where the line integral is independent of the path \( \Gamma \) joining the origin with \((s,t)\). Since the same applies to all the derivatives, \( f(W_{st}) \in H^\infty \).

We will now see how to prove the converse when the process \( \Phi_{st} \in H^\infty \). The proof is a complex version of the proof presented in Cairoli & Walsh [1]. See also Nguyen [3] for local versions of these theorems.

**Proposition 1.4**

If \( \Phi_{st} \) and \( \psi_{st} \in H^{n+1} \) then
\[
E[\Phi_{st} \psi_{st}] = \sum_{j=0}^{n} \frac{(2st)^j}{j!} + (2t)^{n+1} \int_0^s \int_0^t E[\Phi_{ut} \psi_{ut}] duds + \int_0^t \int_0^t E[\Phi_{ut} \psi_{ut}] duds
\]

**Proof**

By Itô's formula on \( f(z,w) = zw \) we get
\[
E[\Phi_{st} \psi_{st}] = \Phi_{00} + 2tE[\Phi_{ut}] du
\]
as in lemma 1.2. Proposition 1.4 then follows by repeated use of this relation.

**Lemma 1.5**

If \( \Phi_{st} \in H^\infty \) then
\[
\lim_{n \to \infty} \frac{(2st)^n}{n!} E|\Phi_{st}^{n+1}|^2 = 0
\]
Proof

Define

\[ g_n(s,t) = E |\phi_{st}^n|^2 \quad n = 0,1,2,\ldots. \]

By proposition 1.4 we have

\[ g_n(s,t) = |\phi_0^n|^2 + 2t \int_0^s g_{n+1}(u,t) du \]

Put \( f_n(x) = g_n(x,1) \), then

\[ f_n'(x) = \frac{\delta g_n}{\delta s}(x,1) = 2g_{n+1}(x,1) = 2f_{n+1}(x) \]

It follows that

\[ f^{(n)}(x) = 2^n f_n(x) \]

By Taylor's formula

\[ f(x) = \sum_{j=0}^{N} f^{(n)}(x_0) \frac{(x-x_0)^n}{n!} + \frac{f^{(N+1)}(\theta)}{(N+1)!} (x-x_0)^{N+1} \]

Since

\[ f^{N+1}(\theta) = 2^{N+1} g_{N+1}(\theta,1) = 2^{N+1} E |\phi_{\theta 1}^{N+1}|^2 > 0 \]

we have

\[ f(2x_0) > \sum_{j=0}^{N} f^{(n)}(x_0) \frac{x_0^n}{n!} \quad \text{for all } N, \text{ all } x_0 > 0 \]

Now \( f(2x_0) = g_0(2x_0,1) = E |\phi_{2x_0}^1|^2 < \infty \). Then the series converges

for all \( x_0 > 0 \) and \( \lim_{n \to \infty} f^{(n)}(x_0) \frac{x_0^n}{n!} = 0 \quad \forall x_0 > 0 \)
It is also true that
\[ \lim_{n \to \infty} f_n(x_0) \frac{(2x_0)^n}{n!} = 0 \quad \forall \ x_0 > 0 \]

It remains to observe that \( E |\Phi_{st}^n|^2 = f_n(s \cdot t) \).

Look at
\[ g_n(s,t) = g_{n0} + 2t \int_0^s g_n(u,t) \, du \]
and the symmetric relation
\[ g_n(s,t) = g_{n0} + t \int_0^t g_{n+1}(s,u) \, du \]

From these one easily sees that
\[ \frac{\delta g_n}{\delta s} = t \frac{\delta g_n}{\delta t} \]

But then
\[ E |\Phi_{st}^n|^2 = g_n(s,t) = g_n(st,1) = f_n(s \cdot t) \]

since \( g_n \) only depends on \((s,t)\) by their product.

**Corollary 1.6**

If \( \Phi_{st} \in H^\infty \) then
\[ E |\Phi_{st}^n|^2 = \sum_{j=0}^{\infty} \frac{(2st)^j}{j!} |\phi_j|^2 \]

**Proof**

By proposition 1.4
\[ E |\Phi_{st}^n|^2 = \sum_{j=0}^{n} \frac{(2st)^j}{j!} |\phi_0|^2 + (2t)^{n+1} \int_0^s \int_0^t \cdots \int_0^s E |\Phi_{ut}^{n+1}|^2 \, du \, ds \]
Since \( \Phi_{st} \in H^\infty \), \( \Phi_{st}^{n+1} \) is a martingale, and \( E|\Phi_{ut}^{n+1}|^2 \) is bounded by
\[
E|\Phi_{st}^{n+1}|^2.
\]
Then the integral term is bounded by \( \frac{(2st)^{n+1}}{(n+1)!} E|\Phi_{st}^{n+1}|^2 \). and this goes to zero by lemma 1.5.

**Proposition 1.7**

If \( \Phi_{st} \in H^{n+1} \) then
\[
E[\Phi_{st}\tilde{W}_st^n] = \Phi_0^n(2st)^n.
\]

**Proof**

Put \( \Phi_{st} = \tilde{W}_st^n \) proposition 1.4. By proposition 1.1 all terms except \( \Phi_{n-n}\frac{(2st)^n}{n!} = \Phi_0^n(2st)^n \) vanish.

**Corollary 1.8**

If \( \Phi \in H^\infty \) with \( E[\Phi\tilde{W}_st^n] = 0 \) for all \( n \), then \( \Phi_{st} \equiv 0 \).

**Proof**

By corollary 1.6 and proposition 1.7
\[
E|\Phi_{st}|^2 = \sum_{j=0}^{\infty} \frac{(2st)^j}{j!} |\Phi_0|^2 = 0.
\]

**Theorem 1.10 (Yor)**

If \( \Phi_{st} \in H^\infty \) then \( \Phi_{st} = f(W_{st}) \) where \( f(z) = \sum_{n=0}^{\infty} \frac{\Phi_0^n}{n!} z^n \) satisfies the conditions in theorem 1.3.
Proof

Put \( f(z) = \sum_{n=0}^{\infty} \frac{\phi_0^n}{n!} z^n \) i.e. \( a_n = \frac{\phi_0^n}{n!} \)

By corollary 1.6

\[
\sum_{n=0}^{\infty} \frac{|a_n|^2 n! t^n}{n!} = \sum_{n=0}^{\infty} \frac{\phi_0^n}{n!} 2^t n! t^n = \sum_{n=0}^{\infty} \frac{(2t \cdot \frac{1}{2})^n}{n!} |\phi_0^n|^2 = E|\phi_1|^2 < \infty
\]

Then \( \psi_{st} = f(W_{st}) \) satisfies the conditions in theorem 1.3. We have \( \psi_{st} \in \mathbb{H}^\infty \) and \( \psi_{0} = \phi_{0}^{n} \) for each \( n \). Since \( \psi_{st} - \psi_{st} \in \mathbb{H}^\infty \) and

\[
E[(\psi_{st} - \psi_{st}) W_{st}^n] = \phi_0^n (2st)^n - \phi_0^n (2st)^n = 0
\]

for each \( n \), \( \Phi_{st} - \psi_{st} = 0 \) by corollary 1.8

Now it only remains to prove that all weakly holomorphic processes are in fact \( \mathbb{H}^\infty \). To prove this we turn our attention to various stochastic integrals.

Part 2 - Stochastic integrals and Green's formula.

We want to define the surface integral \( \int_{R_{st}} \Phi dW \). To define this integral on simple functions, partition \( R_{st} \) into rectangles \( R_{ij} \) with lower left corners \( z_{ij} \) and let the values \( \Phi_{ij} \) on these rectangles be \( \mathcal{G}_{z_{ij}} \)-measurable.

Then

\[
E\left| \sum_{i,j} \Phi_{ij} \Delta_{ij} W \right|^2 = \sum_{i,j} E|\Phi_{ij}|^2 E|\Delta_{ij} W|^2 = \sum_{i,j} E|\Phi_{ij}|^2 \cdot 2 \Delta_{R_{ij}}
\]
Once you have this isometry, the integral can be extended to adapted measurable $\Phi_{st}$ with $E|\Phi_{st}|^2$ bounded on compact sets. This is exactly as in the theory of oneparameter Ito-integrals. If we multiply two such integrals, however, we end up with something new. i.e. let

$$X_{st} = \int \Phi_{st} dW_{R_{st}}$$
$$Y_{st} = \int \Phi_{st} dW_{R_{st}}$$

and look at

$$\sum_{i,j} \Phi_{ij} \Delta_{ij} \int_{k, l} \sum_{k,l} \Phi_{kl} \Delta_{kl} dW_{R_{st}}$$

$$= \sum_{i,j} \left( \sum_{k,l} \Phi_{kl} \Delta_{kl} W_{R_{st}} \Phi_{ij} \right) \Delta_{ij} dW_{R_{st}}$$

$$+ \sum_{k,l} \left( \sum_{i,j} \Phi_{ij} \Delta_{ij} W_{R_{st}} \Phi_{ij} \right) \Delta_{kl} dW_{R_{st}}$$

$$+ \text{remaining terms}$$

In this particular case the $L^2$-norm of the remaining terms can be made uniformly small by choosing the partition fine enough. The terms (1) and (2) can be accounted for as ordinary surface integrals. We cannot, however, include any more terms in these sums as long as we only want to integrate adapted processes. The terms (3) and (4) represents roughly one half on the terms, so they cannot be ignored. At first sight these terms look pretty hopeless, but the particular positioning of the indices turn out to be very convenient. We actually have the following isometry.
If $a_{ijkl}$ is measurable, then

$$E \left| \sum_{i,j,k,l} a_{ijkl} \Delta_{ij} W_{kl} \right|^2 = \sum_{i,j,k,l} E|a_{ijkl}|^2 \Delta R_{ij} \Delta R_{kl}$$

The expression $\sum_{i,j,k,l} a_{ijkl} \Delta_{ij} W_{kl}$ defines the integral $\int a(\xi, \eta) dW_\xi dW_\eta$ on simple four-parameter processes. The isometry above then makes it possible to extend this integral to any four-parameter process $a(\xi, \eta)$ with

(i) $a(\xi, \eta) \in \mathcal{F}_{\xi \vee \eta}$ - measurable

(ii) $a(\xi, \eta) = 0$ unless $\xi \wedge \eta$

(iii) $E|a(\xi, \eta)|^2$ bounded on compact sets in $\mathbb{R}^4$

To account for the term (3) just observe that

$$\sum_{j,k} \sum_{i,j} = \sum_{i,j,k,l}$$

so that this will approximate the term $\int_{\mathbb{R}^4} \Phi \psi dW_\xi dW_\eta$.

The term (4) is accounted for in same way.

The same procedure can be used to define the integrals $\int a dW d\bar{W}$, $\int a d\bar{W} d\bar{W}$ and so on. These are the integrals we will be working with.

$L^2$-martingales

In their paper [4], Wong & Zakai proved that every $L^2$-martingale w.r.t. a 2-parameter Wiener process $B_{st}$, can be represented as a
sum of two stochastic integrals \( \int \Phi dB + \int \Phi dB dB \). This also applies when the \( \sigma \)-algebras are generated by several independent two-parameter Wiener processes. In our case each \( W \)-martingale can be written on the form

\[
\sum_{i=1}^{2} \int_{\mathbb{R}} \Phi_i dB_i + \sum_{i,j} \int_{\mathbb{R} \times \mathbb{R}} \Phi_{ij} dB_i dB_j
\]

If you split the matrices involved into \( \mathbb{C} \)-linear and \( \mathbb{C} \)-antilinear parts, it is easy to see that you have the following representation.

**Theorem 2.1 (Wong & Zakai)**

If \( X \) is a \( \mathbb{C} \)-valued \( L^2 \)-martingale, then

\[
X \equiv X_0 + \int_{\mathbb{R}} \Phi dB + \int_{\mathbb{R} \times \mathbb{R}} \Phi dB dB + \int_{\mathbb{R}} \beta dB dB + \int_{\mathbb{R} \times \mathbb{R}} \gamma dB dB + \int_{\mathbb{R} \times \mathbb{R}} \delta dB dB
\]

It turns out that the terms in this representation are actually orthogonal in \( L^2 \). More exactly we have.

**Proposition 2.2**

Let

\[
I_1 = \int_{\mathbb{R}} \Phi dB, \quad I_2 = \int_{\mathbb{R}} \Phi dB, \quad I_3 = \int_{\mathbb{R} \times \mathbb{R}} \Phi dB dB, \quad I_4 = \int_{\mathbb{R} \times \mathbb{R}} \Phi dB dB,
\]

\[
I_5 = \int_{\mathbb{R} \times \mathbb{R}} \gamma dB dB, \quad I_6 = \int_{\mathbb{R} \times \mathbb{R}} \delta dB dB
\]

Then

\[
E[I_{i} \overline{I}_{i}] = 0 \quad \text{unless} \quad i = j
\]

Also if \( X \equiv X_0 + \int_{T} X' dB \) we get \( E[X_{\mathbb{R}} \overline{I}_{j}] = 0 \) if \( j = 2, 4, 5, 6 \).
Proof
Partition $R_{z_0}$ into rectangles $\Delta_{ij}$ and assume to begin with that all integrands are constant on $\Delta_{ij}$. We first look at

$$E[I_1 I_2] = \sum_{i,j,k,l} E[\Phi_{ij} \Phi_{kl} \Delta_{ij} \Delta_{kl} W W]$$

Here either $\Delta_{ij} W$, $\Delta_{kl} W$ or both are independent from the rest. Since $E[\Delta_{ij} W] = E[\Delta_{ij} W \Delta_{kl} W] = 0$ all terms vanish.

Case two

$$E[I_1 I_3] = \sum_{i,j,k,l,m,n} E[\Phi_{ij} \Phi_{klmn} \Delta_{ij} \Delta_{kl} W W MN]$$

Here either $\Delta_{ij} W$, $\Delta_{kl} W$, $\Delta_{mn} W$ or the pairs $(\Delta_{ij} W \Delta_{kl} W)$, $(\Delta_{ij} W \Delta_{mn} W)$ are independent from the rest. The first three cases are trivial, so let us consider the remaining two. When $(k,l) \wedge (m,n)$ $\Phi_{kl}$ is $\mathcal{F}_{mn}$ measurable and as such independent of $\Delta_{mn} W$. When $\Delta_{ij} W \Delta_{kl} W$ are dependent, but independent from the rest $(i,j) = (k,l)$ and

$$E[\Phi_{ij} \Phi_{klmn} \Delta_{ij} \Delta_{kl} W W MN] = E[\Phi_{kl} \Phi_{klmn}] E[\Delta_{kl} W \Delta_{kl} W] E[\Delta_{mn} W] = 0$$

The first part of the proposition is proved along the same lines and are left to the reader. As for the second part $E[X_{z_0} I_2] = 0$ and $E[X_{z_0} I_6] = 0$ are easy since there are no non-conjugate terms. The two remaining terms require a bit more carefulness. Look at

$$E[X_{z_0} \sum_{i,j,k,l} \Phi_{ijkl} \Delta_{ij} W W \Delta_{kl} W]$$

$(i,j) \wedge (k,l)$
Write
\[ x_{z_0} = x_0 + \int_0^s x'_s \, dW_s = x_0 + \sum_m x'_m (W_{s_{m+1}t_0} - W_{s_mt_0}) \]
where \( s_1, \ldots, s_N \) are the lower \( s \)-coordinates of the partition \( \Delta_{ij} \).

The case with \( x_0 \) is trivial. Consider
\[
\sum_{i,j,k,l} \sum_m \left[ x'_m s_{m+1} t_0 \Delta_{ijkl} (W_{s_{m+1}t_0} - W_{s_mt_0}) \Delta_{kl} \Delta_{kl} \right] \]

The nontrivial case occurs when \( (W_{s_{m+1}t_0} - W_{s_mt_0}) \) and \( \Delta_{ijkl} \) are dependent, but independent from the rest. This only happens when the rectangle \( R_{ij} \) have upper and lower \( s \)-coordinates \( s_{m+1} \) and \( s_m \).
Since \( i < k \), \( X'_m \) is \( \mathcal{F}_{kl} \)-measurable so it is independent from \( \Delta_{kl} \). Then \( \Delta_{kl} \) can also be split out, and we are through.

In general \( X' \) will not be independent from \( \Delta_{kl} \). To prove
\[ E[x_{z_0}^{\bar{1}_{5}}] = 0 \]
you have to use \( x_{z_0} = x_0 + \int_0^{t_0} x'_s \, dW_s \). Then the proof can be carried out along the same lines. The above also explains why you cannot expect to get \( E[x_{z_0}^{\bar{1}_{3}}] = 0 \). In this case you would end up with terms of the form \( E[x'_m s_{m+1} t_0 \Delta_{ijkl} \Delta_{kl}] \) and this may not vanish because \( X'_m s_{m+1} t_0 \) and \( \Delta_{kl} \) may be dependent. E.g. \( X' = \overline{W} \).

**Corollary 2.3**

If \( \Phi_{st} \in H \) then
\[ \Phi_{st} = \Phi_0 + \int R_{st} \Phi dW + \int R_{st} X R_{st} \]
Proof

When $\phi_{st} \in H$, $\phi_{st}$ is clearly an $L^2$-martingale so by theorem 2.1

$$\phi_{st} = \phi_0 + I_1 + I_2 + I_3 + I_4 + I_5 + I_6$$

By proposition 2.2

$$0 = I\phi_{st} - \phi_0 - I_1 - I_2 - I_3 - I_4 - I_5 - I_6 + I_1^2$$

$$= I\phi_{st} - \phi_0 - I_1 - I_3^2 + I_2^2 + I_4^2 + I_5^2 + I_6^2$$

so all the conjugate terms vanish.

The process $J$ and Green's formula.

We define the process $J_z$ by the relation

$$J_z = \int_{R_z \times R_z} \psi dW dW$$

where $\psi(\xi, \eta) = \begin{cases} 1 & \text{if } \xi \wedge \eta \\ 0 & \text{otherwise} \end{cases}$

If you approximate $\psi$ by simple processes and calculate the conditional expectations, if is easy to see that $J_z$ is a martingale with quadratic variation

$$\langle J \rangle_{st} = \int_{R_{st} \times R_{st}} 4|\psi|^2 d\xi d\eta$$

The process $J_z$ gives a connection between surface integrals and line integrals. We first observe the following

Proposition 2.4

$$J_{st} = \int_{H_{st}} W dW - \int_{R_{st}} W dW$$

$$J_{st} = \int_{V_{st}} W dW - \int_{R_{st}} W dW$$
Proof

Partition $R_{s_0 t_0}$ into rectangles $\Delta_{ij} i, j < n$. Then

$$J_{s_0 t_0} = \sum_{i,j,k,l} \Delta_{ij} \Delta_{kl}$$

$$= \sum_{j,k} \sum_{i=1}^{j-1} \Delta_{ij} \Delta_{kl}$$

$$= \sum_{j,k} (w_{kj} - w_{kj+1}) \sum_{i=1}^{j-1} \Delta_{ij} \Delta_{kl}$$

$$= \sum_{j,k} (w_{kj} - w_{kj+1}) \sum_{i=1}^{j-1} \Delta_{ij} \Delta_{kl}$$

The second sum telescopes in $j$ and $k$ and we get

$$= \sum_{j,k} (w_{kj} - w_{kj+1}) \Delta_{kj} \Delta_{kl}$$

$$+ \sum_{j,k} w_{kj} \sum_{i=1}^{j-1} \Delta_{kl} \Delta_{ij}$$

$$- \sum_{j,k} w_{kj} \Delta_{kl} \Delta_{ij}$$

since the first term obviously can be made uniformly small. The proof of the second relation is similar.
A complex version of Greens formula follows almost immediately from proposition 2.4. First you need to observe that it is possible to integrate against \( J_{st} \). The definition on simple processes is of course \( \sum_{i,j} \Phi_{ij} \Delta_{ij} J \) and there is an isometry also in this case. The class of integrable processes against a general \( L^2 \)-martingale \( M_{st} \) depends on the quadratic variation \( <M>_{st} \). In the case of \( J_{st} \), however, the quadratic variation is so small that is suffices to have \( E[\Phi_{st}]^2 \) bounded on compact sets.

Once it is meaningful to integrate against \( J_{st} \) we can state an integration formula.

**Greens formula 2.5**

Assume that \( \Phi'_{st} \) is adapted measurable with \( E|\Phi'_{st}|^2 \) bounded on compact sets. When \( \Phi_{st} = \Phi_0 + \int \Phi' \circ dW \), then for any rectangle \( A \)

\[
\int_{\partial A} \Phi \circ dW = - \int_{V_{st}} \Phi' \circ dW - \int_{A} \Phi' \circ dJ
\]

The integration is counter-clockwise and integration \( \partial_s A \) means that the vertical segments are ignored. With the same conditions on \( \Phi' \), the symmetric relation along the vertical segments is that if

\[
\Phi_{st} = \Phi_0 + \int_{H_{st}} \Phi' \circ dW \text{ then }
\int_{\partial A} \Phi \circ dW = \int_{V_{st}} \Phi' \circ dW + \int_{A} \Phi' \circ dJ
\]

**Proof**

By a standard argument you can reduce to the case where \( \Phi' \) is constant on \( A \). Then the formula follows from proposition 2.4. For details see Cairoli & Walsh [1] p. 151
This Green's formula has two important corollaries.

**Corollary 2.6**

If \( \phi \in H \) then \( \phi \) has a primitive \( \Phi \) s.t

\[
\Phi = \Phi_0 + \int \phi \delta W
\]

**Proof**

By 2.5.

\[
\int_{H_{st}} \phi \delta W = \int_{R_{st}} \phi dW + \int_{R_{st}} \phi dJ = \int_{V_{st}} \phi \delta W
\]

**Corollary 2.7**

If \( \Phi \in H^2 \) then

\[
\Phi_{st} = \Phi_0 + \int_{R_{st}} \phi' dW + \int_{R_{st}} \phi'' dJ
\]

**Proof**

Immediate from 2.5 since

\[
\Phi = \Phi_0 + \int_{R_{st}} \phi' \delta W
\]

Corollary 2.7 is the basic idea to prove that all processes \( \Phi \in H \) are in fact \( H^2 \). From corollary 2.3 we know that

\[
\Phi_{st} = \Phi_0 + \int_{R_{st}} \phi dW + \int_{R_{st}} \alpha dWdW
\]

One can hope to prove that \( \psi = \Phi' \) and that \( \alpha \) represents \( \phi'' \) in some sense. To pursue this further, we need to be able to translate an integral \( \int_{R_{st}} \phi dJ \) to the form \( \int_{R_{st}} \alpha dWdW \).

For this you have the following.
Proposition 2.8.
When $\phi$ is adapted measurable with $E|\phi_{st}|^2$ bounded on compact sets, and

$$\alpha(r,s,t,u) = \begin{cases} \phi_{st} & \text{when } (r,s)^\wedge(t,u) \\ 0 & \text{otherwise} \end{cases}$$

then

$$\int \phi dJ = \int_{R} \int_{R \times R} \alpha dW dW_{st}$$

Proof

Partition $R_{s0}^{t0}$ into rectangles $\Delta_{ij}$, $i,j < n$ and replace $J_{st}$ by its approximation $J_{st}^n$ given by $J_{st}^n = E[\sum_{i,j,k,l} \Delta_{ij} W_{kl}^\Delta | J_{st}]$.

Then the $J_{st}^n$'s are martingales and $J_{st}^n J$ uniformly in $L^2$. When $n$ is fixed according to the partition we have

$$\Delta_{ij} J_{st}^n = \sum_{k,l} \Delta_{kj} W_{il}^\Delta$$

and we get

$$\int \phi dJ = \sum_{i,j} \int_{i-1}^{j-1} \alpha_{ij} \Delta_{ij} J_{st}^n$$

$$= \sum_{i,j,k,l} \phi_{ij} \Delta_{ij} W_{il}^\Delta$$

$$= \sum_{i,j,k,l} \alpha_{ij} \Delta_{ij} W_{il}^\Delta$$

$$= \sum_{i,j,k,l} \alpha_{ij} \Delta_{ij} W_{kl}^\Delta$$

$$= \sum_{i,j,k,l} \alpha_{ij} \Delta_{ij} W_{kl}^\Delta$$

$$= \int \int_{R \times R} \alpha dW dW_{st}$$
Part 3 Weakly holomorphic processes are $H^\infty$

When the processes $X_{st} = \int_{st} \phi dW$ and $Y_{st} = \int_{st} dWdW$ are restricted to the straight line-segments $V_{st}$ and $H_{st}$, they become 1-parameter martingales w.r.t. the $\sigma$-algebras $\mathcal{F}_{st}$ and $\mathcal{F}_{st}$. As such they have unique quadratic variations along these line segments. We denote the variations by $\langle V \rangle_{st}$, $\langle H \rangle_{st}$ and so on. It is fairly straightforward to generalize the quadratic variations from the real case, see Cairoli & Walsh [1] p. 158. The result is the following (we omit the proof).

**Proposition 3.1**

Let $X_{st} = \int_{st} \phi dW$ and $Y_{st} = \int_{st} dWdW$.

Then

$$\langle X^H, X^H \rangle_{st} = \int_{st} 2 \phi^2 d\xi, \quad \langle X^H \rangle_{st} = \int_{st} 2 |\phi|^2 d\xi$$

$$\langle X^H, Y^H \rangle_{st} = \int_{st} 2 \phi(\xi) \int_{\eta=\xi} dW_{\eta} d\xi$$

$$\langle Y^H, Y^H \rangle_{st} = \int_{st} \psi(\xi) \int_{\eta=\xi} d\eta \quad \psi(\eta', \xi) dW_{\eta}, d\xi$$

$$\langle Y^H \rangle_{st} = \int_{st} 2 |\phi|^2 d\xi$$

When $\phi_{st} \in H$ we have

$$\int_{st} \phi dW = \int_{st} \phi dW + \int_{st} \alpha dWdW = \int_{st} \phi' dW$$

Cairoli & Walsh observed that an equality of the above type cannot hold unless $\phi$ and $\alpha$ are intimately related. More exactly we have
Proposition 3.2

If \( \phi \in \mathcal{H} \) then for \((s,t)\) outside a negligible set \( G \subset \mathbb{R}^2 \) the functions

\[ \tau + a(\sigma, t, s, \tau) \] is a.s. essentially constant in \([0, t]\) for a.e. \( \sigma \)
\[ \sigma + a(\sigma, t, s, \tau) \] is a.s. essentially constant in \([0, s]\) for a.e. \( \tau \)

Moreover for \((s, \tau)\) and \((s, \tau')\) outside a negligible set \( F \) and s.t. \( \tau < \tau' < t \) we have

\[ \psi_{st} - \psi_{st'} = \int_{R_{st}} a(u, v, s, \tau') - a(u, v, s, \tau)dW_{uv} \]

and for \((\sigma, t)\), \((\sigma', t)\) outside a negligible set

\[ \phi_{st} - \phi_{st'} = \int_{R_{st}} a(\sigma, t, u, v) - a(\sigma', t, u, v)dW_{uv} \]

Proof

Since

\[ \phi_{st} = \int_{H_{st}} \Phi_{0} \, dW_{st} = \int_{H_{st}} \Phi'_{0} \, dW_{st} \]

we get from Itô's formula that

\[ <\psi^H>_{st_0} = \int_0^s 2t_0 |\Phi'_{ut_0}|^2 \, du \]

and

\[ <\psi^H, \psi^H>_{st_0} = \int_0^s 2t_0 \Phi'_{ut_0} \, du \]

By the first relation we see that

\[ \Phi'_{st_0} = \frac{1}{2t_0} \frac{\phi^H_{st_0}}{0s} <\psi^H, \psi^H>_{st_0} \] for a.e.s

When we insert this in the first relation, we have
\( (*) \quad \langle \phi^H \rangle_{st_0} = \int_0^s \frac{1}{2t_0} \left| \frac{\partial}{\partial u} \langle \phi^H, W^H \rangle_{ut_0} \right|^2 \, du \)

Since
\[
\phi_{st} = \int_{R_{st}} \phi \, dW + \int_{R_{st}} \alpha \, dW \, dW \quad \text{and} \quad W_{st} = \int_{R_{st}} 1 \, dW
\]
we may also compute the quadratic variations from proposition 3.1 i.e.
\[
\langle \phi^H, W^H \rangle_{st_0} = \int_0^s 2 \psi_{uv} + \int_{R_{st}} 2 \alpha(\eta, u, v) \, dW_{\eta} \, dv \, du
\]

Then for a.e.s
\[
\frac{\partial}{\partial s} \langle \phi^H, W^H \rangle_{st_0} = \int_0^t 2 \psi_{sv} + \int_{R_{st}} \alpha(\eta, u, v) \, dW_{\eta} \, dv
\]

The same argument also gives
\[
\langle \phi^H \rangle_{st_0} = 2 \int_{R_{st}} \psi_{uv} + \int_{R_{st}} \alpha(\eta, u, v) \, dW_{\eta} \, dv \left| 2 \, du \right.
\]

When we insert these expressions in (\( *) \) we get
\[
\int_0^t 2 \int_{R_{st}} \psi_{uv} + \int_{R_{st}} \alpha(\eta, u, v) \, dW_{\eta} \, dv \left| 2 \, du \right. = 0
\]

So for a.e.s we must have
\[
\int_0^t \psi_{sv} + \int_{R_{st}} \alpha(\eta, s, v) \, dW_{\eta} \, dv \left| 2 \, dv \right. = \frac{1}{t-0} \int_0^t \psi_{sv} + \int_{R_{st}} \alpha(\eta, s, v) \, dW_{\eta} \, dv \left| 2 \, dv \right.
\]
But by the Cauchy-Schwarz inequality it is easy to see that this can only happen whenever the integral is a.s. constant in \( v \). We get that for a.e. \( s \) there is a random variable \( \rho(s) \) s.t. for a.e. \( v < t \)

\[
\rho(s) = \psi_{sv} + \int_{R_{st_0}} a(\eta, s, v) dW
\]

We can also choose \( \rho \) measurably by averageing. Then outside a negligible set \( F \) with \( \tau < \tau' < t_0 \)

\[
\psi_{st} - \psi_{st'} = \int_{R_{st_0}} a(\eta, s, \tau') - a(\eta, s, \tau) dW
\]

Since the left-hand side is \( \mathcal{F}_{\omega_{\tau'}} \) measurable it is easy to see that we must have \( a(\eta, s, \tau') = a(\eta, s, \tau) \) for a.e. \( \eta \in R_{st_0} - R_{\omega_{\tau}} \).

When this is applied to all possible pairs \((\tau, \tau')\) we get a process

\[
\Psi(u, v, s) \text{ s.t.} \quad \Psi(u, v, s) = a(u, v, s, \tau) \text{ for a.e. } \tau < v.
\]

The second pair of relations is proved along the same lines.

Let us for a moment forget about the negligible sets in proposition 3.2, and let us see what we would have if the relations were true everywhere.

When \( \Phi \in H \) the functions

\[
\tau \to a(\sigma, t, s, \tau) \\
\sigma \to a(\sigma, t, s, \tau)
\]

are essentially constant and \( a \) will essentially satisfy the conditions in proposition 2.8. When we define \( \Psi(s, t) = a(0, t, s, 0) \) we get
We then have
\[ \phi_{st} = \phi_0 + \int \varphi_{dw} + \int \psi_{st} \]

Essentially we also have
\[ \psi_{st} = \psi_0 + \int \psi_{dw} + \int \psi_{st} \]

since everything else vanish outside this set. But we also know that \( a \) essentially doesn't depend to \( u \). i.e. \( a(u,v,s,t) = \psi(s,v) \).

Then we get
\[ (1) \quad \psi_{st}' - \psi_{st} = \int a(\eta,s,t) - a(\eta,s,t') dW_{\eta} \text{ if } t < t' \]
\[ = \int a(u,v,s,t) dW_{uv} \]

By the same way of reasoning we also get
\[ (2) \quad \psi_{st}' - \psi_{st} = \int \psi_{st} dW_{st} - \int \psi_{st} dW_{st} \]

If (1) and (2) were true everywhere it would follow that
\[ \psi_{st} = \psi_0 + \int \psi_{st} \text{ i.e. } \psi = \psi' \]

By Green's formula 2.5 we would also have
\[ \int \psi_{st} dW = \int \psi_{st} dW + \int \psi_{st} dJ = \psi_{st} - \psi_0 \]
\[ \int \psi_{st} dW = \int \psi_{st} dW + \int \psi_{st} dJ = \psi_{st} - \psi_0 \]

This would prove that \( \psi = \psi' \) and that \( \psi = \psi'' \). The process \( \psi \)
then has a second derivative $\phi''$, and by iteration $\phi \in H^\infty$.

In general we need to correct the processes on sets of measure zero, and to extend the equalities using martingale properties and conditional expectation. The details are the same as in Cairoli & Walsh [1], see p. 174, 178, 179. Since these arguments are technical, and have little to do with the complex aspect of this theory, they are left to the reader.
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