Growth properties of the rational solutions of the second order algebraic differential equation

by

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Abstract

The conditions of the exestence and growth properties of the rational solutions of the second order algebraic differential equation

$$\sum_{i=0}^{N} B_{\mu_i}(z) w^{\nu_i} (w')^{\theta_i} (w'')^{\psi_i} = 0$$

where $\mu_i, \nu_i, \theta_i, \psi_i \in \mathbb{N}$ are received.

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We investigate the equation

$$\sum_{i=0}^{N} B_{\mu_i}(z) w^{\nu_i} (w')^{\theta_i} (w'')^{\psi_i} = 0, \tag{1}$$

where μ_i, ν_i, θ_i and ψ_i are integer nonnegative numbers,

$$\sum_{i=0}^{N} (\nu_i^2 + \theta_i^2) \neq 0, \quad \sum_{i=0}^{N} (\nu_i^2 + \psi_i^2) \neq 0, \quad \sum_{i=0}^{N} (\theta_i^2 + \psi_i^2) \neq 0,$$

$$\forall j, i \quad \text{such that} \quad i, j = 0, N, \quad (\nu_i - \nu_j)^2 + (\theta_i - \theta_j)^2 + (\psi_i - \psi_j)^2 \neq 0.$$

$B_{\mu_i}(z)$ are polynomials of a complex variable z, and have the following representation $\beta_{\mu_i}(z) = \beta_i z^{b_i} + \dots$ where $0 \neq \beta_i$, $b_i \in \mathbb{C}$. Following [1,2] we introduce such characteristics of the equation (1) as the dimension of the i-th term of an equation concerning derivatives $\eta_i = \theta_i + \psi_i$, the complete dimension of the i-th term $x_i = \theta_i + \psi_i + \nu_i$, the weight of the i-th term $\Phi_i = \theta_i + 2\psi_i$. The numbers θ_i and ψ_i determine the dimension of the i-th term of the equation (1) concerning the first and the second derivatives accordingly.

We will investigate properties of the rational solution

$$w = \frac{P(z)}{Q(z)} \tag{2}$$

as $z \to \infty$. We propose $P(z) = \alpha_p z^p + \dots + \alpha_0$, $\quad Q(z) = \gamma_q z^q + \dots + \gamma_0$, $\quad \alpha_p \gamma_q \neq 0$, $\quad q \geq 1$ and $P(z)$ and $Q(z)$ are relatively prime.

The rational function (2) is the solution of the equation (1) if and only if

$$\sum_{i=0}^{N} B_{\mu_i}(z) Q^{\tilde{n} - n_i}(z) P^{\nu_i}(z) \tilde{Q}^{\theta_i}(z) \tilde{P}^{\psi_i}(z) \equiv 0, \tag{3}$$

where $\tilde{Q}(z) = P'(z)Q - P(z)Q'(z)$, $\tilde{P}(z) = P''(z)Q^2 - P(z)Q(z)Q''(z) - 2P'(z)Q(z)Q'(z) + 2P(z)(Q')^2(z)$, $\quad n_i = x_i + \Phi_i$, $\quad \tilde{n} = \max_{i=\overline{0,N}}\{n_i\}$.

If $q \neq p$ then $\tilde{Q}(z) = (p-q)\alpha_p \gamma_q z^{p+q-1} + \dots$; if $q = p$, $\Delta_\phi = 0$, $\Delta_k \neq 0$, $\phi = \overline{1, k-1}$, $1 \leq k \leq p$, where

$$\Delta_j = \begin{vmatrix} \alpha_p & \alpha_{p-j} \\ \gamma_p & \gamma_{p-j} \end{vmatrix},$$

and we then have $Q(z) = k\Delta_k z^{2p-(k+1)}$.

If $q \neq p$, $q \neq p-1$ then $\tilde{P}(z) = (p-q)(p-q-1)\alpha_p \gamma_q z \cdot z^{p+2q-2} + \dots$; if $q = p$, $\Delta_\phi = 0$, $\Delta_k \neq 0$, $\phi = \overline{1, k-1}$, $1 \leq k \leq p$, then $\tilde{P}(z) = -k(k+1)\gamma_p \Delta_k z^{3p-(k+2)} + \dots$; if $q = p-1$, $\Lambda_X = 0$, $\Lambda_l \neq 0$, $X = 1, \overline{l-1}$, $1 \leq l \leq p-1$, where

$$\Lambda_j = \begin{vmatrix} \alpha_p & \alpha_{p-1} & \alpha_{p-(j+1)} \\ 0 & \gamma_{p-1} & \gamma_{p-(j+1)} \\ \gamma_{p-1} & \gamma_{p-2} & \gamma_{p-(j+2)} \end{vmatrix}, \; j = \overline{1, p-2}$$

and

$$\Lambda_{p-1} = \begin{vmatrix} \alpha_p & \alpha_{p-1} & \alpha_0 \\ 0 & \gamma_{p-1} & \gamma_0 \\ \gamma_{p-1} & \gamma_{p-2} & 0 \end{vmatrix},$$

we then have $\widetilde{P}(z) = -l(l+1)\Delta_l z^{3p-(l+5)} + ...$

The polynomial $k_i(p, q, \alpha_p, \gamma_q) z^{s_i(p,q)} + ...$, where

$$s_i(p,q) = \begin{cases} x_i(p-q) + \widetilde{n}q + b_i - \Phi_i & \text{, if } q \neq p,\ q \neq p-1; \\ \widetilde{n}p - \eta_i k + b_i - \Phi_i & \text{, if } q = p; \\ \widetilde{n}(p-1) - \psi_i(l+1) + b_i - \Phi_i + x_i & \text{, if } q = p-1 \end{cases}$$

corresponds to the i-th term in the indentical equality (3).

Let

$$K(p, q, \alpha_p, \gamma_q) = \begin{cases} (p-q)^{\eta_i}(p-q-1)^{\psi_i}\beta_i(\alpha_p/\gamma_q)^{x_i}\gamma_q^{\widetilde{n}} & \text{, if } q \neq p,\ q \neq p-1; \\ (-1)^{\psi_i}\beta_i(\alpha_p/\gamma_p)^{\nu_i}\gamma_p^{\widetilde{n}-2\eta_i}k^{\eta_i}(k+1)^{\psi_i}\Delta_k^{\eta_i} & \text{, if } q = p; \\ (-1)^{\psi_i}\beta_i(\alpha_p/\gamma_{p-1})^{x_i-\psi_i}\gamma_{p-1}^{\widetilde{n}-3\psi_i}(l(l+1)\Lambda_l)^{\psi_i} & \text{, if } q = p-1; \end{cases}$$

where the number k is determened by the conditions $\Delta_\phi = 0$, $\Delta_k \neq 0$, $\phi = \overline{1, k-1}$, $1 \leq k \leq p$ and the number l is determened by conditions $\Delta_X = 0$, $\Delta_l \neq 0$, $X = \overline{1, l-1}$, $1 \leq l \leq p-1$.

We will call s_i the function of degree and the function k_i the function of the coefficient of the i-th term of the equation (1).

Lemma. The algebraic differential equation (1) may have rational solutions (2) of the degree (p, q), if functions of degree of two (at least) terms are equal in the point (p, q), and their value is maximal among the values of all functions of degree in this point.

Proof ad absurdem. Let us suppose that the value of only one term is maximal around the values of all functions of degree in this point. In this case the equality (3) is identical only under condition k_j equal to zero. The last situation is inpossible in accordance with (2) and (3).

The rational solution (2) of the equation (1) is called rational solution with special degree (p, q) if we have

$$x_{\tau_0} = x_{\tau_1} = \ldots = x_{\tau_h}, \quad 1 \le h \le N,$$
$$s_{\tau_0}(p,q) = s_{\tau_1}(p,q) = \ldots = s_{\tau_f}(p,q) > s_{\tau_\lambda}(p,q),\ 1 \le f \le h,\ \lambda = \overline{f+1,N} \tag{4A}$$

under conditions $q \neq p$, $q \neq p-1$;

$$\eta_{\tau_0} = \eta_{\tau_1} = \ldots = \eta_{\tau_h}, \quad 1 \le h \le N,$$
$$s_{\tau_0}(p,p,k) = s_{\tau_1}(p,p,k) = \ldots = s_{\tau_f}(p,p,k) > s_{\tau_\lambda}(p,p,k), 1 \le f \le h, \quad \lambda = \overline{f+1,N} \tag{4B}$$

under conditions $q = p$, $\quad \Delta_\phi = 0$, $\quad \Delta_k \neq 0$, $\quad \theta = \overline{1,k-1}$, $\quad 1 \le k \le p$;

$$\psi_{\tau_0} = \psi_{\tau_1} = \ldots = \psi_{\tau_h}, \quad 1 \le h \le N,$$
$$s_{\tau_0}(p,p-1,l) = s_{\tau_1}(p,p-1,l) = \ldots = s_{\tau_f} > s_{\tau_\lambda}(p,p-1,l), \tag{4C}$$
$$1 \le f \le h, \quad \overline{\lambda = f+1,N}$$

under conditions $q = p-1$, $\quad \Lambda_X = 0$, $\quad \Lambda_l \neq 0$, $\quad X = \overline{1,l-1}$, $\quad 1 \le l \le p-1$.

Having seen all possible equalities of the values of the functions of degree with different complete dimensions ($x_i \neq x_j$) we deduce the following results.

Theorem 1. The differences $p-q$ determining the growth of solution (2) of the equation (1) with nonspecial degrees (p,q), $p \neq q$, $\quad q \neq p-1$ belong to the set

$$\{((b_i - \Phi_i) - (b_j - \Phi_j))/(x_j - x_i)\} \tag{5}$$

where $i = \overline{0,N}$, $\quad j = \overline{0,N}$, $\quad i \neq j$. This set is complete under the base of all terms of the equation (1) with nonequal complete dimensions i.e. $x_i \neq x_j \, \forall \, i = \overline{1,N}$, $j = \overline{1,N}$; only integer numbers (except 1 and 0) belong to the set (5).

We have used only the fact that equation (1) must have at least two terms with equal values of functions of degree in a point (p,q). Taking into acount the possibility of the existence of two terms with equal and *maximal* values of the function of degree in a point, we obtain.

Theorem 2. The numbers of the set (5) can determine the growth of the solution (2) of the equation (1) with nonspecific degrees (p,q), where $q \neq p$ and $q \neq p-1$ only under the following conditions

a) there exist $f+1$ terms on the left hand side of the equation (1) such that

$$s_{\tau_0}(p,q) = s_{\tau_1}(p,q) = \ldots = s_{\tau_f}(p,q) > s_{\tau_\lambda}(p,q), \quad \text{where} \quad \lambda = \overline{f+1,N}; \tag{6}$$

b) there exist $n \in \{0,1,\ldots,f\}$, $\quad m \in \{0,1,\ldots,f\}$, such as $x_{\tau_n} \neq x_{\tau_m}$ by $m \neq n$.

So, we have to do the following steps to determine the growth property $p-q$ of rational solutions (2) of equation (1) with nonspecific degrees (p,q) (where $q \neq p$, $q = p-1$): a) to find the set (5); 2) to select the numbers satisfying relation (6) out of the set (5).

The rational solutions of specific degrees (p,q) (where $q \neq p$, $\quad q \neq p-1$) are previded by terms with equal complete dimensions, and there exists at least two numbers among them with equal values. It means that we have the following.

Theorem 3. The equation (1) can have rational solutions (2) of specific degree (p,q) (where $q \neq p$, $\quad q = p-1$) if the equation has a number of terms (block r) with the property

$$\begin{aligned} & x_{\tau_0^r} = x_{\tau_1^r} = \ldots = x_{\tau_{h_r}^r} \neq x_{\tau_{\xi_r}}, \quad \text{where} \quad 1 \leq h_r \leq N, \quad \xi_r = \overline{h_r+1, N}, \\ & b_{\tau_0^r} - \Phi_{\tau_0^r} = b_{\tau_1^r} - \Phi_{\tau_1^r} = \ldots = b_{\tau_{f_r}^r} - \Phi_{\tau_{f_r}^r} > b_{\tau_{\delta_r}^r} - \Phi_{\tau_{\delta_r}^r}, \end{aligned} \tag{7}$$

where $1 \leq f_r \leq h_r$, $\quad \delta_r = \overline{f_r+1, h_r}$.

Spesific degrees (p,q) provided by (7) must satisfy the equation

$$\sum_{\lambda_r=0}^{f_r} k_{\tau_{\lambda_r}}(p,q,\alpha_p,\gamma_p) = 0$$

wich is reduced to the form

$$\sum_{\lambda_r=0}^{f_r} \beta_i (p-q)^{\eta_i} (p-q-1)^{\psi_i} = 0. \tag{8}$$

Theorem 4. Differences $p-q$ determining the growth of all possible solutions (2) with specific degrees (p,q) (where $q \neq p$, $\quad q = p-1$) complying with (7) belong to the set of integer roots of the equation (8) not equal to zero and one.

At least two terms of equation (1) determining the degree of the rational solutions have equal and maximal values of the functions of degree. Taking into acount this observation we have

Theorem 5. An integes root $p-q$ of equation (8) not equal to zero and one when $r = s$ can determine the growth of rational solutions (2) of the equation (1) with nonspecific degrees (p,q) (where $q \neq p$, $\quad q = p-1$) only under conditions

$$s_{\tau_{f_s}}(p,q) > s_{\tau_{\xi_r}}(p,q), \quad \text{where} \quad \xi_r = \overline{h_r+1, N}. \tag{9}$$

Rational solutions (2) with nonspecific degrees (p,q) are only appearing under the base of terms of the equation (1) with nonequal dimension under derivatives. According to the lemma, at least two terms of this equation have the functions of degree with equal and maximal values. Hence, we deduce.

Theorem 6. Equation (1) can have rational solution (2) with nonspecific degrees (p,q), if the set

$$\{((b_i - \psi_i) - (b_j - \psi_j))/(\eta_i - \eta_j)\}, \tag{10}$$

where $i \neq j, \quad i \in \{0,1,...,N\}, \quad j = \{0,1,...,N\}$ and $\eta_i \neq \eta_j$ by $i \neq j$, has a natural $k \leq p$, satisfying the relation

$$s_{\tau_0}(p,p;k) = s_{\tau_1}(p,p,k) = ... = s_{\tau_f}(p,p;k) > s_{\tau_\lambda}(p,p;k), \tag{11}$$

where $\lambda = \overline{f+1,N}$. This relation is used for $f+1$ number of such terms of the equation (1) two of them (at least) have different dimensions under derivatives.

Specific degrees (p,p) of rational solutions (2) satisfy relations, wich are like (4B). Using the lemma we have

Theorem 7. Equation (1) can have rational solutions (2) with specific degrees (p,p) under the following conditions: a) there exist a block r of terms with equal dimensions under derivatives; b) for each block r there exists a natural number $k_r \leq p$ such, that the following relation take place

$$\begin{aligned} &s_{\tau_0^r}(p,p;k_r) = s_{\tau_1^r}(p,p,k_r) = ... = s_{\tau_{f_r}^r}(p,p;k_r) > \\ &> s_{\tau_{\lambda_r}}(p,p,k_r), \quad \text{where} \quad \lambda_r = \overline{0,N}, \quad \lambda_r \neq \tau_\alpha^r, \quad \alpha = \overline{0,f_r}, \quad 1 \leq f_r \leq h_r. \end{aligned} \tag{12}$$

Rational solutions (2) with nonspecific degrees $(p,p-1)$ are found only under base on the terms of equation (1) with nonequal dimensions ψ_i under second derivative, and according to the lemma at least two terms have equal and maximal values of the functions of degree. We have the following

Theorem 8. Equation (1) can have rational solutions (2) with nonspecific terms $(p,p-1)$ if the set

$$\{((b_i + \nu_i - 2\psi_i) - (b_j + \nu_j - 2\psi_j))/(\psi_i - \psi_j\} \tag{13}$$

where $i \neq j, \quad i \in \{0,1,...,N\}, \quad j = \{0,1,...,N\}$ consfructed under base of the terms of the equation (1) with nonequal dimensions under the second derivative has a natural number $l \leq p-1$ such, as

$$s_{\tau_0}(p,p-1;l) = s_{\tau_1}(p,p-1;l) = ... = s(p,p-1;f) > s_{\tau_\lambda}(p,p-1;l) \tag{14}$$

where $\lambda = \overline{f+1,N}$. This relation $f+1$ number of such terms of the equation (1) two of them (at least) have different dimensions under the second derivative.

Specific degrees $(p,p-1)$ of rational solutions (2) satisfy relations wich are like (4B). Using the lemma we have

Theorem 9. Equation (1) can have rational solutions (2) with specific degrees $(p,p-1)$ under the following conditions: a) there exists a block r the terms with equal dimension under the second derivative; b) for each block r exists a natural number $l_r \leq p$ such, that the following relations take place.

$$\begin{aligned} &s_{\tau_0^r}(p,p-1;l_r) = s_{\tau_1^r}(p,p-1;l_r) = \ldots = s_{\tau_{f_r}^r}(p,p-1;l_r) > \\ &> s_{\tau_{\lambda_r}}(p,p-1,l_r), \quad \text{where} \quad \lambda_r = \overline{0,N}, \quad \lambda_r \neq \tau_\alpha^r, \; \alpha = \overline{0,f_r}, \; 1 \leq f_r \leq h_r. \end{aligned} \tag{15}$$

We can deduce from theorems 1, 2, 6 and 7 the well-known properties of Painleve equations, for example about not existence of rational solutions of the first painleve equation [3,4]. $w'' - 6w^2 - z = 0$. The next result obviously follows from these theorems: the second Painleve equation $w'' - 2w^3 - zw + \alpha = 0$ has rational solutions of the degree $(p, p+1)$ by $\alpha \neq 0$.

To complete the investigations of A. Yablonskii, N. Lukashevich, V. Gromak, V. Tsegelnik we can formulate the following results:

The third Painleve equation

$$zww'' - z(w')^2 + ww' - \gamma zw^4 - \alpha w^3 - \beta w - \delta z = 0$$

has a rational solution (2) of degree (p,q) when
a) $\beta = \delta = 0, \quad |\alpha| + |\gamma| = 0$, in this case $p - q \leq 1$;
b) $\alpha = \gamma = 0$, in this case $p - q \geq 1$ and $p - q > 1$, if $|\beta| + |\delta| \neq 0$;
c) $\alpha = \beta = \gamma = \delta = 0$ and $(\gamma = \delta = 0, \quad \alpha\beta \neq 0)$ or $\gamma\delta = 0$, in this case $p = q$.

The fourth Painleve equation

$$2ww'' - (w')^2 - 3w^4 + 8zw^3 - 4(z^2 - \alpha)w^2 - 2\beta = 0$$

has rational solutions of degrees $(p; p+1)$ and $(p, p-1)$. It should be noted, that a rational solution of degree $(p; p+1)$ exists only when $\beta \neq 0$.

The fifth Painleve equation

$$\begin{aligned} &2z^2w^2w'' - 2z^2ww'' - 3z^2w(w')^2 + z^2(w')^2 + 2zw^2w' - \\ &- 2zww' - 2\alpha w^5 + 6\alpha w^4 - 2(\delta z^2 + \gamma z + (3\alpha + \beta))w^3 - \\ &- 2(\delta z^2 - \gamma z - (\alpha + 3\beta))w^2 - 6\beta w + 2\gamma = 0 \end{aligned}$$

has the rational solutions of degree (p,q) when

a) $\delta\beta \neq 0$, in this case $p - q = -1$;
b) $\alpha = \gamma = \delta = 0, \quad \beta \neq 0, \quad \beta \neq -1/2$, in this case $p - q = \sqrt{-2\beta}$;
c) $\beta = \gamma = \delta = 0, \quad \alpha \neq 0$, in this case $p - q = -\sqrt{2\alpha}$;
d) $\alpha = \beta = \gamma = \delta = 0, \quad (|\gamma| + |\delta| \neq 0)$ or $(\gamma = \delta = 0, \quad \alpha \neq 0)$ or $(\gamma = \delta = 0, \beta \neq 0)$, in this case $q = p$;
e) $(\alpha\delta \neq 0)$ or $(\alpha = \delta = \gamma = 0)$, in this case $p - q = 1$.

The sixth Painleve equation

$$
\begin{aligned}
&2(z^4-2z^3+z^2)w^3w''-2(z^5-z^4-z^3+z^2)w^2w''+2(z^5-2z^4+z^3)ww''-\\
&-3(z^4-2z^3+z^2)w^2(w'^2)+2(z^5-z^4-z^3+z^2)w(w')^2-(z^5-2z^4+z^3)(w')^2+\\
&+2(2z^3-3z^2+z)w^3w'-2(z^4+z^3-3z^2+z)w^2w'+2(z^4-z^3)ww'-2\alpha w^6+\\
&4\alpha(z+1)w^5-2((\alpha+\delta)z^2+(4\alpha+\beta+\gamma-\delta)z+(\alpha-\gamma))w^4+4((\alpha+\delta+\gamma+\beta)z^2+\\
&+(\alpha+\beta-\gamma-\delta)z)w^3-2((\beta+\gamma)z^3+(\alpha+4\beta-\gamma+\delta)z^2+(\beta-\delta)z)w^3+\\
&4\beta(z^3+z^2)w-2\beta z^3=0
\end{aligned}
$$

has rational solutions of degree (p,q) when

a) $\alpha=0$, in this case $p-q=1+\sqrt{1-2\delta}$;
b) $\beta=0, \quad \gamma\neq 0$, in this case $p-q=-\sqrt{2\gamma}$;
c) $\alpha=\beta=\gamma=\delta=0$ or $\beta\neq 0$ or $(\beta=\gamma=0, \quad \alpha+\delta\neq 0)$ or $(\beta=\gamma=0, \quad \alpha=-\delta\neq 0)$, in this case $p=q$;
d) for all $\alpha,\beta,\gamma,\delta$, in this case $p-q=1$.

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