AN APPLICATION OF REFLECTED DIFFUSIONS TO THE PROBLEM OF CHOOSING BETWEEN HYDRO AND THERMAL POWER GENERATION

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September 14, 1989

Abstract

It is shown that a certain type of stochastic control problems has a solution (optimal stochastic process) which can be realized as a diffusion with vertical reflection on the boundary of a planar set. The stochastic control problem is motivated by the specific question whether further expansion of the electricity supply system should be based on thermal power (where only fuel costs are taken into account) or on hydro power (where only the initial construction costs are considered), given a stochastic demand.

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†Acknowledgement: The work of B. Øksendal was partially supported by NAVF, Norway.
1 Introduction. Statement of the problem.

The purpose of this paper is to show how diffusions with a certain reflection at the boundary of a (planar) domain arise naturally in a type of stochastic control problems. Moreover, we will use the theory of such diffusions to solve a specific problem of choosing between hydro and thermal power generation under uncertainty.

The background for the interest in this problem is the following:

Norway has abundant energy supplies consisting of hydro power, crude oil and natural gas and is one of the few countries in the world in which more than 99% of the electricity supply is based on hydro power. This reflects that until recently hydro power has been the cheapest source for covering an increasing demand for energy. However, the recent price drop of crude oil plus the fact that the remaining water falls are increasingly expensive to exploit for hydro power generation have changed this situation. Thus the following question has been actualized: Should further expansion of the electricity supply system in Norway be based on thermal power generation or on still unexploited hydro sources?

In our mathematical model of this situation we stylize the difference between the two power sources by representing the cost of hydro power as an everlasting capital, while only the fuel cost of thermal power is taken into consideration. With a constant or deterministic demand function this would be a basically very simple question of marginal cost comparison, but we will consider the more realistic - and intricate - case when the demand is stochastic. Intuitively, the more uncertain the future demand of electricity is, the more careful one should be with the expansion of the everlasting hydro power. We will show that this is indeed the case and we will find explicitly what the optimal choice is at any moment.

We now explain our mathematical model in detail.

We assume that the demand $D_t$ of electricity at time $t$ is a stochastic process of the form

$$D_t = P_t^{-\varepsilon} \cdot \Theta_t,$$

where $\varepsilon > 0$ is some fixed constant, $P_t$ is the (stochastic) price of electricity and $\Theta_t$ is the general "buying power" of the population, taking care of the income effect and other factors which influence demand. We assume that $\Theta_t$ has the structure of a geometric Brownian motion, i.e. it is the solution of a stochastic differential equation of the form

$$d\Theta_t = \alpha \Theta_t dt + \beta \Theta_t dB_t,$$

where $\alpha > 0$ and $\beta$ are known constants and $B_t = B_t(\omega)$ is a 1-dimensional Brownian motion.

The hydro power capacity at time $t$ is denoted by $K_t$. More capacity is available at increasing costs: Let $C(k)$ denote the marginal cost of hydro power (cost/power unit) when the hydro power capacity is at the level $k$. Our control variable is additional hydro power investment $u_t \geq 0$, where
The alternative electricity source is thermal power from natural gas at a price $q$ per power unit. In this article we will assume that $q$ is constant.

With a choice of the control process $u_t(\omega) \geq 0$ the state $Y_t = Y_t^u(\omega) = (t, \Theta_t, K_t)$ of the system at time $t$ is described by the following stochastic differential equation:

$$dY_t(\omega) = \begin{bmatrix}
\frac{dt}{d\Theta_t(\omega)} \\
\frac{dK_t(\omega)}{u_t(\omega)}
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
\alpha \Theta_t(\omega) & \beta \Theta_t(\omega)
\end{bmatrix} dt + \beta \Theta_t(\omega) d\mathbb{B}_t(\omega)$$

$$u_t = \frac{dK_t}{dt}$$

Let $P^{t,\theta,k}$ denote the probability law of $Y_t$ starting at $(t, \theta, k)$ and let $E^{t,\theta,k}$ denote expectation with respect to $P^{t,\theta,k}$. Then the problem is to find a stochastic control $u_t \geq 0$ which maximizes the "performance", i.e. the expected total discounted profit:

$$H(t, \theta, k) = \sup_{u_t} E^{t,\theta,k}[\int_t^\infty (P_t D_s - C(K_s)u_s - q(D_s - K_s))e^{-rs} ds]$$

where $r > 0$ is a given discounting factor. Here the price $P_t$ will be the minimum of the thermal power price $q$ and the equilibrium price with no use of thermal power, i.e.

$$P_t = \min\{(\frac{\Theta_t}{K_t})^\frac{1}{2}, q\}$$

In either case we have

$$P_tD_t - q(D_t - K_t) = P_tK_t$$

Therefore (4) can be written

$$H(t, \theta, k) = \sup_{u_t} E^{t,\theta,k}[\int_t^\infty (P_t K_s - C(K_s)u_s)e^{-rs} ds]$$

It turns out that this supremum is not obtained by any finite choice of the control $u_t$. The optimal control $u^*$ only exists in a generalized sense. Heuristically $u^*$, with its corresponding $K_t^*$, can be described by

$$u_t^* = \begin{cases} 
0 & \text{if } (\Theta_t, K_t^*) \notin \bar{A} \\
\infty & \text{if } (\Theta_t, K_t^*) \in \bar{A}
\end{cases}$$

where $A$ is a certain open subset of the $(\theta, k)$-plane and $\bar{A}$ denotes its closure. See figure 1 below. A detailed description of the set $A$ and a presentation of our candidate for the optimal performance $H(t, \theta, k)$ will be given in section 2.

The precise meaning of such a singular control $u^*$ is that the corresponding process $Y_t^* = (t, \Theta_t, K_t^*)$ should have no increase in the $K_t$ component (i.e. $u^* = 0$) if $Z_t = (\Theta_t, K_t^*)$ is situated outside $\bar{A}$, while its $K_t$-component should jump immediately vertically to the boundary $\partial A$ of $\bar{A}$ if $Z_t$ starts inside $A$. In section 3 we will construct this process explicitly as a Markov process $Z_t$ with horizontal movements outside $A$ and vertical reflection on $\partial A$. Then in section 4 we will solve the stochastic control problem by proving that no finite choice of $u_t$ can give a better performance than the function
Figure 1:

$H$ found in section 2 and that our reflection process $Z_t$ constructed in section 3 actually obtains this performance.

2 The Hamilton - Jacobi - Bellman equation

The Hamilton - Jacobi - Bellman equation associated to the system (3) and the stochastic control problem (6) is (see e.g. [2], Ch. X)

$$\sup_{v \geq 0} [(pk - C(k)v)e^{-rt} + A^vH(t, \theta, k)] = 0, \quad p = \min([\theta^2], q)$$

(7)

(the sup being taken over all numbers $v \geq 0$), where in general

$$A^u(t, \theta, k) = \frac{\partial f}{\partial t} + \alpha \frac{\partial f}{\partial \theta} + u(t, \theta, k) \frac{\partial f}{\partial k} + \frac{1}{2}(\beta \theta)^2 \frac{\partial^2 f}{\partial \theta^2}$$

(8)

is the generator of the diffusion $Y^u$ resulting from the Markov control $u_t = u(Y_t)$. This gives

$$\sup_{v \geq 0} [v(\frac{\partial H}{\partial k} - C(k)e^{-rt}) + pk e^{-rt} + \frac{\partial H}{\partial t} + \alpha \theta \frac{\partial H}{\partial \theta} + \frac{1}{2}(\beta \theta)^2 \frac{\partial^2 H}{\partial \theta^2}] = 0$$

(9)

If we try to put

$$H(t, \theta, k) = e^{-rt}G(\theta, k)$$

equation (9) becomes

$$\sup_{v \geq 0} [v(\frac{\partial G}{\partial k} - C(k)) + pk - rG + \alpha \theta \frac{\partial G}{\partial \theta} + \frac{1}{2}(\beta \theta)^2 \frac{\partial^2 G}{\partial \theta^2}] = 0$$

(10)
We see that this implies
\[
\frac{\partial G}{\partial k} - C(k) \leq 0 \text{ everywhere and } v^* = 0 \text{ if } \frac{\partial G}{\partial k} - C(k) < 0,
\] (11)
where \( v^* \) is the value of \( v \) which maximizes (10). Thus if \( v^* \) exists, we must in either case have
\[
v^* \left( \frac{\partial G}{\partial k} - C(k) \right) = 0
\] (12)
Therefore (10) simplifies to
\[
pr - rG + \alpha \frac{\partial G}{\partial k} + \frac{1}{2} (\beta \theta)^2 \frac{\partial^2 G}{\partial \theta^2} = 0, \quad p = \min\{[\frac{\theta}{k}]^4, q\}
\] (13)
We now consider two regions separately:

2.1 Region I: \( \theta < kq^e \)

In this region \( p = \lceil [\frac{\theta}{k}]^4 \rceil \) and the general solution of (13) satisfying the boundary requirement
\[
G(0, k) = 0
\]
is of the form
\[
G_1(\theta, k) = \xi \theta^e k^{1 - \frac{1}{3}} + R(k) \theta^e
\] (14)
Here
\[
\xi = \frac{\varepsilon}{\varepsilon r - \alpha - \zeta} \text{ and } \zeta = \frac{1}{2} \beta^2 \left( \frac{1}{\varepsilon} - 1 \right),
\] (15)
\( \gamma_2 < 0 < \gamma_1 \) are the roots of the equation
\[
\frac{1}{2} \beta^2 \gamma^2 + (\alpha - \frac{1}{2} \beta^2) \gamma - r = 0
\] (16)
and \( R(k) \) is a function depending only on \( k \).

2.2 Region II: \( \theta > kq^e \)

Here \( p = q \) and the general solution of (13) is of the form
\[
G_2(\theta, k) = \frac{ab}{r} + S_1(k) \theta^e + S_2(k) \theta^e
\] (17)
where \( S_1, S_2 \) are functions depending only on \( k \).

If we require that \( G_1, G_2 \) should have the same values on the line \( \theta = kq^e \) (the common boundary of the two regions) we get
\[ \xi kq + R(k)k^{\eta_1}q^{\eta_1} = \frac{qk}{r} + S_1(k)k^{\eta_1}q^{\eta_1} + S_2(k)k^{\eta_2}q^{\eta_2} \]  

(18)

If we also require that \( \frac{\partial G_1}{\partial \theta} = \frac{\partial G_2}{\partial \theta} \) on the line \( \theta = kq^{\theta} \) we get

\[ \frac{1}{\varepsilon} \xi \theta^{\frac{1}{\varepsilon} - 1} k^{\frac{1}{\varepsilon} - 1} + \gamma_1 R(k)q^{\eta_1 - 1} = \gamma_1 S_1(k)q^{\eta_1 - 1} + \gamma_2 S_2(k)q^{\eta_2 - 1} \quad \text{for} \quad \theta = kq^{\theta} \]

i.e.

\[ \frac{1}{\varepsilon} \xi kq + \gamma_1 R(k)k^{\eta_1}q^{\eta_1} = \gamma_1 \cdot S_1(k)k^{\eta_1}q^{\eta_1} + \gamma_2 \cdot S_2(k)k^{\eta_2}q^{\eta_2} \]  

(19)

Multiply (18) by \( \gamma_1 \) and subtract from (19). The result is

\[ \xi kq(\gamma_1 - \frac{1}{\varepsilon}) = \frac{qk \gamma_1}{r} + (\gamma_1 - \gamma_2) S_2(k)k^{\eta_2}q^{\eta_2}, \]

i.e.

\[ S_2(k) = \frac{1}{\gamma_1 - \gamma_2}(\xi(\gamma_1 - \frac{1}{\varepsilon}) - \frac{\gamma_1}{r}) q^{1-\eta_2} k^{1-\eta_2} \]

or

\[ S_2(k) = -\eta_2 \cdot q^{1-\eta_2} \cdot k^{1-\eta_2}, \quad \text{where} \quad \eta_2 = \frac{\gamma_1(\alpha + \xi) - r}{r(\gamma_1 - \gamma_2)(\alpha + \xi - \varepsilon r)} \]  

(20)

From (13) we see that if \( \frac{\partial G}{\partial \theta} \) is continuous, then so is \( \frac{\partial^2 G}{\partial \theta^2} \). Therefore, if we choose \( S_2(k) \) as in (20) we also know that \( \frac{\partial^2 G}{\partial \theta^2} = \frac{\partial^2 G_2}{\partial \theta^2} \) for \( \theta = kq^{\theta} \). Thus we still have two functions \( R(k) \) and \( S_1(k) \) to be determined. To do this we must take into account the boundary conditions and (11), which says that \( \frac{\partial G}{\partial \theta} - C(k) \leq 0 \) always, and \( u^* \) is non-zero only when \( \frac{\partial G}{\partial \theta} - C(k) = 0 \). To get an idea of the situation let us consider the problem of pathwise maximizing the integral in (6), i.e. we consider the problem to find, for each \( \omega \),

\[ M = \sup_{K} \int_{\Gamma} (P_s K_s - C(K_s)K'_s)e^{-r_s}ds = \sup_{K} \int_{\Gamma} (P_s K_s - C(K_s)K'_s)e^{-r_s}ds e^{-rt} \]  

(21)

Integration by parts gives

\[ \int_{\Gamma} C(K_s)K'_s e^{-r_s}ds = \int_{\Gamma} \Gamma(K_s) e^{-r_s} + r \int_{\Gamma} \Gamma(K_s) e^{-r_s}ds, \]

where \( \Gamma(x) = \int_x^\infty C(y)dy \) is the antiderivative of \( C \) vanishing at the origin. Therefore, with \( k = K_0 \),

\[ M = \sup_{K} \int_{\Gamma} (P_s K_s - r \Gamma(K_s)) e^{-r_s}ds + \Gamma(k)e^{-rt} \]

(22)

The maximum value of the function
\[ f(k) = q \cdot k - r \Gamma(k); \quad k \geq 0 \]
is obtained when

\[ f'(k) = q - r C(k) = 0, \]
i.e. when \( k = K_{\text{max}} \) given implicitly by

\[ C(K_{\text{max}}) = \frac{q}{r} \tag{23} \]

Thus if we put \( P_t \equiv q \), which approximates the situation when \( \theta \) is very large, then the solution of (21) is the function \( K_t^* \) which jumps immediately to \( K_{\text{max}} \) and stays there if \( K_0^* = k < K_{\text{max}} \).

With the general form of \( P_t \), but assuming \( K_0 = k \geq K_{\text{max}} \) a quick calculation shows that

\[ g(k) = p(\theta, k)k - r \Gamma(k), \quad \text{where} \quad p(\theta) = \min\{\left(\frac{\theta}{k}\right)^2, q\}, \]
is decreasing, so in this case it is always optimal to choose \( K_t^* \equiv k \), i.e. \( u_t^* \equiv 0 \). Since this solves the problem pathwise, it also solves (6). We conclude that if \( k \geq K_{\text{max}} \) the solution of (6) is to choose \( u^* \equiv 0 \) with corresponding \( K_t^* \equiv k \) for all \( t \), which gives

\[ H(t, \theta, k) = e^{-rt}k \cdot E^k \left[ \int_0^\infty P_s e^{-rs}ds \right], \quad \text{where} \quad P_t = \min\left\{ \left(\frac{\theta}{k}\right)^2, q \right\}. \]

Note that

\[ \lim_{\theta \to \infty} H(t, \theta, k) = e^{-rt}kq \quad \text{if} \quad k \geq K_{\text{max}}. \tag{24} \]

From now on we will only consider the remaining case \( k < K_{\text{max}} \). By the above it is enough then to consider processes \( K_t(\omega) \) such that

\[ K_t \leq K_{\text{max}} \quad \text{for all} \quad (t, \omega) \tag{25} \]

2.2.1 Some heuristic arguments

The optimal solution we found above by putting \( P_t \equiv q \) should be approximately optimal in the general case, for large \( \theta \). This indicates that the optimal process \( K_t^* \) in (6) should jump to approximately \( K_{\text{max}} \) if \( \theta \) is large. If \( \theta \) is decreased then it gets more likely that \( P_t < q \) occurs in which case the function

\[ g(k) = p(\theta, k)k - r \Gamma(k) \]

obtains a maximum at a value \( k_0 < K_{\text{max}} \). Thus for smaller \( \theta \) the optimal \( K_t^* \) should not jump as high as for large \( \theta \).

This argument indicates that there exists a "forbidden" region \( \mathcal{A} \) in the \((\theta, k)\)-plane, such that the optimal process \((\Theta_t, K_t^*)\) jumps vertically up to the boundary of \( \mathcal{A} \) if starting inside \( \mathcal{A} \) and has no vertical movement outside the closure \( \overline{\mathcal{A}} \). Moreover, \( \mathcal{A} \) should have the form
\[ A = \{ (\theta, k); \quad k < \phi(\theta), \quad \theta \geq 0 \} \]

for some function \( \phi(\theta) \) satisfying:

\[
\phi(\theta) \text{ is continuous, increasing,} \\
\phi(0) = 0 \quad \text{and} \quad \lim_{\theta \to \infty} \phi(\theta) = K_{\max}
\]

Returning to the Hamilton-Jacobi-Bellman equation (11), (13) it is therefore natural to guess that the complement of \( A \) — where no movement in \( K^* \) occurs, i.e. \( u^* = 0 \) — should coincide with the set

\[
\{ (\theta, k); \quad \frac{\partial G}{\partial k} - C(k) < 0 \}
\]

In other words, we try to define the curve \( k = \phi(\theta) \) bounding \( A \) by putting

\[
\frac{\partial G}{\partial k}(\theta, k) - C(k) = 0 \quad \text{when} \quad k = \phi(\theta)
\]

On the other hand, since (heuristically) \( u^* = \infty \) in \( A \) we must have

\[
\frac{\partial G}{\partial k}(\theta, k) - C(k) = 0 \quad \text{for all} \quad (\theta, k) \in A
\]

This condition is not compatible with the solution (14), (17) we have found. This argument indicates that the Hamilton-Jacobi-Bellman equation (7) has no solution and that an optimal control \( u^* \) does not exist (as a finite function). We will therefore settle
with a weaker version of the Hamilton-Jacobi-Bellman equation, which nevertheless is strong enough for us to solve the problem:

To obtain (26) we must modify $G$ inside $A$. Since $K^*_t$ jumps immediately from a point $(\theta, k)$ inside $A$ to the point $(\theta, \phi(\theta))$ on $\partial A$, we see from (26) that $G$ should be modified to

$$
\tilde{G}(\theta, k) = \begin{cases} 
G(\theta, \phi(\theta)) + \Gamma(k) - \Gamma(\phi(\theta)) & \text{for } (\theta, k) \in A \\
G(\theta, k) & \text{for } (\theta, k) \notin A
\end{cases}
$$

(27)

Then it is clear that $\frac{\partial \tilde{G}}{\partial k} - C(k) = 0$ in $A$, as desired, but with this modification we can no longer expect (13) to hold in $A$. However, as will be explained in section 4, it is sufficient that

$$
pk - r\tilde{G} + \alpha \theta \frac{\partial \tilde{G}}{\partial \theta} + \frac{1}{2} (\beta \theta)^2 \frac{\partial^2 \tilde{G}}{\partial \theta^2} \leq 0 \text{ in } A
$$

(28)

By (27), (26) and (13) the left hand side of (28) is

$$
pk - rG(\theta, \phi(\theta)) - r\Gamma(k) + r\Gamma(\phi(\theta)) + \alpha \theta \left[ \frac{\partial C}{\partial \theta} + \frac{\partial \phi'}{\partial k} \right]_{k=\phi(\theta)} + \frac{1}{2} (\beta \theta)^2 \left[ \frac{\partial^2 \phi}{\partial k^2} + \frac{\partial^2 \tilde{G}}{\partial \theta \partial k} \cdot \phi' \right]_{k=\phi(\theta)}
$$

$$
= pk - \rho \phi(\theta) - r\Gamma(k) + r\Gamma(\phi(\theta)) + \frac{1}{2} (\beta \theta)^2 \frac{\partial C}{\partial k}(\theta, \phi(\theta)) \phi'
$$

$$
= \gamma(k), \text{ say.}
$$

Thus (28) requires that for each $\theta$ we should have

$$
\gamma(k) < 0 \text{ if } k < \phi(\theta) \text{ (i.e.} (\theta, k) \in A) \quad (29)
$$

and that

$$
\gamma(k) \to 0 \text{ as } k \uparrow \phi(\theta)
$$

(30)

This leads to the condition that

$$
\frac{\partial^2 \tilde{G}}{\partial \theta \partial k}(\theta, \phi(\theta)) = 0 \quad (31)
$$

To sum up we try to determine the two functions $R(k), S_1(k)$ from (14), (17) and the curve $k = \phi(\theta)$ such that (26) and (31) hold. It turns out to be more convenient to work with the inverse function $\psi = \phi^{-1}$. Recall that by (14), (17) and (20)

$$
G(\theta, k) = \begin{cases} 
\frac{\xi}{r} \theta^{1-q} + R(\theta) \theta^{q} & \text{if } \theta < k q^e \\
\frac{\eta}{r} + S_1(k) \theta^{q} - \eta_2 q^{1-q} k^{1-q} \theta^{q_2} & \text{if } \theta \geq k q^e
\end{cases}
$$

(32)

So (26) gives

$$
C(k) = \begin{cases} 
(1 - \frac{1}{q}) \xi \theta^{1-q} + R'(\theta) \theta^{q} & \text{if } \theta = \psi(k) < k q^e \\
\frac{\eta}{r} + S_1'(k) \theta^{q} - \eta_2 (1 - q_2) q^{1-q} k^{-q} \theta^{q_2} & \text{if } \theta = \psi(k) > k q^e
\end{cases}
$$

(33)
while (31) gives

\[ \frac{1}{\epsilon}(1 - \frac{1}{\epsilon})\xi\theta^{\frac{1}{x}-1}k^{-\frac{1}{x}} + \gamma_1 C(k)\theta^{\gamma_1-1} \quad \text{if} \quad \theta = \psi(k) < kq^x \]

\[ \gamma_1 S_1'(k)\theta^{\gamma_1-1} - \eta_2(1 - \gamma_2)q^{1-\gamma_2}k^{-\gamma_2}\theta^{\gamma_2-1} \quad \text{if} \quad \theta = \psi(k) > kq^x \] 

(34)

Subtract \( \theta \cdot (34) \) from \( \gamma_1 \cdot (33) \). The result is

\[ \gamma_1 \cdot C(k) = \begin{cases} 
\xi(1 - \frac{1}{\epsilon})[\gamma_1 - \frac{1}{\epsilon}]\big(\frac{\epsilon}{\xi}\big)^{\frac{1}{\epsilon}} & \text{if} \quad \theta = \psi(k) < kq^x \\
\frac{\eta_2}{\gamma_2}(1 - \gamma_2)(\gamma_1 - \gamma_2)q^{1-\gamma_2}k^{-\gamma_2}\theta^{\gamma_2} & \text{if} \quad \theta = \psi(k) > kq^x 
\end{cases} \] 

(35)

Solving for \( \theta = \psi(k) \) in these equations we get, see the Appendix,

\[ \theta = \psi(k) = \begin{cases} 
\frac{[r(\epsilon - 1)C(k)]^x \cdot k}{\epsilon - 1} & \text{for} \quad \theta < kq^x \\
\left[\frac{r(1-\gamma_2)}{C(k)}\big(\frac{q}{\gamma_1 - \gamma_2}\big)^{\gamma_2}k^{\gamma_2}q^{\gamma_2}\right]^{\frac{1}{\gamma_2}}k \cdot q^x & \text{for} \quad \theta \geq kq^x 
\end{cases} \] 

(36)

Note that if \( \epsilon \leq 1 \) the curve \( \theta = \psi(k) \) lies entirely in the region \( \theta \geq kq^x \). If \( \epsilon > 1 \) the point \( \hat{k} \) where the curve \( \theta = \psi(k) \) meets the straight line \( \theta = kq^x \) is of special interest. From (36) we find
\[
\hat{k} = C^{-1}\left(\frac{q(e-1)}{r(e - \frac{1}{\gamma_2})}\right)
\]

We also note that if \( \varepsilon > 1 \) then in general \( \psi(k) \) is not differentiable at \( k = \hat{k} \). To see this, we apply (36) and get

\[
\psi'(k) = \begin{cases} 
(\varepsilon \frac{C'(k)}{C(k)} + \frac{1}{\hat{k}})\psi(k) & \text{if } k < \hat{k} \\
(-\frac{1}{\gamma_2} \cdot \frac{C'(k)}{r - C(k)} + \frac{1}{\hat{k}})\psi(k) & \text{if } k > \hat{k}
\end{cases}
\]

Substituting \( C(\hat{k}) = \frac{q(e-1)}{r(e - \frac{1}{\gamma_2})} \) we see that

\[
\frac{\psi'_+(\hat{k}) - q \varepsilon}{\psi'_- (\hat{k}) - q \varepsilon} = \frac{-\frac{1}{\gamma_2} \cdot \frac{r(e - \frac{1}{\gamma_2})}{q^{(e-1) - (1 - \frac{1}{\gamma_2})}} \cdot C'(\hat{k})\hat{k}}{1 - \frac{1}{\gamma_2} < 1,
\]

since \( \varepsilon > 1 \) and \( \gamma_2 < 0 \).

It remains to determine \( R(k) \) and \( S_1(k) \): From the lower parts of (33) and (35) we have

\[
C(k) = \frac{q}{r} + S'_1(k)\theta_1 - \eta_2(1 - \gamma_2)(\frac{\varepsilon}{kq^*})^{\gamma_2}q
\]

\[
= \frac{q}{r} + S'_1(k)\theta_1 - \frac{\eta_2}{(\eta_1 - \eta_2)\gamma_1} + \frac{\theta_1}{\gamma_1 - \gamma_2} \cdot C(k)
\]

i.e.

\[
S'_1(k) = \left(\frac{q}{r} - C(k)\right)\left(\frac{\gamma_1}{\gamma_1 - \gamma_2} - 1\right)\psi(k)^{-\gamma_1}
\]

Since \( S_1(K_{\text{max}}) = 0 \) by (24) we conclude that

\[
S_1(k) = \frac{-\gamma_2}{\gamma_1 - \gamma_2} \cdot \int_k^{K_{\text{max}}} \left(\frac{q}{r} - C(y)\right)\psi(y)^{-\gamma_1}dy, \quad k \leq K_{\text{max}}
\]

(37)

where \( \psi(y) \) is given by the lower part of (36). Note that \( S_1(k) > 0 \) and that \( S_1(k) \) decreases to 0 as \( k \uparrow K_{\text{max}} \).

Finally we note that since \( S_1(k) \) and \( S_2(k) \) are given by (37) and (20) we can use (18) to determine \( R(k) \). The result is

\[
R(k) = \left(\frac{1}{r} - \xi\right)kq(kq^*)^{-\gamma_1} + S_1(k) + S_2(k) \cdot (kq^*)^{-(\gamma_1 - \gamma_2)}
\]

(38)

To summarize, we have now proved the following:

Define

\[
h(t, \theta, k) = e^{-rt}\tilde{G}(\theta, k),
\]

with \( \tilde{G} \) given by (27), in which

\[
G(\theta, k) = \begin{cases} 
G_1(\theta, k) & \text{given by (14)} \text{ if } \theta < kq^* \\
G_2(\theta, k) & \text{given by (17)} \text{ if } \theta \geq kq^*
\end{cases}
\]

(39)
where \( R(k), S_1(k) \) and \( S_2(k) \) are given by (38), (20) and (37) and \( k = \phi(\theta) \) is the inverse of \( \theta = \psi(k) \) given by (36). Then

\[
pke^{-rt} + \frac{\partial h}{\partial t} + \alpha \theta \frac{\partial h}{\partial \theta} + \frac{1}{2} (\beta \theta)^2 \frac{\partial^2 h}{\partial \theta^2} \begin{cases} = 0 & \text{for } k \geq \phi(\theta) \\ < 0 & \text{for } k < \phi(\theta) \end{cases}
\]

and

\[
\frac{\partial h}{\partial k} - C(k) \begin{cases} < 0 & \text{for } k > \phi(\theta) \\ = 0 & \text{for } k \leq \phi(\theta) \end{cases}
\]

Thus \( h \) satisfies our original Hamilton-Jacobi-Bellman equation only partially. Nevertheless we shall prove in section 4 that (40) - (41) are sufficient to conclude that \( h = H \). Before doing this we must construct rigorously the optimal jumping process \( Z_t = (\Theta_t, K_t^*) \) described heuristically in this section and show that it has the properties we need. This will be done in the next section.

3 Diffusions with vertical reflection on the boundary of a planar set.

In this section we shall prove the existence and uniqueness of the process \( K^* \). More precisely, given an initial hydro power capacity \( K_0 \), we shall show that there is exactly one continuous, increasing process \( K_t^* \) such that

(i) \( K_t^* = K_0 \) and \( K_t^* \geq \phi(\Theta_t) \) for all \( t \)

(ii) \( K_t^* \) increases only when \( K_t^* = \phi(\Theta_t) \).

Not surprisingly, this problem is intimately related to the existence of one-dimensional, reflected Brownian motion, and we shall follow the ideas of El Karoui and Chaleyat-Maurel [1] quite closely. An interesting aspect of this approach is that it has essentially nothing to do with probability theory, but reduces the problem to an abstract reflection property of real functions.

**Proposition 1** Assume that \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) is a strictly increasing, continuous function. Given a continuous function \( \theta : \mathbb{R}_+ \to \mathbb{R}_+ \) and a real number \( k_0 \geq f(\theta(0)) \), then

\[
k(t) = k_0 \lor \max_{s \leq t} f(\theta(s))
\]

is the only function \( k : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfying the following conditions:

(i) \( k \) is an increasing, continuous function with \( k(0) = k_0 \)

(ii) \( k(t) \geq f(\theta(t)) \) for all \( t \in \mathbb{R}_+ \)

(iii) \( k \) only increases when \( k(t) = f(\theta(t)) \), i.e.

\[
\int_0^t [k(s) - f(\theta(s))]dk(s) = 0 \quad \text{for all } t
\]

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Proof: It's trivial to check that

\[ k(t) = k_0 \vee \max_{s \leq t} f(\theta(s)) \]

satisfies (i)-(iii), and hence we concentrate on the uniqueness. Assume that \( k_1 \) and \( k_2 \) are two solutions. Then

\[
[k_1(t) - k_2(t)]^2 = 2 \int_0^t [k_1(s) - k_2(s)] d(k_1 - k_2)(s)
= 2 \int_0^t [(k_1(s) - f(\theta(s))) - (k_2(s) - f(\theta(s)))] d(k_1 - k_2)(s)
= 2 \int_0^t [k_1(s) - f(\theta(s))] dk_1(s) - 2 \int_0^t [k_1(s) - f(\theta(s))] dk_2(s)
- 2 \int_0^t [k_2(s) - f(\theta(s))] dk_1(s) + 2 \int_0^t [k_2(s) - f(\theta(s))] dk_2(s)
\]

By condition (iii), the first and the last term are zero, and since \( k_1(s) - f(\theta(s)) \) and \( k_2(s) - f(\theta(s)) \) are nonnegative and \( k_2 \) and \( k_1 \) increases, the two middle terms cannot be positive. Hence \( [k_1(t) - k_2(t)]^2 \leq 0 \) for all \( t \), and we conclude that \( k_1 = k_2 \).

Applying this result to our process \( \Theta_t \) and \( K_t^* \), we get

\[ K_t^* = K_0^* \vee \max_{0 \leq s < t} \phi(\Theta_s) \text{ for } t > 0 \]  \hspace{1cm} (43)

Note that (43) holds even if \( K_0^* < \phi(\Theta_0) \).

To prove rigorously that \( K^* \) is optimal, we shall need a lemma:

**Lemma 1** \( Z_t = (\Theta_t, K_t^*) \) is a strong Markov process whose infinitesimal generator \( \hat{A} \) has the following property: If \( f(\theta, k) \) is continuously differentiable with respect to \( k \) and twice continuously differentiable with respect to \( \theta \), and if in addition

\[ \frac{\partial f}{\partial k} = 0 \text{ when } k \leq \phi(\theta), \]  \hspace{1cm} (44)

then \( f \in D[\hat{A}] \) and

\[ \hat{A}f = \alpha \theta \frac{\partial f}{\partial \theta} + \frac{1}{2} (\beta \theta)^2 \frac{\partial^2 f}{\partial \theta^2} \]  \hspace{1cm} (45)

Proof: In order to prove the Markov property, first observe that by (43) the processes \( Z_t \) and \( \Theta_t \) generate the same filtration. Since \( \Theta_t \) is a strong Markov process, we need only check \( K_t^* \). But for any stopping time \( r \), we see from (43) that

\[ K_{r^+}^* = K_r^* \vee \max_{t \leq r} \phi(\Theta_{r+t}) \]

and the Markov property follows immediately.

Turning to the proof of formula (45), we first note that if \( k < \phi(\theta) \), then \( \hat{A}f(\theta, k) = \hat{A}f(\theta, \phi(\theta)) \); this follows from (44) and the fact that \( Z \) jumps immediately from \( (\theta, k) \) to \( (\theta, \phi(\theta)) \). Hence we may assume that \( k \geq \phi(\theta) \). Applying Itô's formula to \( f(Z_t) \) and recalling that \( d\Theta_t = \alpha \Theta_t dt + \beta \Theta_t dB_t \), we get:
By condition (44), \( \frac{\partial f}{\partial k}(Z_t) \) is zero whenever \( K_t^* \) increases, and hence the last term on the right hand side is zero. Moreover, if we take expectations, the martingale term drops out, and we are left with

\[
E[f(Z_t) - f(Z_0)] = E[\int_0^t \alpha \Theta_s \frac{\partial f}{\partial \theta}(Z_s) ds + \int_0^t (\beta \Theta_s)^2 \frac{\partial^2 f}{\partial \theta^2}(Z_s) ds + \int_0^t \frac{1}{2} \frac{\partial f}{\partial k}(Z_s) dK_t^*]
\]

Dividing by \( t \) and letting \( t \) go to zero, we obtain (45).

4 Solution of the problem

We are now ready to show that the process \( Y_t^* = (t, Z_t) \), where \( Z_t = (\Theta_t, K_t^*) \) is the reflecting process constructed in section 2 indeed solves our problem. More precisely, we will find a function \( h(t, \theta, k) \) such that

\[
E^{t, \theta, k}[\int_t^\infty (P_s K_s - C(K_s) u_s) e^{-rs} ds] \leq h(t, \theta, k)
\]

for all choices of the control function \( u_s = \frac{dK_s}{dt} \geq 0 \), and such that the performance \( h(t, \theta, k) \) is actually obtained if we use the process \( Y_t^* \). Since \( Y_t^* \) (heuristically) corresponds to the singular control function

\[
u_t = \begin{cases} \infty & \text{if } Z_t \in \bar{A} \\ 0 & \text{if } Z_t \notin \bar{A}, \end{cases}
\]

it is necessary to interpret what we mean by the left hand side of (46) in this case. Using integration by parts we can write

\[
\int_t^\infty C(K_s) e^{-rs} u_s ds = \int_t^\infty C(K_s) e^{-rs} K'_s ds = \int_t^\infty C(K_s) e^{-rs} dK_s
\]

\[
= \int_t^\infty \Gamma(K_s) e^{-rs} + r \int_t^\infty \Gamma(K_s) e^{-rs} ds
\]

\[
= -\Gamma(K_t) e^{-rt} + r \int_t^\infty \Gamma(K_s) e^{-rs} ds,
\]

where \( \Gamma(x) = \int_0^x C(y) dy \) is the antiderivative of \( C \) vanishing at the origin. Thus (46) gets the form

\[
E^{t, \theta, k}[\int_t^\infty (P_s K_s - r \Gamma(K_s)) e^{-rs} ds] + \Gamma(k) e^{-rt} \leq h(t, \theta, k)
\]

for all processes \( Y_t = (t, \Theta_t, K_t) \) coming from some choice of the control function \( u_t = \frac{dK_t}{dt} \geq 0 \).

By saying that the performance \( h(t, \theta, k) \) is actually obtained if we use the process \( Y_t^* = (t, \Theta_t, K_t^*) \), we then mean that
\[ E^{t,\theta,k}[\int_t^\infty (P_\ast^\ast K_\ast^{\ast \ast} - r\Gamma(K_\ast^{\ast \ast}))e^{-rs}ds + \Gamma(k)e^{-rt}] = h(t, \theta, k), \] (49)

where \( P_\ast^\ast = \min((\Theta_{K_{\ast}})^\frac{1}{2}, q). \)

To prove (48) and (49) we choose \( h(t, \theta, k) \) to be the solution that we found in section 3 of the Hamilton-Jacobi-Bellman equation, i.e.

\[ h(t, \theta, k) = e^{-rt}\tilde{G}(\theta, k) \] (50)

where

\[ \tilde{G}(\theta, k) = \begin{cases} G(\theta, k) & \text{if } k \geq \phi(\theta) \\ G(\theta, \phi(\theta)) + \Gamma(k) - \Gamma(\phi(\theta)) & \text{if } k < \phi(\theta) \end{cases} \] (51)

and

\[ G(\theta, k) = \begin{cases} \zeta \theta \cdot \frac{k^{1-\frac{1}{2}} + R(k)\theta^{\gamma_1}}{r} & \text{if } \theta \leq kq^e \\ \frac{8}{\gamma} + S_1(k)\theta^{\gamma_1} + S_2(k)\theta^{\gamma_2} & \text{if } \theta > kq^e \end{cases} \] (52)

with \( \zeta, \gamma_1, \gamma_2, R(k), S_1(k), S_2(k) \) as in section 2.

Then by (40 - 41) we know that

\[ pke^{-rt} + \frac{\partial h}{\partial t} + \alpha \theta \frac{\partial h}{\partial \theta} + \frac{1}{2}(\beta \theta)^2 \frac{\partial^2 h}{\partial \theta^2} - C(k)e^{-rt} < 0 \] (53)

for \( k > \phi(\theta) \)

and

\[ pke^{-rt} + \frac{\partial h}{\partial t} + \alpha \theta \frac{\partial h}{\partial \theta} + \frac{1}{2}(\beta \theta)^2 \frac{\partial^2 h}{\partial \theta^2} - C(k)e^{-rt} = 0 \] (54)

for \( k < \phi(\theta) \)

So for all \( (t, \theta, k) \) and all \( v \geq 0 \) we have

\[ pke^{-rt} + v(\frac{\partial h}{\partial k} - C(k)e^{-rt}) + \frac{\partial h}{\partial t} + \alpha \theta \frac{\partial h}{\partial \theta} + \frac{1}{2}(\beta \theta)^2 \frac{\partial^2 h}{\partial \theta^2} + \frac{1}{2} \leq 0 \] (55)

Therefore, if \( u_t \geq 0 \) is some chosen control with corresponding process \( Y_t = Y_t^* = (t, \Theta_t, K_t) \) and generator

\[ A^u = \frac{\partial}{\partial t} + u \frac{\partial}{\partial k} + \alpha \theta \frac{\partial}{\partial \theta} + \frac{1}{2}(\beta \theta)^2 \frac{\partial^2}{\partial \theta^2} \] (56)

we conclude from (55) that

\[ A^u h \leq -(pk - uC(k))e^{-rt} \text{ for all } (t, \theta, k) \] (57)

since \( h \) is continuously differentiable with respect to \( t \) and \( k \) and twice continuously differentiable with respect to \( \theta \). So by Dynkin’s formula (see [2]) and (57) we have, for all constant \( T > t \):

\[ E^{t,\theta,k}[h(Y_T)] = h(t, \theta, k) + E^{t,\theta,k} [\int_t^T A^u h(Y_s)ds] \leq h(t, \theta, k) - E^{t,\theta,k} [\int_t^T (P_\ast^\ast K_\ast^{\ast \ast} - uC(K_\ast))e^{-rs}ds], \] (58)
or

\[ E^{t,\theta,k}[\int_t^T (P_s K_s - u_s C(K_s)) e^{-r_s} ds] \leq h(t, \theta, k) - E^{t,\theta,k}[h(Y_T)] \tag{59} \]

As noted earlier (see (23)) we can rule out the processes for which \( K_t \) is not bounded. So letting \( T \to \infty \) in (59) we obtain (46) and therefore (48).

It remains to prove that (49) holds. If we let \( A^* \) denote the generator of \( Y^*_t = (t, Z_t) \), then \( A^* = \frac{\partial}{\partial t} + \hat{A} \), where \( \hat{A} \) is the generator of \( Z_t \) as described in Lemma 1. The function

\[ g(t, \theta, k) = h(t, \theta, k) - \Gamma(k) e^{-rt} \tag{60} \]

satisfies

\[ \frac{\partial g}{\partial k} - C(k) e^{-rt} = 0 \quad \text{for} \quad k \leq \phi(\theta), \tag{61} \]

and hence Dynkin's formula and Lemma 1 tell us that

\[ E^{t,\theta,k}[g(Y^*_T)] = g(t, \theta, k) + E^{t,\theta,k}[\int_t^T A^* g(Y^*_s) ds] \]

\[ = g(t, \theta, k) + E^{t,\theta,k}[\int_t^T \left( \frac{\partial h}{\partial k} + C(k) e^{-rt} \right) ds] \]

\[ = g(t, \theta, k) + E^{t,\theta,k}[\int_t^T (-P^*_s K^*_s + r \Gamma(K^*_s)) e^{-rs} ds], \tag{62} \]

where we have used that \( Z_s \) lives in the region \( k \geq \phi(\theta) \) so that (53) holds when \( (\theta, k) = Z_s \) for some \( s \geq 0 \). Letting \( T \to \infty \) we conclude from (62) that

\[ E^{t,\theta,k}[\int_t^\infty (P^*_s K^*_s - r \Gamma(K^*_s)) e^{-rs} ds] = g(t, \theta, k), \]

which is the statement (49) that we wanted to prove.

Finally, we need to show that the function \( h \) actually coincides with the supremum of the expected discounted profits taken over all \( \text{finite} \) controls \( u_s \), i.e. that \( h = H \), where \( H \) is defined by (4). To this end, choose a natural number \( m \) and let \( u_s = u^{(m)}_s \) be the control

\[ u_s(t, \theta, k) = \begin{cases} m & \text{if} \quad (\theta, k) \in A \\ 0 & \text{if} \quad (\theta, k) \not\in A \end{cases} \]

Let \( Y_t = Y^{u}_t \) be the corresponding diffusion. Proceeding as above we compute, for each \( T < \infty \),

\[ E^{t,\theta,k}[g(Y^{u}_T)] = g(t, \theta, k) + E^{t,\theta,k}[\int_t^T A^u g(Y^{u}_s) ds] \]

\[ = g(t, \theta, k) + E^{t,\theta,k}[\int_t^T (-P^*_s K^*_s + r \Gamma(K^*_s)) e^{-rs} (1 - \chi) ds] \]

\[ + E^{t,\theta,k}[\int_t^T A^u g(Y^{u}_s) \chi ds] \]

where

\[ \chi = I_{Y^*_s \in A} = \begin{cases} 1 & \text{if} \quad Y^*_s \in A \\ 0 & \text{if} \quad Y^*_s \not\in A \end{cases} \]

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Letting $T \to \infty$ we conclude that

$$E_{t,e,k}^\infty \left[ \int_t^\infty \left( P_s K_s - r \Gamma(K_s) \right) e^{-rs} (1 - \chi) ds \right] = g(t, \theta, k) + E_{t,e,k}^\infty \left[ \int_t^\infty A^u g(Y_s^u) \chi ds \right]$$  

(63)

Now we claim the following:

(i) The total amount of time that $Y_s^u$ spends in $A$ is at most

$$\frac{K_{\max} - k}{m}$$

(64)

and

(ii) There exists $M < \infty$ such that

$$A^u g \geq -M \text{ in } A$$

(65)

Suppose (64) and (65) are proved. Then from (63) we conclude that

$$E_{t,e,k}^\infty \left[ \int_t^\infty \left( P_s K_s - r \Gamma(K_s) \right) e^{-rs} ds \right] =

\left( g(t, \theta, k) + E_{t,e,k}^\infty \left[ \int_t^\infty \left[ A^u g(Y_s^u) + (P_s K_s - r \Gamma(K_s)) e^{-rs} \right] \chi ds \right] \right) \geq g(t, \theta, k) - (M + qK_{\max} + r \Gamma(K_{\max})) \cdot \frac{K_{\max} - k}{m} \Rightarrow g(t, \theta, k) \text{ as } m \to \infty,$$

which proves that $h = H$.

It remains to verify (i) and (ii): To prove (i) note that $Y_s \in A \Rightarrow K_s$ is increasing with rate $m$. This gives

$$\int_t^\infty I_{Y_s \in A} ds \leq \int_t^\infty I_{K_s = m} ds \leq \frac{1}{m} \int_t^\infty K_s ds \leq \frac{K_{\max} - k}{m}$$

To prove (ii) we substitute

$$g(t, \theta, k) = h(t, \theta, k) - \Gamma(k)e^{-rt} = (\tilde{G}(\theta, k) - \Gamma(k))e^{-rt}

= (G(\theta, \phi(\theta)) - \Gamma(\phi(\theta)))e^{-rt} \text{ in } A$$

From the computation of (28) we deduce that

$$A^u g = \frac{\partial g}{\partial t} + u \frac{\partial g}{\partial x} + \alpha \theta \frac{\partial g}{\partial \theta} + \frac{1}{2} (\beta \theta)^2 \frac{\partial^2 g}{\partial \theta^2}$$

$$= (r \tilde{G}(\theta, \phi(\theta)) + r \Gamma(\phi(\theta)) + \alpha \theta \frac{\partial \tilde{G}}{\partial \theta}(\theta, \phi(\theta)) + \frac{1}{2} (\beta \theta)^2 \frac{\partial^2 \tilde{G}}{\partial \theta^2}(\theta, \phi(\theta)) e^{-rt}

= (r \Gamma(\phi(\theta)) - p(\phi(\theta)))e^{-rt}.$$

We conclude that

$$|A^u g| \leq (r \Gamma(k) + qK_{\max})e^{-rt} \text{ in } A$$

which implies (ii).

To summarize we have now proved the following:
Theorem 1 a) Suppose there exists a continuous function \( k = \phi(\theta) \geq 0 \) and a function \( h(t, \theta, k) \) which is continuously differentiable in \( t \) and \( k \) and twice continuously differentiable in \( \theta \), such that the following holds:

\[
pke^{-rt} + \frac{\partial h}{\partial t} + \alpha \theta \frac{\partial h}{\partial \theta} + \frac{1}{2} (\beta \theta)^2 \frac{\partial^2 h}{\partial \theta^2} \begin{cases} = 0 & \text{if } k \geq \phi(\theta) \\ \leq 0 & \text{if } k < \phi(\theta) \end{cases}
\]  

(66)

and

\[
\frac{\partial h}{\partial k} - C(k)e^{-rt} \begin{cases} \leq 0 & \text{if } k > \phi(\theta) \\ = 0 & \text{if } k \leq \phi(\theta) \end{cases}
\]  

(67)

Let \( Y_t = Y^u_t = (t, \Theta_t, K^u_t) \) be the diffusion associated to the Markov control \( u \) and let \( Y^*_t = (t, \Theta_t, K^*_t) \) be the diffusion with vertical reflection at \( k = \phi(\theta) \), as constructed in section 3. Suppose that

\[
\lim_{T \to \infty} E^{t, \theta, k}[h(Y^*_T)] = \lim_{T \to \infty} E^{t, \theta, k}[h(Y^*_T)] = \lim_{T \to \infty} E^{t, \theta, k}[\Gamma(K^*_T)e^{-rT}] = 0
\]  

(68)

Then \( h \) and \( Y^*_t \) are optimal for the stochastic control problem

\[
H(t, \theta, k) = \sup_u E^{t, \theta, k}\left[ \int_t^\infty (P_s K_s - C(K_s) u_s) e^{-rs} ds \right] = \Gamma(k)e^{-rt} + \sup_u E^{t, \theta, k}\left[ \int_t^\infty (P_s K_s - r\Gamma(K_s)) e^{-rs} ds \right]
\]  

(69)

in the following sense:

\[
E^{t, \theta, k}\left[ \int_t^\infty (P_s K_s - C(K_s) u_s) e^{-rs} ds \right] \leq h(t, \theta, k)
\]  

(70)

for all Markov controls \( u \),

\[
H(t, \theta, k) = h(t, \theta, k)
\]  

(71)

and

\[
E^{t, \theta, k}\left[ \int_t^\infty (P_s K^*_s - r\Gamma(K^*_s)) e^{-rs} ds \right] = h(t, \theta, k)
\]  

(72)

b) The function \( h(t, \theta, k) \) defined by (50), (51), (52) satisfies (66), (67), (68) and so solves the control problem (69) in the sense of (70), (71) and (72).

5 Discussion

Having proved that the formulas we derived in Section 2 really describe the solution of our problem, it may be worthwhile to take a closer look at them. In particular, it may be interesting to compare the solution of the stochastic problem to the solution of the deterministic problem obtained by putting \( \beta = 0 \). Intuitively, one would expect that the larger \( \beta \) is, i.e. the more uncertain the future demand for energy is, the more restrictive one should be in expanding (irreversibly) the hydro power system. We shall now show that this is really the case.

Recall that the boundary of the forbidden region \( A \) is the curve \( \theta = \psi(k) \) given by
Figure 5:

\[ \psi(k) = \begin{cases} \left[ \frac{r(e-1/\gamma_2)}{e-1} C(k) \right]^k & \text{for } \theta < kq^e \\ \left[ \frac{r(1-e\gamma_2)}{e(1-\gamma_2)} (q - C(k)) \right]^{1/\gamma_2} kq^e & \text{for } \theta \geq kq^e \end{cases} \]  

and that the critical point \( \psi(\hat{k}) = \hat{k} \) is given by

\[ \hat{k} = C^{-1}\left( \frac{q(e-1)}{r(e-1/\gamma_2)} \right). \]  

The only quantity in these formulas which depends on \( \beta \) is \( \gamma_2 \), which - by definition - is the negative root of the quadratic equation \( \frac{e^2}{2} \gamma^2 + (\alpha - e^{\frac{e}{2}}) \gamma - r = 0 \). It is easy to check that \( \gamma_2 \rightarrow -\infty \) as \( \beta \rightarrow 0 \), and consequently that \( \psi(k) \) decreases to

\[ \psi(k) = \begin{cases} \left[ \frac{r e C(k)}{e-1} \right]^k & \text{for } \theta < kq^e \\ kq^e & \text{for } \theta \geq kq^e \end{cases} \]  

and that \( \hat{k} \) increases to

\[ \hat{k} = C^{-1}\left( \frac{q}{r(1-1/e)} \right). \]  

when \( \beta \downarrow 0 \). This confirms our intuitive feeling that higher uncertainty should lead to a more restrictive policy for hydro power expansion. Figure 4 shows \( \psi(k) \) for \( \beta = 0 \) and for a positive value of \( \beta \).

Let us finally take a critical look at our model. In order to obtain an explicit solution, we have had to make several simplifying assumptions. The most unrealistic one is probably our assumption that the gas price \( q \) is constant. To get a more reasonable
model we could let \( q \) be a geometric Brownian motion independent of \( \Theta \), but unfortunately this leads to a much more complicated situation where the ordinary differential equations in (13) are replaced by partial differential equations. Nevertheless, we hope to be able to treat this problem in the future. Another simplifying assumption is that the elasticity \( \varepsilon \) is constant, but it does not seem too difficult to modify our approach to allow more general price-demand dynamics. We hope to return to this problem as well.
APPENDIX

In order to derive (36), we need to prove the following conjecture:

$$\frac{r}{\varepsilon} \left( \varepsilon - \frac{1}{\gamma_1} \right) \left( \varepsilon - \frac{1}{\gamma_2} \right) = r\varepsilon - \alpha - \zeta$$

where $\zeta$ is given in (15). Proof: Since $\gamma_1$ and $\gamma_2$ are roots of the characteristic equation

$$\frac{1}{2} \beta^2 \gamma^2 + \left( \alpha - \frac{1}{2} \beta^2 \right) \gamma - r = 0$$

we have that

$$\gamma_1 + \gamma_2 = \frac{-\alpha - \frac{\beta^2}{2}}{\beta^2} = \frac{\beta^2 - 2\alpha}{\beta^2}$$

and

$$\gamma_1 \gamma_2 = \frac{r}{\beta^2} = \frac{-2r}{\beta^2}.$$ 

Using these restrictions in the conjecture gives

$$\frac{\xi}{\varepsilon} \left( \varepsilon - \frac{1}{\gamma_1} \right) \left( \varepsilon - \frac{1}{\gamma_2} \right) = \frac{\xi}{\varepsilon} \left( \varepsilon^2 - \frac{\gamma_1 + \gamma_2}{\gamma_1 \gamma_2} \varepsilon + \frac{1}{\gamma_1 \gamma_2} \right)$$

$$= r\varepsilon + \frac{\beta^2 - 2\alpha}{2\varepsilon} \gamma - \frac{\beta^2}{2\varepsilon} = r\varepsilon - \alpha + \frac{\beta^2}{2} \left( 1 - \frac{1}{\varepsilon} \right)$$

$$= r\varepsilon - \alpha - \zeta$$

QED

By the conjecture, the expression for $\xi$, given by (15) can be rewritten

$$\xi = \frac{\varepsilon}{\varepsilon r - \alpha - \zeta} = \frac{\varepsilon}{\varepsilon} \frac{\varepsilon}{\varepsilon \left( \gamma_1 - \frac{1}{\gamma_1} \right) \left( \gamma_2 - \frac{1}{\gamma_2} \right)}$$

We the use this to rewrite $\eta_2$ as given by (20):

$$\eta_2 = \frac{\gamma_1(\alpha + \gamma - \varepsilon \gamma)}{\alpha(\alpha + \gamma - \varepsilon \gamma)}$$

$$= \frac{1}{\gamma_1 - \gamma_2} \left( \xi \left( \gamma_1 - \frac{1}{\gamma_1} \right) - \gamma \right) = \frac{1}{\gamma_1 - \gamma_2} \left( \xi \left( \gamma_1 - \frac{1}{\gamma_1} \right) \left( \varepsilon - \frac{1}{\gamma_1} \right) \gamma \right)$$

$$= \frac{\gamma_1}{\gamma_1 - \gamma_2} \left( \varepsilon \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \gamma - 1 \right) = \frac{\gamma_1}{\gamma_1 - \gamma_2} \left( \varepsilon \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \gamma - 1 \right)$$

We can now derive (36) from (35). Solving for $\theta$ in (35) gives, in the case of $\theta = \psi(k) < kq^*:

$$\theta = \psi(k) = \left[ \frac{\gamma_1 C(k) \varepsilon}{\left( \varepsilon - 1 \right) \left( \gamma_1 - \frac{1}{\gamma_1} \right)} \right]^\varepsilon k$$

$$= \left[ \frac{r C(k) \varepsilon}{\left( \varepsilon - 1 \right) \left( \gamma_1 - \frac{1}{\gamma_1} \right)} \right]^\varepsilon k$$

$$= \left[ \frac{r \varepsilon}{\varepsilon - 1} C(k) \right]^\varepsilon k$$
In the case of $\theta = \psi(k) > kq^e$ we get:

$$\theta = \psi(k) = \left[ \frac{\eta_1(q-C(k))}{\eta_2q(1-\eta_2)(\eta_1-\eta_2)} \right]^{\frac{1}{\eta_2}} kq^e$$

$$= \left[ \frac{\eta_1(q-C(k))}{\eta_1-\eta_2(1-\eta_2)q(1-\eta_2)} \right]^{\frac{1}{\eta_2}} kq^e$$

$$= \left[ \frac{r(1-\eta_2)}{q(1-\eta_2)} \left( q - C(k) \right) \right]^{\frac{1}{\eta_2}} kq^e$$

References
