The K-Theories of Gersten/Swan and Waldhausen Agree on most Simplicial Rings.

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0. INTRODUCTION.

The aim of this article is to show that the algebraic K-theory of simplicial rings without unit can be interpreted in terms of the left derived functors of the degreewise general linear group functor. This is an extension of the result of Gersten [4] and Anderson [1] which showed that this is the case when dealing with ordinary rings, that is, the theories of Gersten/Swan and Quillen coincided. More precisely they showed that for a ring without unit, A, the fiber of the plus map $BGL(A) \to (BGL(A))^+$ is homotopy equivalent to $BGL(F.)$ where $F.$ is a free simplicial resolution of A. Keune [6] essentially proved that $A \mapsto \pi_* GL(F.)$ could be considered as some sort of left derived functor of $A \mapsto GL(A)$.

In the following outline technicalities are suppressed and the terminology used will be explained in the text proper. For simplicity we will assume that k is a field, and A is a simplicial k-algebra without unit.

The simplicial case is somewhat different as it is not defined by means of the degreewise extension of GL, but rather of Waldhausen's $\widehat{GL}(-)$. $\widehat{GL}(A)$ is not a simplicial group, but a grouplike simplicial monoid. On the other hand $\widehat{GL}(-)$ is a homotopy functor, that is, if two simplicial rings A and B are homotopy equivalent $\widehat{GL}(A)$ and $\widehat{GL}(B)$ will also be homotopy equivalent. More generally, a method for obtaining functors factoring through the homotopy category is taking (if possible) the derived functor in the sense of Quillen [7] of some given functor. The surprising fact is now that if you take the derived functor of the degreewise extension of GL, say $\mathbf{L}(GL)(A)$, then $B\mathbf{L}(GL)(A)$ will be homotopy equivalent to the fiber of the plus map $B\widehat{GL}(A) \to B\widehat{GL}(A)^+$.

That is, in the homotopy category there is a map $f\colon B\mathbf{L}(GL)(A) \to B\widehat{GL}(A)$, where it might be reasonable to call $\pi_{q-1}(f)$ the "qth K functor of Gersten/Swan". The main theorem is then that this is exactly Waldhausen's K-theory. Hence:

THEOREM. *Let k and A be as above. Then there is a long exact sequence*

$$\begin{array}{l} \dots \longrightarrow \pi_q(\mathbf{L}(GL)(A)) \longrightarrow M(\pi_q(A)) \longrightarrow K_{q+1}(A) \longrightarrow \pi_{q-1}(\mathbf{L}(GL)(A)) \longrightarrow \\ \dots \longrightarrow \pi_1(\mathbf{L}(GL)(A)) \longrightarrow M(\pi_1(A)) \longrightarrow K_2(A) \\ \quad \longrightarrow \pi_0(\mathbf{L}(GL)(A)) \longrightarrow GL(\pi_0(A)) \longrightarrow K_1(A) \longrightarrow 0 \end{array}$$

∎

The first section gives an outline of the facts we will need about simplicial k-algebras. After a short explanation of the notation used, we will set up the machinery needed for defining the derived functor of GL. We recognize the category of simplicial k-algebras as a closed simplicial model category and prove that the total left derived functor of the general linear group functor exists.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

Section two is a straight forward extension of Anderson and Gersten's result to the degreewise case: if A is a simplicial algebra without unit then

$$B\mathbf{L}(GL)(A) \simeq fiber\{BGL(A) \to BGL(A)^+\}.$$

The bridge between the degreewise case and Waldhausen's construction is the existense of simplicial rings for which the two approaches coincide. The idea used here stems from [8] where Yongjin Song gives an outline of a method for solving a problem with the Volodin construction for simplicial rings in [5] credited Fiedorowicz. The problem is then to show that all homotopy classes of simplicial rings contain such representatives. Under some flatness hypothesis we show in section three that this is always the case.

One might suspect that the limitations set by this prosess are really not of a fundamental character. We therefore include in section four a partial result not using any flatness hypothesis showing that at least the fundamental group term is correct.

1. Simplicial k-algebras.

Let k be a commutative ring, and let $\underline{A}$ (resp. $\underline{A}^u$) be the category of associative k-algebras without unit (resp. with unit).

We have a functor $(-)^+ : \underline{A} \to \underline{A}^u$ given by $A \mapsto A^+$ where A^+ is the augmented k-algebra which is $A \oplus k$ as a k-module and where multiplication is given by $(a,p)\cdot(b,q) = (ab + aq + pb, pq)$. We have a split exact sequence $0 \to A \to A^+ \to k \to 0$, and a group valued functor $G : \underline{A}^u \to \underline{Gr}$ induces a functor $\underline{A} \to \underline{Gr}$, denoted by abuse of notation by G as well, by

$$G(A) = ker\{G(A^+) \to G(k)\}.$$

If G is product preserving this yields the original functor on $\underline{A}^u \subset \underline{A}$.

Examples. the general linear group functor GL, the group of elementary matrices E, the Steinberg group functor St and finally the K-functors K_n. If $A \in \underline{A}$ we will denote the product in $GL(A)$ by $\star$, i.e. $\alpha \star \beta = \alpha \cdot \beta + \alpha + \beta$ for $\alpha, \beta \in GL(A)$.

Let Δ be the category with objects the ordered sets $[n] = \{0, 1, \dots, n\}$ for each integer $n \geq 0$ and morphisms (weakly) monotone maps. For any category $\underline{C}$, we will denote the associated simplicial category by $s\underline{C}$, that is the category of functors $\Delta^o \to \underline{C}$. If $G : \underline{A}(^u) \to \underline{C}$ is a functor the induced functor $s\underline{A}(^u) \to s\underline{C}$ given by applying G degreewise is also denoted by G. The only exception to this is $\{q \mapsto K_n(A_q)\}$ which we will call $K_n^s(A)$ as we will restrict the symbol $K_n(A)$ to Waldhausen's algebraic K-theory.

A map of simplicial algebras $f : A \to B$ is called *free* if there is a collection of sets X_q, $q \geq 0$ such that B_q is isomorphic to $A_q \coprod F(X_q)$ where $F(X_q)$ is free (with or without unit as is the case) on X_q, such that f_q is the inclusion on the first factor and such that $s_j(X_q) \subseteq X_{q+1}$ for $0 \leq j \leq q$. $A \in s\underline{A}(^u)$ is said to be *free* if $0 \to A$ (resp. $k \to A$) is free. The results of this section are true for both $\underline{A}$ and $\underline{A}^u$, but we will give them only for $\underline{A}$ and leave the unital case to the reader. In the language of [7] $s\underline{A}$ has a natural structure as a simplicial closed model category. More precisely:

PROPOSITION 1.1. *Define a map $f: A \to B$ in $s\underline{A}$ to be a fibration (resp. weak equivalence) if it is so in $s\underline{Ens}$, and a cofibration if it is a retract of a free map, i.e. it can be imbedded in a diagram*

$$\begin{array}{ccc} A & \xrightarrow{\text{free}} & C \\ {\scriptstyle f}\downarrow & \nearrow & \downarrow{\scriptstyle \text{trivial fibration.}} \\ B & = & B \end{array}$$

With these definitions $s\underline{A}$ is a closed simplicial model category.

PROOF: Quillen [7, Theorem 4, II p 4.1] has proved that $s\underline{A}$ is a closed simplicial model category with the above cofibrations. The only thing left is to recognize the fibrations and weak equivalences. Let $\underline{\mathrm{Hom}}_{s\underline{A}}(-,-)$ be the functor $(s\underline{A})^o \times (s\underline{A}) \to s\underline{Ens}$ given by $\underline{\mathrm{Hom}}_{s\underline{A}}(X,Y)_n = \mathrm{Map}(X \times \Delta[n], Y)$ where $X, Y \in s\underline{A}$, $n \geq 0$ and $\Delta[n]$ the simplicial set represented by $[n] \in \Delta$ (See [7, II p 1.7]). In [7] a map $f : A \to B \in s\underline{A}$ is a fibration (resp. weak equivalence) if the map of simplicial sets

$$\underline{\mathrm{Hom}}_{s\underline{A}}(P, f): \underline{\mathrm{Hom}}_{s\underline{A}}(P, A) \to \underline{\mathrm{Hom}}_{s\underline{A}}(P, B)$$

is so for every projective P. In fact, it suffices to consider the case where P is the free algebra on one generator, but then one may check that there is a natural isomorphism $\underline{\mathrm{Hom}}_{s\underline{A}}(P, A) \cong A$ as simplicial sets. ▮

As simplicial algebras are simplicial (abelian) groups we have other characterizations of fibrations and weak equivalences [7, II, 3 Proposition 3 and Lemma 6].

COROLLARY 1.2. *A map $f: A \to B \in s\underline{A}$ is a fibration if $A_p \to B_p \coprod_{\pi_0(B)} \pi_0(A)$ is an epimorphism for all $p \geq 0$. In particular all objects are fibrant. f is a homotopy equivalence if $\pi_*(f)$ is an isomorphism.* ▮

Example. For any map $A \to B \in s\underline{A}$ we can describe its factorization into a free map followed by a trivial fibration explicitly. Let $R \in \underline{A}$ and let $U: R\backslash\underline{A} \to \underline{Ens}$ be the forgetful functor assigning to $R \to S \in R\backslash\underline{A}$ the underlying set of S, and let $F_R: \underline{Ens} \to R\backslash\underline{A}$ be the coadjoint functor assigning to a set X the inclusion

$$R \to F_R(X) = R \coprod F(X)$$

where $F(X)$ is the free k-algebra on X. This constitutes a cotriple $\Phi_R = F_R \circ U$ in $R\backslash\underline{A}$ and hence an augmented simplicial object $\Phi_R S \to S \in sR\backslash\underline{A}$ where $(\Phi_R S)_p = \Phi_R^{p+1} S$ and face maps (resp. degeneracies) defined by the adjunction $F_R \circ U \to 1_{R\backslash\underline{A}}$ (resp. $1_{\underline{Ens}} \to U \circ F_R$). $\Phi_R S \to S$ is an equivalence as $U(\Phi_R S) \to U(S)$ has an extra degeneracy. For $A \to B$ as above we get a factorization

$$A \xrightarrow{i} \Phi_A B \xrightarrow{p} B$$

where $\Phi_A B$ is the diagonal of the bisimplicial set $\{\Phi_{A_q}^{p+1} B_q\}$. p is a weak equivalence as $\Phi_{A_q} B_q \to B_q$ is a homotopy equivalence for each $q \geq 0$. i is by construction a free map, and p is both an epimorphism and a weak equivalence, and hence by corollary 1.2 a trivial fibration. If we choose $A = 0$ we just get what we will call a *free resolution* of B, that is a free simplicial algebra together with a trivial fibration onto B.

We now turn to the proof of the existence of the total left derived functors of the general linear group functor. This will follow from the more general considerations below. Let $\underline{B}$ and $\underline{C}$ be any categories for which $s\underline{B}$ and $s\underline{C}$ are closed simplicial model categories (e.g. $\underline{A}(^u)$, $\underline{Ens}$ or $\underline{Gr}$). The full subcategory of cofibrant (resp. both fibrant and cofibrant) objects in $s\underline{B}$ is denoted by $cs\underline{B}$ (resp. $cfs\underline{B}$). Recall that $\gamma_B : s\underline{B} \to \mathrm{Ho}\, s\underline{B}$ is the localization of $s\underline{B}$ with respect to the class of weak equivalences. For any functor $F: s\underline{B} \to s\underline{C}$ we mean by the *total left derived functor of F* a functor $\mathbf{L}(F): \mathrm{Ho}\, s\underline{B} \to \mathrm{Ho}\, s\underline{C}$ together with a natural transformation $\epsilon: \mathbf{L}(F) \circ \gamma_B \to \gamma_C \circ F$ with the following universal property. Given any other functor $G: \mathrm{Ho}\, s\underline{B} \to \mathrm{Ho}\, s\underline{C}$ with natural transformation $\eta: G \circ \gamma_B \to \gamma_C \circ F$, then η factors through ϵ by a unique natural transformation $G \to \mathbf{L}(F)$.

PROPOSITION 1.3. *Assume all objects of $s\underline{B}$ are fibrant. For any functor $G : \underline{B} \to \underline{C}$, the total left derived functor*

$$\mathbf{L}(G): \mathrm{Ho}\, s\underline{B} \to \mathrm{Ho}\, s\underline{C}$$

exists, and for any cofibrant $A \in s\underline{B}$ $G(A)$ is isomorphic to $\mathbf{L}(G)(A)$.

PROOF: By [**7**, I p 4.4 Corollary] we have to show that G carries weak equivalences in $cs\underline{B} = cfs\underline{B}$ to weak equivalences in $s\underline{C}$. But weak equivalences in $cfs\underline{B}$ are composed of (strict) simplicial homotopy equivalences which are respected by functors defined by degreewise extension. The simplicial homotopy relation is finer than weak equivalence, so weak equivalences in $cs\underline{B}$ are carried to weak equivalences in $s\underline{C}$. ∎

COROLLARY 1.4. *The total left derived functor of the degreewise general linear group functor $GL: s\underline{A} \to s\underline{Gr}$ exists.* ∎

Lastly, denote the diagonal of the nerve construction from simplicial monoids to simplicial sets by $N = \mathrm{diag}\, N: s\underline{Mon} \to s\underline{Ens}$. This is a homotopy functor for if $M \xrightarrow{\sim} M'$ is a weak equivalence, then $N_p(M) = M^p \xrightarrow{\sim} M'^p = N_p(M')$ is also a weak equivalence, and hence so is $\{p \mapsto N_p(M)\} \to \{p \mapsto N_p(M')\}$. The geometric realization also factors through the homotopy categories so it makes sense to take the classifying space of the left derived functor of a simplicial group valued functor. We will denote $|N(-)|$ by B.

2. The degreewise case

If $A \in s\underline{A}$, we first want to compute the groups $\pi_q((BGLA)^+)$ where X^+ is the Quillen plus construction of X. This is done by means of the fibration

$$E^1: \qquad \Omega((BGLA)^+) \to \mathrm{A}_\infty(BGLA) \to BGLA$$

which we will identify below, where A_∞ is Dror's acyclic functor [**3**]. This fibration is up to homotopy uniquely characterized by the following lemma.

Lemma 2.1. *Let $F \to E \xrightarrow{p} B$ be a fibration such that*

(1) *E is acyclic,*
(2) *$\pi_1(E)$ acts trivially on $\pi_*(F)$ and*
(3) *$im\{\pi_1(p)\} = P\pi_1(B)$, the maximal normal perfect subgroup of $\pi_1(B)$.*

Then p is fiber homotopy equivalent to $\Omega(B^+) \to \mathrm{A}_\infty B \xrightarrow{f} B$.

Proof: Let $\tilde{B}$ be the covering space of B with fundamental group $\pi_1(\tilde{B}) = P\pi_1(B)$. p lifts to a nilpotent fibration $\bar{F} \to E \to \tilde{B}$ where $\bar{F}$ is a path component of F. By the universal property of $\mathrm{A}_\infty B = \mathrm{A}_\infty \tilde{B}$ we have a lifting $a\colon E \to \mathrm{A}_\infty B$ over $\tilde{B}$ unique up to homotopy. As both $E \to \tilde{B}$ and $\mathrm{A}_\infty B \to \tilde{B}$ are nilpotent this is a nilpotent map which is a homology equivalence (as both spaces are acyclic), and hence a homotopy equivalence. Any homotopy inverse $b'\colon \mathrm{A}_\infty B \to E$ may be deformed into a homotopy inverse over B in the following manner. Let $H\colon \mathrm{A}_\infty B \times I \to \mathrm{A}_\infty B$ be the homotopy with $H \circ i_0 = a \circ b'$ and $H \circ i_1 = 1_{\mathrm{A}_\infty B}$. The diagram

$$\begin{array}{ccc} \mathrm{A}_\infty B & \xrightarrow{b'} & E \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{A}_\infty B \times I & \xrightarrow{f \circ H} & B \end{array}$$

has a lifting l. Then $b = l \circ i_1$ is a homotopy inverse to a over B. ∎

We now want to show that $B\mathbf{L}(GL)(A)$ is homotopy equivalent to $\mathrm{A}_\infty BGL(A)$. Using the results of Gersten and Anderson in the absolute case this is indeed easy to prove (assuming k is regular and Noetherian): choose a free bisimplicial resolution $F_{..} \to A$ (i.e. $F_{.p} \to A_p$ is a free simplicial resolution for each $p \geq 0$). For each $p \geq 0$ we get a homotopy equivalence $BGL(F_{.p}) \simeq \mathrm{A}_\infty BGL(A_p)$, and hence a homotopy equivalence of realizations

$$B\mathbf{L}(GL)(A) \simeq |p \mapsto BGL(F_{.p})| \simeq |p \mapsto \mathrm{A}_\infty BGL(A_p)| \simeq \mathrm{A}_\infty BGL(A).$$

The (following almost) direct proof is not much longer and contains information about group actions which we shall need later on.

LEMMA 2.2. *Let k be regular and Noetherian. Then $B\mathbb{L}(GL)(A)$ is acyclic.*

PROOF: Let $F \xrightarrow{\sim} A$ be a free resolution. Consider the spectral sequence of the bisimplicial complex $\{N_p GL(F_q)\}$

$$II^2_{pq} = H_q(H_p(BGLF)) \Rightarrow H_{p+q}(BGLF).$$

Gersten [4] has shown that $H_p(BGL(F_q)) = 0$ as F_q is free, so the sequence collapses. ∎

Furthermore, Gersten's result implies that if F is cofibrant, then $GL(F) = E(F)$, where E is induced from the functor $E: \underline{A}^u \to \underline{Gr}$ giving the group of elementary matrices. Thus if $F \to A$ is a resolution of $A \in s\underline{A}$ and $R = ker\{F \to A\}$ the rows of

$$\begin{array}{ccccccccc} 0 & \longrightarrow & GLR & \longrightarrow & EF & \longrightarrow & EA & \longrightarrow & 0 \\ & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & \downarrow & & \\ 0 & \longrightarrow & GLR & \longrightarrow & GLF & \longrightarrow & GLA & \longrightarrow & K_1^s(A) & \longrightarrow & 0 \end{array}$$

are exact (recall that E preserves epimorphisms). Hence we have a fibration

$$E^2: \qquad BGLR \times K_1^s(A) \to BGLF \to BGLA.$$

To show that this really is E^1 we need the following useful lemma due to Anderson [1].

LEMMA 2.3. *Let $A \in s\underline{A}$ and $I \in Id(A)$ (two sided ideal) with $\pi_q(I) = 0$ for some $q \geq 0$. Then $\pi_0(GLA)$ acts trivially (by conjugation) on $\pi_q(GLI)$.*

PROOF: For $R \in \underline{A}^u$ and $J \in Id(R)$ let $E(R, J)$ be the smallest normal subgroup of $E(R)$ containing all elementary matrices with entries in J (in particular $E(J) = E(J^+, J)$). Let $H_{q+1}I = \cap_{j>0} ker\{d_j : I_{q+1} \to I_q\}$ and $Z_q I = \cap_{j\geq 0} ker\{d_j : I_q \to I_{q-1}\}$. As $0 = \pi_q(I) = coker\{d_0 : H_{q+1}I \to Z_q I\}$ and as $E(-,-)$ preserves epimorphisms we have exactness in

$$\begin{array}{ccccccc} E(A^+_{q+1}, H_{q+1}I) & \xrightarrow{d_0} & E(A^+_q, Z_q I) & \longrightarrow & 0 & & \\ {\scriptstyle incl}\downarrow & & {\scriptstyle incl}\downarrow & & \downarrow & & \\ GL(H_{q+1}I) & \xrightarrow{d_0} & GL(Z_q I) & \longrightarrow & \pi_q(GL(I)) & \longrightarrow & 0 \end{array}$$

By the relative Whitehead lemma $[GL(A^+_q), GL(Z_q I)] \subseteq E(A^+_q, Z_q I)$, and so $GL(A_q) \subseteq GL(A^+_q)$ acts trivially on $\pi_q GL(I)$. ∎

Lemma 2.2 and lemma 2.3 now combines to show that E^2 satisfies the conditions of lemma 2.1 so we get:

THEOREM 2.4. *Let k be regular and Noetherian. Then $E^1 \simeq E^2$, and in particular $B\mathbb{L}(GL)(A) \simeq \mathrm{A}_\infty BGL(A)$.* ∎

3. 0-COMPLETENESS

The aim of this article is to describe $A_\infty B\widehat{GL}(A) = fiber\{B\widehat{GL}(A) \to B\widehat{GL}(A)^+\}$ where $\widehat{GL}(A) = \varinjlim \widehat{GL}_n(A)$ is Waldhausen's grouplike simplicial monoid of matrices invertible up to homotopy given by the pullback diagram

$$\begin{array}{ccc} \widehat{GL}_n(A) & \longrightarrow & M_n(A) \\ \downarrow & & \downarrow \\ GL_n(\pi_0(A)) & \longrightarrow & M_n(\pi_0(A)) \end{array} .$$

Note that this works just as well without unit and that $\widehat{GL}_n(A) = ker\{\widehat{GL}_n(A^+) \to GL(k)\}$. There is an induced map $GL(A) \to \widehat{GL}(A)$ which is generally not a weak equivalence, but for a certain class of simplicial algebras it even is an isomorphism.

DEFINITION 3.1. *Let $A \in s\underline{A}$ and $\tilde{A} = ker\{A \to \pi_0(A)\}$. The 0-completion of A, $\hat{A}$, is the (degreewise) $\tilde{A}$-adic completion $\varprojlim (A/\tilde{A}^n)$. A is 0-complete if $A = \hat{A}$.*

Note that $\hat{A}$ and any $K(A_0, 0) \in s\underline{A}$ are 0-complete. The following observation of Fiedorowicz justifies the definition.

LEMMA 3.2. *If $A \in s\underline{A}$ is 0-complete, then $GL(A) = \widehat{GL}(A)$.*

PROOF: We have to show that $GL(A_q) \to (\widehat{GL}(A))_q$ is surjective. If $\alpha \in (\widehat{GL}_n(A))_q$ there is a $\beta \in (\widehat{GL}_n(A))_q$ such that

$$\alpha \star \beta = \alpha \cdot \beta + \alpha + \beta \in ker\{(\widehat{GL}_n(A))_q \to GL(\pi_0(A))\} = M(\tilde{A}_q).$$

But as A is 0-complete

$$M = \sum_{1 \le k} (-1)^k (\alpha \star \beta)^k \in M(\tilde{A}_q)$$

and $(\alpha \star \beta) \star M = 0$, and hence $\alpha \in GL(A_q)$. ∎

As $\widehat{GL}(-)$ is a homotopy functor Theorem 2.4 gives

THEOREM 3.3. *Assume k is regular and Noetherian and let $A \in s\underline{A}$. If A is weakly equivalent to a 0-complete simplicial algebra then $A_\infty B\widehat{GL}(A) \simeq B\mathrm{L}(GL)(A)$.* ∎

Now the interesting question is: when does a given isomorphism class in $\mathrm{Ho}\, s\underline{A}$ contain a 0-complete element? At present we have only obtained the following two partial results:

LEMMA 3.4. *Let $A \in s\underline{A}$. If $\tilde{A} = ker\{A \to \pi_0(A)\}$ is degreewise flat as an A^+ module then $A \to \hat{A}$ is a weak equivalence.*

PROOF: Let J be an $n-1$ connected two sided ideal in A. As $\tilde{A}_q$ is a flat A_q^+-module for every $q \geq 0$ $\mathrm{Tor}_n^{A_q^+}(\tilde{A}_q, -) = 0$ for $n > 0$, so by [7, II Theorem 6 and Corollary] there is a spectral sequence

$$E_{pq}^2 = \pi_p(\tilde{A} \otimes_{A^+} \pi_q(J)) \Rightarrow \pi_{p+q}(\tilde{A} \otimes_{A^+} J) \cong \pi_{p+q}(\tilde{A} \cdot J),$$

where $\tilde{A} \otimes_{A^+} J \cong \tilde{A} \cdot J$ as $\mathrm{Tor}_1^{A_q^+}(\tilde{A}_q, A^+/J) = 0$. Now as J is $n-1$ connected $E_{pq}^2 = 0$, and hence $\pi_q(\tilde{A} \cdot J) = 0$, for $q < n$ and $\pi_n(\tilde{A} \cdot J) \cong \pi_0(\tilde{A} \otimes_{A^+} \pi_n(J))$.

Let $\omega = \sum \alpha_i \otimes \beta_i \in \tilde{A}_0 \otimes_{A_0^+} \pi_n(J)$. As $\pi_0(\tilde{A}) = 0$ there are $\gamma_i \in \tilde{A}_1$ such that $(d_0 - d_1)\gamma_i = \alpha_i$, and so $\omega = (d_0 \otimes id - d_1 \otimes id) \sum \gamma_i \otimes \beta_i$, hence

$$\pi_n(\tilde{A} \cdot J) = \pi_0(\tilde{A} \otimes_{A^+} \pi_n(J)) = 0.$$

Setting $J = \tilde{A}^n$ this proves by induction that $\tilde{A}^n$ is $n-1$ connected. By

$$0 \to \varprojlim_n{}^{(1)} \pi_{q+1}(A/\tilde{A}^n) \to \pi_q(\hat{A}) \to \varprojlim_n \pi_q(A/\tilde{A}^n) \to 0$$

this proves that $A \to \hat{A} = \varprojlim_n (A/\hat{A}^n)$ is a weak equivalence. ∎

LEMMA 3.5. *Let $A \in s\underline{A}$. If $\pi_0 A$ is flat as a k-module then $A \in \mathrm{Ho}\, s\underline{A}$ is isomorphic to a 0-complete simplicial algebra.*

PROOF: We may without loss of generality assume that A is free (take a free resolution), so A_n^+ $n \geq 0$ is a free k-algebra with unit. For each n we have a spectral sequence [2, XVI §5]

$$E_{pq}^2 = HH_q(A_n^+, \mathrm{Tor}_p^k(-, \pi_0(A)^+) \Rightarrow \mathrm{Tor}_{p+q}^{A_n^+}(\pi_0(A)^+, -)$$

where HH_q denotes Hochschild homology. As $\mathrm{Tor}_p^k(-, \pi_0(A)^+) = 0$ for $p \neq 0$ this collapses to

$$HH_q(A_n^+, - \otimes_k \pi_0(A)^+) \cong \mathrm{Tor}_q^{A_n^+}(\pi_0(A)^+, -).$$

But as A_n^+ is free $HH_q(A_n^+, - \otimes_k \pi_0(A)^+) = 0$ for $q > 1$. If $\tilde{A} = ker\{A \to \pi_0(A)\}$ we get that $\mathrm{Tor}_p^{A_n^+}(\tilde{A}_n, -) = \mathrm{Tor}_{p+1}^{A_n^+}(\pi_0(A)^+, -) = 0$ for $p > 0$, and consequently the proof of lemma 3.4 goes through. ∎

All in all this gives:

COROLLARY 3.6. *Let k be regular and Noetherian and $A \in s\underline{A}$. If either $\pi_0 A$ is flat as a k-module or $ker\{A \to \pi_0 A\}$ is flat as an A^+ module then $B\mathbf{L}(GL)(A) \simeq \mathrm{A}_\infty B\widehat{GL}(A)$.* ∎

This can of course be carried into the domain of simplicial augmented algebras, if one prefers this setting. Any such is of the form A^+ where $A \in s\underline{A}$. If $F \xrightarrow{\sim} A$ is a free resolution, then $F^+ \xrightarrow{\sim} A^+$ is a free simplicial resolution of A^+ in $s\underline{A}^u$, and so the homotopy fibers of $B\mathbf{L}(GL)(A) \to B\widehat{GL}(A)$ and $B\mathbf{L}(GL)(A^+) \to B\widehat{GL}(A^+)$ are homotopy equivalent where the latter $\mathbf{L}$ denotes the total left derived functor in $s\underline{A}^u$. Hence:

COROLLARY 3.7. *Let k be regular and Noetherian, $A \in s\underline{A}$ (resp. A a simplicial augmented k-algebra) and let X be the homotopy fiber of $B\mathbf{L}(GL)(A) \to B\widehat{GL}(A)$. If either $\pi_0 A$ is flat as a k-module or $ker\{A \to \pi_0 A\}$ is flat as an A^+-module (resp. A-module) then*

$$K_n(A) \cong \pi_{n-1}(X)$$

$$(\textit{resp.}\ \ K_n(A) \cong \pi_{n-1}(X) \oplus K_n(k)\ \ ).$$

∎

Examples. If k is a field, then for any $A \in s\underline{A}$ $B\mathbf{L}(GL)(A) \simeq \mathrm{A}_\infty B\widehat{GL}(A)$.

Another example is a simplicial monoid algebra $A = k[M]$, where k is any regular Noetherian commutative ring and M is a simplicial monoid. Then $\pi_0(A) = k[\pi_0(M)]$ is a free k-module. Perhaps the most important case is when $k = \mathbf{Z}$ and $A = \mathbf{Z}[GX]$, where $X \in s\underline{Ens}$ and G is Kan's loop group functor. Then

$$K_n(\mathbf{Z}[GX]) \cong \pi_{n-1}(fiber\{B\mathbf{L}(GL)(\mathbf{Z}[GX]) \to B\widehat{GL}(\mathbf{Z}[GX])\}) \oplus K_n(\mathbf{Z}).$$

As to the problem with the Volodin construction in [**5**] also noted by Song, it is unclear how to avoid some sort of flatness condition. It is easy to see that the degreewise Volodin construction $X(A)$, $A \in s\underline{A}^u \subset s\underline{A}$ is equivalent to $B\mathbf{L}(GL)(A)$: let $F_{\cdot\cdot} \to A$ be a free bisimplicial resolution of A. The results of Gersten/Anderson and Suslin prove that

$$\begin{array}{ccccc} B\mathbf{L}(GL)(A) & \simeq & |p \mapsto BGL(F_{\cdot p})| & \simeq & |p \mapsto \mathrm{A}_\infty BGL(A_p)| \\ & \simeq & |p \mapsto X(A_p)| & = & X(A) \end{array}.$$

4. TOWARDS COMPLETE RESULTS.

It may be that all weak homotopy types have 0-complete representatives, but I am not able to prove this. It is therefore interesting to see what we can prove about $\mathbf{L}(GL)$ independently of the completion machine. An indication that the equation $B\mathbf{L}(GL)(A) \simeq \mathrm{A}_\infty B\widehat{GL}(A)$ holds true for all $A \in s\underline{A}$ is given by the following lemma.

LEMMA 4.1. *Assume k is regular and Noetherian and let $A \in s\underline{A}$. Then*

$$\pi_0(\mathbf{L}(GL)(A)) \cong St(\pi_0 A) \cong \pi_1(\mathrm{A}_\infty B\widehat{GL}(A)).$$

PROOF: We may assume that A is free, so that $GL(A) = E(A)$ ($\simeq \mathbf{L}(GL)(A)$) maps to $E(\pi_0(A)) = \pi_1(fiber\{B\widehat{GL}(A) \to BK_1(\pi_0 A)\}$. If $\tilde{A} = ker\{A \to \pi_0(A)\}$ we have an exact sequence

$$0 \to GL(\tilde{A}) \to GL(A) \to E(\pi_0(A)) \to 0$$

which induces

$$0 \to \pi_0 GL(\tilde{A}) \to \pi_0 GL(A) \to E(\pi_0(A)) \to 0.$$

By lemma 2.3 this is a central extension. If $H_1(\pi_0 GL(A)) = H_2(\pi_0 GL(A)) = 0$ ($\pi_0 GL(A)$ is superperfect) it is the universal central extension, which is exactly $St(\pi_0(A))$. But $BGL(A)$ is acyclic so $H_1(\pi_0 GL(A)) = H_1(\pi_1 BGL(A)) = H_1(BGL(A)) = 0$, and $0 = H_2(BGL(A))$ surjects onto $H_2(\pi_1 BGL(A)) = H_2(\pi_0 GL(A))$. ∎

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