A FRACTIONAL STEPS METHOD FOR SCALAR CONSERVATION LAWS WITHOUT THE CFL CONDITION

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ABSTRACT. We present a numerical method for the n–dimensional initial value problem for the scalar conservation law $u(x_1, \ldots, x_n, t)_t + \sum_{i=1}^{n} f_i(u)_{x_i} = 0$, $u(x_1, \ldots, x_n, 0) = u_0(x_1, \ldots, x_n)$. Our method is based on the use of dimensional splitting and Dafermos' method to solve the one dimensional equations. This method is unconditionally stable in the sense that the time-step is not limited by the space discretization. Furthermore we show that this method produces a sub-sequence which converges to the weak entropy solution as both the time and space discretization go to zero.

0. Introduction. Scalar conservation laws have, due to their wide range of applications, been studied extensively over the years, both from a mathematical, physical and numerical point of view.

Fundamental problems are the emergence of discontinuous solutions of the partial differential equation with the subsequent call for weak solutions, which again results in subtle uniqueness questions.

Existence and uniqueness was first proved for the general Cauchy problem by Conway and Smoller [1], and later on by Kuznetsov [8], Volpert [9], Kružkov [7] who used a viscocity

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method. We will here use Kružkov's formulation of the entropy condition, which is a mechanism to identify the unique physical solution.

We here study the Cauchy problem

$$u_t + \sum_{i=1}^{n} f_i(u)_{x_i} = 0 \tag{0.1}$$
$$u(x_1, \dots, x_n, 0) = u_0(x_1, \dots, x_n).$$

Kružkov's definition of the entropy weak solution reads as follows: u is the entropy weak solution if for all constants k, all $\phi \in C_0^1$, $\phi \geq 0$, the inequality

$$\int_{\mathbb{R}^n} \int_{t>0} \left[\phi_t |u-k| + \operatorname{sign}(u-k) \sum_{i=1}^{n} (f_i(u) - f_i(k)) \phi_{x_i} \right] d^n x dt + \int_{\mathbb{R}^n} |u_0 - k| \phi(x_1, \dots, x_n, 0)\, d^n x \geq 0 \tag{0.2}$$

holds.

The method of fractional steps, or dimensional splitting, was introduced by Godunov [4] in connection with gas dynamics, and later modified and extended by various authors.

Let us briefly describe the method of fractional steps due to Godunov for the case $n = 2$. Let $u(x,y,t) = S(t)u_0(x,y)$ denote the entropy solution of

$$u_t + f(u)_x + g(u)_y = 0 \tag{0.3}$$
$$u(x,y,0) = u_0(x,y)$$

at time t. Similarly let $v(x,y,t) = S^{f,x}(t)v_0(x,y)$ denote the entropy solution of

$$v_t + f(u)_x = 0 \tag{0.4}$$
$$v(x,y,0) = v_0(x,y)$$

at time t, when y is considered a parameter. The idea is then alternatively to apply the operators $S^{f,x}$ and $S^{g,y}$ (defined as $S^{f,x}$, but with y as a parameter) for small timesteps Δt to approximate $u(x,y,t)$, viz.

$$u(x,y,t) = (S(t)u_0)(x,y) \approx [S^{f,x}(\Delta t) S^{g,y}(\Delta t)]^n u_0(x,y) \tag{0.5}$$

with $n\Delta t = t$.

When solving the one-dimensional problem (0.5), one may choose from the diversity of methods available. Crandall and Majda [2] analyze rigorously the fractional steps method for monotone schemes, the Glimm method, and the Lax-Wendroff scheme.

We here propose another scheme which has the advantage of yielding an unconditionally stable approximation in the sense that the time-step is not limited by the space-step used in the discretization, i.e. one does not need the Courant–Friedrichs–Lewy (CFL) condition. Our method is based on an idea by Dafermos [3] of approximating the flux function by a polygon, i.e. a continuous, piecewise linear function. Furthermore the initial data are approximated by step functions, thereby yielding (multiple) Riemann problems. This has the advantage of replacing rarefaction waves by shocks in the solution and thus the solution will be a step function in x for each t. Holden, Holden, and Høegh-Krohn [5], [6] extended this method into a numerical method for $n = 1$.

Finally we will give a brief resyme of the paper. Let $\delta > 0$ denote the parameter measuring the polygonal approximation of the flux function in the sense of (1.4), and fix a grid in the x, y-plane. We then use the Dafermos scheme in the x-direction for a small timestep Δt. The solution is then projected back onto the original grid before we apply the Dafermos scheme in the y-direction for a timestep Δt, using the solution computed in the x-direction as initial data. After each time we apply the Dafermos scheme we project the function onto the original grid, thereby obtaining a sequence of functions indexed by the number of interations and the mesh size.

In a series of lemmas we then prove that this sequence is uniformly bounded by the initial data in the L^∞-norm, the $T.V.$-norm, and has L_1 norm which is Lipschitz continuous in the time variable. Helly's theorem then gives a convergent sub-sequence which is finally proved to satify the Kružkov entropy condition (0.3).

1. Construction of approximate solutions. For simplicity of notation we will consider (0.1) in two dimensions, since generalization to more than two dimensions is straightforward. In two dimensions (0.1) reads

$$\begin{aligned} u_t + f(u)_x + g(u)_y &= 0 \\ u(x, y, 0) &= u_0(x, y) \end{aligned} \tag{1.1}$$

where f and g are continuous functions $\mathbb{R} \to \mathbb{R}$ that are also in $BV_{\text{loc}}(\mathbb{R}) \cap L^1_{\text{loc}}(\mathbb{R})$.

We wish to construct a numerical approximation of the solution u based on dimensional splitting, and where the one-dimensional solution operators are constructed by Dafermos' [3] method. Now we will give a brief description of Dafermos' method as used in [6] and as we will use it here.

Let u_0 be some given real number and let $u_i = u_0 + i\delta$ for $i = 1 \ldots, N$, let $f_i = f(u_i)$. We then define $f_\delta(u)$ by

$$u \in [u_i, u_{i+1}] \Rightarrow f_\delta(u) = \frac{f_{i+1} - f_i}{u_{i+1} - u_i}(u - u_i) + f_i, \quad i = 0, \ldots, N-1 \tag{1.2}$$

and

$$u \leq u_0 \Rightarrow f_\delta(u) = u_0, \quad u \geq u_N \Rightarrow f_\delta(u) = u_N. \tag{1.3}$$

Consider the Riemann problem with $u_l = u_0$ and $u_r = u_N$. Let f_c denote the lower convex envelope of f_δ on $[u_l, u_r]$. Then also f_c is piecewise linear and continuous. Let $\tilde{u}_0 < \tilde{u}_1 < \cdots < \tilde{u}_M$ be such that

$$\tilde{u}_0 = u_0, \quad \tilde{u}_M = u_N, \quad \{\tilde{u}_0, \ldots, \tilde{u}_M\} \subseteq \{u_0, \ldots, u_N\}, \tag{1.4}$$

and such that f_c is linear on each interval $[\tilde{u}_i, \tilde{u}_{i+1}]$. The solution of the one dimensional Riemann problem with left state u_0 and right state u_N is now given by:

$$u(x,t) = \begin{cases} u_l, & \text{for } x \leq \tilde{s}_0 t \\ \tilde{u}_i, & \text{for } \tilde{s}_{i-1}t < x \leq \tilde{s}_i t, \quad i = 1, \ldots, M-1 \\ u_r, & \text{for } x > \tilde{s}_{M-1} t \end{cases} \tag{1.5}$$

where

$$\tilde{s}_i = \frac{\tilde{f}_{i+1} - \tilde{f}_i}{\tilde{u}_{i+1} - \tilde{u}_i}, \quad i = 0, \ldots, M-1. \tag{1.6}$$

There is a similar formula involving the upper convex envelope for the solution of the Riemann problem in the case where the left initial value is larger than the right. In particular, we see that the solution in each case is a step function in x/t. Dafermos' method as used in [6] and others involves approximating the initial function by a step function and thereby defining a series of Riemann problems. The solutions of these will define a function which can be defined for $t > 0$ until two discontinuities interact. The interacting discontinuities will then define a Riemann problem. This Riemann problem is solved and the solution can be continued in this fashion up to any positive time. For a complete description of this procedure we refer the reader to [5], [6].

Let Δx and Δy be given (small) numbers and let π be a projection from $BV(\mathbb{R}^2)$ to functions that are constant on each square

$$z_{ij} = \{(x,y); \quad i\Delta x < x < (i+1)\Delta x, \quad j\Delta y < y < (j+1)\Delta y\} \tag{1.7}$$

for $i, j \in \mathbb{Z}$. The projection π should satisfy

$$\begin{gathered} \lim_{\Delta x \to 0 \Delta y \to 0} \pi u(x,y) = u(x,y) \\ \iint |\pi u - u| \, dx dy = O(\max(\Delta x, \Delta y)) \\ (\pi u)_{ij} \Delta x \Delta y = \int_{i\Delta x}^{(i+1)\Delta x} \int_{j\Delta y}^{(j+1)\Delta y} \pi u \, dx dy = \int_{i\Delta x}^{(i+1)\Delta x} \int_{j\Delta y}^{(j+1)\Delta y} u \, dx dy, \end{gathered} \tag{1.8}$$

where we write $(\pi u)_{ij}$ for $\pi u|_{z_{ij}}$. Furthermore the value of πu in z_{ij} should only depend on u in z_{ij}. In addition the projection is required to satisfy $\min_{(x,y) \in z_{ij}} u \leq (\pi u)_{ij} \leq \max_{(x,y) \in z_{ij}} u$.

The canonical choice would be to let π denote the grid average, i.e.,

$$\pi u(x,y) = \mu(z_{ij})^{-1} \int_{z_{ij}} d\mu\,(\tilde{x}, \tilde{y}) u(\tilde{x}, \tilde{y}), \qquad (x,y) \in z_{ij}. \tag{1.9}$$

for some measure μ. Since we will use Dafermos' method in each direction, we define f_δ and g_δ to be piecewise linear continuous approximations to f and g respectively. The approximations should be good both in the $T.V.$ norm and in L_1, i.e.,

$$\begin{gathered} \lim_{\delta \to 0} |f(u) - f_\delta(u)|_{T.V.} = 0 \\ \lim_{\delta \to 0} |f(u) - f_\delta(u)|_{L_1} = 0, \end{gathered} \tag{1.10}$$

similarly for g. If $v_0(x)$ is a piecewise constant function taking a finite number of values, we can use Dafermos' method to calculate the solution to the initial value problem:

$$v_t + f_\delta(v)_x = 0, \qquad v(x,0) = v_0(x). \tag{1.11}$$

We will write $v(x,t) = S_\delta^{f,x}(t)v_0(x)$ to indicate that $v(x,t)$ is the weak entropy solution of (1.11).

If, for each fixed x, $u(x,y)$ is a piecewise constant function in y on the intervals $\langle j\Delta y, (j+1)\Delta y\rangle$, $j \in \mathbb{Z}$ we write

$$u_j(x) = u|_{j\Delta y < y < (j+1)\Delta y}(x,y). \tag{1.12}$$

Similarly,

$$u_i(y) = u|_{i\Delta x < x < (i+1)\Delta x}(x,y) \tag{1.13}$$

for functions that are constant in x for each y. Furthermore

$$u_j(x,t) = S_\delta^{f,x}(t)u_j(x), \qquad u_i(y,t) = S_\delta^{g,y}(t)u_i(y). \tag{1.14}$$

Dimensional splitting consists in first applying the solution operator $S_\delta^{f,x}$ to u_j for each j, then projecting the solution back onto the grid, and subsequently applying the solution operator $S_\delta^{g,y}$ to u_i for each i. Finally the result of this is projected onto the grid, and the process repeated. In "computer code" this looks like

```
t := 0
n := 0
u^0(x,y) := π ∘ u_0(x,y)
do while t < T
   do j := -N step 1 to N
      u_j^{n+1/2}(x) := S_δ^{f,x}(Δt)u^n(x,(j+1/2)Δy)
   enddo
   u^{n+1/2}(x,y) := π ∘ u^{n+1/2}(x,y)
   do i := -N step 1 to N
      u_i^{n+1}(y) := S_δ^{g,y}(Δt)u^{n+1/2}((i+1/2)Δx,y)
   enddo
   u^{n+1}(x,y) := π ∘ u^{n+1}(x,y)
   t := t + Δt
   n := n + 1
enddo
```

Here N is a constant that is chosen so large that u^n is constant outside the square bounded by $\pm N\Delta x$ and $\pm N\Delta y$ in the time interval $[0,T]$.

2. Convergence. For convenience we will from now on assume that $\Delta x = \Delta y = c\Delta t$ for some $c \neq 0$. Then we have three main lemmas which ensure the existence of a convergent sub-sequence.

Lemma 1.

$$||u^n(x,y)||_\infty \leq ||u_0(x,y)||_\infty \tag{2.1}$$

Proof. This is true since $S_\delta^{f,x}$ and $S_\delta^{g,y}$ do not introduce new maxima or minima, and neither does the projection π. □

Lemma 2.

$$T.V._{(x,y)}\big(u^n(x,y)\big) \leq T.V._{(x,y)}\big(u_0(x,y)\big) \tag{2.2}$$

Proof. Recall that for a function $h(x,y)$, $T.V._{(x,y)}h(x,y)$ is defined as

$$T.V._{(x,y)}h(x,y) = \int T.V._x\big(h(x,y)\big)dy + \int T.V._y\big(h(x,y)\big)dx. \tag{2.3}$$

The lemma will hold inductively if we show that $T.V.\big(u^{n+1}(x,y)\big) \leq T.V.\big(u^n(x,y)\big)$.

¿From [6] we know that if u and v are two weak solutions of

$$u_t + f(u)_x = 0 \tag{2.4}$$

with initial values u_0 and v_0 respectively, then

$$\int |u-v|dx \leq \int |u_0 - v_0|dx. \tag{2.5}$$

We now have that $u_j^n(x,\Delta t)$ and $u_{j+1}^n(x,\Delta t)$ are step functions that are constant on some intervals $\{[x_k, x_{k+1}\rangle\}$. Thus if $\tilde{x}_k \in [x_k, x_{k+1}\rangle$

$$\int |u_{j+1}^n(x,\Delta t) - u_j^n(x,\Delta t)|dx = \\ \sum_k |u_{j+1}^n(\tilde{x}_k,\Delta t) - u_j^n(\tilde{x}_k,\Delta t)|(x_{k+1}-x_k) \leq \sum_i |u_{i,j+1}^n - u_{i,j}^n|\Delta x, \tag{2.6}$$

using (2.5). But by the construction of the projection π,

$$\sum_k |u_{j+1}^n(\tilde{x}_k,\Delta t) - u_j^n(\tilde{x}_k,\Delta t)|(x_{k+1}-x_k) = \sum_i |u_{i,j+1}^{n+1/2} - u_{i,j}^{n+1/2}|\Delta x. \tag{2.7}$$

Therefore

$$\sum_i |u_{i,j+1}^{n+1/2} - u_{i,j}^{n+1/2}|\Delta x \le \sum_i |u_{i,j+1}^n - u_{i,j}^n|\Delta x. \tag{2.8}$$

If again u is a weak solution of (2.4) then from [5] we have

$$T.V._x(u) \le T.V._x(u_0). \tag{2.9}$$

By this it follows that

$$\sum_k |u_j^n(\tilde{x}_{k+1}, \Delta t) - u_j^n(\tilde{x}_k, \Delta t)| \le \sum_i |u_{i+1,j}^n - u_{i,j}^n|. \tag{2.10}$$

Now let $h = h(x) \in BV$ be any piecewise constant function, and let h_c be a continuous approximation to h defined as follows. In a small neighborhood of each jump we let h_c be a linear interpolation between the two constant values. Then $T.V.(h) = T.V.(h_c) \ge T.V.(\pi h)$ since πh is a particular partition of h_c.

This implies

$$\sum_i |u_{i+1,j}^{n+1/2} - u_{i,j}^{n+1/2}| \le \sum_i |u_{i+1,j}^n - u_{i,j}^n|. \tag{2.11}$$

Multiplying (2.11) by Δy and summing over j, and summing (2.8) over j, and then adding the results, we obtain

$$T.V._{(x,y)}\big(u^{n+1/2}(x,y)\big) \le T.V._{(x,y)}\big(u^n(x,y)\big). \tag{2.12}$$

The desired result then follows by applying $S_\delta^{g,y}$. □

Lemma 3.

$$\sum_{i,j} |u_{i,j}^m - u_{i,j}^n|\Delta x \Delta y = (C\Delta t + h(\Delta x, \Delta y, u))(m-n), \tag{2.13}$$

where h is such that $\lim_{\Delta x \to 0, \Delta y \to 0} h(\Delta x, \Delta y, u) = 0$.

Proof. If we again turn to the one-dimensional equation and let u be the solution of (2.4) then

$$\int |u(x,t_2) - u(x,t_1)|dx \le C(t_2 - t_1), \tag{2.14}$$

for some constant C. In our notation this reads

$$\sum_k |u_j^{n+1/2}(\tilde{x}_{k+1}, \Delta t) - u_j^n(\tilde{x}_k)|(x_{k+1} - x_k) \le C\Delta t, \tag{2.15}$$

where the intervals $\{[x_k, x_{k+1}\rangle\}$ are chosen such that both $u_j^n(x,\Delta t)$ and $u_j^n(x)$ are constant on $[x_k, x_{k+1}\rangle$ and $\tilde{x}_k \in [x_k, x_{k+1}\rangle$. Now

$$(2.16)\quad |u_{i,j}^{n+1} - u_{i,j}^n| \le |u_{i,j}^{n+1} - u_j^n(x,\Delta t)| + \\ |u_j^n(x,\Delta t) - u_{i,j}^{n+1/2}| + |u_{i,j}^{n+1/2} - u_j^{n+1/2}(x,\Delta t)| + |u_j^{n+1/2}(x,\Delta t) - u_{i,j}^n|,$$

for $i\Delta x \le x < (i+1)\Delta x$. Integrating (2.16) in both the x and y direction and using (2.15) gives

$$(2.17)\qquad \sum_{i,j} |u_{i,j}^{n+1} - u_{i,j}^n|\Delta x\Delta y \le 4NC\Delta t + \iint |\pi v - v|dxdy + \iint |\pi w - w|dxdy$$

where

$$(2.18)\qquad \begin{aligned} w(x,y) &= u_j^n(x,\Delta t) \\ v(x,y) &= u_j^{n+1/2}(x,\Delta t), \end{aligned}$$

and N is such that $u_{i,j}^n$ is constant outside the square bounded by $\pm N\Delta x$ and $\pm N\Delta y$. Due to (1.8) the last two terms on the righthand side of (2.17) will be of order $O(\Delta x) = O(\Delta t)$ as Δx and Δy tend to zero. The lemma now follows by induction. □

Denote $u^n(x,y)$ by $u_\eta^n(x,y)$ where $\eta = (\delta, \Delta x)$. Now by using lemmas 1–3 and Helly's theorem as in e.g. [1], one shows the existence of a convergent sub-sequence of u_η^n (which we for simplicity also will call $u_\eta^n(x,y,t)$). Furthermore this sequence converges uniformly in $L_1(\mathbb{R}^2 \times [0,T])$ for any $T > 0$, and the limit takes the correct initial value. We will denote this limit by $u(x,y,t)$.

Lemma 4. *The limit $u(x,y,t)$ is a weak entropy solution of (2.1).*

Proof. We always have that $u_j^n(x,\Delta t)$ is a weak entropy solution of the problem

$$(2.19)\qquad u_t + f(u)_x = 0 \qquad u(x, n\Delta t) = u_j^n(x).$$

Therefore

$$(2.20)\quad \int_{\mathbb{R}}\int_{n\Delta t}^{(n+1)\Delta t} \phi_t|u_j^n(x,t) - k| + \phi_x \operatorname{sign}\big(u_j^n(x,t) - k\big)\Big(f_\delta\big(u_j^n(x,t)\big) - f_\delta(k)\Big)dtdx \\ - \int_{\mathbb{R}} \phi(x,(n+1)\Delta t)|u_j^n(x,\Delta t) - k|dx + \int_{\mathbb{R}} \phi(x,n\Delta t)|u_j^n(x) - k|dx \ge 0$$

for any constant k. Since $u_j^n(x,t)$ is a step function in x the integration with respect to x can be approximated by a Riemann sum of $u_{i,j}^{n+1/2}$. Therefore for any small $\epsilon > 0$ we may find a corresponding η such that

$$(2.21)\quad \int_{n\Delta t}^{(n+1)\Delta t} \sum_i \phi(i\Delta x, j\Delta y, t)_t |u_{i,j}^{n+1/2} - k| + \\ \phi_{i,j}(t)_x \operatorname{sign}\big(u_{i,j}^{n+1/2} - k\big)\Big(f_{i,j}^{n+1/2} - f(k)\big)\Delta x dt \\ - \sum_i \phi_{i,j}^{n+1}|u_{i,j}^{n+1/2} - k|\Delta x + \sum_i \phi_{i,j}^n |u_{i,j}^n - k|\Delta x > -\epsilon,$$

where $f_{i,j}^n = f(u_{i,j}^n)$ and $\phi_{i,j}^n = \phi(i\Delta x, j\Delta y, n\Delta t)$. Here we have used (1.8) when replacing $u_j^n(x,\Delta t)$ by $u_j^{n+1/2}$ and $f_\delta(u_j^n(x,\Delta t))$ by $f(u_j^{n+1/2})$. Furthermore we can approximate the differentiation with respect to t by a difference, and the integration with respect to t by a multiplication with Δt. Thus for any $\epsilon_1 > 0$ we can find η such that

(2.22)

$$\sum_i \left\{ \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} |u_{i,j}^{n+1/2} - k| + \left(\phi_{i,j}^{n+1}\right)_x \operatorname{sign}\left(u_{i,j}^{n+1/2} - k\right)\left(f_{i,j}^{n+1/2} - f(k)\right) \right\} \Delta x \Delta t \\ - \sum_i \phi_{i,j}^{n+1} |u_{i,j}^{n+1/2} - k| \Delta x + \sum_i \phi_{i,j}^n |u_{i,j}^n - k| \Delta x > -\epsilon_1.$$

Similarly we get

(2.23)

$$\sum_j \left\{ \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} |u_{i,j}^{n+1/2} - k| + \left(\phi_{i,j}^{n+1}\right)_y \operatorname{sign}\left(u_{i,j}^{n+1/2} - k\right)\left(g_{i,j}^{n+1/2} - g(k)\right) \right\} \Delta y \Delta t \\ - \sum_j \phi_{i,j}^{n+1} |u_{i,j}^{n+1} - k| \Delta y + \sum_i \phi_{i,j}^n |u_{i,j}^{n+1/2} - k| \Delta y > -\epsilon_2$$

for any $\epsilon_2 > 0$ and for some sufficiently small η. Multiplying (2.22) by Δy and adding for all j, and multiplying (2.23) by Δx and adding for all i, and finally adding the results we get

$$\sum_{i,j} \left\{ \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{\Delta t} |u_{i,j}^{n+1/2} - k| \right. \\ \left. + \operatorname{sign}\left(u_{i,j}^{n+1/2} - k\right)\left(\left(\phi_{i,j}^{n+1}\right)_x \left(f_{i,j}^{n+1/2} - f(k)\right) + \left(\phi_{i,j}^{n+1}\right)_y \left(g_{i,j}^{n+1/2} - g(k)\right)\right) \right\} \Delta x \Delta y \Delta t \\ - \sum_{i,j} \phi_{i,j}^{n+1} |u_{i,j}^{n+1} - k| \Delta x \Delta y + \sum_{i,j} \phi_{i,j}^n |u_{i,j}^n - k| \Delta x \Delta y > -L(\epsilon_1 + \epsilon_2) = -L\epsilon, \tag{2.24}$$

where $L = N\Delta x = N\Delta y$, and N is such that $\operatorname{supp}(\phi) \subset \{|x| < N/2, |y| < N/2\} \times [0,T]$. Summing (2.24) over n and letting $\eta \to 0$ we get that u is an entropy weak solution of (2.1). □

The generalization of this to higher dimensions is straightforward. We define

$$G_\delta(t) = \pi S_\delta^{f_n, x_n} \cdots \pi S_\delta^{f_1, x_1} \tag{2.25}$$

and let η denote the 'grid spacing', i.e., $\eta = (\delta, \Delta x_1, \ldots, \Delta x_n, \Delta t)$. The approximate solution is denoted

$$u_\eta(x_1, \ldots, x_n, m\Delta t) = \left(G_\delta(\Delta t)\right)^m u_0(x_1, \ldots, x_n). \tag{2.26}$$

Theorem. *Let $f_1, \ldots, f_n$ be continuous functions that are in $BV_{\text{loc}}(\mathbb{R}) \cap L^1_{\text{loc}}(\mathbb{R})$. Define by (2.26) a sequence of approximate solutions of (0.1) indexed by η. As $\eta \to 0$, a subsequence of u_η converges to the unique entropy weak solution (0.2) of (0.1).*

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