Partial investment under uncertainty

T. Ø. Kobila *†

January 29, 1990

1 Introduction

The importance of irreversibility in investment decisions is currently receiving renewed attention in the literature. Recently developed stochastic optimization techniques offer new insight into the combined effect of uncertainty and irreversibility. Important and diverse applications include Brennan and Schwartz (1985), McDonald and Siegel (1986) and Pindyck (1988). Pindyck (1988) solves the problem of optimal capacity choice and capacity expansion under uncertainty in future demand and irreversibility of investment. He explicitly states the problem of evaluating a marginal unit of capacity as an option value problem and establishes the link to financial option value techniques.

In this paper we analyze a general class of investment problems under uncertainty and irreversibility. We consider the optimal investment in irreversible capacity for a profit - maximizing firm with the profit function subject to random fluctuations. This general representation of uncertainty includes the cases of demand uncertainty, stochastic product price, input price uncertainty or random disturbances in the production function or cost function. The term "partial investment" is coined to describe the optimal investment strategy where the firm incrementally expands capacity, in contrast to lumpy investments where the entire production capacity is installed at once.

In Section 2 we specify the uncertainty as a geometric Brownian motion and derive a generalized Hamilton-Jacobi- Bellman equation to characterize the optimal investment decision. The optimal investment rule turns out to be singular, with the rate of capacity expansion equal to zero or infinity depending on the random fluctuations and the level of capacity already installed.

In Section 3 we discuss the results of Section 2 in the light of neoclassical investment theory. We find that the optimal capacity is smaller when uncertainty

*This is an acronym for Iulie Aslaksen, Olav Bjerkholt and Kjell Arne Brekke, all at the Central Bureau of Statistics, Box 8131 Dep, N-0033 Oslo, Norway, and Tom Lindstrøm and Bernt Øksendal, both at the Institute of Mathematics, University of Oslo, Box 1053 Blindern, N-0316 Oslo 3, Norway.

†*Acknowledgement:* The work of B. Øksendal was partially supported by NAVF, Norway. Kobila thanks Robert McDonald for comments to the first version of this paper.

and irreversibility are taken into account. This is in accord with the general results obtained under certainty in Arrow (1968) and Nickell (1978). The conclusion that optimal capacity is lower depends critically on the assumption of irreversible investment. When investment is reversible, uncertainty in itself may lead to an increase in optimal capacity, see e.g. Abel (1983). Abel's argument is analogous to that of option value theory, in that optimal capacity is higher, just as a call option on a stock is worth more, the more volatile is the price of the stock. Combined with irreversibility, however, the effect of uncertainty is to lower optimal investment. Finally, we discuss the relationship between our stochastic control model and an option value model, and show that the optimal investment strategy of Pindyck (1988) is a special case of our model.

2 The model

Consider a firm with profit depending on a stochastic parameter Θ_t and capital K_t. Let $\pi(\theta, k)$ denote the profit function, exclusive of capacity acquisition, and suppose that the returns to capacity are increasing in θ and decreasing in k, i.e. $\frac{\partial^2 \pi}{\partial \theta \partial k} \geq 0$ and $\frac{\partial^2 \pi}{\partial k^2} \leq 0$. Assume that capacity can be increased instantaneously, and that the capacity expansion is irreversible. Assume furthermore that the cost of capacity expansion $C(K_t)$ is a function of existing capacity, K_t, with $C'(K_t) > 0, C''(K_t) \geq 0$. The optimization problem of the firm is then:

$$H(t, \theta, k) = \sup_{u_s} E^{t,\theta,k}[\int_t^\infty (\pi(\Theta_s, K_s) - C(K_s)u_s)e^{-rs}ds] \tag{1}$$

where u_s is the rate of capacity expansion such that $dK_s = u_s ds$. Assume furthermore that Θ_t is an Itô diffusion,

$$d\Theta_t = a(\Theta_t)dt + b(\Theta_t)dB_t \tag{2}$$

where B_t is a Brownian motion.

The solution to (1) turns out to be of a singular type with a "forbidden region" where the process cannot stay for any positive amount of time. The usual sufficient condition for optimality that H is C^2 and solves the Hamilton-Jacobi-Bellman (HJB) equation turns out to be too restrictive.

We will derive a modified optimality condition in the form of a generalized HJB-equation. First, we will briefly outline the structure of the solution. Since there is no upper bound on the size of investment, optimal investment will either be 0 or ∞. Under reasonable assumptions we cannot have $u_t = \infty$ on a time interval of positive length, hence, the solution will be of the following form: There exists an open subset $\mathcal{A}$ of the (θ, k)-plane such that $u_t = \infty$ if and only if $(\Theta_t, K_t) \in \mathcal{A}$. If the process starts in $\mathcal{A}$, it cannot stay there for a time interval of positive length, hence, it will immediately be thrown out of this area. Similarly, it is impossible for the process to ever enter the interior of this area, the process will thus "live" outside $\mathcal{A}$.

It turns out that a sufficient condition for optimality is that H solves the generalized Hamilton-Jacobi- Bellman equation. To formulate this generalization we define the operator:

$$\mathcal{L}^u h = u\frac{\partial h}{\partial k} + \frac{\partial h}{\partial t} + a(\theta)\frac{\partial h}{\partial \theta} + \frac{1}{2}b^2(\theta)\frac{\partial^2 h}{\partial \theta^2} \tag{3}$$

The traditional HJB-equation states that if h satisfies the equation:

$$\sup_{v \geq 0}[\mathcal{L}^v h + (\pi(\theta, k) - C(k)v)e^{-rt}] = 0 \tag{4}$$

then $h = H$, where H is the optimal value function defined in (1). This can be generalized to the following theorem, where the relation between the region $\mathcal{A}$ and the function $k = \phi(\theta)$ is given by

$$\mathcal{A} = \{(\theta, k) : \; k < \phi(\theta), \; \theta \geq 0\}. \tag{5}$$

Theorem 1 *Suppose there exists a bounded continuous function $k = \phi(\theta)$ and a function $h(t, \theta, k)$ which is C^1 in t and k, and C^2 in θ, and such that the following conditions are fulfilled:*

$$\sup_{v \geq 0}[\mathcal{L}^v h(t, \theta, k) + (\pi(\theta, k) - vC(k))e^{-rt}] \begin{cases} = 0 & \text{if } k \geq \phi(\theta) \\ \leq 0 & \text{if } k < \phi(\theta) \end{cases} \tag{6}$$

and

$$\frac{\partial h}{\partial k}(t, \theta, k) - C(k)e^{-rt} \begin{cases} < 0 & \text{if } k > \phi(\theta) \\ = 0 & \text{if } k \leq \phi(\theta). \end{cases} \tag{7}$$

Moreover, suppose that there exists $M < \infty$ such that for all $v \geq 0$, $t \geq 0$ and $k < \phi(\theta)$

$$\mathcal{L}^v h(t, \theta, k) + (\pi(\theta, k) - vC(k))e^{-rt} \geq -M \tag{8}$$

and that for all t, θ, k and all controls $u \geq 0$

$$\lim_{T \to \infty} E^{t,\theta,k}[h(Y_T^u)] = 0 \tag{9}$$

where the state of the system, Y_t, is defined by $Y_t = (t, \Theta_t, K_t)$. Then $h = H$, where H is the optimal value function defined in (1), and the optimal policy is

$$u^*(t, \theta, k) = \begin{cases} 0 & \text{if } k \geq \phi(\theta) \\ \infty & \text{if } k < \phi(\theta). \end{cases} \tag{10}$$

Proof:

Note that equation (6) gives the inequality

$$\mathcal{L}^v h(t, \theta, k) \leq -(\pi(\theta, k) - vC(k))e^{-rt} \tag{11}$$

By Dynkin's formula, see e.g. Øksendal (1989), we have for all $T > t$, and for any control u:

$$\begin{aligned} E^{t,\theta,k}[h(T, \Theta_T, K_T)] &= h(t, \theta, k) + E^{t,\theta,k}[\int_t^T \mathcal{L}^u h(s, \Theta_s, K_s) ds] \\ &\leq h(t, \theta, k) - E^{t,\theta,k}[\int_t^T (\pi(\Theta_s, K_s) - u(s, \Theta_s, K_s)C(K_s))e^{-rs} ds] \end{aligned} \tag{12}$$

Rearranging this gives the inequality:

$$E^{t,\theta,k}[\int_t^T (\pi(\Theta_s, K_s) - u(s, \Theta_s, K_s)C(K_s))e^{-rs}ds] \leq h(t, \theta, k) - E^{t,\theta,k}[h(T, \Theta_T, K_T)] \tag{13}$$

As $T \to \infty$, using (9), the last term in this inequality vanishes, and hence, h dominates the expected profit for any policy u. In other words, $h \geq H$. To prove equality we proceed as follows: Choose a control $u_s \geq 0$ and $T < \infty$. Then by Dynkin's formula and (6) we have

$$\begin{aligned} E^{t,\theta,k}[h(Y_T^u)] &= h(t,\theta,k) + E^{t,\theta,k}[\textstyle\int_t^T \mathcal{L}^u h(Y_s^u)ds] \\ &= h(t,\theta,k) - E^{t,\theta,k}[\textstyle\int_t^T (\pi(\Theta_s, K_s) - C(K_s)u_s)e^{-rs}(1 - \chi(Y_s^u))ds] \\ &\quad + E^{t,\theta,k}[\textstyle\int_t^T \mathcal{L}^u h(Y_s^u)\chi(Y_s^u)ds] \end{aligned}$$

where

$$\chi(y) = \chi(t, \theta, k) = \begin{cases} 1 & \text{if } k < \phi(\theta) \\ 0 & \text{if } k \geq \phi(\theta). \end{cases}$$

Defining $J^u(t, \theta, k)$ by the next equation and letting $T \to \infty$ we get

$$\begin{aligned} J^u(t,\theta,k) &= E^{t,\theta,k}[\textstyle\int_t^\infty (\pi(\Theta_s, K_s) - C(K_s)u_s)e^{-rs}ds] \\ &= h(t,\theta,k) + E^{t,\theta,k}[\textstyle\int_t^\infty \{\mathcal{L}^u h(Y_s^u) + (\pi(\Theta_s, K_s) - C(K_s)u_s)e^{-rs}\}\chi(Y_s^u)ds] \\ &\geq h(t,\theta,k) - M \cdot E^{t,\theta,k}[\textstyle\int_t^\infty \chi(Y_s^u)ds] \end{aligned} \tag{14}$$

We now choose $u = w$ defined by

$$w = \begin{cases} m & \text{if} \quad k < \phi(\theta) \\ 0 & \text{if} \quad k \geq \phi(\theta) \end{cases}$$

where m is a large integer. Then observe that if $Y_s^w \in \mathcal{A}$, then K_s increases with speed m. The total amount of time that Y_s^w spends in $\mathcal{A}$ is at most $(\hat{k} - k)/m$, where $\hat{k} = \sup_{\theta \geq 0} \phi(\theta)$. Substituting in (14) with $u = w$ we get

$$J^w(t, \theta, k) \geq h(t, \theta, k) - \frac{M \cdot (\hat{k} - k)}{m} \to h(t, \theta, k) \text{ as } m \to \infty,$$

which shows that $h \leq \sup_u J^u = H$.

QED

The precise meaning of the singular control u^* as given by (10) is that the corresponding process $Y_t^* = (t, \Theta_t, K_t^*)$ should have no increase in the K_t component (i.e. $u^* = 0$) if (Θ_t, K_t^*) is situated outside $\bar{\mathcal{A}}$, while its K_t component should immediately jump vertically to the boundary $\partial\mathcal{A}$ of $\mathcal{A}$ if (Θ_t, K_t^*) starts inside $\mathcal{A}$. In Kobila (1989) we have shown that (Θ_t, K_t^*) is a Markov process with horizontal movements outside $\mathcal{A}$ and vertical reflection on $\partial\mathcal{A}$.

The generalized Hamilton-Jacobi-Bellman equation (6) can be written as:

$$\sup_{v\geq 0}[v(\frac{\partial H}{\partial k}-C(k)e^{-rt})+\pi(\theta,k)e^{-rt}+\frac{\partial H}{\partial t}+a(\theta)\frac{\partial H}{\partial \theta}+\frac{1}{2}b^2(\theta)\frac{\partial^2 H}{\partial \theta^2}]\leq 0 \quad (15)$$

where equality only is required for $(\theta,k)\notin \mathcal{A}^o$ ($\mathcal{A}^o$ denotes the interior of $\mathcal{A}$). Note that the equations (7) and (10) implies that $\frac{\partial H}{\partial k}-C(k)e^{-rt}\leq 0$, and that $v=0$ for $\frac{\partial H}{\partial k}-C(k)e^{-rt}<0$. In both cases the HJB equation (15) can be written:

$$\pi(\theta,k)e^{-rt}+\frac{\partial H}{\partial t}+a(\theta)\frac{\partial H}{\partial \theta}+\frac{1}{2}b^2(\theta)\frac{\partial^2 H}{\partial \theta^2}\leq 0 \quad (16)$$

Suppose $H(t,\theta,k)=G(\theta,k)e^{-rt}$, then the HJB equation can be written:

$$\pi(\theta,k)-rG+a(\theta)\frac{\partial G}{\partial \theta}+\frac{1}{2}b^2(\theta)\frac{\partial^2 G}{\partial \theta^2}\leq 0 \quad (17)$$

In $\mathcal{A}$ we have that $\frac{\partial G}{\partial k}=C(k)$. (Remember that $\frac{\partial G}{\partial k}<C(k)$ otherwise.) Since marginal profit is increasing in θ, a natural requirement for $\mathcal{A}$ is that it is bounded by the concave curve $k=\phi(\theta)$. This justifies the definition of $\mathcal{A}$ given by (5). Since we know the partial derivative $\frac{\partial G}{\partial k}$ in $\mathcal{A}$, we can express the unknown function G in $\mathcal{A}$ by the value at the boundary by integrating from k to $\phi(\theta)$. Hence,

$$G(\theta,k)=G(\theta,\phi(\theta))-\int_k^{\phi(\theta)}C(x)dx. \quad (18)$$

Equation (18) has an intuitive economic interpretation. In the area $\mathcal{A}$ we want to increase the capacity to $\phi(\theta)$. This can be done instantaneously, hence the value function at this point is the value in the point we immediately adjust to, less the cost of increasing the capacity to this point.

Using (18) in (17) we need to consider $\frac{\partial^2 G}{\partial \theta^2}$, which is given by

$$\frac{\partial^2 G}{\partial \theta^2}=\begin{cases}\frac{\partial^2 G}{\partial \theta^2}(\theta,k) & \text{for } k>\phi(\theta)\\ \frac{\partial^2 G}{\partial \theta^2}(\theta,\phi(\theta))+\frac{\partial^2 G}{\partial \theta\partial k}(\theta,\phi(\theta))\phi'(\theta) & \text{otherwise.}\end{cases}$$

In order for $\frac{\partial^2 G}{\partial \theta^2}$ to be continuous at the boundary $k=\phi(\theta)$, we must require that $\frac{\partial^2 G}{\partial \theta\partial k}(\theta,\phi(\theta))=0$. In this case G must satisfy

$$-rG+a(\theta)\frac{\partial G}{\partial \theta}+\frac{1}{2}b^2(\theta)\frac{\partial^2 G}{\partial \theta^2}=-\tilde{\pi}(\theta,k) \quad (19)$$

where

$$\tilde{\pi}(\theta,k)=\begin{cases}\pi(\theta,k) & \text{for } k>\phi(\theta)\\ \pi(\theta,\phi(\theta))-r\int_k^{\phi(\theta)}C(x)dx & \text{otherwise.}\end{cases} \quad (20)$$

To be able to solve the equation for G we must specify the Itô diffusion Θ_t. Assume from now on that

$$d\Theta_t=\alpha\Theta_t dt+\beta\Theta_t dB_t \quad (21)$$

i.e., Θ_t is a geometric Brownian motion. Then the differential equation for G becomes

$$-rG + \alpha\theta\frac{\partial G}{\partial \theta} + \frac{1}{2}(\beta\theta)^2\frac{\partial^2 G}{\partial \theta^2} = -\tilde{\pi}(\theta, k). \tag{22}$$

Note that in this differential equation we only have derivatives in θ. k is only introduced at the right hand side, so we have one differential equation for each k. Hence, we only need boundary conditions for $\theta = 0$ and for $\theta = \infty$. Under the assumption of geometric Brownian motion $\Theta_t = 0$ is an absorbing state. A natural requirement is that no capacity expansion should take place when $\theta = 0$. This gives the restriction $\pi'_k(0, k) < rC(k)$. From (22) we have that $G(0, k) = \frac{\tilde{\pi}(0,k)}{r} = \frac{\pi(0,k)}{r}$. The other boundary condition is more complicated. Define

$$\pi'_k(\infty, k) = \lim_{\theta\to\infty} \pi'_k(\theta, k) \tag{23}$$

Since π'_k by assumption is increasing in θ this limit exists but may be infinite.

Suppose first that the limit is finite, $\pi'_k(\infty, k) < \infty$. We need to consider the behavior of k for large θ. In Kobila (1989) the existence of an upper bound $k_{\max}$ was secured by a constant opportunity cost. We will here justify the existence of an upper bound $k_{\max}$ for K_t by the following heuristic argument.

Consider an infinitesimal capacity expansion Δk. A minimum requirement for profitable capacity expansion is that the expected marginal increase in profit minus investment cost is positive. Hence, we consider the following expression:

$$\begin{aligned}
& \lim_{\theta\to\infty} E^\theta \textstyle\int_0^\infty [\pi(\Theta_t, k + \Delta k) - \pi(\Theta_t, k)]e^{-rt}dt - C(k)\Delta k \\
\approx\ & \Delta k \lim_{\theta\to\infty} E^\theta \textstyle\int_0^\infty \pi'_k(\Theta_t, k)e^{-rt}dt - C(k)\Delta k \\
=\ & \Delta k(\frac{\pi'_k(\infty,k)}{r} - C(k)) > 0
\end{aligned}$$

The last equality above follows since as $\theta \to \infty$, future revenues are almost certain, and we can disregard the expectation operator and use (23). Hence, we obtain the following inequality as a minimum requirement for profitable capacity expansion,

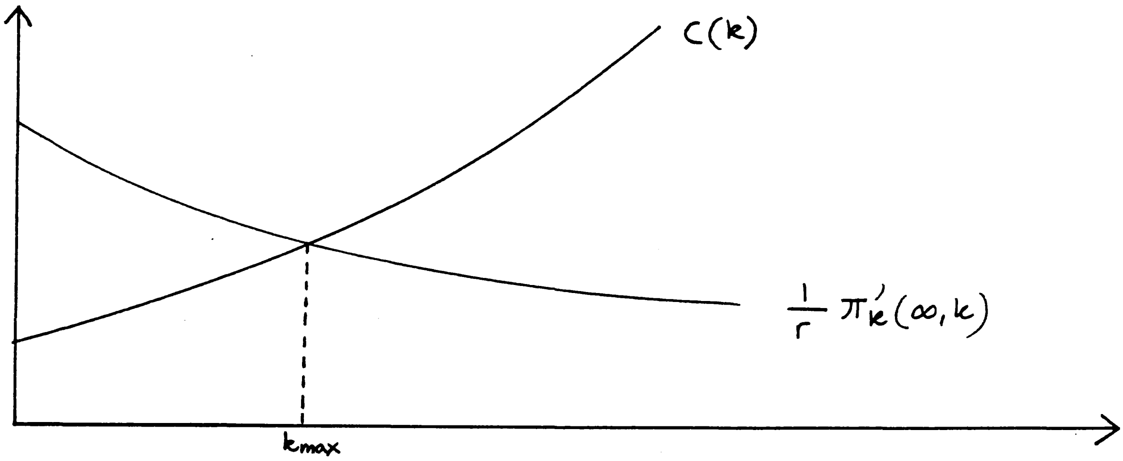
$$C(k) < \frac{1}{r}\pi'_k(\infty, k)$$

Since $C(k)$ is increasing and π'_k is decreasing the function $\eta(k)$ defined by

$$\eta(k) = C(k) - \frac{1}{r}\pi'_k(\infty, k) \tag{24}$$

is increasing with exactly one point $k_{\max}$ such that $\eta(k_{\max}) = 0$, see the figure below.

We will interpret $k_{\max}$ as the upper bound for K_t as $\theta \to \infty$.



It is not profitable to expand capacity beyond $k_{\max}$, since the cost of this expansion is higher than the upper bound on the expected present value of net future income from the expansion. But as $\theta \to \infty$ it will be optimal to expand the capacity up to a capacity infinitesimally less than $k_{\max}$, since future revenues in this case are almost certain.

Secondly, consider the case when $\pi'_k(\infty, k) = \infty$. Then the existence of an upper bound $k_{\max}$ cannot be justified by a similar heuristic argument. Nevertheless, we will also in this case assume that K_t is bounded by $K_t \leq k_{\max}$.

Define $\pi(\infty, k) = \lim_{\theta \to \infty} \pi(\theta, k)$. Assume now that $\pi(\infty, k) < \infty$. A reasonable boundary condition in this case is then

$$\begin{aligned} G(\infty, k) &= \lim_{\theta \to \infty} G(\theta, k) \\ &= \lim_{\theta \to \infty} E^{\theta}\{\int_0^{\infty} \pi(\Theta_t, k_{\max}) e^{-rt} dt\} - \int_k^{k_{\max}} C(x) dx \\ &= \frac{\pi(\infty, k_{\max})}{r} - \int_k^{k_{\max}} C(x) dx \\ &= \frac{\bar{\pi}(\infty, k)}{r} \end{aligned} \tag{25}$$

We have here used that when $\theta \to \infty$, $K_t \to k_{\max}$, and future revenues are almost certain so that the expectation operator can be disregarded.

In the case where $\pi(\infty, k) = \infty$, the boundary condition can only be explicitly stated in specific cases. As before we need that K_t is bounded by $K_t \leq k_{\max}$. Consider the following specification of the profit function,

$$\pi(\theta, k) = \theta \lambda(k) - \xi(k) \tag{26}$$

where $\lambda'(k) > 0, \lambda''(k) < 0$ and $\xi'(k) > 0$. Here $\xi(k)$ represents unit cost. As explained above, for large θ the optimal policy is to invest until $k \approx k_{\max}$, and the value function becomes

$$\begin{aligned} G(\theta, k) &\approx \bar{\lambda} \cdot E^{\theta}\{\int_0^{\infty} \Theta_t e^{-rt} dt\} - \int_0^{\infty} \xi(k) e^{-rt} dt - \int_k^{k_{\max}} C(x) dx \\ &= \bar{\lambda} \frac{\theta}{r - \alpha} - N \end{aligned} \tag{27}$$

where N is independent of θ, and we use the notation $\bar{\lambda} = \lambda(k_{\max})$. We have here used that the expected value, $E^\theta[\Theta_t]$, when Θ_t is given by (21), is $\theta e^{\alpha t}$. Dividing by θ and taking the limit for $\theta \to \infty$, we get

$$\lim_{\theta\to\infty} \frac{G(\theta,k)}{\theta} = \frac{\bar{\lambda}}{r-\alpha} \text{ for } k \leq k_{\max} \tag{28}$$

The solution to (22) with boundary conditions (25) and (28) is given by the following lemma.

Lemma 1 *Assume that $f : [0,\infty) \to \mathbb{R}$ is a continuous function such that either i) $\lim_{\theta\to\infty} f(\theta) = f(\infty)$ exists or ii) $\lim_{\theta\to\infty} \frac{f(\theta)}{\theta} = -\bar{\lambda}$ exists. Then there exists a unique solution to the differential equation:*

$$-rg + \alpha\theta g'(\theta) + \frac{1}{2}(\beta\theta)^2 g''(\theta) = f(\theta) \tag{29}$$

such that

$$g(0) = -\frac{f(0)}{r}$$

and

$$\lim_{\theta\to\infty} g(\theta) = -\frac{f(\infty)}{r} \qquad \textit{in case i)}$$

or

$$\lim_{\theta\to\infty} \frac{g(\theta)}{\theta} = \frac{\bar{\lambda}}{r-\alpha} \qquad \textit{in case ii)}\ .$$

The solution is:

$$g(\theta) = \frac{2}{(\gamma_1-\gamma_2)\beta^2}[\theta^{\gamma_1}\int_\infty^\theta \frac{f(s)}{s^{\gamma_1+1}}ds - \theta^{\gamma_2}\int_0^\theta \frac{f(s)}{s^{\gamma_2+1}}ds] \tag{30}$$

where $\gamma_1 > 0 > \gamma_2$ are the roots of the characteristic equation

$$-r + (\alpha - \frac{1}{2}\beta^2)\gamma + \frac{1}{2}(\beta\gamma)^2 = 0 \tag{31}$$

Proof:
That $g(\theta)$ is a solution to (29) is easily established by inserting the derivatives into (29) and using the equality (31). To check the limits when $\theta \to \infty$, we use L'Hôpital's rule. In case (i) we get

$$\lim_{\theta\to\infty} \theta^{\gamma_1}\int_\infty^\theta \frac{f(s)}{s^{\gamma_1+1}}ds = \lim_{\theta\to\infty} \frac{\int_\infty^\theta \frac{f(s)}{s^{\gamma_1+1}}ds}{\theta^{-\gamma_1}} = \lim_{\theta\to\infty} \frac{\frac{f(\theta)}{\theta^{\gamma_1+1}}}{-\gamma_1\theta^{-\gamma_1-1}} = -\frac{f(\infty)}{\gamma_1}$$

$$\lim_{\theta\to\infty} \theta^{\gamma_2}\int_0^\theta \frac{f(s)}{s^{\gamma_2+1}}ds = \lim_{\theta\to\infty} \frac{\int_0^\theta \frac{f(s)}{s^{\gamma_2+1}}ds}{\theta^{-\gamma_2}} = \lim_{\theta\to\infty} \frac{\frac{f(\theta)}{\theta^{\gamma_2+1}}}{-\gamma_2\theta^{-\gamma_2-1}} = -\frac{f(\infty)}{\gamma_2}$$

Hence,

$$\lim_{\theta\to\infty} g(\theta) = \frac{2}{(\gamma_1-\gamma_2)\beta^2}\left(-\frac{f(\infty)}{\gamma_1}+\frac{f(\infty)}{\gamma_2}\right) = \frac{2f(\infty)}{\gamma_1\gamma_2\beta^2} = -\frac{f(\infty)}{r} \tag{32}$$

since $\gamma_1\gamma_2 = -\frac{2r}{\beta^2}$. The condition $g(0) = -\frac{f(0)}{r}$ is checked in the same way. Similarly, in case (ii) we find

$$\lim_{\theta\to\infty}\frac{1}{\theta}\theta^{\gamma_1}\int_\infty^\theta \frac{f(s)}{s^{\gamma_1+1}}ds = \lim_{\theta\to\infty}\frac{\int_\infty^\theta \frac{f(s)}{s^{\gamma_1+1}}ds}{\theta^{-(\gamma_1-1)}} = \lim_{\theta\to\infty}\frac{\frac{f(\theta)}{\theta^{\gamma_1+1}}}{-(\gamma_1-1)\theta^{-\gamma_1}} = \lim_{\theta\to\infty}\frac{1}{1-\gamma_1}\frac{f(\theta)}{\theta} = \frac{\bar\lambda}{\gamma_1-1}$$

$$\lim_{\theta\to\infty}\frac{1}{\theta}\theta^{\gamma_2}\int_0^\theta \frac{f(s)}{s^{\gamma_2+1}}ds = \lim_{\theta\to\infty}\frac{\int_0^\theta \frac{f(s)}{s^{\gamma_2+1}}ds}{\theta^{-(\gamma_2-1)}} = \lim_{\theta\to\infty}\frac{\frac{f(\theta)}{\theta^{\gamma_2+1}}}{-(\gamma_2-1)\theta^{-\gamma_2}} = \frac{1}{1-\gamma_2}\frac{f(\theta)}{\theta} = \frac{\bar\lambda}{\gamma_2-1}$$

Hence,

$$\lim_{\theta\to\infty}\frac{g(\theta)}{\theta} = \frac{2}{(\gamma_1-\gamma_2)\beta^2}\left(\frac{\bar\lambda}{\gamma_1-1}-\frac{\bar\lambda}{\gamma_2-1}\right) = -\frac{2(\gamma_2-\gamma_1)\bar\lambda}{(\gamma_1-\gamma_2)\beta^2(r-\alpha)\frac{2}{\beta^2}} = \frac{\bar\lambda}{r-\alpha} \tag{33}$$

Here we have used that

$$(\gamma_1-1)(\gamma_2-1) = -\frac{2}{\beta^2}(r-\alpha)$$

which follows from

$$\gamma_1\gamma_2 = -\frac{2r}{\beta^2} \quad \text{and} \quad \gamma_1+\gamma_2 = 1-\frac{2\alpha}{\beta^2}.$$

QED

In order to completely characterize the solution, we need to determine the boundary $k=\phi(\theta)$. It is more convenient to consider the inverse function. Let $\psi(k)$ denote the inverse of $\phi(\theta)$, and note that for $\theta=\psi(k)$ we have

$$C(k) = G'_k(\psi(k),k) \tag{34}$$

Assume that $f(\theta) = -\tilde\pi(\theta,k)$ satisfies (i) or (ii) of Lemma 1, for each k. Then we can apply Lemma 1 to (22) and insert the solution given by (30). Using (20), this gives

$$\begin{aligned}
C(k) &= G'_k(\psi(k),k) \\
&= \tfrac{2}{(\gamma_1-\gamma_2)\beta^2}\left[-\psi(k)^{\gamma_1}\int_\infty^{\psi(k)}\tfrac{\tilde\pi'_k(s,k)}{s^{\gamma_1+1}}ds + \psi(k)^{\gamma_2}\int_0^{\psi(k)}\tfrac{\tilde\pi'_k(s,k)}{s^{\gamma_2+1}}ds\right] \\
&= \tfrac{2}{(\gamma_1-\gamma_2)\beta^2}\left[\psi(k)^{\gamma_1}\int_{\psi(k)}^\infty\tfrac{\tilde\pi'_k(s,k)}{s^{\gamma_1+1}}ds + \psi(k)^{\gamma_2}\int_0^{\psi(k)}\tfrac{\tilde\pi'_k(s,k)}{s^{\gamma_2+1}}ds\right] \qquad (35)\\
&= \tfrac{2}{(\gamma_1-\gamma_2)\beta^2}\left[\psi(k)^{\gamma_1}\int_{\psi(k)}^\infty\tfrac{rC(k)}{s^{\gamma_1+1}}ds + \psi(k)^{\gamma_2}\int_0^{\psi(k)}\tfrac{\pi'_k(s,k)}{s^{\gamma_2+1}}ds\right] \\
&= \tfrac{2}{(\gamma_1-\gamma_2)\beta^2}\left[\tfrac{rC(k)}{\gamma_1} + \psi(k)^{\gamma_2}\int_0^{\psi(k)}\tfrac{\pi'_k(s,k)}{s^{\gamma_2+1}}ds\right]
\end{aligned}$$

where the last equality follows from

$$\int_{\psi(k)}^{\infty} s^{-\gamma_1-1}ds = \psi(k)^{-\gamma_1}/\gamma_1.$$

In order to simplify this expression, we rewrite (35) and obtain

$$C(k)[1-\frac{2r}{(\gamma_1-\gamma_2)\beta^2\gamma_1}] = \frac{2}{(\gamma_1-\gamma_2)\beta^2}\psi(k)^{\gamma_2}\int_0^{\psi(k)}\frac{\pi_k'(s,k)}{s^{\gamma_2+1}}ds$$

Rearranging, we get

$$C(k) = \frac{2\gamma_1}{(\gamma_1-\gamma_2)\beta^2\gamma_1-2r}\psi(k)^{\gamma_2}\int_0^{\psi(k)}\frac{\pi_k'(s,k)}{s^{\gamma_2+1}}ds$$

Using the equality $\gamma_1\gamma_2 = -2r/\beta^2$ from Lemma 1, we obtain

$$C(k) = \frac{2}{\beta^2\gamma_1}\psi(k)^{\gamma_2}\int_0^{\psi(k)}\frac{\pi_k'(s,k)}{s^{\gamma_2+1}}ds. \tag{36}$$

We state this important equation as a theorem.

Theorem 2 *Let Θ_t be a geometric Brownian motion as given by (21). Assume that $\pi(\infty,k)<\infty$ and that K_t is bounded by $K_t \le k_{\max}$. Moreover, assume that there exists $M<\infty$ such that*

$$\tilde{\pi}(\theta,k) - \pi(\theta,k) \le M. \tag{37}$$

where $\tilde{\pi}(\theta,k)$ is defined in (20). Then the solution to

$$H(t,\theta,k) = \sup_{u_s} E^{t,\theta,k}[\int_t^{\infty}(\pi(\Theta_s,K_s) - C(K_s)u_s)e^{-rs}ds] \tag{38}$$

is given by $H=h$, where

$$h(t,\theta,k) = \frac{2e^{-rt}}{(\gamma_1-\gamma_2)\beta^2}[-\theta^{\gamma_1}\int_{\infty}^{\theta}\frac{\tilde{\pi}(s,k)}{s^{\gamma_1+1}}ds + \theta^{\gamma_2}\int_0^{\theta}\frac{\tilde{\pi}(s,k)}{s^{\gamma_2+1}}ds] \tag{39}$$

The corresponding optimal control is

$$u^*(t,\theta,k) = \begin{cases} 0 & \textit{for } \theta \le \psi(k) \\ \infty & \textit{otherwise} \end{cases} \tag{40}$$

where ψ is determined by the equation

$$C(k) = \frac{2}{\beta^2\gamma_1}\psi(k)^{\gamma_2}\int_0^{\psi(k)}\frac{\pi_k'(s,k)}{s^{\gamma_2+1}}ds. \tag{41}$$

Proof:
As shown in (25) the function $f(\theta) = -\tilde{\pi}(\theta,k)$ satisfies (i) of Lemma 1 for each k. So for each k the function $g(\theta,k)$ defined by

$$h(t,\theta,k) = e^{-rt}g(\theta,k) \tag{42}$$

solves (22) by Lemma 1. We have shown that (22) is equivalent to (6) and (7) of Theorem 1. So (39) and (40) follow from Theorem 1 and then (41) follows from (36) once we have established that (8) holds.
Now if $k < \phi(\theta)$ we have

$$\begin{aligned} \mathcal{L}^v h + (\pi - vC)e^{-rt} &= [-rg + \alpha\theta\tfrac{\partial g}{\partial \theta} + \tfrac{1}{2}(\beta\theta)^2\tfrac{\partial^2 g}{\partial \theta^2} + \pi v(\tfrac{\partial g}{\partial k} - C)]e^{-rt} \\ &= [-\tilde{\pi}(\theta,k) + \pi(\theta,k)]e^{-rt}, \end{aligned} \tag{43}$$

because $\frac{\partial g}{\partial k} = C(k)$, by (42) and the definition of h. Hence (8) follows from (37) and the proof is complete.**QED**

Despite the general form of Theorem 2 it gives an investment strategy with a simple intuitive interpretation. The optimal investment rate is zero as long as the random variable Θ_t is below a critical level $\psi(K_t)$. The critical level depends on $\pi'_k(\theta,k)$, which represents the expected future income potential from a marginal capacity expansion. When the random variable Θ_t is sufficiently high, capacity is increased according to the infinite investment rate.

When $\pi(\theta,k)$ is a linear function in θ, (37) of Theorem 2 is not satisfied, and we need to impose a different restriction. We state the result for the linear case in a separate theorem.

Theorem 3 *Let Θ_t be a geometric Brownian motion as given by (21). Suppose that the function $\pi(\theta,k)$,*

$$\pi(\theta,k) = \theta\lambda(k) - \xi(k). \tag{44}$$

is linear in θ, with $\lambda'(k) > 0, \lambda''(k) < 0$ and $\xi'(k) > 0$. Suppose that the stochastic process is restricted by $K_t \leq k_{\max}$. Then the solution to

$$H(t,\theta,k) = \sup_{u_s} E^{t,\theta,k}[\int_t^\infty (\pi(\Theta_s,K_s) - C(K_s)u_s)e^{-rs}ds]$$

is given by $H = h$, where

$$h(t,\theta,k) = \frac{2e^{-rt}}{(\gamma_1-\gamma_2)\beta^2}[-\theta^{\gamma_1}\int_\infty^\theta \frac{\tilde{\pi}(s,k)}{s^{\gamma_1+1}}ds + \theta^{\gamma_2}\int_0^\theta \frac{\tilde{\pi}(s,k)}{s^{\gamma_2+1}}ds] \tag{45}$$

The corresponding optimal control is

$$u^*(t,\theta,k) = \begin{cases} 0 & \text{for } \theta \leq \psi(k) \\ \infty & \text{otherwise} \end{cases} \tag{46}$$

where ψ is determined by the equation

$$C(k) = \frac{2}{\beta^2\gamma_1}\psi(k)^{\gamma_2}\int_0^{\psi(k)} \frac{\pi'_k(s,k)}{s^{\gamma_2+1}}ds. \tag{47}$$

Proof:
The proof is parallel to that of Theorem 2, but using case (ii) of Lemma 1. Furthermore, we cannot invoke the results of Theorem 1 directly, since condition (37) is not satisfied. By inspection of the proof of Theorem 1, however, we see that it suffices to show that for controls

$$u = \begin{cases} m & \text{if } k < \phi(\theta) \\ 0 & \text{if } k \geq \phi(\theta) \end{cases} \tag{48}$$

we have

$$\limsup_{m\to\infty} E^{t,\theta,k}\left[\int_t^\infty [\mathcal{L}^u h(Y_s^u) + (\pi(\Theta_s, K_s) - u_s C(K_s))e^{-rs}]\chi(Y_s^u)ds\right] \geq 0 \tag{49}$$

We use (43) to reformulate this in terms of π and $\tilde{\pi}$.

$$\liminf_{m\to\infty} E^{t,\theta,k}[\int_t^\infty [\tilde{\pi}(\Theta_s, K_s) - \pi(\Theta_s, K_s)]\chi(Y_s)e^{-rs}ds] \leq 0 \tag{50}$$

Note that

$$\tilde{\pi}(\theta, k) - \pi(\theta, k) \leq \theta(\lambda(\hat{k}) - \lambda(k)). \tag{51}$$

Thus it suffices to prove that:

$$\liminf_{m\to\infty} E^{t,\theta,k}[\int_t^\infty \Theta_s\chi(Y_s)e^{-rs}ds] \leq 0 \tag{52}$$

Using Hölder's inequality and changing the order of integration we get

$$\begin{array}{rl} & E^{t,\theta,k}[\int_t^\infty \Theta_s\chi(Y_s)e^{-rs}ds] \\ \leq & [\int_t^\infty E^{t,\theta,k}(\Theta_s e^{-rs})^p]ds]^{(1/p)}E^{t,\theta,k}[\int_t^\infty(\chi(Y_s))^q ds]^{(1/q)} \end{array} \tag{53}$$

with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Using Itô's lemma we see that $(\Theta_s)^p$ is a geometric Brownian motion with drift $p\alpha + \frac{1}{2}p(p-1)\beta^2$. Hence,

$$E^{t,\theta,k}[\Theta_s e^{-rs}]^p = \theta e^{-\nu s} \tag{54}$$

with $\nu = p(r-\alpha) - \frac{1}{2}p(p-1)\beta^2$. Since $\nu > 0$ for p sufficiently close to 1, the first term is finite. For the second term we have:

$$E^{t,\theta,k}[\int_t^\infty (\chi(Y_s))^q ds]^{(1/q)} \leq \left[\frac{\hat{k} - k}{m}\right]^{(1/q)} \to 0 \tag{55}$$

QED

3 A discussion in relation to investment theory.

Standard neoclassical investment theory analyzes the investment decision of a profit-maximizing firm with a concave production function. A general conclusion under full certainty, if demand is growing over time, is that capacity should be expanded until the cost of capacity expansion equals the marginal increase in profit. This conclusion can also be extended to the case of uncertain future profit, given that investments are reversible. A corresponding decision rule under uncertainty and irreversibility can be derived from Theorem 2 of the previous section. The concavity of the profit function is captured by Theorem 2. Integrating (41) by parts gives

$$rC(k) = \pi_k'(\psi(k), k) - \int_0^{\psi(k)} (\frac{\psi(k)}{s})^{\gamma_2} \pi_{\theta k}''(s, k) ds. \tag{56}$$

Under full certainty and reversible investments we have the familiar optimality condition that $rC(k) = \pi_k'$. The second term in (56) expresses the effect of uncertainty and irreversibility, which contributes towards a lower optimal capacity. Under full certainty (i.e. $\beta = 0$) we have that $\gamma_2 = -\infty$, hence, the integral in (56) is zero, and the condition $rC(k) = \pi_k'$ is obtained as the limiting case.

Note that in the special case of $\pi_{\theta k}'' = 0$, the integral in (56) is zero, and the optimal investment strategy under uncertainty and irreversibility coincides with the certainty case.

In the case of a profit function which is linear in θ we obtain an optimality condition which is similar to (56). Integrating (47) using (44) gives

$$\begin{aligned} rC(k) &= \pi_k'(\psi(k), k) - \lambda'(k)\tfrac{1}{1-\gamma_2}\psi(k) \\ &= \lambda'(k)\psi(k)\tfrac{-\gamma_2}{1-\gamma_2} + \xi'(k) \end{aligned} \tag{57}$$

Compared to the case of full certainty the marginal revenue is adjusted by the factor $-\gamma_2/(1-\gamma_2) < 1$.

Rewriting (57) we find the following expression for $\psi(k)$,

$$\psi(k) = (rC(k) + \xi'(k))\frac{1}{\lambda'(k)}\frac{1-\gamma_2}{-\gamma_2} \tag{58}$$

The interpretation of (58) is that the reservation price $\psi(k)$ equals the sum of the rental cost $rC(k)$ and marginal variable cost $\xi'(k)$, weighed by the marginal productivity $\lambda'(k)$, and weighed by the factor $(1-\gamma_2)/(-\gamma_2) > 1$, which represents risk-adjustment and the irreversibility premium. This is a certainty equivalent result in the sense that the optimality condition under uncertainty and irreversibility is of the same form as the optimality condition under full certainty. The difference consists of the factor $(1-\gamma_2)/(-\gamma_2) > 1$, which leads to a smaller optimal capacity under uncertainty and irreversibility. In the limiting case of $\beta = 0$ (full certainty), $\gamma_2 = -\infty$ and the risk-adjustment vanishes.

Finally, we will discuss the relationship between the solution in Theorem 3 and the investment model in Pindyck (1988), and show that his solution is a special case of our model. Pindyck (1988) analyzes the problem of optimal capacity choice and capacity expansion under uncertainty in future demand and irreversibility of investment. He explicitly states the problem of evaluating a marginal unit of capacity as an option value problem and establishes the link to financial option value techniques.

Consider a firm facing a linear demand function where the product price P_t depends on a random variable Θ_t and the capacity K_t,

$$P_t = \Theta_t - \mu K_t \tag{59}$$

where $\mu > 0$. The investment cost function is assumed quadratic,

$$C(K_t) = c_1 K_t + \frac{1}{2} c_2 K_t^2. \tag{60}$$

For a given (θ, k) the profit function becomes

$$\begin{aligned} \pi(\theta, k) &= pk - C(k) \\ &= \theta k - \mu k^2 - c_1 k - \tfrac{1}{2} c_2 k^2 \end{aligned} \tag{61}$$

In order to use Theorem 3, we have to assume a capacity constraint $K_t \leq k_{\max}$. We then get

$$\pi'_k(\theta, k) = \max[0, \theta - \varsigma(k)] \tag{62}$$

where $\varsigma(k) = (2\mu + c_2)k + c_1$. Since (62) is linear for large θ, we can show that we can apply Theorem 3 to this specification of marginal profit. Due to the maximum operator in (62), the following expression for the critical value $C(k)$ will differ slightly from what we found above in (57). Using (47) with (62) and the relations between the roots of the characteristic equation, we find

$$\begin{aligned} C(k) &= \tfrac{2}{\gamma_1 \beta^2} \psi(k)^{\gamma_2} \int_{\varsigma(k)}^{\psi(k)} \tfrac{s - \varsigma(k)}{s^{\gamma_2 + 1}} ds \\ &= \tfrac{2}{\gamma_1 \beta^2} \left[\tfrac{\psi(k)}{1 - \gamma_2} - \psi(k)^{\gamma_2} \tfrac{\varsigma(k)^{1-\gamma_2}}{\gamma_2 (1 - \gamma_2)} + \tfrac{\varsigma(k)}{\gamma_2} \right] \\ &= -\psi(k) \tfrac{\gamma_2}{r(1-\gamma_2)} + \psi(k)^{\gamma_2} \varsigma(k)^{1-\gamma_2} \tfrac{1}{r(1-\gamma_2)} - \tfrac{\varsigma(k)}{r} \end{aligned} \tag{63}$$

We will now show that the critical value for investment, $C(k)$, as given by (63), coincides with the critical value found by Pindyck (1988). In our notation Pindyck's solution (his eq. (11)) is given by

$$C(k) = \psi(k) \frac{(\gamma_1 - 1)}{\gamma_1 (r - \alpha)} + \psi(k)^{\gamma_2} \varsigma(k)^{1-\gamma_2} \frac{(r - \gamma_1 \alpha)}{\gamma_1 r (r - \alpha)} - \frac{\varsigma(k)}{r}. \tag{64}$$

Note that we use a risk-free discount rate whereas Pindyck uses a risk-adjusted discount rate derived from the Capital Asset Pricing Model. To show that (64) equals (63) note that the coefficients of $\varsigma(k)$ coincide, and check the coefficients of $\psi(k)$ and $\psi(k)^{\gamma_2} \varsigma(k)^{1-\gamma_2}$, again using the relations between the roots of the characteristic equation, cf. the proof of Lemma 1. In particular, note that $r(\gamma_1 - 1)(\gamma_2 - 1) = \gamma_1 \gamma_2 (r - \alpha)$. By comparing (63) and (64) term by term we find:

$$\text{For } \psi(k): \quad -\frac{\gamma_2}{r(1-\gamma_2)} = \frac{\gamma_2 \gamma_1 (\gamma_1 - 1)}{r \gamma_1 (\gamma_1 - 1)(\gamma_2 - 1)} = \frac{\gamma_1 - 1}{\gamma_1 (r - \alpha)}$$

$$\text{For } \psi(k)^{\gamma_2} \varsigma(k)^{1-\gamma_2}: \quad \frac{1}{r(1-\gamma_2)} = \frac{1 - \gamma_1}{r(1-\gamma_1)(1-\gamma_2)} = \frac{\gamma_1 (\gamma_2 + \frac{2\alpha}{\beta^2})}{\gamma_1^2 \gamma_2 (r - \alpha)} = \frac{r - \gamma_1 \alpha}{\gamma_1 r (r - \alpha)}$$

Hence, (63) and (64) coincide, and we have shown that Pindyck's solution corresponds to our Theorem 3.

We have thus shown that a stochastic dynamic programming approach yields the same solution as an explicit option value problem. The intuition behind this result is quite straightforward. Since the control problem is linear in u_t, the rate of investment, the solution becomes singular, i.e. the investment rate is either zero or infinity. The interpretation is that the stochastic control problem degenerates into what is formally an optimal stopping problem, or equivalently, an option value problem.

References

[1] Abel, A.B. (1983): Optimal Investment under Uncertainty, *American Economic Review*, 73, 228–33.

[2] Arrow, K.J. (1968): Optimal Capital Policy with Irreversible Investment, in J.N. Wolfe (ed:): *Value, Capital and Growth, Papers in Honour of Sir John Hicks*, Edinburgh University Press.

[3] Brennan, M.J. and E.S. Schwartz (1985): Evaluating Natural Resource Investments, *Journal of Business*, 58, 135–157.

[4] Kobila, T. Ø. (1989): An Application of Reflected Diffusions to the Problem of Choosing between Hydro and Thermal Power Generation, *Preprint* no. 5, Institute of Mathematics, University of Oslo.

[5] McDonald, R. and D. Siegel (1986): The Value of Waiting to Invest, *Quarterly Journal of Economics*, 101, 707–727.

[6] Nickell, S.J. (1978): *The Investment Decisions of Firms*, Cambridge University Press.

[7] Øksendal, B. (1989): *Stochastic Differential Equations. An Introduction with Applications.* Second edition. Springer–Verlag.

[8] Pindyck, R.S. (1988): Irreversible Investment, Capacity Choice and the Value of the Firm, *American Economic Review*, 78, 969–85.