A decomposition theorem for regular extensions of von Neumann algebras

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Abstract

We show that regular extensions of von Neumann algebras by groups may be decomposed via normal subgroups and quotient groups. An application within the theory of II_1-factors is also given.

1 Introduction

Ever since the pioneering work of Murray and von Neumann, crossed products have been a central theme in the theory of operator algebras. The concept of a regular extension of a von Neumann algebra M by a locally compact (separable) group G was introduced by Sutherland in [13] as a generalization of the twisted crossed product construction considered in [14] (for discrete groups) and in [12]. For a finite factor M and a discrete group G, it had first been studied by Nakamura and Takeda in [7]. In an ordinary twisted crossed product, the twist is produced by a two-cocycle of the group taking values in the unitaries of the center of the algebra. In a regular extension, the two cocycle is now allowed to take unitary values in the whole algebra. When $M = \mathbf{C}$, both constructions coincide and reduce to the von Neumann algebra generated by a projective left regular representation of the group. At last, in an ordinary crossed product, the two-cocycle is just the trivial one.

In [1; proposition 3], we proved that an ordinary crossed product may be decomposed as the iteration of the induced crossed product from a normal (closed) subgroup followed by a regular extension of the quotient group. On the other hand, Packer and Raeburn defined in [9] the twisted crossed product of a twisted C^*_- dynamical system and proved a general decomposition theorem ([9; theorem 4.1]). This suggests that the same decomposition result should hold for regular extensions of von Neumann algebras and, in fact, it does. However, it is not quite obvious how our proof of [1; proposition 3] should be altered to handle the more general situation. As this result is of some importance from a structural point of view, we present a proof in this paper. For the sake of clarity, we restrict ourselves to discrete groups, the proof for locally compact (separable) groups being then essentially a routine matter.

As an illustration of how this decomposition theorem may be used, we shall prove the following result: Suppose that N is a separable II_1-factor which contains a regular subfactor M with trivial relative commutant. Suppose further that M has property Γ and the inclusion

$M \subset N$ is amenable in the sence of Popa [10; 3.2.1]. Then N has also property Γ.

This answers partially of question of Popa [10; 3.3.2], where neither regularity of the subfactor nor triviality of its relative commutant is assumed. It also generalizes [1; theorem A], where we showed that the crossed product of a separable II_1-factor with property Γ by a free action of a countable amenable group has property Γ. Another proof of this last theorem was recently given by Bisch ([2; theorem 2.1]) and one should note that our extended result may alternatively be derived from his work.

Our notation will be as in [1].

2 Decomposition of regular extensions

Let M denote a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. A cocycle crossed action of a (discrete) group G on M is a pair (α, u), where $\alpha : G \to \mathrm{Aut}(M)$ and $u : G \times G \to \mathcal{U}(M)$ satisfy for $g, h, k \in G$:

$$\alpha_g \alpha_h = ad(u(g,h))\alpha_{gh} \,,$$

$$u(g,h)u(gh,k) = \alpha_g(u(h,k))u(g,hk) \,,$$

$$u(g,1) = u(1,h) = 1 \,.$$

The regular extension of M by G, $M \times_{(\alpha,u)} G$, is then defined as the von Neumann algebra acting on $l^2(G,\mathcal{H})$ generated by $\pi_\alpha(M)$ and $\lambda_u(G)$, where π_α is the faithful normal representation of M on $l^2(G;\mathcal{H})$ defined by

$$(\pi_\alpha(x)\xi)(g) = \alpha_{g^{-1}}(x)\xi(g)\,,$$

while, for each $h \in G$, $\lambda_u(h)$ is the unitary operator on $l^2(G,\mathcal{H})$ defined by

$$(\lambda_u(h)\xi)(g) = u(g^{-1},h)\xi(h^{-1}g)$$

$$(x \in M,\ \xi \in l^2(G,\mathcal{H}),\ g \in G)\,.$$

It is well-known that the algebraic structure of $M \times_{(\alpha,u)} G$ is independent of the Hilbert space $\mathcal{H}$ and that the following formulas hold for all $g, h \in G,\ x \in M$:

$$\pi_\alpha(\alpha_g(x)) = ad(\lambda_u(g))(\pi_\alpha(x))$$

$$\lambda_u(g)\lambda_u(h) = \pi_\alpha(u(g,h))\lambda_u(gh)$$

Our aim is to establish the following W^*-algebraic version of [9; theorem 4.1]:

Theorem 1: Let $1 \to H \to G \xrightarrow{\pi} K \to 1$ denote an exact sequence of (discrete) groups and (α, u) a cocycle crossed action of G on a von Neumann algebra M acting on $\mathcal{H}$. Identify H with its copy in G and denote by (α', u') the restriction of (α, u) to H.

Then there exists a cocycle crossed action (β, v) of K on $M \times_{(\alpha',u')} H$ such that

$$M \times_{(\alpha,u)} G \text{ is } *\text{-isomorphic to } (M \times_{(\alpha',u')} H) \times_{(\beta,v)} K\,.$$

Proof: We divide the proof into three lemmas.

Lemma 1: For each $g \in G$, there exists $\gamma_g \in \mathrm{Aut}(M \times_{(\alpha',u')} H)$ such that

i) $\gamma_g(\pi_{\alpha'}(x)) = \pi_{\alpha'}(\alpha_g(x)) \quad (x \in M)$

ii) $\gamma_g(\lambda_{u'}(h)) = \pi_{\alpha'}(u(g,h)u(ghg^{-1},g)^*)\lambda_{u'}(ghg^{-1}) \quad (h \in H)$

Proof: Without loss of generality, we may assume that each α_g is implemented by a unitary operator $a(g)$ on $\mathcal{H}$. (Otherwise, proceed as in the proof of [1; proposition 3, claim 1].) Then define, for each $g \in G$, the operator $b(g) \in \mathcal{B}(l^2(H,\mathcal{H}))$ by

$$(b(g)\xi)(h) = u(h^{-1},g)a(g^{-1})^* u(g^{-1},h^{-1}g)\xi(g^{-1}hg)$$

$$(\xi \in l^2(H,\mathcal{H}),\ h \in H).$$

Then one checks easily that $b(g)$ is a unitary operator on $l^2(H,\mathcal{H})$.

Further we have that

$$b(g)\pi_{\alpha'}(x) = \pi_{\alpha'}(\alpha_g(x))b(g)$$

$$b(g)\lambda_{u'}(h) = \pi_{\alpha'}(u(g,h)u(ghg^{-1},g)^*)\lambda_{u'}(ghg^{-1})b(g)$$

$$(g \in G,\ x \in M,\ h \in H).$$

Indeed, for $\xi \in l^2(H,\mathcal{H})$, $p \in H$, we compute (using the cocycle equations for (α,u)):

$$\begin{aligned}
&(b(g)\pi_{\alpha'}(x)\xi)(p) \\
&\quad = u(p^{-1},g)a(g^{-1})^* u(g^{-1},p^{-1}g)[(\pi_{\alpha'}(x)\xi)(g^{-1}pg)] \\
&\quad = u(p^{-1},g)a(g^{-1})^* u(g^{-1},p^{-1}g)\alpha_{g^{-1}p^{-1}g}(x)\xi(g^{-1}pg)
\end{aligned}$$

$$= u(p^{-1},g)a(g^{-1})^*\alpha_{g^{-1}}(\alpha_{p^{-1}g}(x))u(g^{-1},p^{-1}g)\xi(g^{-1}pg)$$

$$= u(p^{-1},g)\alpha_{p^{-1}g}(x)a(g^{-1})^*u(g^{-1},p^{-1}g)\xi(g^{-1}pg)$$

$$= \alpha_{p^{-1}}(\alpha_g(x))u(p^{-1},g)a(g^{-1})^*u(g^{-1},p^{-1}g)\xi(g^{-1}pg)$$

$$= [\pi_{\alpha'}(\alpha_g(x))b(g)\xi](p),$$

while

$$(b(g)\lambda_{u'}(h)\xi)(p)$$

$$= u(p^{-1},g)a(g^{-1})^*u(g^{-1},p^{-1}g)[(\lambda_{u'}(h)\xi)(g^{-1}pg)]$$

$$= u(p^{-1},g)a(g^{-1})^*u(g^{-1},p^{-1}g)u(g^{-1}p^{-1}g,h)\xi(h^{-1}g^{-1}pg)$$

$$= u(p^{-1},g)a(g^{-1})^*\alpha_{g^{-1}}(u(p^{-1}g,h))u(g^{-1},p^{-1}gh)\xi(h^{-1}g^{-1}pg)$$

$$= u(p^{-1},g)u(p^{-1}g,h)a(g^{-1})^*u(g^{-1},p^{-1}gh)\xi(h^{-1}g^{-1}pg)$$

$$= \alpha_{p^{-1}}(u(g,h))u(p^{-1},gh)a(g^{-1})^*u(g^{-1},p^{-1}gh)\xi(h^{-1}g^{-1}pg)$$

$$= \alpha_{p^{-1}}(u(g,h))u(p^{-1},gh)u(p^{-1}ghg^{-1},g)^*b(g)\xi(gh^{-1}g^{-1}p)$$

$$= \alpha_{p^{-1}}(u(g,h))u(p^{-1},gh)u(p^{-1}ghg^{-1},g)^*u(p^{-1},ghg^{-1})^*[(\lambda_{u'}(ghg^{-1})b(g)\xi)(p)]$$

$$= \alpha_{p^{-1}}(u(g,h))u(p^{-1},gh)(\alpha_{p^{-1}}(u(ghg^{-1},g))u(p^{-1},gh))^*[(\lambda_{u'}(ghg^{-1})b(g)\xi)(p)]$$

$$= \alpha_{p^{-1}}(u(g,h)u(ghg^{-1},g)^*)[\lambda_{u'}(ghg^{-1})b(g)\xi(p)]$$

$$= (\pi_{\alpha'}(u(g,h)u(ghg^{-1},g)^*)\lambda_{u'}(ghg^{-1})b(g)\xi)(p).$$

Thus $\mathrm{ad}(b(g))$ restricted to $M\times_{(\alpha',u')}H$ has the required properties of γ_g, which ends the proof of lemma 1.

Lemma 2: Let $n : K \to G$ be a section for π with $n(1)=1$, and define

$\beta : K \to \mathrm{Aut}(M\times_{(\alpha',u')}H)$ by $\beta = \gamma \circ n$.

Further, define $m : K\times K \to H$ by $m(k,l) = n(k)n(l)n(kl)^{-1}$

and $v : K\times K \to \mathcal{U}(M \times_{(\alpha',u')} H)$ by

$v(k,l) = \pi_{\alpha'}(u(n(k),n(l))u(m(k,l),n(kl))^*)\lambda_{u'}(m(k,l)).$

Then (β,v) is a cocycle crossed action of K on $M \times_{(\alpha',u')} H$.

Proof: Apart from some notational changes, the computations required are precisely those effectued in [9; p. 306–307].

Lemma 3: Define $\Lambda : l^2(K,l^2(H,\mathcal{H}))\to l^2(G,\mathcal{H})$ by

$(\Lambda\xi)(g) = u(g^{-1}n(\pi(g^{-1}))^{-1}, n(\pi(g^{-1})))^*[(\xi(\pi(g)))(n(\pi(g^{-1}))g)]$

$(\xi\in l^2(K,l^2(H,\mathcal{H}), g\in G)$.

Then Λ is a unitary operator such that

i) $\Lambda\pi_\beta(\pi_{\alpha'}(x))\Lambda^* = \pi_\alpha(x)$, $x\in M$

ii) $\Lambda\pi_\beta(\lambda_{u'}(h))\Lambda^* = \lambda_u(h)$, $h\in H$

iii) $\Lambda\lambda_v(k)\Lambda^* = \lambda_u(n(k))$, $k\in K$

Proof: It is easy to check that Λ is unitary.

Now, let $\xi \in l^2(K,l^2(H,\mathcal{H}))$, $g \in G$ and set $l = \pi(g) \in K$, so that

$l^{-1}=\pi(g^{-1})$,

and $w=u(g^{-1}n(l^{-1})^{-1}, n(l^{-1}))^* \in\mathcal{U}(M)$.

As a sample, we prove iii) and leave the proof of i) and ii) as an exercise for masochistic readers.

For $k\in K$, we have that

$(\Lambda\lambda_v(k)\xi)(g)$

$= w[((\lambda_v(k)\xi)(l))(n(l^{-1})g)]$

$= w[(v(l^{-1},k)\xi(k^{-1}l))(n(l^{-1})g)]$

$$= w[(\pi_{\alpha'}(u(n(l^{-1}), n(k))u(m(l^{-1}, k), n(l^{-1}k))^*)\lambda_{u'}(m(l^{-1}, k))\xi(k^{-1}l))$$

$$(n(l^{-1})g)]$$

$$= w\alpha_{g^{-1}n(l^{-1})^{-1}}(u(n(l^{-1}), n(k))u(m(l^{-1}, k), n(l^{-1}k))^*)$$

$$[\lambda_{u'}(m(l^{-1}, k)\xi(k^{-1}l)](n(l^{-1})g)$$

$$= u(g^{-1}, n(k))u(g^{-1}n(l^{-1})^{-1}, n(l^{-1})n(k))^*\alpha_{g^{-1}n(l^{-1})^{-1}}(u(m(l^{-1}, K),$$

$$(*) \quad n(l^{-1}k)))^*u(g^{-1}n(l^{-1})^{-1}, m(l^{-1}, k))[(\xi(k^{-1}l))(m(l^{-1}, k)^{-1}n(l^{-1})g)]$$

$$= u(g^{-1}, n(k))u(g^{-1}n(k)n(l^{-1}k)^{-1}, n(l^{-1}k))^*[(\xi(k^{-1}l))(n(l^{-1}k)n(k)^{-1}g)]$$

(**)

$$= u(g^{-1}, n(k))\Lambda\xi(n(k)^{-1}g)$$

$$= (\lambda_u(n(k))\Lambda\xi)(g),$$

where we have used that $u(a,b)^*\alpha_a(u(b,c)) = u(ab,c)u(a,bc)^*$ with $a = g^{-1}n(l^{-1})^{-1}$, $b = n(l^{-1})$, $c = n(k)$, to obtain equality at (*), and that $\alpha_a(u(b,c))^*u(a,b) = u(a,bc)u(ab,c)^*$ with $a = g^{-1}n(l^{-1})^{-1}$, $b = m(l^{-1}, k)$, $c = n(l^{-1}k)$ at (**). Thus, lemma 3 is proved.

Now, since $(M \times_{(\alpha',u')} H) \times_{(\beta,v)} K$ is generated by

$$\{\pi_\beta(\pi_{\alpha'}(x)), \pi_\beta(\lambda_{v'}(h)), \lambda_v(k); x \in M, h \in H, k \in K\}$$

while $M \times_{(\alpha,u)} G$ is generated by

$$\{\pi_{\alpha'}(x)), \lambda_u(h)), \lambda_u(n(k)); x \in M, h \in H, k \in K\},$$

it is clear that Λ implements the desired *-isomorphism between these two algebras, and this ends the proof of the theorem.

Corollary 2: Let $1 \to H \to G \to K \to 1$ denote an exact sequence of (discrete) groups and $u : G \times G \to T$ (the circle group) a two-cocycle of G. Denote by u' the restriction of u to H, and by $L_u(G)$ (resp. $L_{u'}(H)$) the von Neumann algebra generated by the projective left u-regular (resp. u'-regular) representation of G (resp. H) on $l^2(G)$ (resp. $l^2(H)$). Then $L_u(G)$ may be written as a regular extension of $L_{u'}(H)$ by K.

Proof: Set $M = \mathbf{C}$ in the theorem.

Corollary 2 generalizes [13; proposition 3.1.7], where left regular representations are considered.

3 II_1-factors and Property Γ

In [1; theorem A], we proved that the crossed product of a separable II_1-factor with property Γ by a free action of a countable amenable group has property Γ. With theorem 1 at hand, this result extends to regular extensions. As the proof is in the same vein as the one used to prove [1; theorem B], we sketch it briefly.

Theorem 3: Let M denote a separable II_1-factor and (α, u) a free cocycle crossed action of a countable amenable group G. Then $N \times_{(\alpha,u)} G$ is a II_1-factor, which has property Γ whenever M has property Γ.

Proof: It is well-known that $MX_{(\alpha,u)}G$ is a II_1-factor ([7; theorem 1]). Suppose that M has property Γ. Let $H = \{h \in G | \alpha_h$ is centrally trivial on $M\}$. Then M is a normal subgroup of G and $K = G/H$ is amenable. Let (β, v) denote the cocycle crossed action of K on $M \times_{(\alpha',u')} H$ obtained from the theorem, which by construction is centrally free. By Ocneanus 2-cohomology vanishing result ([8; theorem 1.1]), we may perturb (β, v) to an action $\tilde{\beta}$ of K on N. Now, it follows easily from the covariance formula that N has property Γ when M has. Thus we have that $M \times_{(\alpha,u)} G \simeq N \times_{(\beta,v)} K \simeq N \times_{\tilde{\beta}} K$ has property Γ by invoking theorem 1 and [1; theorem A].

We note that theorem 3 may also be derived from Bisch's [2; theorem 1.1]. Further, the McDuff-version of theorem 3 follows by the same pattern of proof, or from the slightly more general result of Matsumoto ([6; theorem 3.1]). At last, Popa has recently shown that the 2-cohomology vanishes for all cocycle crossed actions of discrete groups with subsexponential growth on II_1-factors ([11; theorem 2.1]), while Connes and Jones have proved that groups such as $SL(3, \mathbf{Z})$ may have non vanishing 2-cohomology on II_1-factors ([4; theorem 5]).

From theorem 3, we will now deduce the result announced in the introduction:

Theorem 4: Let N be a separable II_1-factor and M a regular subfactor of N with $M' \cap N = \mathbf{C}$. If M has property Γ and the inclusion $M \subset N$ is amenable (in the sense of Popa ([10; 3.2.1])), then N has property Γ.

Proof: By Choda's characterization ([3; theorem 4], see also [5]), there exist a countable discrete group G on M such that $N \simeq M \times_{(\alpha,u)} G$. Furthermore, the isomorphism sends $M \subset N$ onto $\pi_\alpha(M) \subset M \times_{(\alpha,u)} G$. Now, by [10; 3.2.4], this last inclusion is amenable if and only if G is amenable. Hence, theorem 3 gives the result.

The basic definition of the amenability of the inclusion $M \subset N$ in theorem 4 requires some knowledge of the notion of correspondences between von Neumann algebras. One equivalent formulation in our setting is the following: There exists a state on the (Jones) extension of N by M which contains N in its centralizer ([10; 3.2.3]).

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