A STOCHASTIC APPROACH TO MOVING BOUNDARY PROBLEMS

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Summary

Moving boundary problems arise for example in the study of fluid flow in porous media. Using optimal stopping of an associated diffusion, a (stochastic) weak concept of a solution of a moving boundary problem is introduced. This allows the use of methods from stochastic analysis to investigate weak/variational and classical solutions.

In particular, we prove the existence of a variational solution $\{p(x,t), W_t\}_{t \geq 0}$ of the moving boundary problem

$$\begin{cases} div(k(x)\nabla_x p(x,t)) = -f(x,t) & \text{for } x \in W_t, \ t \geq 0 \\ p(\cdot,t) \in H_0^1(W_t) & \text{for } t \geq 0 \\ \theta_0 |\partial W_t \cdot \frac{d}{dt}(\partial W_t) = -k\nabla_x p|\partial W_t & \text{for } t \geq 0 \end{cases}$$

where W_0 is a given bounded open set in $\mathbb{R}^n$, $\theta_0(x)$ and $f(x,t)$ are bounded measurable functions and we assume that

(i) $k(x) \geq$ is a Muckenhoupt A_2 weight. (In particular, this allows k to have zeroes)

and

(ii) $W_0 \subset W_t$ for all $t \geq 0$. (This holds, for example, if $f(x,t) \geq 0$ for all x,t)

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§1. Introduction

To describe the flow of an incompressible fluid in a porous medium the following basic equation is used:

(1.1)   Darcy's law:

$$\vec{q} = -\frac{k}{\mu} \nabla p$$

where $\vec{q} = \vec{q}(x,t)$ denotes the seepage - velocity of the fluid at the point $x \in \mathbb{R}^3$ and at time t, $k = k(x) \geq 0$ is the permeability of the medium, μ is the viscosity of the fluid and $p = p(x,t)$ is the pressure of the fluid. (Here and throughout ∇ denotes the gradient with respect to x).

(1.2)   We also use the continuity equation

$$\frac{\partial \theta}{\partial t} = -\operatorname{div}(\rho \vec{q}) + f$$

where $\theta = \theta(x,t)$ is the degree of saturation (i.e. the fluid weight per unit volume of the medium) ρ is the density of the fluid and $f = f(x,t)$ gives the fluid source/sink (depending on the sign of f) rate at the point x and at time t. Combining (1.1) and (1.2) we get

(1.3)
$$\frac{\partial \theta}{\partial t} = \operatorname{div}(k \nabla p) + f,$$

where for simplicity we have set the value of the two constants ρ, μ equal to 1.

In addition we need a relation between θ and p. We will assume that at every instant t $\theta(x,t)$ assumes one of only 2 possible values, 0 or $\theta_0(x) > 0$, corresponding to zero saturation ("dry" region) or complete saturation ("wet" region). Thus we put

(1.4)
$$W_t = \{x; \theta(x,t) = \theta_0(x)\} \text{ (the wet region)}$$

As we will explain below the interpretation of (1.3) then becomes:

(1.5)
$$Lp := \operatorname{div}(k \nabla p) = -f \text{ for } x \in W_t$$

(1.6)
$$p = 0 \text{ for } x \in \partial W_t \quad \text{(the boundary of } W_t)$$

(1.7)
$$\theta_0|_{\partial W_t} \frac{d}{dt}(\partial W_t) = -k \nabla p|_{\partial W_t} \quad (t \geq 0)$$

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Thus *the moving boundary problem* is the following:

Given measurable functions $k : \mathbb{R}^n \to [0, \infty), \theta_0 : \mathbb{R}^n \to (0, \infty)$ and $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and given a bounded initial domain W_0, find a function $p(x, t)$ and a family of domains $\{W_t\}_{t \geq 0}$ such that (1.5)-(1.7) hold. Following Gustafsson [17] we call a solution (p, W_t) of (1.5)-(1.7) a *classical* solution.

Remark. The exact meaning of the "expansion velocity" $\frac{d}{dt}(\partial W_t)$ in (1.7) is the following: Suppose that locally at some time t_0 and some point $y \in \partial W_{t_0}$ the domains W_t can be described by

$$W_t = \{x; \phi(x) < t\},$$

where ϕ is some smooth level function with $\nabla \phi \neq 0$. Then we define

(1.8)
$$\frac{d}{dt}(\partial W_t) = \frac{\nabla \phi}{|\nabla \phi|^2} \text{ at } y.$$
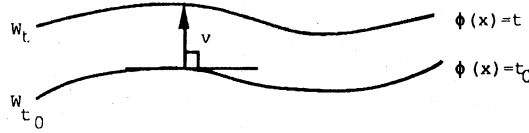
$$v \approx (t - t_0) \frac{\nabla \phi}{|\nabla \phi|^2}$$

In this paper we are primarily interested in the case where the permeability $k(x)$ is allowed to vary rapidly from point to point, so we want to impose as few restrictions on $k(x)$ as possible. In particular, we do not want to assume in the set-up that $k(x)$ is smooth, only that $k(x)$ is (Borel) measurable (and that $k(x) \geq 0$).

This means that (1.3) and (1.5)-(1.7) should be interpreted in the distribution sense. More precisely, let $H_0 = H_0(W_t)$ denote the closure of $C_0^\infty(W_t)$ in the norm

$$\|\psi\|_{H_0}^2 = \int_{W_t} |\psi|^2 k dx + \int_{W_t} |\nabla \psi|^2 k dx$$

The natural variational (or distributional) interpretation of (1.3) is that $p(\cdot, t) \epsilon H_0(W_t)$ (where W_t is open) for each t and

(1.9)
$$-\int\int \theta(x, t)\psi(x)\phi'(t) dx dt$$
$$= -\int\int \nabla p \cdot \nabla \psi \cdot k\phi \, dx dt + \int\int f\psi\phi \, dx dt,$$

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for all $\phi(t) \in C_0^\infty(\mathbb{R}), \psi(x) \in C_0^\infty(\mathbb{R}^3)$

(Here - and in the rest of this paper - $dx, dt \cdots$ means Lebesgue measure on $\mathbb{R}^n$ for the appropriate n. In general we let $C_0^\infty(U)$ denote the family of infinitely differentiable functions with compact support in the set $U \subset \mathbb{R}^n$, while C_0^∞ means $C_0^\infty(\mathbb{R}^n)$ for the appropriate n).

Using (1.4) we rewrite the left hand side of (1.9) as

$$(1.10) \qquad -\int(\int_{W_t} \theta_0(x)\psi(x)dx)\phi'(t)dt = \int \frac{d}{dt}(\int_{W_t} \theta_0(x)\psi(x)dx)\phi(t)dt$$

Since this holds for all $\phi \in C_0^\infty$ we conclude that

$$(1.11) \qquad \frac{d}{dt}(\int_{W_t} \theta_0(x)\psi(x)dx) = -\int \nabla p \cdot \nabla\psi \cdot k dx + \int f\psi dx$$

for all $\psi \in C_0^\infty$.

In particular, suppose $t \to W_t$ is *left continuous*, in the sense that

(1.12) 　　for all compact sets $K \subset W_t$ there exists $\epsilon > 0$ such that $K \subset W_s$ for all $s \in (t - \epsilon, t]$.

Then by (1.11) we get

$$(1.13) \qquad \int \nabla p \cdot \nabla\psi k dx = \int f\psi dx \quad \text{for all } \phi \in C_0(W_t),$$

which is the variational interpretation of (1.5).

To see that (1.11) also contains (1.7), we assume that ∂W_t is smooth and write (for $\psi \in C_0^\infty$)

$$(1.14) \qquad \int_{W_t} \nabla p \cdot \nabla\psi k dx = \int_{\partial W_t} \psi \cdot \frac{\partial p}{\partial n} k d\sigma - \int_{W_t} \psi \cdot Lp dx,$$

where $d\sigma$ denotes surface measure on ∂W_t. Since $Lp = -f$ in W_t, the substitution of (1.14) in (1.11) gives

$$(1.15) \qquad \frac{d}{dt}(\int_{W_t} \theta_0(x)\psi(x)dx) = -\int_{\partial W_t} \psi \cdot \frac{\partial p}{\partial n} \cdot k d\sigma, \quad \forall \psi \in C_0^\infty$$

which is the variational formulation of (1.7).

Thus we may regard both (1.5) and (1.7) as consequences of the one condition (1.3), i.e. of (1.11). However, it is convenient to split (1.11) into the corresponding "inner" and "boundary" part, thereby also allowing a localization of the boundary part. This leads to the following:

DEFINITION 1.1. Let $k : \mathbb{R}^n \to (0, \infty), \theta_0 : \mathbb{R}^n \to (0, \infty)$ and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be measurable functions, $k \in L^1(dx)$ and θ_o and f bounded, and let W_0 be a bounded domain in $\mathbb{R}^n$. We say that $\{p(x,t), W_t\}_{t\geq 0}$ is a variational solution of the moving boundary problem if the following, (1.16)-(1.18), hold:

(1.16)
$$\int_{W_t} \nabla p \cdot \nabla \psi k dx = \int_{W_t} f \psi dx \quad \text{for all} \quad \psi \in C_0^\infty(W_t), t \geq 0$$

(1.17)
$$W_t \subset \mathbb{R}^n \text{ is open and } p(\cdot, t) \in H_0(W_t) \text{ for all } t \geq 0$$

(1.18) For all $t \geq 0$ and all $x_0 \in \partial W_t$ there exists a ball B centered at x_0 such that

$$\frac{d}{dt}(\int_{W_t} \theta_0(x)\psi(x)dx) = -\int \nabla p(x,t) \cdot \nabla \psi(x)k(x)dx + \int f(x,t)\psi(x)dx$$

for all $\psi \in C_0^\infty(B)$.

If (1.18) is interpreted in distribution sense with respect to t, we call the solution *weak variational*. If (1.18) is interpreted in the (strong) sense that the t-derivative exists for each $t \in I$, we call the solution *strong* variational.

The purpose of this paper is twofold. First, we introduce the concept of a *stochastic solution*, defined quite generally. Second, we prove that under mild conditions the stochastic solution actually constitutes a variational solution outside W_0 and we give conditions which guarantee that the stochastic solution is a classical solution.

Regarding the first part, it is now well known through the works of Baiocchi [1], Duvaut [6], Elliott [9], Gustafsson [16], [17] and subsequently Begehr & Gilbert [2], [3] that if $p(x,t) \geq 0$ is a classical solution and we define

(1.19)
$$u(x,t) = \int_0^t p(x,s)ds$$

then for each t the function $u(\cdot, t)$ is a solution of a certain variational inequality denoted by $\mathcal{U}_t$. In general, if for each t a solution $u(\cdot, t)$ of $\mathcal{U}_t$ exists, then $u(x,t)$ is called a *weak solution* of the moving boundary problem. This concept was introduced by Gustaffson in [16], [17]. We refer to these works for more information about such weak solutions. We

remark that it is well known that under certain conditions optimal stopping problems and variational inequalities are equivalent [4], so in such cases our concept of a stochastic weak solution coincides with the weak one. However, as mentioned before we are interested in studying moving boundary problems with as few regularity/ellipticity conditions on the permeability $k(x)$ as possible and in such a general set-up we no longer necessarily have this equivalence. Moreover, by introducing a stochastic approach we can benefit from an efficient machinery from stochastic analysis and Dirichlet forms.

In particular, we prove the existence of a variational solution of the moving boundary problem under the assumptions that (see Theorem 3.4 below)

(1.20) $k(x)$ is Muckenhoupt A_2 weight (see (2.3) below)

(in particular, this allows $k(x)$ to have zeroes)

$$(1.21) \qquad\qquad W_0 \subset W_t \qquad \text{for alle } t$$

(A sufficient, but not necessary, condition for (1.21) is that $f(x,t) \geq 0$ for all x, t).

Remark. This result extends directly to the anistropic case where $k(x) \geq 0$ is replaced by a matrix $[k_{ij}(x)]_{1 \leq i,j \leq n}$, provided that (1.20) is replaced by

(1.20)' There exists a Muckenhoupt A_2 weight $w(x) \geq 0$ and a constant $C > 0$ such that

$$\frac{1}{C} w(x)|\xi|^2 \leq \sum_{i,j} \xi_i \xi_j k_{ij}(x) \leq C w(x)|\xi|^2$$

for all $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$ and all x.

Theorem 3.4 represents a substantial extension of the (isotropic case of the) main result of Begehr & Gilbert in [3], where the corresponding assumptions are

$$(1.22) \qquad\qquad k(x) \in C^{1+\alpha},$$

and

$$(1.23) \qquad\qquad (\exists \lambda > 0) \quad \lambda \leq k(x) \leq \lambda^{-1} \qquad \text{for all } x$$

and

$$(1.24) \qquad\qquad f(x,t) = f(x) \geq 0 \qquad \text{for all } x$$

§2. Construction of the stochastic solution.

Let

$$(2.1) \qquad L\phi = \text{div}(k \bigtriangledown \phi)$$

be the operator in (1.5). First we construct a diffusion $(X_s, P^x, \zeta); s \geq 0, x \in \mathbb{R}^n$ whose generator is L. Here P^x denotes the law of $\{X_s\}_{s \geq 0}$ starting at x and $\zeta \leq \infty$ is the life time of X_s.

A sufficient condition for the existence of such a stochastic process is the Hamza condition

$$(2.2) \qquad \frac{1}{k(x)} \in L^1_{loc}(dx) \; a.e.(dx) \; \text{outside} \; Z,$$

where

$$Z = \{x; k(x) = 0\}, dx = \text{Lebesgue measure on } \mathbb{R}^n.$$

If (2.2) holds then the symmetric bilinear form

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^n} \bigtriangledown u^T \bigtriangledown v k dx; \quad u,v \in C_0^\infty$$

(regarded as a densely defined form on $K = L^2(\mathbb{R}^n, dx)$) is *closable* and hence constitutes a regular Dirichlet form with generator L, i.e,

$$\mathcal{E}(u,v) = -(Lu,v)_K; \quad u \in \mathcal{D}(L), v \in C_0^\infty$$

(Here $(\cdot,\cdot)_K$ denotes the usual inner product in K, i.e.

$$(\phi, \psi)_K = \int_{\mathbb{R}^n} \phi(x)\psi(x)dx; \quad \phi, \psi \in C_0^\infty)$$

Therefore there exists a diffusion $\{X_s\}$ whose generator coincides with L. See [14] for details.

Note that condition (2.2) is very weak. For example, it suffices that $k(x)$ is lower semicontinuous. In such a generality we cannot get enough information about X_t for our purposes, so we will from now on assume that $k(x)$ satisfies the following stronger condition

$$k(x) \text{ is a } Muckenhoupt \; A_2 \; weight, \text{i.e.}$$

$$(2.3) \qquad \sup_B \left(\frac{1}{|B|} \int_B k(x)dx\right)\left(\frac{1}{|B|} \int_B \frac{1}{k(x)}dx\right) < \infty,$$

the sup being taken over all balls $B \subset \mathbb{R}^n$ and $|B| = \int_B dx$ is the volume of B.

The A_p weights were originally introduced by Muckenhoupt in connection with weighted maximal function inequalities. The concept has turned out to be important in many other connections also, for example in the potential theory for degenerate elliptic equations. For example, in [10], [11] and [12] several fundamental results are established regarding the potential theory for

$$L\phi = \text{div}(k \triangledown \phi)$$

when k is an A_2-weight as in (2.3). We will apply some of these results in this paper.

Fix a bounded open set $U \subset \mathbb{R}^n$ such that $U \supset W_0$ and let ζ denote the first exit time for U for X_t. From now on we assume that there exists $T > 0$ such that

$$(2.4) \qquad W_t \cup \text{supp } f(\cdot, t) \subset U \text{ for all } t < T$$

and we consider only the time interval $0 \le t < T$. The *Green operator* (of X_t in U), is defined by

$$(2.5) \qquad Gv(x) = G_U v(x) = E^x\left[\int_0^\zeta v(X_t)dt\right] \text{ for } v \text{ lower bounded,}$$

where E^x denotes expectation with respect to P^x.

This definition is the stochastic equivalent to the variational definition of the Green operator in [11, Theorem 1]. There $G = G_U : H_0(U)^* \to H_0(U)$ is defined by the relation

$$(2.6) \qquad \mathcal{E}_U(G(F), v) = F(v) \quad \text{for } v \in H_0(U), F \epsilon H_0(U)^*,$$

where $\mathcal{E}_U(\phi, \psi) = \int_U \triangledown\phi \cdot \triangledown\psi \cdot kdx; \phi, \psi \in C_0^\infty(v)$ is the Dirichlet form corresponding to the process X killed at the first exit time from U and $H_0(U)^*$ denotes the dual of $H_0(U)$. Now $H_0(U)^*$ can be identified with the space of distributions

$$F = f_0 - \text{div}\vec{f}$$

where $\vec{f} = (f_1, \cdots, f_n)$ and $\frac{f_i}{k} \in L^2(U, kdx), i = 0, 1, \cdots, n$, with the action of F on $v \in H_0(U)$ given by

$$F(v) = \int_U vf_0 dx + \sum_{i=1}^n \int_U f_i \frac{\partial v}{\partial x_i}dx$$

In particular, choosing $T = f(\cdot, t)$ we get from (2.6) the useful relation

$$(2.7) \qquad \mathcal{E}_U(G_U f(\cdot, t), v) = \int_U f(\cdot, t)v(x)dx, \quad v \in H_0(U).$$

For a more detailed explanation of the equivalence of the two definitions (2.5) and (2.6) see for example [20,§3].

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We can now apply an important result about the Green operator from [10, Lemma 3.6]:

(2.8)   If v is bounded, then Gv is continuous.

By a version of the Hille-Yosida theorem (see e.g. [5, p. 252]) it follows that

(2.9) X_t is *Feller-continuous*,

i.e. $x \to E^x[u(X_t)]$ is continuous for each $t \geq 0$ and for each bounded, continuous function u.

We are now ready to start the construction of the (weak) stochastic solution of the moving boundary problem:

Define

$$(2.10) \qquad\qquad \eta = \begin{cases} 0 & \text{on} & W_0 \\ \theta_0 & \text{outside} & W_0 \end{cases}$$

Then by (2.8) the function $G\eta$ is continuous. Choose a continuous function g_0 such that

$$(2.11) \qquad\qquad \begin{cases} g_0 & < & G\eta & \text{on} & W_0 \\ g_0 & = & G\eta & \text{outside} & W_0 \end{cases}$$

Before we proceed we recall some basic notions and results from the theory of optimal stopping:

If g is a lower bounded, continuous function on $\mathbb{R}^n$ (*the reward function*) we define

$$g^*(x) = \sup_\tau E^x[g(X_\tau)],$$

the sup being taken over all $\{\mathcal{M}_t\}$-stopping times τ, where $\mathcal{M}_t$ is the σ-algebra generated by $\{X_s(\cdot); s \leq t\}$.

A function, denoted by $\hat{g}$, is called *the least X_t-superharmonic majorant* of g if $\hat{g}$ is X_t-superharmonic (i.e. $\hat{g}$ is lower semi continuous and $\hat{g}(x) \geq E^x[g(X_\tau)]$ for all M_t-stopping times τ), $\hat{g} \geq g$ and if h is any X_t-superharmonic function such that $h \geq g$ then $h \geq \hat{g}$.

If $V \subset \mathbb{R}^n$ is a Borel set we let

$$\tau_V = \inf\{t > 0; X_t \notin V\}$$

denote the first exit time from V for X_t.

The fundamental theorem of optimal stopping states the following:

THEOREM A. Assume that (2.4) holds. Then we have:
 (i) $\hat{g} = g^*$
 (ii) Define $D = \{x; g(x) < \hat{g}(x)\}$ (the continuation region). Assume that $\tau_0 < \infty$ a.s.
   P^x and that g is bounded. Then τ_D is an optimal stopping time in the sense that

$$g^*(x) = E^x[g(X_{\tau_D})]$$

A proof of Theorem A (which only requires that the process is Feller continuous) can be found in [19, Th. 10.9]. Note that the existence of $\hat{g}$ (which is not obvious) is a part of the statement (i) and that since $\hat{g}$ is lower semicontinuous the set D must be open.

We now apply this to our function g_0 above:

LEMMA 2.1.

$$\hat{g}_0 = G\eta.$$

Proof. Since $G\eta$ is X_t-superharmonic and $G\eta \geq g_0$ we clearly have

$$(2.12) \qquad\qquad G\eta \geq \hat{g}_0$$

On the other hand, by the strong Markov property we have that $G\eta$ is X_t-harmonic in W_0, so

$$G\eta(x) = E^x[G\eta(X_\sigma)]$$

where $\sigma = \tau_{W_0}$. Therefore

$$(2.13) \qquad\qquad G\eta(x) \leq \sup_\tau E^x[g_0(X_\tau)] = g_0^*(x)$$

By Theorem A (i) we have $g_0^* = \hat{g}$ so Lemma 2.1 follows from (2.12) and (2.13).

We conclude that we can identity the starting region W_0 as the continuation region for the optimal stopping problem for g_0:

COROLLARY 2.2.

$$W_0 = \{x; g_0(x) < \hat{g}_0(x)\}$$

The idea is to extend this identification to work for all $t \geq 0$:

For $t \geq 0$ define

$$(2.14) \qquad\qquad g_t(x) = g_0(x) - \int_0^t Gf(x,s)ds$$

and let

$$(2.15) \qquad\qquad g_t^*(x) = \sup_\tau E^x[g_t(X_\tau)]$$

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We now have all the ingredients for the weak stochastic solution concept:

DEFINITION 2.3. Let

$$w(x,t) = g_t^*(x) - g_t(x)$$

and put

$$D_t = \{x; g_t(x) < g_t^*(x)\}; \quad t \geq 0.$$

Then $(w(x,t), D_t)_{t \geq 0}$ is called *the stochastic solution* of the moving boundary value problem.

§3. When does a stochastic solution give a variational solution?

To justify the terminology of Definition 2.3 we now proceed to show that under reasonable conditions we have that

$$p(x,t) := \frac{\partial w}{\partial t}(x,t) \text{ and } W_t := D_t$$

actually is a variational solution of the moving boundary problem (1.16)-(1.18).

First we establish some useful (basically well known) auxiliary results.

LEMMA 3.1. Let $h > 0$. Then

$$g_t^* - \int\limits_t^{t+h} G|f|(x,s)ds \leq g_{t+h}^* \leq g_t^* + \int\limits_t^{t+h} G|f|(x,s)ds$$

Proof. Note that if $h > 0$ then

$$\hat{g}_t = (g_{t+h} + \int\limits_t^{t+h} Gf(x,s)ds)^\wedge \leq \hat{g}_{t+h} + \int\limits_t^{t+h} G|f|(x,s)ds$$

and

$$\hat{g}_{t+h} = (g_t - \int\limits_t^{t+h} Gf(x,s)ds)^\wedge \leq \hat{g}_t + \int\limits_t^{t+h} G|f|(x,s)ds$$

COROLLARY 3.2. If $f(x,t) \geq 0$ for all x, t then $w(x,t)$ and D_t are increasing in t

Proof. Choose $h > 0$. Then by Lemma 3.1

$$w(x, t+h) = g_{t+h}^*(x) - g_{t+h}(x) \geq g_t^*(x) - \int\limits_t^{t+h} Gf(x,s)ds - g_0(x) + \int\limits_0^{t+h} Gf(x,s)ds$$

$$= g_t^*(x) - g_t(x) = w(x,t).$$

It follows that $D_t = \{x; w(x,t) > 0\} \subset \{x; w(x,t+h) > 0\} = D_{t+h}$.

COROLLARY 3.3. $w(x, \cdot)$ is Lipschitz-continuous for each x.

Proof. $|w(x,t+h) - w(x,t)| \leq 2 \int\limits_{t}^{t+h} G|f|(x,s)ds \leq hM$ for all $h > 0$, for some constant M depending only on f and the domain U, using (2.8).

In particular, $\frac{\partial w}{\partial t}(x,t)$ exists for a.a.t. We proceed to show that $\frac{\partial w}{\partial t}$, if it exists, is related to p in (1.13) or (1.5):

LEMMA 3.4. Assume that

$$\frac{\partial w(x,t)}{\partial t} \text{ exists for } t = t_0 \text{ and some } x \in D_{t_0}.$$

Then

(3.1) $$(\frac{\partial w}{\partial t})_{t=t_0} = Gf(x,t_0) - (Gf)^{(t_0)}(x,t_0),$$

where in general $h^{(t)}(x) = E^x[h(X_{\tau_t})]$ denotes the X-harmonic extension of a given function h on ∂D_t to the interior D_t and we have put

$$\tau_t = \inf\{s > 0; X_s \notin D_t\}(= \tau_{D_t}).$$

Proof. For all $t \geq 0$ define the X-harmonic measure $\lambda_x^{(t)}$ on ∂D_t by

$$\lambda_x^{(t)}(F) = P^x[X_{\tau_t} \in F]; F \subset \partial D_t$$

Then if $\frac{\partial w}{\partial t}(x,t_0)$ exists, we have

$$(\frac{\partial w}{\partial t})_{t=t_0} = \frac{\partial}{\partial t}(g_t^*)_{t=t_0} - \frac{\partial}{\partial t}(g_t)_{t=t_0}$$

$$= \frac{\partial}{\partial t}(\int\limits_{\partial D_t} g_t(y)d\lambda_x^{(t)}(y))_{t=t_0} + Gf(x,t_0)$$

$$= \int\limits_{\partial D_{t_0}} \frac{d}{dt}(g_t(y))_{t=t_0}d\lambda_x^{(t_0)}(y) + \frac{d}{dt}(\int\limits_{\partial D_t} g_{t_0}(y)d\lambda_x^{(t)}(y))_{t=t_0} + Gf(x,t_0)$$

$$= -\int\limits_{\partial D_{t_0}} Gf(y)d\lambda_x^{(t_0)}(y) + Gf(x,t_0) = Gf(x,t_0) - (Gf)^{(t_0)}(x,t_0).$$

In this argument we have used that

$$t \to \int\limits_{\partial D_t} g_{t_0}(y)d\lambda_x^{(t)}(y)$$

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is maximal for $t = t_0$ (since D_{t_0} is the continuation region for g_{t_0}) and therefore the derivative is zero if it exists.

We now define

$$(3.2) \qquad p(x,t) = Gf(x,t) - (Gf)^{(t)}(x,t) \quad \text{for all } x \text{ and } t$$

To see that $p(\cdot,t)$ actually solves the boundary value problem (1.13) with $W_t := D_t$, we use the strong Markov property to rewrite p as follows: (Put $\tau_t = \tau$ for simplicity)

$$
\begin{aligned}
p(x,t) &= E^x[\int_0^\zeta f(X_s)ds] - E^x[E^{X_\tau}[\int_0^\zeta f(X_r)dr]] \\
(3.3) \qquad &= E^x[\int_0^\zeta f(X_s)ds] - E^x[E^x[\int_\tau^\zeta f(X_s)ds]] \\
&= E^x[\int_0^\tau f(X_s)ds] = G_{D_t}f(x,t),
\end{aligned}
$$

using the notation of (2.5).

Thus $p(\cdot,t)$ coincides with the Green operator for the domain D_t applied to f and this function is by construction in $H_0(D_t)$ (see (2.6)). Moreover by (2.7) and (3.3) we get

$$\int_{W_t} \nabla p \cdot \nabla \psi k dx = \mathcal{E}_t(p,\psi) = \mathcal{E}_t(G_{D_t}f,\psi) = \int_{D_t} f\psi dx$$

for all $\psi \in C_0^\infty(D_t)$, where $\mathcal{E}_t = \mathcal{E}_{D_t}$. Hence (1.15) holds.

Note that from Corollary 3.3 and Lemma 3.4 we get that for all x $\frac{\partial w}{\partial t}(x,t)$ exists for a.a.t and

$$(3.4) \qquad w(x,t) = \hat{g}_0 - g_0 + \int_0^t G_{D_s}f(x,s)ds \quad \text{for all } t$$

We are now ready to prove one of the main results of this paper:

THEOREM 3.4. Let θ_0, f be as in Definition 1.1, U as in (2.4) and suppose that $k \in A_2$, i.e. k satisfies (2.3).

Suppose that

$$(3.5) \qquad W_0 \subset D_t \quad \text{for all } \quad t \geq 0.$$

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Then
$$p(x,t) := G_{D_t}f(x,t) \text{ and } W_t := D_t = \{x; w(x,t) > 0\}$$

constitute a (weak) variational solution of the moving boundary problem (1.16)-(1.18).

Proof. We have already established that (1.16) and (1.17) hold and that $D_0 = W_0$ (Corollary 2.2). It remains to prove that (1.18) holds.

Fix $t \geq 0$, $x_0 \in \partial D_t$, let B be a ball centered at x_0 and choose $\psi \in C_0^\infty(B)$.

As above put $\mathcal{E}_t(\cdot, \cdot) = \mathcal{E}_{D_t}(\cdot, \cdot)$ and let $(\cdot, \cdot)_t$ denote the inner product in $K_t = L^2(D_t; dx)$. Then since $\eta = \theta_0$ outside W_0, $\eta = 0$ on W_0 and $W_0 \subset D_t$ we have

$$
\begin{aligned}
\int_{D_t} \theta_0 \psi dx - \int_{W_0} \theta_0 \psi dx &= \int_{D_t} \eta \psi dx = (\eta, \psi)_t = \mathcal{E}_t(G\eta, \psi) \\
&= \mathcal{E}_t(g_0, \psi) + \mathcal{E}_t(G\eta - g_0, \psi) \\
&= \mathcal{E}_t(g_t, \psi) + \mathcal{E}_t(\int_0^t Gf ds, \psi)
\end{aligned}
$$

(3.6)

Now if $\psi \in \mathcal{D}_L$ (the domain of the generator L) then

$$
\begin{aligned}
\mathcal{E}_t(\int_0^t Gf ds, \psi) &= -(L\psi, \int_0^t Gf ds)_t = -\int_0^t (L\psi, Gf)_t ds \\
&= \int_0^t \mathcal{E}_t(Gf, \psi) ds = \int_0^t (f, \psi)_t ds,
\end{aligned}
$$

(3.7)

and since $\mathcal{D}_L$ is dense in C_0^∞ (in the $H_0(U)$-norm) (3.7) extends to all $\psi \in C_0^\infty$.

By Theorem A we have $g_t^*(x) = E^x[g_t(X_{\tau_{D_t}})]$, which is X-harmonic in D_t. Therefore

$$g_t^*(x) = E^x[g_t(X_\sigma)] = E^x[\overline{g}_t(X_\sigma)] \text{ for } x \in D_t \setminus \overline{D}_0,$$

where

$$\sigma = \inf\{s > 0; X_s \notin D_t \setminus \overline{D}_0\} \text{ and } \overline{g}_t = G\eta - \int_0^t Gf ds.$$

Hence $g_t^* = G_{D_t \setminus \overline{D}_0}(L\overline{g}_t) + \overline{g}_t$ in $D_t \setminus \overline{D}_0$, so

$$
\begin{aligned}
\mathcal{E}_t(g_t^*, \psi) &= \mathcal{E}_t(G_{D_t \setminus \overline{D}_0}(L\overline{g}_t), \psi) + \mathcal{E}_t(\overline{g}_t, \psi) \\
&= (L\overline{g}_t, \psi) + \mathcal{E}_t(\overline{g}_t, \psi) \\
&= -\mathcal{E}_t(\overline{g}_t, \psi) + \mathcal{E}_t(\overline{g}_t, \psi) = 0
\end{aligned}
$$

(3.8)

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Therefore, using (3.4) we may write

$$\mathcal{E}_t(g_t, \psi) = \mathcal{E}_t(g_t, \psi) - \mathcal{E}_t(g_t^*, \psi) = -\mathcal{E}_t(w_t, \psi)$$

$$(3.9) \qquad = -\mathcal{E}_t(\hat{g}_0 - g_0, \psi) - \mathcal{E}_t\left(\int_0^t \frac{\partial w}{\partial s}(x, s)ds, \psi\right)$$

$$= -\mathcal{E}_t(\hat{g}_0 - g_0, \psi) - \int_0^t \mathcal{E}_t\left(\frac{\partial w}{\partial s}, \psi\right)ds,$$

by adapting the same argument as in (3.7).

Substituting (3.9) and (3.7) in (3.6) we get

$$(3.10) \qquad \int_{D_t} \theta_0 \psi dx - \int_{W_0} \theta_0 \psi dx = -\int_0^t \mathcal{E}_t\left(\frac{\partial w}{\partial s}, \psi\right)ds + \int_0^t (f, \psi)_t ds,$$

and (1.18) follows.

∎

The proof above also leads to a sufficient condition that the stochastic solution is a *strong* variational solution. From (3.10) we see that the left hand side is differentiable for all t if the functions $f(x, s)$ and $G_{D_s} f(x, s)$ are both s-continuous for each x. If we assume that $f(x, \cdot)$ is continuous then a sufficient condition for the continuity of $s \to G_{D_s} f(x, s)$ is the following:

(3.11) $\quad t \to \tau_t(\omega)$ is continous in probability (measure) with respect to P^x, for all x.

Condition (3.11) states that in some weak (stochastic) sense the domains D_t vary continuously with t. A situation where (3.11) does not hold- and where one cannot expect to find a classical solution of the moving boundary problem - is indicated on the figure below.
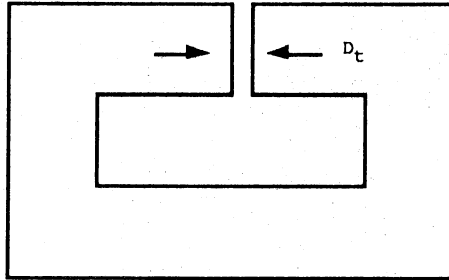
This gives the following result

THEOREM 3.5. Let θ_0, f, U, k, D_t be as in Theorem 3.4 and assume in addition that

(3.11) $\qquad s \to \tau_s(w)$ is continuous in P^x-measure, for all x

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and

(3.12) $\qquad\qquad s \to f(x,s)$ is continuous for all x

Then $s \to w(x,s)$ is continuously differentiable for all x and

$$p(x,t) = \frac{\partial w}{\partial t}(x,t) = G_{D_t}f(x,t), W_t = D_t = \{x; w(x,t) > 0\}$$

constitute a strong variational solution of (1.16)-(1.18).

Finally we return to the classical problem (1.5)-(1.7) and ask for sufficient conditions that our general stochastic solution gives rize to a classical solution. Applying Theorem 3.4 we obtain the following result:

(In the following $C^{k,\alpha}$ will denote the functions whose derivatives up to $k'th$ order are Hölder continuous with exponent $\alpha > 0$)

THEOREM 3.6 Suppose the following holds:

(3.13) $\qquad\qquad k(x) \in C^{1,\alpha}$ for some $\alpha > 0$

(3.14) $\qquad\qquad \inf_{x \in K} k(x) > 0$ for all compacts $K \subset \mathbb{R}^n$

(i.e. $L = \text{div}(k\text{grad})$ is (locally) uniformly elliptic)

(3.15) $\qquad\qquad$ For all $t \geq 0$ there exists $\alpha > 0$ such that $f(\cdot,t) \in C^{0,\alpha}$

In addition, assume that (3.5) holds and that

(3.16) $\qquad\qquad D_t$ is a $C^{2,\alpha}$ domain for each t

Then $t \to w(x,t)$ is continuously differentiable for all x and the function

$$p(x,t) = \frac{\partial w}{\partial t}(x,t) = G_{D_t}f(x,t)$$

together with the sets $D_t = \{x; w(x,t) > 0\}$ solve the classical moving boundary problem (1.15)-(1.17).

Proof of Theorem 3.6: Properties (1.5)-(1.7) follow from known regularity results of solutions of elliptic boundary value problems, (see e.g. [15] or [18]) combined with Theorem 3.5.

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