

# STOCHASTIC MODELLING OF FLUID FLOW IN POROUS MEDIA

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## Abstract

The purpose of this paper is to give a non-technical survey of a stochastic approach to fluid flow in porous media. A key feature of such flow is that a basically *microscopic* property of the medium (the permeability) gives rise to the *macroscopic* property of the flow. To describe this mathematically the permeability is modelled by a certain distribution valued stochastic processes ("positive noise") and then the flow is the solution of the corresponding stochastic partial differential equation. To make this equation mathematically meaningful, the product of the distribution valued processes involved is interpreted as a *Wick product*. This interpretation is motivated both from classical Ito calculus and from the usual renormalization techniques (the removal of terms due to selfinteraction) in quantum statistics.

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## §1. Introduction and motivation.

The equations for the flow of an incompressible fluid in a porous medium are the following:

$$(1.1) \quad \text{Darcy's law:} \quad \vec{q} = -\frac{k}{\mu} \nabla p$$

where  $\vec{q} = \vec{q}(x, t)$  is the *velocity* of the fluid at the point  $x$  and at time  $t$ ,  $p = p(x, t) \geq 0$  is the *pressure* of the fluid at  $x$ ,  $\mu$  (constant) is the *viscosity* of the fluid and  $k = k(x) \geq 0$  is the *permeability* of the medium at  $x$ . The gradient is taken with respect to  $x$ .

$$(1.2) \quad \text{The continuity equation:} \quad \frac{\partial \theta}{\partial t} = -\text{div}(\rho \vec{q}) + f$$

where  $\theta = \theta(x, t)$  is the *saturation* of the fluid at the point  $x$  and at time  $t$ ,  $\rho$  (constant) is the *density* of the fluid and  $f = f(x, t)$  is the *source* or *sink rate* (depending on the sign of  $f$ ) of the fluid.

If we put  $\rho = \mu = 1$  and combine (1.2) and (1.3) we get

$$(1.3) \quad \frac{\partial \theta}{\partial t} = \text{div}(k \nabla p) + f$$

We now make the following simplifying assumption:

$$(1.4) \quad \text{For each } (x, t) \text{ the saturation is either } 0 \\ \text{or equal to complete saturation } \theta_0(x) > 0.$$

With this assumption we define *the wet region* at time  $t$ ,  $D_t$ , as the set of points  $x$  with complete saturation:

$$(1.5) \quad D_t = \{x; \theta(x, t) = \theta_0(x)\}$$

Then we get the following 3 equations

$$(1.6) \quad \text{div}(k \nabla p) = -f \quad \text{in } D_t$$

$$(1.7) \quad p(x, t) = 0 \quad \text{for } x \notin D_t$$

$$(1.8) \quad \theta_0 \cdot \frac{d}{dt}(\partial D_t) = -k \nabla p \quad \text{for } x \in \partial D_t$$

where  $\partial D_t$  denotes the boundary of  $D_t$ .

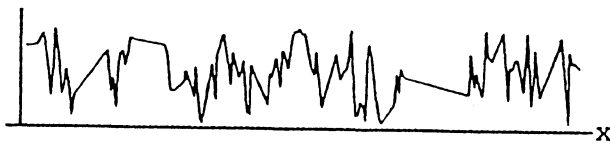
In other words, we have a *moving boundary problem*. If  $k$  is constant or a known, "reasonable" nonnegative function, several methods and results are known regarding this problem. See [G] and the references there, as well as [BG 1-2] and [Ø].

A major difficulty with this setup is that in reality, for example in the important special case of oil/water flow in a porous rock, the permeability function  $k(x)$  is often hard to measure and varies rapidly from point to point. Actual measurements of  $k(x)$  in a porous rock sample in a given direction is shown in Figure 1.

a)



b)



c)

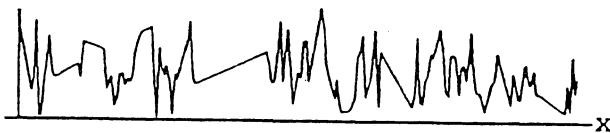


Figure 1

Measurements of permeability in a porous rock. ( A linear interpolation is used on intervals where the values are missing )

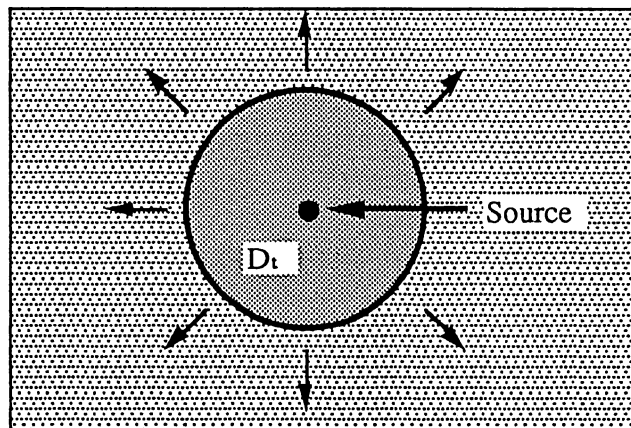


Figure 2

A constant  $k$  leads to solutions of the moving boundary problem consisting of expanding spheres with centre at the injection hole.

One could argue that it would be sufficiently accurate to replace  $k(x)$  by its (constant) average value  $\bar{k}$  in the equations above. However, if  $k(x)$  is constant then the solution of the moving boundary problem (1.6)-(1.8) with  $f(x)$  equal to a point source will be expanding spheres  $D_t$  (Figure 2) for  $t > 0$ .

This is, however, far from what actual experiments with fluid flow in a porous rock show. See Figure 3, which is from an experiment of Knut Jørgen Måløy. See also [OMBAFJ]. In fact, it has been estimated by P. Meakin and others that  $\partial D_t$  is a fractal with Hausdorff dimension about 2.5.



Figure 3

*A physical experiment showing the fractal nature of the wet region ( dark area ). Courtesy of Knut Jørgen Måløy.*

Thus it is necessary to take into account the fluctuations of  $k(x)$  in order to get a good mathematical description of the flow. The crux of the matter is that the permeability is basically a *microscopic* property of the medium (the sizes and shapes of the pores etc.) while we want to describe the *macroscopic* properties of the flow. How can microscopic and macroscopic quantities operate in the same mathematical equation? The answer is

that one should regard  $k$  as a (Schwartz) *distribution*. This means that  $k$  is a function which has average values over macroscopic sets, but no values at specific points: The permeability value only exists as a macroscopic average, it does not make sense to ask for the permeability value at a specific molecule. In fact, returning to Figure 1, the graph shown is the result of taking averages between  $x$  and  $x + h$  ( $h > 0$  fixed) and varying  $x$ . A smaller  $h$  would give an even more erratic picture.

Even if we accept that  $k$  should be represented as a distribution, we are still faced with the problem that we don't know what these average values of  $k$  are. This lack of information suggests that one should use a stochastic approach:  $k$  should be represented as a *distribution valued stochastic process* with certain basic probabilistic properties and one should seek solutions of the corresponding stochastic partial differential equations, also in terms of distribution valued stochastic processes. Properties of the solutions are then found by again taking averages.

This motivates that the right space to work in is the space of tempered distributions, equipped with a natural probability measure. This mathematical model can be used to obtain information about the probabilistic properties of the solution (see §7). In addition we believe it can also be used to gain insight into possible numerical approaches. This is explained more closely in the next section.

## §2. Numerical methods.

A natural numerical approach to the stochastic version of equations like (1.6) is the following: Partition  $D = D_t$  into a high number of disjoint boxes  $D^j$ . On each  $D^j$  pick a value  $k_j(\omega)$  for the permeability  $k$  according to a lognormal distribution, with  $k_i$  independent of  $k_j$  when  $i \neq j$ . Then solve numerically (for each  $\omega$ ) the boundary value problem

$$(2.1) \quad \begin{cases} \operatorname{div} (k(x, \omega) \nabla p) = -f & \text{in } D \\ p = 0 & \text{on } \partial D \end{cases}$$

where  $k(x, \omega) = \sum_j k_j(\omega) \chi_{D^j}(x)$ .

The idea is that by repeating this for a large number of experiments  $\omega$  one should get information about the probabilistic behaviour of the solution  $p = p(x, \omega)$ .

However, as the next simple example illustrates, there is no guarantee that such a numerical approach will give the effects one seeks.

### EXAMPLE

Consider the 1-dimensional version of the related problem

$$(2.2) \quad \begin{cases} (k(x)u'(x))' = 0 & x > 0 \\ u(0) = 0, u'(0) = 1 \end{cases}$$

We split the interval  $[0, 1]$  into  $M$  subintervals and put  $k = k_j(\omega) = e^{N_j(\omega)}$  on subinterval  $I_j$ , where  $\{N_j\}$  are independent normal  $N(0, 1)$  random variables. Then we compute the corresponding solution  $u_M(x, \omega)$  of (2.2). A numerical simulation of  $k$  using  $M = 50$  subintervals is shown on Figure 4, together with the solution  $u = u_{50}(x, \omega)$  associated to this  $k$ .

The splitting of the interval was of course just a numerical device and one would expect that by increasing the number  $M$  of subintervals the solution  $u_M$  should approach the "true" stochastic solution  $u(x, \omega)$ . However, in this case the effect of increasing  $M$  is to kill the stochastics altogether! Figure 5 shows a simulation with  $M = 250$  subintervals.

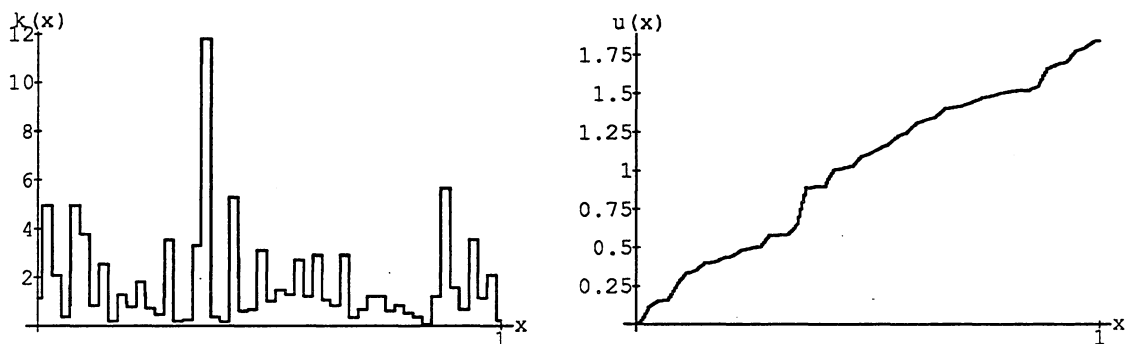


Figure 4

Simulation of  $k(x)$  using  $M = 50$  subintervals, and the corresponding solution  $u(x)$ .

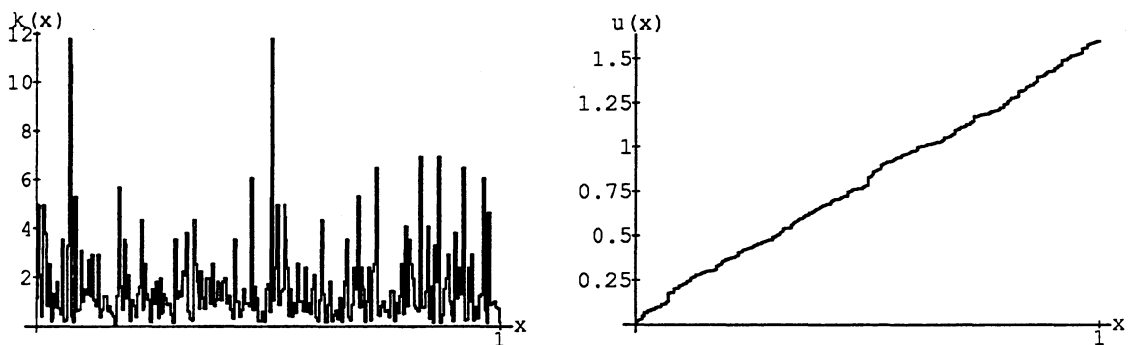


Figure 5

As in Figure 4 but with  $M = 250$  subintervals

If we examine this example more closely we see that the expected value of  $u_M(x) = u_M(x, \omega)$  is given by

$$(2.3) \quad E[u_M(x)] = x$$

and the variance of  $u_M(x)$  satisfies the inequality

$$(2.4) \quad E[|u_M(x) - E[u_M(x)]|^2] \leq \frac{6x}{M}$$

This shows that if one increases  $M$  the solution  $u_M(x, \omega)$  approaches the trivial deterministic solution  $u(x) = x$  corresponding to the constant value  $k(x, \omega) = 1$ . As explained in the introduction, this is certainly not what we are interested in for the fluid flow in porous media. A numerical approach as above would have to consider a different scaling depending on the number  $M$  of subdivisions of the domain. The question is *what* scaling and how to prove convergence when  $M \rightarrow \infty$ .

To get a mathematical platform for handling these and other questions we now turn to the stochastic model indicated in §1.

### §3. The white noise probability space.

Let  $d$  denote the parameter dimension (in the fluid flow we have  $d = 3$ ) and let  $\mathcal{S} = \mathcal{S}(\mathbf{R}^d)$  be the Schwartz space of rapidly decreasing smooth ( $C^\infty$ ) functions on  $\mathbf{R}^d$ . The dual  $\mathcal{S}' = \mathcal{S}'(\mathbf{R}^d)$  is the space of *tempered distributions* on  $\mathbf{R}^d$ .

According to the *Bochner-Minlos theorem* there exists a probability measure  $\mu$  on  $\mathcal{S}'$  such that

$$(3.1) \quad \int_{\mathcal{S}'} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2} \quad \text{for all } \phi \in \mathcal{S}$$

where  $\|\phi\|^2 = \|\phi\|_{L^2(\mathbf{R}^d)}^2 = \int_{\mathbf{R}^d} |\phi(x)|^2 dx$  and  $\langle \omega, \phi \rangle = \omega(\phi)$  is the action of  $\omega \in \mathcal{S}'$  on  $\phi \in \mathcal{S}$ .

We give  $\mathcal{S}'$  the weak star topology and let  $\mathcal{B}$  be the Borel sets on  $\mathcal{S}'$ . The triple  $(\Omega, \mathcal{B}, \mu)$  is called *the white noise probability space*. See [H] or [HKPS].

The *white noise process*  $W : \mathcal{S} \times \Omega \rightarrow \mathbf{R}$  is defined by

$$(3.2) \quad W_\phi(\omega) = \langle \omega, \phi \rangle ; \omega \in \Omega, \phi \in \mathcal{S}.$$

One can prove that

$$(3.3) \quad E_\mu[\langle \cdot, \phi \rangle^2] = \|\phi\|^2 ; \phi \in \mathcal{S}$$

(where  $E_\mu$  denotes expectation w.r.t.  $\mu$ ) and this can be used to define  $W_\psi$  more generally for  $\psi \in L^2(\mathbf{R}^d)$  as follows:

$$(3.4) \quad W_\psi = \lim_{n \rightarrow \infty} W_{\phi_n}$$

the limit being taken in  $L^2(\mu)$ , where  $\phi_n \in \mathcal{S}$  and  $\phi_n \rightarrow \psi$  in  $L^2(\mathbf{R}^d)$ . In particular, if we put

$$(3.5) \quad \tilde{B}_x(\omega) = \tilde{B}_{x_1, x_2, \dots, x_d}(\omega) = \langle \omega, \chi_{[0, x_1] \times \dots \times [0, x_d]} \rangle$$

then there exists an  $x$ -continuous version  $B_x(\omega)$  of  $\tilde{B}_x$  and this version  $B_x$  is a  $d$ -parameter *Brownian motion* (Brownian sheet). In terms of stochastic integration w.r.t.  $B_x$  we can write

$$(3.6) \quad W_\phi(\omega) = \int_{\mathbf{R}^d} \phi(x) dB_x(\omega)$$

In distribution sense this means that we can regard  $W_x$  as the distributional derivative of  $B_x$ :

$$(3.7) \quad W_x = \frac{\partial^d B_x}{\partial x_1 \dots \partial x_d}$$

#### §4. The Wiener-Ito chaos theorem and (white noise) functional processes.

As usual we define  $L^p(\mu) = \{f : \mathcal{S}' \rightarrow \mathbf{R}; \int |f|^p d\mu < \infty\}$ . If  $p > 1$  then  $L^p(\mu) \subset (\mathcal{S}^*)$ , the space of *generalized white noise functionals* (see [HKPS]), but there are important functions in  $L^1(\mu)$  which do not belong to  $(\mathcal{S}^*)$ . (See §7 and also [HLØUZ 1-2]).

Of special interest is the space  $L^2(\mu)$ , which can be given the following representation:

#### **THEOREM 3.1** (The Wiener-Ito chaos theorem)

Every  $f \in L^2(\mu)$  can be written uniquely on the form

$$(4.1) \quad f(\omega) = \sum_{n=0}^{\infty} \int_{(\mathbf{R}^d)^n} f_n(u_1, \dots, u_n) dB_u^{\otimes n}$$

where  $f_n \in \hat{L}^2((\mathbf{R}^d)^n)$  (i.e.  $f_n \in L^2((\mathbf{R}^d)^n)$  and  $f_n$  is *symmetric* in the sense that  $f_n(x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_n}) = f_n(x_1, \dots, x_n)$  for every permutation  $\sigma$  of  $(1, 2, \dots, n)$ ) and  $dB_u^{\otimes n} = dB_{u_1} \dots dB_{u_n}$  denotes Ito's  $n$ -multiple Brownian differential on  $(\mathbf{R}^d)^n$  (see [I]). Moreover, if  $E_\mu$  denotes expectation with respect to  $\mu$ ,

$$(4.2) \quad \|f\|_{L^2(\mu)}^2 := E_\mu[f^2] = \sum_{n=0}^{\infty} n! \|f_n\|_{L^2((\mathbf{R}^d)^n)}^2$$



In the same paper [I] Ito also showed:

If  $\{e_1, e_2, \dots\}$  is a fixed orthonormal basis of  $L^2(\mathbf{R}^d)$  and  $\alpha = (\alpha_1, \dots, \alpha_m)$  is a multiindex of natural numbers then

$$(4.3) \quad \int_{(\mathbf{R}^d)^n} e_1^{\otimes \alpha_1} \otimes e_2^{\otimes \alpha_2} \dots \otimes e_m^{\otimes \alpha_m} dB^{\otimes n} = \prod_{j=1}^m h_{\alpha_j}(\theta_j)$$

where  $\otimes$  denotes the *symmetric* tensor product,

$$(4.4) \quad \theta_j = \theta_j(\omega) = \int_{\mathbf{R}^d} e_j(x) dB_x(\omega) \quad ; \quad j = 1, 2, \dots$$

$n = |\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_m$  and

$$(4.5) \quad h_n(x) := (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} (e^{-\frac{x^2}{2}}) ; \quad n = 1, 2, \dots$$

are the *Hermite polynomials*.

Using that the tensor products of  $\{e_j\}$  constitute a basis of  $L^2((\mathbf{R}^d)^n)$  and expanding each  $f_n$  a sum of such tensor products we see that by combining (4.1) and (4.3) we can write every  $f \in L^2(\mu)$  uniquely on the form

$$(4.6) \quad f(\omega) = \sum_{\alpha} c_{\alpha} H_{\alpha}(\omega)$$

where the sum is taken over all multi-indices  $\alpha$  and

$$(4.7) \quad H_{\alpha}(\omega) = \prod_{j=1}^m h_{\alpha_j}(\theta_j) \quad \text{if } \alpha = (\alpha_1, \dots, \alpha_m)$$

Moreover, we have

$$(4.8) \quad \|f\|_{L^2(\mu)}^2 = \sum_{\alpha} \alpha! c_{\alpha}^2$$

where  $\alpha! = \alpha_1! \alpha_2! \dots \alpha_m!$  if  $\alpha = (\alpha_1, \dots, \alpha_m)$ .

Referring to our discussion in the introduction we now return to the definition of the distribution valued stochastic processes  $X$  we want to work with. Basically these are stochastic variables (i.e. measurable functions on  $S'$ ) for each test function  $\phi$ , i.e.  $X = X(\phi, \omega)$ . Rather than taking derivatives of  $X$  with respect to  $\phi$  in distribution sense we find it conceptually simpler to adopt the Colombeau point of view [C]: For fixed test function or "window"  $\phi$  consider the  $x$ -shifted window  $\phi_x$  defined by

$$\phi_x(y) = \phi(y - x)$$

Then consider derivatives with respect to  $x$  of  $X(\phi_x, \omega)$ . More generally, we consider the following objects:

**DEFINITION.** Let  $p \geq 1$ . An  $L^p$  (white noise) *functional process* is a function

$$X : \mathcal{S} \times \mathbf{R}^d \times \mathcal{S}' \rightarrow \mathbf{R}$$

such that

(i) the map  $x \rightarrow X(\phi, x, \omega)$  is (Borel) measurable for each  $\phi \in \mathcal{S}$  and a.a.  $\omega \in \Omega$

and

(ii) the map  $\omega \rightarrow X(\phi, x, \omega)$  is in  $L^p(\mu)$  for each  $\phi \in \mathcal{S}$  and each  $x \in \mathbf{R}^d$ .

In particular, an  $L^2$  functional process can be written

$$(4.9) \quad X(\phi, x, \omega) = \sum_{\alpha} c_{\alpha}(\phi, x) H_{\alpha}(\omega)$$

where

$$\sum_{\alpha} \alpha! c_{\alpha}^2(\phi, x) < \infty \quad \text{for each } x \in \mathbf{R}^d \quad \text{and each } \phi \in \mathcal{S}$$

and

$$x \rightarrow c_{\alpha}(\phi, x) \quad \text{is measurable for each } \phi \in \mathcal{S}.$$

To summarize we may regard  $X(\phi, x, \omega)$  as *the value of  $X$  obtained by using the window  $\phi$  shifted to the point  $x$  in the experiment  $\omega$* . For example, the  $d$ -parameter white noise  $W_{\phi}$  is an  $L^2$  functional process given by

$$(4.10) \quad W(\phi, x, \omega) = \int_{\mathbf{R}^d} \phi_x(u) dB_u(\omega)$$

Computer simulations of  $W_x = W(\phi, x, \omega)$  when  $d = 1$  and the window is

$$\phi(u) = \frac{1}{h} \cdot \chi_{[0, h]}(u)$$

with  $h = 0.1$ ,  $h = 0.01$  and  $h = 0.001$  are shown in Figure 6.

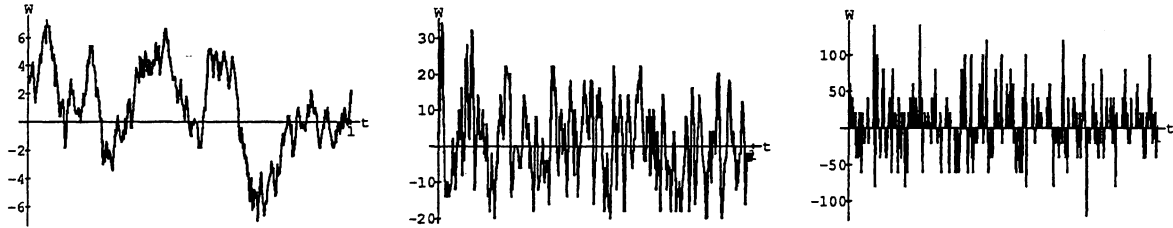


Figure 6

Computer simulations of white noise ( $h = 0.1$ ,  $h = 0.01$  and  $h = 0.001$ )

## §5. Wick multiplication

### a) Wick products of functions in $L^2(\mu)$

We have argued that in equation (1.6) both  $k$  and  $p$  should be modelled by functional processes. Since these are distribution valued stochastic processes this raises the question how to define the product " $k \cdot \nabla p$ ", since in general there is no good product definition for distributions. In this stochastic setting, however, it is natural to adopt a renormalization principle from quantum statistics:

*When we consider the interaction of two classes of particles, we should subtract terms due to interaction of a particle with itself.*

Let us interpret multiplication as a kind of interaction and adopt this principle to give a definition of such a "renormalized product" or *Wick product*, denoted by  $\diamond$ : First consider the special case when

$$(5.1) \quad X(\omega) = \int_{\mathbf{R}^d} \phi(x) dB_x(\omega) \quad \text{and} \quad Y(\omega) = \int_{\mathbf{R}^d} \psi(x) dB_x(\omega)$$

with  $\phi, \psi \in \mathcal{S}$ . Then renormalization leads to the definition

$$(5.2) \quad X \diamond Y = X \cdot Y - \int_{\mathbf{R}^d} \phi(x) \psi(x) dx,$$

because  $\int_{\mathbf{R}^d} \phi(x) \psi(x) dx$  is a natural representation of the "selfinteraction" term. (If, for example,  $\phi$  and  $\psi$  have disjoint support, there is no "selfinteraction"). See [HLØU] for a discrete analogue. Using Ito's multiple integral we can write (5.2) as follows:

$$(5.3) \quad X \diamond Y = \int_{(\mathbf{R}^d)^2} \phi \otimes \psi(x, y) dB_{x,y}^{\otimes 2}$$

when  $X, Y$  are given by (5.1).

Thus the direct consequence of this definition is to define more generally *the Wick product of two functions*  $X, Y \in L^2(\mu)$  represented by

$$X = \sum_{n=0}^{\infty} \int_{(\mathbb{R}^d)^n} f_n dB^{\otimes n} \quad \text{and} \quad Y = \sum_{m=0}^{\infty} \int_{(\mathbb{R}^d)^m} g_m dB^{\otimes m}$$

as follows:

$$(5.4) \quad X \diamond Y = \sum_{n,m} \int_{(\mathbb{R}^d)^{(n+m)}} f_n \otimes g_m dB^{\otimes(n+m)},$$

whenever the series on the right converges in  $L^1$ .

An equivalent way of stating this is, using (4.6):

If  $X(\omega) = \sum_{\alpha} a_{\alpha} H_{\alpha}(\omega)$  and  $Y(\omega) = \sum_{\beta} b_{\beta} H_{\beta}(\omega)$  then

$$(5.5) \quad X \diamond Y(\omega) = \sum_{\alpha, \beta} a_{\alpha} b_{\beta} H_{\alpha+\beta}(\omega)$$

whenever the series on the right converges in  $L^1$ .

Another motivation for the use of this Wick product comes from ordinary (i.e.  $d = 1$ ) Ito stochastic differential equations: In [LØU 2, Corollary 3.4] it is proved that if  $Y_t$  is a "reasonable" adapted stochastic process then in fact

$$(5.6) \quad \int_0^T Y_t(\omega) dB_t(\omega) = \int_0^T Y_t \diamond W_t(\omega) dt$$

in a sense which is made precise in [LØU 2]. (There are several versions of this result. See also [AP]). This means that if one - as is customary - interprets "multiplication by white noise" in a stochastic differential equation as a differential version of "integral with respect to Brownian motion" one is really interpreting the product in the Wick sense. So from this point of view the use of Wick products in our setting is a natural extension from the classical, 1-dimensional case.

We also point out that if one of the two factors is deterministic, then the Wick product reduces to the usual product.

b) *Wick products of functions in  $L^1(\mu)$ .*

It turns out that solution of equations like (1.6) will not be in  $L^2(\mu)$  (in fact not even in  $(\mathcal{S}^*)$ ), but only in  $L^1(\mu)$  (see §7 and [HLØU 1-2]). Therefore it is necessary to extend the

definition of Wick product to  $L^1(\mu)$ -functions. The most natural - and naive - way to do this is the following:

$$(5.7) \quad \text{If } X, Y \in L^1(\mu), \text{ choose } X_n, Y_n \in L^2(\mu)$$

such that

$$X_n \rightarrow X \text{ in } L^1(\mu), Y_n \rightarrow Y \text{ in } L^1(\mu)$$

Define

$$X \diamond Y = \lim_{n \rightarrow \infty} X_n \diamond Y_n \quad (\text{limit in } L^1(\mu)),$$

provided the limit on the right exists in  $L^1(\mu)$ .

It can be proved that this definition does not depend on the choice of the approximating sequences  $\{X_n\}, \{Y_n\}$  (see [HLØUZ 1]).

In fact, if  $X, Y \in L^1(\mu)$  and  $X \diamond Y = \lim_{n \rightarrow \infty} X_n \diamond Y_n$  exists in  $L^1(\mu)$  then

$$(5.8) \quad \mathcal{F}(X \diamond Y)(\phi) = e^{-\frac{1}{2}\|\phi\|^2} \mathcal{F}(X)(\phi) \cdot \mathcal{F}(Y)(\phi)$$

where

$$(5.9) \quad \mathcal{F}(Z)(\phi) = \int_{S'} e^{i\langle \phi, \omega \rangle} Z(\omega) d\mu(\omega) \quad (\phi \in S)$$

denotes the Fourier transform of  $Z \in L^1(\mu)$ . In particular, choosing  $\phi = 0$  in (5.8) we get

$$(5.10) \quad E[X \diamond Y] = E[X] \cdot E[Y] \quad ([HLØUZ 1])$$

## §6. A mathematical model for permeability: Positive noise.

The remaining problem in finding a mathematical model for equation (1.6) is to represent the permeability  $k(x)$  as a suitable functional process  $K(\phi, x, \omega)$ . One obvious property that such a process should have is the *positivity*, i.e.

$$(6.1) \quad K(\phi, x, \omega) \geq 0$$

for all  $\phi \in S, x \in \mathbb{R}^d$  and a.a.  $\omega \in S'$ .

Therefore white noise  $W_\phi = W(\phi, x, \omega)$  will not do. And neither will the Wick square of white noise, since

$$(6.2) \quad E_\mu[W_\phi^{\diamond 2}] = E_\mu\left[\int_{(\mathbb{R}^d)^2} \phi \otimes \phi dB^{\otimes 2}\right] = 0$$

However, if we define the Wick exponential of white noise by

$$(6.3) \quad \text{Exp}[W_\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} W_\phi^{\circ n}$$

then one can prove that in fact

$$(6.4) \quad \text{Exp}[W_\phi] = \exp(W_\phi - \frac{1}{2}\|\phi\|^2), \quad \phi \in \mathcal{S}. \quad (\text{see [LØU 1]}).$$

This shows that  $\text{Exp}[W_\phi]$  is positive. Computer simulations of  $\text{Exp}[W_\phi]$  are shown in Figure 7.

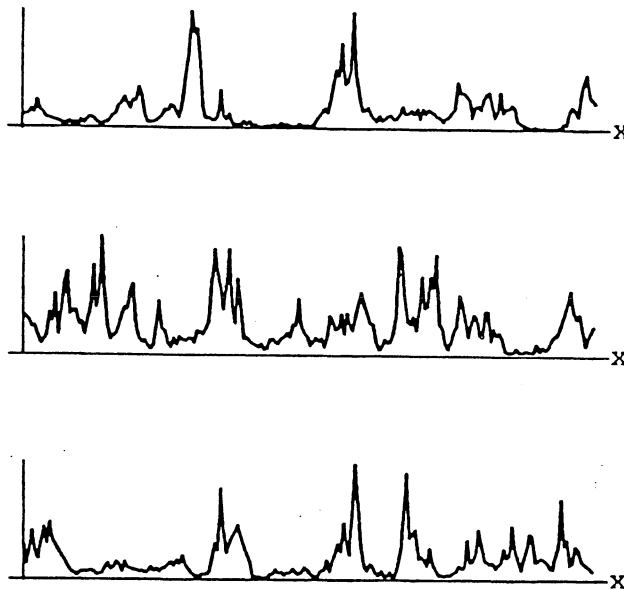


Figure 7

*Computer simulations of the Wick exponential of white noise*

Comparing the simulations in Figure 7 with the actual measurements in Figure 1 it is not unreasonable to accept  $\text{Exp}[W_\phi]$  as our model for the permeability process  $K(\phi, x, \omega)$ . However, at the present stage this should be regarded as just one of several possibilities, and the important question of finding the best representation for  $K$  is a problem of future research.

More generally, if  $N(\phi, x, \omega)$  is a functional process such that

$$(6.5) \quad N(\phi, x, \omega) \geq 0$$

for all  $\phi \in \mathcal{S}, x \in \mathbf{R}^d$  and a.a.  $\omega \in \mathcal{S}'$  we call  $N$  a *positive noise*. A characterization of an  $L^2$ -positive noise in terms of its *Hermite transform* is given in [LØU 1] (see also [HLØUZ 1]). A consequence of this characterization is that if  $X$  and  $Y$  are  $L^1$ -positive noises, then  $X \diamond Y$  is positive also (when defined). This result is important as a justification of the mathematical model we have constructed.

### §7. The stochastic pressure equation for fluid flow in porous media.

We now return to the original equations (1.6) - (1.8) governing fluid flow in porous media. At this stage we consider only equations (1.6) and (1.7) for a given  $t$ . According to our discussion above we interpret these equations as follows:

Let  $D \subset \mathbf{R}^d$  be a given bounded domain and let  $f$  be a bounded continuous function on  $D$ . Find an  $L^1$ -functional process  $X(x) = X(\phi, x, \omega)$  such that

$$(7.1) \quad \operatorname{div}(\operatorname{Exp} W(x) \diamond \nabla X(x)) = -(f * \phi)(x) \text{ in } D$$

and

$$(7.2) \quad X(x) = 0 \quad \text{on } \partial D$$

where  $*$  denotes convolution:

$$(f * \phi)(x) = \int_{\mathbf{R}^d} f(y)\phi(y-x)dy$$

and  $\operatorname{div}, \nabla$  are taken with respect to  $x$ .

Here  $W(x) = W_{\phi_x}(\omega)$  as before and the equation (7.1) is to be regarded as a differential equation in  $x$ , for each  $\phi$  and a.a.  $\omega$ . The difference from a deterministic boundary value problem (for each fixed  $\phi$  and  $\omega$ ) is that ordinary multiplication is replaced by Wick multiplication. (See [LØU 2] for a further discussion about this).

With this in mind we arrive at the following candidate for a solution:

$$(7.3) \quad X(\phi, x, \omega) = \frac{1}{2} \operatorname{Exp}\left[-\frac{1}{2}W(x)\right] \diamond \hat{E}^x \left[ \int_0^{\tau_D} (f * \phi)(b_t) \operatorname{Exp}\left\{-\frac{1}{2}W(b_t)\right\} \right. \\ \left. - \frac{1}{4} \int_0^t \left[ \frac{1}{2} \nabla W^{\diamond 2}(b_s) + \Delta W(b_s) \right] ds \right] dt$$

Here  $(b_t, \hat{P}^x)$  is an auxiliary (1-parameter) Brownian motion in  $\mathbf{R}^d$ ,  $\tau_D = \inf\{t > 0; b_t \notin D\}$  and  $\hat{E}^x$  denotes expectation with respect to the law of  $\hat{P}^x$  of  $b_t$  starting at  $x$  ( $\hat{P}^x(b_0 = x) =$

1), while  $\nabla W(b_s)$  means  $(\nabla_x W_{\phi_x})_{x=b_s}$ , etc. In order to verify that  $X$  given by (7.3) is the solution of (7.1), one must prove that  $X \in L^1(\mu)$ , that  $\text{Exp}W \diamond \nabla X$  is well-defined (in the sense of (5.7)), and that  $X$  satisfies the partial differential equation. For these and other properties of the solution we refer to the forthcoming paper [HLØUZ 2].

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## REFERENCES

- [AP] J. Asch and J. Potthoff: Ito's lemma without non-anticipatory conditions. *Probab. Th. Rel. Fields* 88 (1991), 17-46.
- [BG1] H. Begehr and R.P. Gilbert: Hele-Shaw type flows in  $\mathbb{R}^n$ . *Nonlinear Analysis* 10 (1986), 65-86.
- [BG2] H. Begehr and R.P. Gilbert: Non-Newtonian Hele-Shaw flows in  $n \geq 2$  dimensions. *Nonlinear Analysis* 11 (1987), 17-47.
- [C] J.F. Colombeau: Generalized functions. *Bull. Amer. Math. Soc.* 23 (1990), 251-268.
- [G] B. Gustafsson: Applications of variational inequalities to a moving boundary problem for Hele-Shaw flows. *Siam J. Math. Anal.* 16 (1985), 279-300.
- [H] T. Hida: *Brownian Motion*. Springer-Verlag 1980.
- [HKPS] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit: *White Noise Analysis*. (Forthcoming book).
- [HLØU] H. Holden, T. Lindstrøm, B. Øksendal and J. Ubøe: Discrete Wick calculus and stochastic functional equations. Manuscript, University of Oslo 1991.
- [HLØUZ 1] H. Holden, T. Lindstrøm, B. Øksendal, J. Ubøe and T.-S. Zhang: Stochastic boundary value problems. A white noise functional approach. Manuscript, University of Oslo 1991.
- [HLØUZ 2] H. Holden, T. Lindstrøm, B. Øksendal, J. Ubøe and T.-S. Zhang: On the equation  $\text{div}(K \text{ grad } X) = -f$  when  $K$  is positive noise. (In preparation).



- [I] K. Ito: Multiple Wiener integral. *J. Math. Soc. Japan* 3 (1951), 157-169.
- [LØU 1] T. Lindstrøm, B. Øksendal and J. Ubøe: Stochastic differential equations involving positive noise. To appear in M. Barlow and N. Bingham (editors): *Stochastic Analysis*. Cambridge Univ. Press 1991/92.
- [LØU 2] T. Lindstrøm, B. Øksendal and J. Ubøe: Wick multiplication and Ito-Skorohod stochastic differential equations. To appear in S. Albeverio et. al. (editors): *Ideas and Methods in Mathematical Analysis, Stochastics and Applications*. Cambridge Univ. Press 1991/92.
- [Ø] B. Øksendal: A stochastic approach to moving boundary value problems. In M. Pinsky (editor): *Diffusion Processes and Related Problems in Analysis, Volume 1*, pp. 201-218. Birkhäuser 1990.
- [OMBAFJ] U. Oxaal, M. Murat, F. Boger, A. Aharony, J. Feder and T. Jøssang: Viscous fingering on percolation clusters. *Nature* 329 (1987), 32-37.