WICK MULTIPLICATION AND ITO-SKOROHOD STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract

We show that there is a close connection between deterministic differential equations of the form

$$\frac{d\xi_t}{dt} = b(\xi_t) + \sigma(\xi_t) \cdot \sum_k \zeta_k(t) z_k$$

(where $z_k = x_k + iy_k$ are complex parameters) and Ito-Skorohod stochastic differential equations of the form

$$dX_t = b^\diamond(X_t)dt + \sigma^\diamond(X_t)\delta B_t,$$

where $b^\diamond, \sigma^\diamond$ denote the *Wick versions* of the functions b, σ.

The connection is provided by the Hermite transform $\mathcal{H}$, which maps L^2 stochastic processes X_t into (deterministic) analytic functions $\mathcal{H}(X_t)(z_1, z_2, \cdots)$ on

$$\mathbb{C}_0^{\mathbb{N}} = \{(z_1, z_2, \cdots); z_k \in \mathbb{C} \text{ and } \exists M \text{ with } z_j = 0 \text{ for } j > M\},$$

and by its inverse $\mathcal{H}^{-1}$, which can be given an explicit form.

§1. Introduction.

The purpose of this paper is to establish a link between deterministic differential equations and Ito-Skorohod stochastic differential equations. If the coefficients are analytic functions the connection becomes particularly simple. The key to the link is to replace ordinary products in the deterministic equation by Wick products $\diamond$ in the corresponding Ito-Skorohod equation. More generally, the given ordinary functions f should be replaced by their *Wick versions* $f^{\diamond}$.

The proof of this connection is provided by the use of the ***Hermite transform*** $\mathcal{H}$ and its (left) inverse $\mathcal{H}^{-1}$. The Hermite transform associates to a given (generalized) stochastic process X_t on the white noise probability space $(\mathcal{S}', \mathcal{F}, \mu)$ (see definition in §2) an analytic function $\mathcal{H}(X_t)(z_1, z_2, \cdots) = \tilde{X}_t(z_1, z_2, \cdots)$ on $\mathbb{C}_0^{\mathbb{N}}$. This particular transform was introduced by us in [LØU], but the general idea of associating analytic functions to functions on $\mathcal{S}'$ is much older. See [H], [HKPS] and the survey [MY] and the references there. An important property of $\mathcal{H}$ is that it transforms Wick products into ordinary complex products and this explains its role in the link above.

Another crucial property of $\mathcal{H}$ is that it has a (left) inverse $\mathcal{H}^{-1}$ which can be computed explicitly as an integral with respect to the infinite product of the normalized Gaussian measures on $\mathbb{R}$. This gives a useful method for solving Ito-Skorohod stochastic (possibly anticipating) differential equations involving Wick versions.

A key result (Theorem 3.3) states that if $\int \cdot \delta B_t$ denotes Skorohod integral (B_t is Brownian motion) then

$$\int_0^T Y_t \delta B_t = \int_0^T Y_t \diamond W_t dt$$

for all Skorohod integrable processes Y_t, where W_t denotes the white noise (generalized) process. Thus Wick multiplication appears naturally when Ito or Skorohod stochastic differential equations are used to model dynamical systems with noise. Ordinary and Wick multiplication coincide for deterministic processes, but for stochastic processes the products differ and we would like to stress that it is not obvious what type of product one should use to get the best model. As an example - and an illustration of our main result - we discuss (§4) the following model for population growth in a crowded, random environment:

$$dX_t = rX_t \diamond (N - X_t)dt + \alpha X_t \diamond (N - X_t)\delta B_t$$

§2. Some preliminaries.

Since white noise is so fundamental for our construction, we recall some basic facts about this generalized (i.e. distribution valued) process:

For $n = 1, 2, \cdots$ let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all rapidly decreasing smooth (C^∞)

functions on $\mathbb{R}^n$. Then $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space under the family of seminorms

$$\|f\|_{N,\alpha} = \sup_{x\in\mathbb{R}}(1+|x|^N)|\partial^\alpha f(x)|,$$

where $N \geq 0$ is an integer and $\alpha = (\alpha_1, \cdots, \alpha_k)$ is a multi-index of non-negative integers α_j. The space of *tempered distributions* is the dual $\mathcal{S}'(\mathbb{R}^n)$ of $\mathcal{S}\mathbb{R}^n)$, equipped with the weak star topology.

Now let $n = 1$ for the rest of this section and put $\mathcal{S} = \mathcal{S}'(\mathbb{R})$. By the Bochner-Minlos theorem (see e.g. [GV]) there exists a probability measure μ on $(\mathcal{S}', \mathcal{F})$ (where $\mathcal{F}$ denotes the Borel subsets of $\mathcal{S}'$) such that

$$E^\mu[e^{i<\omega,\phi>}] := \int_{\mathcal{S}'} e^{i<\omega,\phi>} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2} \text{ for all } \phi \in \mathcal{S}, \tag{2.1}$$

where $\|\phi\|^2 = \|\phi\|^2_{L^2(\mathbb{R})}$ and $<\omega, \phi> = \omega(\phi)$ for $\omega \in \mathcal{S}'$. We call $(\mathcal{S}', \mathcal{F}, \mu)$ the *white noise probability space.*

It follows from (2.1) that

$$\int_{\mathcal{S}'} f(<\omega,\phi>)d\mu(\omega) = (2\pi\|\phi\|^2)^{-\frac{1}{2}} \int_{\mathbb{R}} f(t)e^{-\frac{t^2}{2\|\phi\|^2}} dt; \phi \in \mathcal{S}, \tag{2.2}$$

for all f such that the integral on the right converges. (It suffices to prove (2.2) for $f \in C_0^\infty$, i.e. f smooth with compact support. Such a function f is the inverse Fourier transform of its Fourier transform $\hat{f}$ and we obtain (2.2) by (2.1) and the Fubini theorem). In particular, if we choose $f(t) = t^2$ we get from (2.2)

$$E^\mu[<\omega,\phi>^2] = \|\phi\|^2; \phi \in \mathcal{S}. \tag{2.3}$$

This allows us to extend the definition of $<\omega,\phi>$ from $\phi \in \mathcal{S}$ to $\phi \in L^2(\mathbb{R})$ for a.a. $\omega \in \mathcal{S}'$, as follows:

$$<\omega,\phi> := \lim_{k\to\infty} <\omega,\phi_k> \text{ for } \phi \in L^2(\mathbb{R}), \tag{2.4}$$

where ϕ_k is any sequence in $\mathcal{S}$ such that $\phi_k \to \phi$ in $L^2(\mathbb{R})$ and the limit in (2.4) is in $L^2(\mathcal{S}', \mu)$.

In particular, if we define

$$\tilde{B}_t(\omega) := <\omega, \chi_{[0,t]}> \tag{2.5}$$

then we see that $(\tilde{B}_t, \mathcal{S}', \mu)$ becomes a Gaussian process with mean 0 and covariance

$$E^\mu[\tilde{B}_t(\omega)\tilde{B}_s(\omega)] = \int_{\mathcal{S}'} <\omega, \chi_{[0,t]}> \cdot <\omega, \chi_{[0,s]}> d\mu(\omega)$$

$$= \int_{\mathbb{R}} \chi_{[0,t]}(x) \cdot \chi_{[0,s]}(x)dx = \min(s,t), \text{ using (2.3).}$$

Therefore $\tilde{B}_t$ is essentially a Brownian motion, in the sense that there exists a t-continuous version B_t of $\tilde{B}_t$:

$$\mu(\{\omega; B_t(\omega) = \tilde{B}_t(\omega)\}) = 1 \text{ for all } t.$$

If $u \in L^2(\mathbb{R})$ we define, using (2.4)

$$\int_{-\infty}^{\infty} \phi(t) dB_t(\omega) = <\omega, \phi>,$$

which coincides with the classical Ito integral if supp $\phi \subset [0, \infty)$.

If we define the *white noise process* W_ϕ by

$$W_\phi(\omega) = <\omega, \phi> \text{ for } \phi \in \mathcal{S}, \omega \in \mathcal{S}' \tag{2.6}$$

then the *white noise process* W_ϕ may be regarded as the distributional derivative of B_t, in the sense that, if $\phi \in \mathcal{S}$

$$\begin{aligned}
<\frac{d}{dt}B_t(\omega), \phi> &= -\int_{-\infty}^{\infty} \phi'(t) B_t(\omega) dt = \int_{-\infty}^{\infty} \phi(t) dB_t(\omega) \\
&= \lim_{\Delta t_j \to 0} \sum_j \phi(t_j)(B_{t_{j+1}} - B_{t_j}) = \lim_{\Delta t_j \to 0} \sum_j \phi(t_j) <\omega, \chi_{(t_j, t_{j+1}]}> \\
&= \lim_{\Delta t_j \to 0} <\omega, \sum_j \phi(t_j)\chi_{(t_j, t_{j+1}]}> = <\omega, \phi> = W_\phi(\omega),
\end{aligned}$$

where the second identity is based on integration by parts for Ito integrals.

By the Wiener-Ito chaos theorem (see e.g. [I] and [HKPS]), we can write any function $f \in L^2(\mu)(= L^2(\mathcal{S}', \mu))$ on the form

$$f = \sum_{n=0}^{\infty} \int f_n dB^{\otimes n}, \tag{2.7}$$

where

$$f_n \in \hat{L^2}(\mathbb{R}^n, dx), \tag{2.8}$$

i.e. $f_n \in L^2(\mathbb{R}^n, dx)$ and f_n is *symmetric* (in the sense that $f_n(x_{\sigma_1}, x_{\sigma_2}, \cdots, x_{\sigma_n}) = f(x_1, \cdots, x_n)$ for all permutations σ of $(1, 2, \cdots, n)$) and

$$\begin{aligned}
\int f_n dB^{\otimes n} &= \int_{\mathbb{R}^n} f_n(u) dB_u^{\otimes n} \\
&= n! \int_{-\infty}^{\infty} \left(\int_{-\infty}^{u_n} \cdots \int_{-\infty}^{u_3} \left(\int_{-\infty}^{u_2} f_n(u_1, \cdots, u_n) dB_{u_1} \right) dB_{u_2} \cdots dB_{u_{n-1}} \right) dB_{u_n}
\end{aligned} \tag{2.9}$$

for $n \geq 1$, while $n = 0$ term in (3.1) is just a constant f_0.

For a general (non-symmetric) $f \in L^2(\mathbb{R}^n)$ we define

$$\int f dB^{\otimes n} := \int \hat{f} dB^{\otimes n} \tag{2.10}$$

where $\hat{f}$ is the symmetrization of f, defined by

$$\hat{f}(u_1, \cdots, u_n) = \frac{1}{n!} \sum_{\sigma} f(u_{\sigma_1}, \cdots, u_{\sigma_n}), \tag{2.11}$$

the sum being taken over all permutations σ of $(1, 2, \cdots, n)$.
With f, f_n as in (2.7) we have

$$\|f\|^2_{L^2(\mu)} = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2(\mathbb{R}^n)} \tag{2.12}$$

Note that (2.12) follows from (2.7) and (2.9) by the Ito isometry, since

$$E[(\int_{\mathbb{R}^n} f_n dB^{\otimes n})(\int_{\mathbb{R}^m} f_m dB^{\otimes m})] = 0 \text{ for } n \neq m$$

and

$$E[(\int_{\mathbb{R}^n} f_n dB^{\otimes n})^2] = (n!)^2 E[(\int_{-\infty}^{\infty} \cdots (\int_{-\infty}^{u_2} f_n(u_1, \cdots, u_n) dB_1) \cdots dB_{u_n})^2]$$

$$= (n!)^2 \cdot \int_{-\infty}^{\infty} \cdots (\int_{-\infty}^{u_2} f_n^2(u_1, \cdots, u_n) du_1) \cdots du_n = n! \int_{\mathbb{R}^n} f_n^2(u) du$$

Here $B_t(\omega); t \geq 0, \omega \in \mathcal{S}'$ is the 1-dimensional Brownian motion associated with the white noise probability space $(\mathcal{S}', \mu)$ as explained above.

Now suppose that $X_t = X(t, \omega) : \mathbb{R} \times \mathcal{S}' \to \mathbb{R}$ is an $\mathcal{B} \times \mathcal{F}$-measurable stochastic process such that $E[X_t^2] < \infty$ for all t. (Here $\mathcal{B}$ denotes the Borel σ-algebra on $\mathbb{R}$). Then by the above there exist $f_n(t, \cdot) \in \hat{L}^2(\mathbb{R}^n)$ such that

$$X_t(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n(t, u_1, \cdots, u_n) dB_u^{\otimes n}(\omega) \tag{2.13}$$

Moreover, each f_n can be chosen measurable in all its variables (see [NZ]). Fix $T > 0$. Let $\hat{f}_n$ denote the symmetrization of $f_n \cdot \chi_{0 \leq t \leq T}$ with respect to its $n + 1$ variables. Suppose

$$E[\int_0^T X_t^2 dt] + \sum_{n=0}^{\infty} (n+1)! \|\hat{f}_n\|^2_{L^2(\mathbb{R}^{n+1})} < \infty \tag{2.14}$$

Then *the Skorohod integral* of X_t, denoted by $\int_0^T X_t \delta B_t$, is defined by

(2.15) $$\int_0^T X_t \delta B_t = \sum_{n=0}^{\infty} \int_{\mathbb{R}^{n+1}} \hat{f}_n(t,u) dB^{\otimes(n+1)}$$

The Skorohod integral is an extension of the Ito integral in the following sense:

(2.16) If Y_t is adapted and $E[\int_0^T Y_t^2 dt] < \infty$ then the Skorohod integral of Y exists and

$$\int_0^T Y_t \delta B_t = \int_0^T Y_t dB_t.$$

(See [NZ]).

As is customary we let $H^s = H^s(\mathbb{R}^n)$ denote the Sobolev space

$$H^s(\mathbb{R}^n) = \{\psi \in \mathcal{S}'(\mathbb{R}^n); \|\psi\|^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\hat{\psi}(y)|^2 (1+|y|^2)^s dy < \infty\},$$

where $\hat{\psi}$ denotes the Fourier transform of ψ and $s \in \mathbb{R}$. Then the dual of H^s is H^{-s} for all $s \in \mathbb{R}$. For notational simplicity we put

$$H^{-\infty} = \bigcup_{k=1}^{\infty} H^{-k},$$

so that if $F \in H^{-\infty}$ then $F \in H^{-k}$ for some k.

We now recall the definition of *functional processes*, which were introduced in [LØU]:

DEFINITION 2.1. A *functional process* $\{X(\cdot,\omega)\}_{\omega \in \mathcal{S}'}$ is a sum of distribution valued processes of the form

(2.17) $$X_\phi(\omega) = X(\phi,\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} F^{(n)}(\phi^{\otimes n}) dB^{\otimes n}(\omega); \phi \in \mathcal{S}, \omega \in \mathcal{S}'$$

where

$$F^{(n)}(\cdot) \in H^{-\infty}(\mathbb{R}^n; L^2(\mathbb{R}^n)) \text{ for all } n \geq 1$$

and

$$F^{(0)}(\cdot) \in H^{-\infty}(\mathbb{R}).$$

Moreover, we assume that

$$E[|X(\phi,\omega)|^2] = \sum_{n=0}^{\infty} n! \int_{\mathbb{R}^n} < F^{(n)}, \phi^{\otimes n} >^2 (u)du < \infty \tag{2.18}$$

for all $\phi \in \mathcal{S}$ with $\|\phi\|_{L^2}$ sufficiently small.

To make the notation more suggestive we often write the functional process $X(\phi,\omega)$ on the form

$$X_t(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} F^{(n)}_{t,\cdots,t}(u)dB^{\otimes n}_u(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} F^{(n)}_t dB^{\otimes n}, \tag{2.19}$$

where each $F^{(n)}_t(u)$ is really an L^2-valued distribution in the t-variable, $t = (t_1, \cdots, t_n)$. The distributional derivative of X_t with respect to t is then defined by

$$\frac{dX_t}{dt}(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} \frac{d}{dt} F^{(n)}_{t,\cdots,t}(u)dB^{\otimes n}_u(\omega) \tag{2.20}$$

where

$$\frac{d}{dt} F^{(n)}_{t,t,\cdots,t} = (\sum_{j=1}^{n} \frac{\partial F^{(n)}}{\partial x_j})_{x=(t,\cdots,t)},$$

$\frac{\partial}{\partial x_j}$ denoting the usual distributional derivative with respect to x_j, i.e.

$$< \frac{\partial F^{(n)}}{\partial x_j}, \psi >= - < F^{(n)}, \frac{\partial \psi}{\partial x_j} > \text{ for } \psi = \psi(x_1, \cdots, x_n) \in \mathcal{S}(\mathbb{R}^n).$$

EXAMPLE 2.2. The *white noise process* W_t can be represented as a functional process as follows:

$$W_t = \int_{-\infty}^{\infty} \delta_t(u)dB_u \tag{2.21}$$

where $\delta_t(u)$ is the usual Dirac measure, i.e.

$$< \delta_t(u), \phi(t) >= \phi(u)$$

To see this note that, according to the definition above, (2.21) means that

$$W_\phi(\omega) = \int \phi(u)dB_u(\omega) \tag{2.22}$$

for suitable constants $c_\alpha^{(n)} = c_\alpha^{(n)}(\phi)$, where $\alpha = (\alpha_1, \cdots, \alpha_m), |\alpha| = \alpha_1 + \cdots + \alpha_m$ and

$$\zeta^{\otimes\alpha} = \zeta_1^{\otimes\alpha_1} \otimes \zeta_2^{\otimes\alpha_2} \otimes \cdots \otimes \zeta_m^{\otimes\alpha_m}$$

This gives the (unique) representation

$$X_\phi = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} c_\alpha^{(n)} \int \zeta^{\otimes\alpha} dB^{\otimes n} = \sum_\alpha c_\alpha \int \zeta^{\otimes\alpha} dB^{\otimes|\alpha|} \tag{2.26}$$

where $c_\alpha^{(n)}(\cdot) \in H^{-\infty}(\mathbb{R}^n)$ for $n \geq 1, c_\alpha^{(0)}(\cdot) \in H^{-\infty}(\mathbb{R})$.

DEFINITION 2.4. Let X_ϕ be the functional process with the representation (2.26). Then the *Hermite transform* (or $\mathcal{H}$-transform) of X_ϕ is the formal power series in infinitely many complex variables $z_1, z_2, \cdots$ given by

$$\mathcal{H}(X_\phi)(z) = \tilde{X}_\phi(z) = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} c_\alpha z^\alpha = \sum_\alpha c_\alpha z^\alpha, \tag{2.27}$$

where $z = (z_1, z_2, \cdots), z^\alpha = z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots z_m^{\alpha_m}$ if $\alpha = (\alpha_1, \cdots, \alpha_m)$.

The main properties of the Hermite transform can be summarized as follows:

THEOREM 2.5 [LØU].
(a) For each integer N put

$$z^{(N)} := (z_1, \cdots, z_N, 0, 0, \cdots) \text{ if } z = (z_1, \cdots, z_N, z_{N+1}, \cdots) \in \mathbb{C}^{\mathbb{N}}$$

and define

$$\tilde{X}_\phi^{(N)}(z) = \tilde{X}_\phi(z^{(N)})$$

Then the power series for $\tilde{X}_\phi^{(N)}$ converges uniformly on compacts in $\mathbb{C}^N$ and hence represents an analytic function in $\mathbb{C}^N$, for each N.

(b) (Inverse $\mathcal{H}$-transform). Define the measure λ on the product σ-algebra on $\mathbb{R}^{\mathbb{N}}$ by
(2.28)

$$\int f(y)d\lambda(y) = \int_{-\infty}^{\infty} \cdots \Big(\int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} f(y_1, \cdots, y_n) e^{-\frac{1}{2}y_1^2} \frac{dy_1}{\sqrt{2\pi}}\Big) e^{-\frac{1}{2}y_2^2} \frac{dy_2}{\sqrt{2\pi}}\Big) \cdots e^{-\frac{1}{2}y_n^2} \frac{dy_n}{\sqrt{2\pi}}$$

if $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a bounded function depending only on finitely many variables $y_1, \cdots, y_n$. (This defines λ as a premeasure on the algebra generated by finite products of sets in $\mathbb{R}$ and so λ extends uniquely to a measure on the product σ-algebra of $\mathbb{R}^{\mathbb{N}}$). Then

$$X_\phi = \mathcal{H}^{-1}(\tilde{X}_\phi) := \Big[\int \tilde{X}_\phi(x + iy) d\lambda(y)\Big]_{x=\int \zeta dB} \tag{2.29}$$

where $x + iy = (x_1 + iy_1, x_2 + iy_2, \cdots)$ and "$x = \int \zeta dB$" is a short-hand notation for the substitution $x_k = \int \zeta_k dB, k = 1, 2, \cdots$.

(c) Suppose $p, q \geq 1$ is such that $\frac{1}{p} + \frac{1}{q} = 1$. Let X_ϕ, Y_ϕ be functional processes such that

$$\tilde{X}_\phi \in L^{2p}(\lambda \times \lambda), \tilde{Y}_\phi \in L^{2q}(\lambda \times \lambda)$$

for all $\phi \in \mathcal{S}$ sufficiently small. Then $X_\phi \diamond Y_\phi$ is a functional process and

$$\mathcal{H}(X_\phi \diamond Y_\phi) = \mathcal{H}(X_\phi) \cdot \mathcal{H}(Y_\phi),$$

where the product on the right is the usual complex product in the complex variables z_j (and a tensor product in the coefficients (as functions of ϕ)).

§3. Wick multiplication and Ito-Skorohod stochastic differential equations.

In this section we establish a connection between deterministic and Ito-Skorohod differential equations. The key to this connection is that ordinary multiplication should be replaced by Wick multiplication or, more generally, given functions should be replaced by their *Wick versions.* This will be explained more closely below. Heuristically, our main result could be formulated as follows:
Ito-Skorohod calculus using usual multiplication is equivalent to usual calculus using Wick multiplication.

First we make some remarks about the Hermite transform and its inverse, explained in §2:

DEFINITION 3.1. Let $u_t(z) = u(t; z_1, z_2, \cdots) : [0, \infty) \times \mathbf{C}_0^{\mathbf{N}} \to \mathbf{C}$ be measurable and satisfy the conditions

(3.1) (Antisymmetry) $\qquad u_t(\overline{z}) = \overline{u_t(z)}$ for all t, z,

where $\overline{}$ denotes complex conjugation, and

(3.2)
$$\int_0^T \int \int |u_t(z)|^2 d\lambda(x) d\lambda(y) dt < \infty$$

for all $T < \infty$, where $z = x + iy$ as before.

Then we say that $u_t(t)$ is a *generalized Hermite transform.* The family of such functions is denoted by $\mathcal{G}$. If - in addition - $u_t(z)$ satisfies the requirement

(3.3) $\qquad u_t(\cdot)$ is analytic in each $z_k \in \mathbf{C}, k = 1, 2, \cdots$

we call $u_t(z)$ an *analytic Hermite transform.* The family of such functions is denoted by $\mathcal{A}$.

Note that if $u \in \mathcal{A}$ we can write

$$u_t(z) = \sum_\alpha c_\alpha(t) z^\alpha \quad \text{for } z \in \mathbf{C}_0^{\mathbf{N}}$$

and we see that

$$u_t(z) = \tilde{X}_t(z)$$

where

$$X_t = \sum_\alpha c_\alpha(t) \int \zeta^{\otimes\alpha} dB^{\otimes|\alpha|} = \mathcal{H}^{-1}(u_t).$$

is the inverse Hermite transform of $u_t(\cdot)$. So u is indeed the Hermite transform of a functional process X.

However, if we start with a general $v_t(z) \in \mathcal{G}$ and apply the inverse Hermite transform

$$Y_t := \mathcal{H}^{-1}(v_t) = [\int v_t(z) d\lambda(y)]_{x=\int \zeta dB}$$

we get a functional process Y_t whose Hermite transform $\tilde{Y}_t = \mathcal{H}(Y_t)$ does not necessarily coincide with v_t. For example, if

$$v_t(z) = c(t)|z_1|^2 \quad z = (z_1, z_2, \cdots)$$

then

$$Y_t = [\int c(t)[x_1^2 + y_1^2] d\lambda(y_1)]_{x_1=\int \zeta_1 dB} = c(t)[(\int \zeta_1 dB)^2 + 1].$$

This can be written in canonical form

$$Y_t = c(t)[\int \zeta_1^{\otimes 2} dB^{\otimes 2} + 2],$$

from which we see that

$$\tilde{Y}_t(z) = c(t)[z_1^2 + 2].$$

But this argument shows that to any given $v_t \in \mathcal{G}$ we can always find a (unique) *analytic* $f_t = \mathcal{H}(\mathcal{H}^{-1}(v_t)) \in \mathcal{A}$ with the same inverse Hermite transform as that of v_t. We call f_t the *analytic representative* of v_t.

We will need the following result:

LEMMA 3.2. Let $v_t \in \mathcal{G}$ and let $f_t \in \mathcal{A}$ be the analytic representative of v_t. Then

$$\int v_t(z) z_k d\lambda(y) = \int f_t(z) z_k d\lambda(y) \text{ for } k = 1, 2, \cdots \tag{3.4}$$

Proof: $\int v_t(z)z_k d\lambda(y) = x_k \int v_t(z)d\lambda(y) + i\int v_t(z)y_k d\lambda(y)$ and similarly for $\int f_t(z)d\lambda(y)$. Since v_t and f_t have the same $\mathcal{H}^{-1}$ transform we have

$$\int v_t(z)d\lambda(y) = \int f_t(z)d\lambda(y)$$

So to prove (3.4) it suffices to prove that

$$\int_{-\infty}^{\infty} v_t(z)y_k e^{-\frac{1}{2}y_k^2}dy_k = \int_{-\infty}^{\infty} f_t(z)y_k e^{-\frac{1}{2}y_k^2}dy_k = 0 \text{ for all } k$$

and this is a direct consequence of the antisymmetry relation (3.1).

We proceed to prove the following basic relation between Ito/Skorohod integrals and white noise calculus: (As usual we let $W_t = \int \delta_t(u)dB_u$ denote the white noise functional process and

$$\tilde{W}_t(z) = \sum_{j=1}^{\infty} \zeta_j(t)z_j$$

its Hermite transform)

THEOREM 3.3. Let $T > 0$ and let Y_t be a stochastic process such that

$$\int_0^T \Big(\int\int |\tilde{Y}_t(t)\cdot\tilde{W}_t(z)|^2 d\lambda(x)d\lambda(y)\Big)dt < \infty.$$

Then its Skorohod integral $\int_0^T Y_t(\omega)\delta B_t$ exists and

$$\int_0^T Y_t(\omega)\delta B_t = \int_0^T \Big[\int \tilde{Y}_t(z)\cdot\tilde{W}_t(z)d\lambda\Big]_{x=\int \zeta dB} dt \quad (z = x+iy) \tag{3.5}$$

In particular, if $\{Y_t\}$ is $\{\mathcal{F}_t\}$-adapted, then

$$\int_0^T Y_t(\omega)dB_t = \int_0^T \Big[\int \tilde{Y}_t(z)\cdot\tilde{W}_t(z)d\lambda\Big]_{x=\int \zeta dB} dt \tag{3.6}$$

Proof. If we put $Y_t = 0$ for $t \notin [0,T]$ and write

$$Y_t = \sum_\alpha c_\alpha(t)\int \zeta^{\otimes\alpha}dB^{\otimes|\alpha|} = \sum_{a,k}(c_\alpha,\zeta_k)\zeta_k(t)\int \zeta^{\otimes\alpha}dB^{\otimes|\alpha|},$$

then we get (all $d\lambda$-integrals are evaluated at $x = \int \zeta dB$)

$$\begin{aligned}
\int_{-\infty}^{\infty} Y_t \delta B_t &= \sum_{\alpha,k} (c_\alpha, \zeta_k) \int \zeta_k \otimes \zeta^{\otimes\alpha} dB^{\otimes|\alpha|+1} \\
&= \sum_{\alpha,k} (c_\alpha, \zeta_k) \int z_k z^\alpha d\lambda \\
&= \sum_{\alpha,k,j} (c_\alpha, \zeta_k) \int z_j z^\alpha d\lambda \cdot \int_{-\infty}^{\infty} \zeta_k(t)\zeta_j(t) dt \\
&= \int_{-\infty}^{\infty} \Big(\int \sum_{\alpha,k} (c_\alpha, \zeta_k)\zeta_k(t) z^\alpha \big(\sum_j \zeta_j(t) z_j\big) d\lambda \Big) dt \\
&= \int_{-\infty}^{\infty} \Big(\int \tilde{Y}_t(z) \cdot \big(\sum_j \zeta_j(t) z_j\big) d\lambda \Big) dt = \int_{-\infty}^{\infty} \Big(\int \tilde{Y}_t(z) \tilde{W}_t(z) d\lambda \Big) dt,
\end{aligned}$$

as claimed.

A more striking way of stating Theorem 3.3 is the following:

COROLLARY 3.4. Let Y_t be as in Theorem 3.3. Then

$$\int_0^T Y_t(\omega) \delta B_t = \int_0^T Y_t \diamond W_t dt \tag{3.7}$$

If, in addition, Y_t is $\mathcal{F}_t$-adapted then

$$\int_0^T Y_t(\omega) dB_t = \int_0^T Y_t \diamond W_t dt \tag{3.8}$$

In other words: *Ito integration is equivalent to Wick multiplication by white noise followed by usual (Lebesgue) integration.*

Remark: Using (3.8) repeatedly we see that the Wiener-Ito chaos formula (2.7) may be written

$$f = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} f_n \diamond W_t^{\diamond n} dt^{\otimes n},$$

which is strikingly similar to the Taylor expansion of a real analytic function. See [St] for a discussion about this.

From (3.8) we see that if we model a white noise differential equation

$$\frac{dX_t}{dt} = b(X_t) + \sigma(X_t) \cdot \text{ ``white noise''}$$

by the Ito stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t,$$

we are actually interpreting the product

$$\sigma(X_t) \cdot \quad \text{``white noise''} \quad \text{as the Wick product} \quad \sigma(X_t) \diamond W_t.$$

This raises the question whether it may be more appropriate to interpret other nonlinear terms in the equation in the "Wick sense" as well. For example, as a model for population growth in a crowded - and random - environment we could use the equation

$$\frac{dX_t}{dt} = (r + \alpha W_t)X_t(N - X_t)$$

in the "traditional" sense, i.e.

$$dX_t = rX_t(N - X_t)dt + \alpha X_t(N - X_t)dB_t \tag{3.9}$$

or we could use the Wick version of the products:

$$dX_t = rX_t \diamond (N - X_t)dt + \alpha X_t \diamond (N - X_t)dB_t. \tag{3.10}$$

Which model gives the best description of the situation? We will examine this example more closely in §4. (We remark that it follows from the main result in [LØU] that if $0 \le X_s \le N$ for $s \le t$ then $X_s \diamond (N - X_s) \ge 0$ for $s \le t$.)

First we introduce the general concept of *the Wick version* $f^\diamond$ of a given real function f:

DEFINITION 3.5. Let $f : \mathbf{C} \to \mathbf{C}$ be measurable. If X_t is a stochastic process in $L^2(\mu)$ such that

$$\int\int |f(\tilde{X}_t(x + iy))|^2 d\lambda(x)d\lambda(y) < \infty$$

then

$$Y_t := [\int f(\tilde{X}_t(x + iy))d\lambda(y)]_{x=\int \varsigma dB} = \mathcal{H}^{-1}(f(\mathcal{H}(X_t))) \tag{3.11}$$

defines a stochastic process in $L^2(\mu)$. This process Y_t is denoted by $f^\diamond(X_t)$ and called *the Wick version* of $f(X_t)$.

EXAMPLE 3.6. If $f(z) = \sum_{k=0}^{n} a_k z^k$ is a complex polynomial then

$$f^\diamond(X_t) = \sum_{k=0}^{n} a_k X_t^{\diamond k},$$

i.e. the $\diamond$-polynomial obtained by replacing the usual powers by Wick powers (assuming the latter exist).

THEOREM 3.7. Let $b : \mathbb{C} \to \mathbb{C}$ and $\sigma : \mathbb{C} \to \mathbb{C}$ be measurable functions, $b(\overline{z}) = \overline{b(z)}$ and $\sigma(\overline{z}) = \overline{\sigma(z)}$. Suppose there exists $T = T(x_1, x_2, \cdots) > 0$ such that for all $k \in \mathbb{N}$ and all $(z_1, \cdots, z_k) = (x_1 + iy_1, \cdots, x_k + iy_k)$ there is a unique solution $\xi_t^{(k)} \in L^2(\chi_{[0,T]} dt \times d\lambda \times d\lambda)$ of the (deterministic) differential equation

$$(3.12) \qquad \begin{aligned} \frac{d\xi_t^{(k)}}{dt} &= b(\mathcal{H}(\mathcal{H}^{-1}(\xi_t^{(k)}))) + \sigma(\mathcal{H}(\mathcal{H}^{-1}(\xi_t^{(k)}))) \cdot \sum_{j=1}^{k} \zeta_j(t) z_j; \\ \xi_0^{(k)} &= \xi_0 \in L^2(\lambda \times \lambda) \text{ given.} \end{aligned}$$

Define $\xi_t(z)$ for $z \in \mathbb{C}_0^{\mathbb{N}}$ by putting

$$(3.13) \qquad \xi_t(z) = \xi_t^{(k)}(z) \text{ if } z = (z_1, \cdots, z_k, 0, \cdots).$$

Assume that $b(\mathcal{H}(\mathcal{H}^{-1}(\xi_t)))$ and $\sigma(\mathcal{H}(\mathcal{H}^{-1}(\xi_t))) \cdot \tilde{W}_t$ belong to $L^2(\chi_{[0,T]} dt \times d\lambda \times d\lambda)$.

Define

$$(3.14) \qquad X_t(\omega) = \mathcal{H}^{-1}(\xi_t) := [\int \xi_t(z) d\lambda(y)]_{x = \int \zeta dB(\omega)} \quad \text{for } t < T(\int \zeta dB).$$

Then the process X_t solves the Ito-Skorohod stochastic differential equation

$$(3.15) \qquad dX_t = b^\diamond(X_t)dt + \sigma^\diamond(X_t)\delta B_t \quad ; \quad X_0 = \mathcal{H}^{-1}(\xi_0).$$

Proof. If $z^{(k)} = (z_1, \cdots, z_k, 0, \cdots)$ when $z = (z_1, \cdots, z_k, z_{k+1}, \cdots)$ we have by uniqueness

$$\xi_t^{(k)}(z^{(k)}) = \xi_t^{(m)}(z^{(k)}) \quad \text{for all } m > k.$$

This shows that $\xi_t(z)$ is well-defined. Moreover, note that by antisymmetry of b and σ we have

$$(3.16) \qquad \xi_t(\overline{z}) = \overline{\xi_t(z)} \quad \text{for all } z \in \mathbb{C}_0^{\mathbb{N}}.$$

With X_t defined by (3.14) we have

$$\xi_t(z) = \xi_0 + \int_0^t b(\tilde{X}_s) ds + \int_0^t \sigma(\tilde{X}_s) \sum_j \zeta_j(s) z_j ds \text{ for } t < T(x)$$

for all $z \in \mathbb{C}_0^{\mathbb{N}}$. We integrate this identity with respect to $d\lambda(y)$ and apply the Fubini theorem to obtain

$$(3.17)\quad X_t(\omega) = a + \int_0^t b^\diamond(X_s)ds + \int_0^t \Big(\int \sigma(\tilde{X}_s)\tilde{W}_s d\lambda(y)\Big)_{x=\int \zeta dB} ds \quad \text{for } t < T\Big(\int \zeta dB\Big)$$

By Lemma 3.2 we may replace $\sigma(\tilde{X}_s)$ by its analytic representative and by Theorem 3.3 we obtain (3.15).

If b and σ are analytic then $\xi_t(\cdot)$ becomes analytic and hence ξ_t coincides with its analytic representative $\mathcal{H}(\mathcal{H}^{-1}(\xi_t))$. So in this case Theorem 3.7 simplifies to:

THEOREM 3.8. Let $b : \mathbb{C} \to \mathbb{C}$ and $\sigma : \mathbb{C} \to \mathbb{C}$ be *analytic* functions satisfying $b(\overline{z}) = \overline{b(z)}$ and $\sigma(\overline{z}) = \overline{\sigma(z)}$. Suppose that there exists $T = T(x_1, x_2, \cdots) > 0$ such that for all $z = (z_1, z_2, \cdots) = (x_1 + iy_1, x_2 + iy_2, \cdots)$ there is a unique solution $\xi_t(z) \in L^2(\chi_{[0,T]}dt \times d\lambda \times d\lambda)$ of the equation

$$(3.18)\qquad \frac{d\xi_t}{dt} = b(\xi_t) + \sigma(\xi_t)\cdot \tilde{W}_t(z); \xi_0 \in L^2(\lambda \times \lambda)$$

for $t < T$, where $\xi_0(z_1, z_2, ...)$ is analytic.

Moreover, assume $b(\xi_t)$ and $\sigma(\xi_t)\cdot\tilde{W}_t$ belong to $L^2(\chi_{[0,T]}dt \times d\lambda \times d\lambda)$.

Then

$$(3.19)\qquad X_t := \mathcal{H}^{-1}(\xi_t)$$

solves the Ito-Skorohod stochastic differential equation

$$(3.20)\qquad dX_t = b^\diamond(X_t)dt + \sigma^\diamond(X_t)\delta B_t \quad ; X_0 = \mathcal{H}^{-1}(\xi_0)$$

for $t < T(\omega) := T(\int \zeta_1 dB, \int \zeta_2 dB, \cdots)$. Moreover, this is the unique solution satisfying

$$(3.21)\qquad \int\int\Big(\int_0^T |\tilde{X}_t(z)|^2 dt\Big)d\lambda(x)d\lambda(y) < \infty \quad (z = x + iy)$$

Proof. It only remains to prove uniqueness. If X_t and Y_t both solve (3.20) then $\tilde{X}_t$ and $\tilde{Y}_t$ both solve (3.18). By uniqueness of the solution of (3.18) we have $\tilde{X}_t = \tilde{Y}_t$ and hence

$$X_t = \mathcal{H}^{-1}(\tilde{X}_t) = \mathcal{H}^{-1}(\tilde{Y}_t) = Y_t.$$

Remark. Note that X_0 may be anticipating, so Theorems 3.7 and 3.8 provide a new approach to (this type of) anticipating Skorohod stochastic differential equations. See [P] and the references there for more information about such equations.

§4. Application to population growth in a crowded, stochastic environment.

To illustrate Theorem 3.8 we consider example (3.10) in detail, i.e. we consider the following Ito-Skorohod stochastic differential equation

$$dX_t = rX_t \diamond (1 - X_t)dt + \alpha X_t \diamond (1 - X_t)\delta B_t; X_0 = x \tag{4.1}$$

where x, r, α are constants, r is positive and where we for simplicity have put $N = 1$ and assume $x > 0$.

In view of Theorem 3.8 we are led to consider the following deterministic equation

$$\frac{d\xi_t}{dt} = r\xi_t(1 - \xi_t) + \alpha\xi_t(1 - \xi_t) \cdot \tilde{W}_t(t); \xi_0 = x > 0. \tag{4.2}$$

Put $c = \frac{1-x}{x}$.
First assume $x > \frac{1}{2}$, i.e. $|c| < 1$. Then the (unique) solution of (4.2) is

$$\xi_t(z) = [1 + c\exp(-rt - \alpha F(t, z))]^{-1} \text{ for } t < T \tag{4.3}$$

where

$$F(t, z) = \int_0^t \tilde{W}_s(z)ds = \sum_k Z_k(t)z_k, \tag{4.4}$$

with $Z_k(t) = \int_0^t \zeta_k(s)ds$ and $T = T(x_1, \cdots) = \inf\{t > 0; c\exp(-rt - \alpha \sum_k Z_k(t)x_k) = 1\}$

For $t < T$ we have

$$\begin{aligned} &\int \xi_t(z)d\lambda(y) = \\ &\sum_{m=0}^{\infty}(-1)^m c^m \exp(-rmt - \alpha m \sum_k Z_k(t)x_k) \cdot \int \exp(-im\alpha \sum_k Z_k y_k)d\lambda(y) \end{aligned} \tag{4.5}$$

Now

$$\int \exp(-im\alpha Z_k y_k - \frac{1}{2}y_k^2)\frac{dy_k}{\sqrt{2\pi}} = \exp(-\frac{1}{2}m^2\alpha^2 Z_k^2) \tag{4.6}$$

So

$$\int \xi_t(z)d\lambda(y) = \sum_{m=0}^{\infty}(-1)^m c^m \exp(-rmt - \alpha m \sum_k Z_k(t)x_k - \frac{1}{2}m^2\alpha^2 \sum_k Z_k^2(t))$$

Substituting $x_k = \int \zeta_k dB$ we conclude that (4.1) has the solution

$$X_t = X_t^{(1)} = \sum_{m=0}^{\infty} (-1)^m c^m \exp(-(rm + \frac{1}{2}\alpha^2 m^2)t - \alpha m B_t) \tag{4.7}$$

This is the solution if $\frac{1}{2} < x < 1$ and $t < T(\omega) = \inf\{t > 0; c\exp(-rt - \alpha B_t) = 1\}$ $(B_0 = 0)$.

Since the series in (4.7) actually converges for all t (for a.a.ω) it is natural to define X_t for all t by this formula. With this definition we see that

$$E^\mu[X_t^{(1)}] = x_t, \tag{4.8}$$

where x_t is the solution of (4.1) in the deterministic case $(\alpha = 0)$. Moreover,

$$\lim_{t\to\infty} X_t^{(1)} = 1 \quad \text{a.s.,} \tag{4.9}$$

although for all $t > 0$ we have

$$P^\mu[X_t^{(1)} > 1] > 0 \quad (\text{if } \alpha \neq 0) \tag{4.10}$$

Thus in this stochastic model there is always a positive probability that the population will exceed the limiting value 1.

However, since

$$E^\mu[(X_t^{(1)})^2] = \infty$$

for all t (if $\alpha \neq 0$) $X_t^{(1)}$ is not a global solution of (4.1) in our (L^2) sense. But we shall show below that $X_t^{(1)}$ is a global solution in a weaker sense.

Next assume $0 < x < \frac{1}{2}$, *i.e.* $c > 1$.
Then the unique solution of (4.2) can be written

$$\xi_t(z) = c^{-1}\exp(rt + \alpha F(t,z))[1 + c^{-1}\exp(rt + \alpha F(t,z))]^{-1} \tag{4.11}$$

for $t < T(x)$.

For $t < T(x)$ we have, by a similar calculation as above,

$$\int \xi_t(z) d\lambda(y) = \sum_{m=1}^{\infty} (-1)^{m+1} c^{-m} \exp(rmt + \alpha m \sum_k Z_k(t) x_k - \frac{1}{2} m^2 \alpha^2 \sum_k Z_k^2(t))$$

Substituting $x_k = \int \zeta_k dB$ we get the solution

$$X_t = X_t^{(2)} = \sum_{m=1}^{\infty} (-1)^{m+1} c^{-m} \exp((rm - \frac{1}{2}\alpha^2 m^2)t + \alpha m B_t) \tag{4.12}$$

if $0 < x < \frac{1}{2}$ and $t < T(\omega)$.

Again we note that (4.12) actually converges for all t (for a.a. ω) so we define $X_t^{(2)}$ for all t by this formula. In this case ($0 < x < \frac{1}{2}$) we still get

$$E^\mu[X_t^{(2)}] = x_t \tag{4.13}$$

and

$$P^\mu[X_t^{(2)} > 1] > 0 \quad \text{for all} \quad t > 0 \quad (\alpha \neq 0) \tag{4.14}$$

However, in contrast with (4.9) we now have

$$\lim_{t \to \infty} X_t^{(2)} = 0 \quad \text{a.s. if} \quad r - \frac{1}{2}\alpha^2 < 0 \tag{4.15}$$

Now define X_t by

$$X_t = \begin{cases} X_t^{(1)} & if\ \frac{1}{2} < x \qquad \text{i.e. } |c| < 1 \\ X_t^{(2)} & if\ 0 < x < \frac{1}{2} \quad \text{i.e. } c > 1 \end{cases} \tag{4.16}$$

We claim that X_t actually solves (4.1) for all t, in the sense that X_t is $\mathcal{F}_t$-adapted,

$$P^\mu[\int_0^T |X_t \diamond (1 - X_t)|^2 dt < \infty \quad \text{for all } T] = 1 \tag{4.17}$$

and

$$X_T = x + r\int_0^T X_t \diamond (1 - X_t)dt + \alpha \int_0^T X_t \diamond (1 - X_t)dB_t \qquad \text{for all } T \tag{4.18}$$

To verify this we have to compute $X_t \diamond (1 - X_t)$. If $X_t = X_t^{(1)}$ we have by (4.7)

$$X_t \diamond (1 - X_t) =$$

$$-\sum_{\substack{m=0\\n=1}}^{\infty} (-c)^{n+m} \exp(-[r(n+m) + \frac{1}{2}\alpha^2(n^2 + m^2)]t) \exp(-\alpha m B_t) \diamond \exp(-\alpha n B_t) \tag{4.19}$$

To compute the last Wick product we rewrite the last two exponentials as *Wick exponentials*:

Define

$$\mathrm{E}xp(V_t) := \sum_{n=o}^{\infty} \frac{1}{n!} V_t^{\diamond n} \tag{4.20}$$

A similar computation verifies (4.17),(4.18) in the case when $X_t = X_t^{(2)}$.

Remark 1 Note that both the computation for $X_t = X_t^{(1)}$ and for $X_t = X_t^{(2)}$ actually still works if we put $c = 1$, as long as $t > 0$ (and $\alpha \neq 0$). But neither of them converges for $t = 0$ with this value of c. It is an interesting question if equation (4.18) has any solution at all with $x = \frac{1}{2}$ (if $\alpha \neq 0$) and if so, whether it is unique or not. The difficulty at this starting point $x = \frac{1}{2}$ reflects the fact that the corresponding (complex) deterministic equation (4.2) does not have a solution for all $z_k \in \mathbb{C}$. In view of (4.9) and (4.15) it is natural to regard $x = \frac{1}{2}$ as a kind of "stochastic bifurcation point".

Remark 2. It is interesting to note that our solution X_t is closely related to the classical Θ-function. The latter is defined by

$$\Theta(w,\tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n w) \quad ; \tag{4.24}$$

where $w \in \mathbb{C}$ and $\tau \in H = \{z \in \mathbb{C};\ Im\ z = 0\}$ (See e.g. [M]). So, for example, if we choose $c = 1$ (and $t > 0$) in (4.7) and (4.12) we have

$$\begin{aligned} X_t^{(1)} - X_t^{(2)} &= \sum_{n=-\infty}^{\infty} (-1)^n \exp(-(rn + \frac{1}{2}\alpha^2 n^2)t - \alpha n B_t) \\ &= \sum_{n=-\infty}^{\infty} \exp(-\frac{1}{2}\alpha^2 n^2 t + n(\pi i - rt - \alpha B_t)) = \Theta(w,\tau) \end{aligned} \tag{4.25}$$

with

$$w = \frac{1}{2} + \frac{i}{2\pi}(rt + \alpha B_t), \quad \tau = \frac{i}{2\pi}\alpha^2 t \tag{4.26}$$

Remark 3. Note that the (unique) solutions $X_t^{(1)}$, $X_t^{(2)}$ in (4.7), (4.12) are *not* Markov. For example, if $0 < x < \frac{1}{2}$ we have

$$E^x[X_{t+h}^{(2)} | \mathcal{F}_t] = \sum_{m=1}^{\infty} (-1)^{m+1} c^{-m} \exp((rm - \frac{1}{2}\alpha^2 m^2)t + \alpha m B_t) \exp(rmh) \tag{4.27}$$

while

$$E^{X_t^{(2)}}[X_h^{(2)}] = \sum_{m=1}^{\infty} (-1)^{m+1} \gamma_t^{-m} \exp(rmh), \tag{4.28}$$

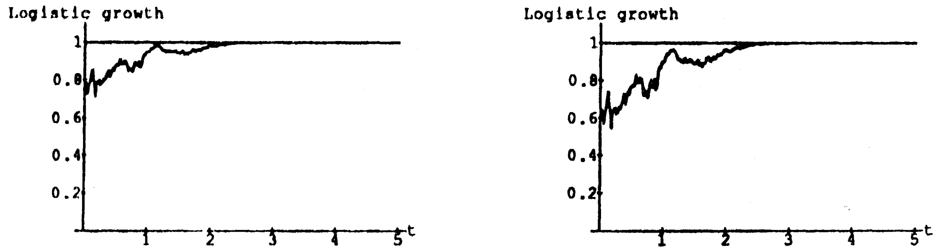
where $c = (1-x)/x$, $\gamma_t = (1 - X_t^{(2)})/X_t^{(2)}$ and $\mathcal{F}_t$ is the σ-algebra generated by $\{B_s(\cdot)\}_{s \leq t}$. The equality of (4.27) and (4.28) would imply that

$$\frac{\gamma_t}{c} = \exp(-rt + \frac{1}{2}\alpha^2 mt - \alpha B_t)$$

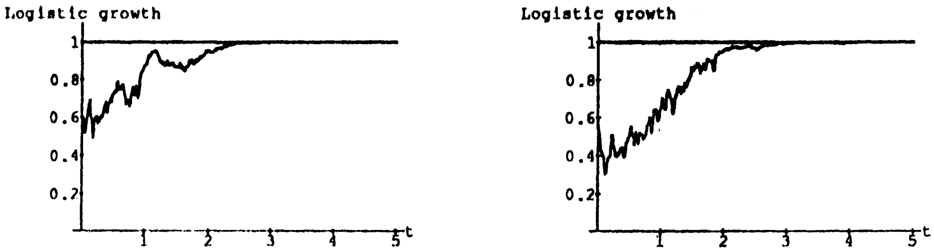
for all m, which is impossible unless $\alpha = 0$.

Logistic paths

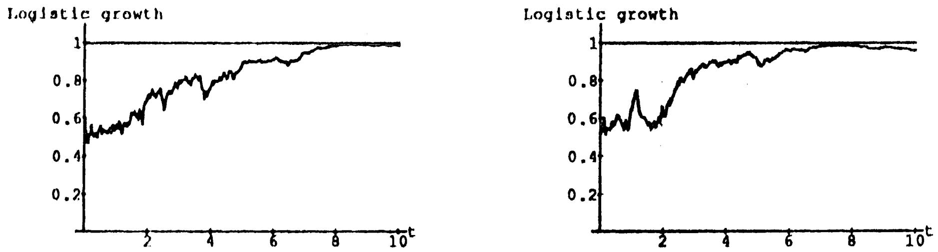
The same sample path with $r = 1$, $\alpha = 1$. Starting points: 0.75, 0.6



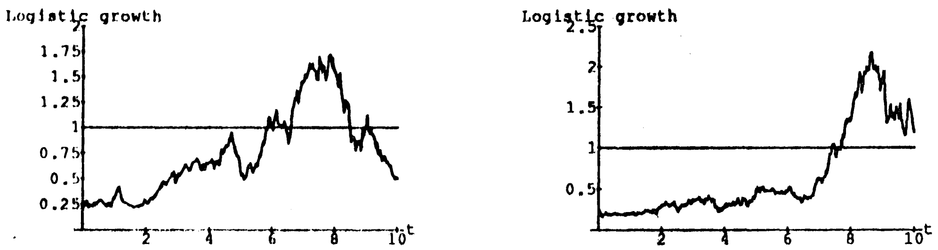
Different sample paths with $r = 1$, $\alpha = 1$. Starting point: 0.55



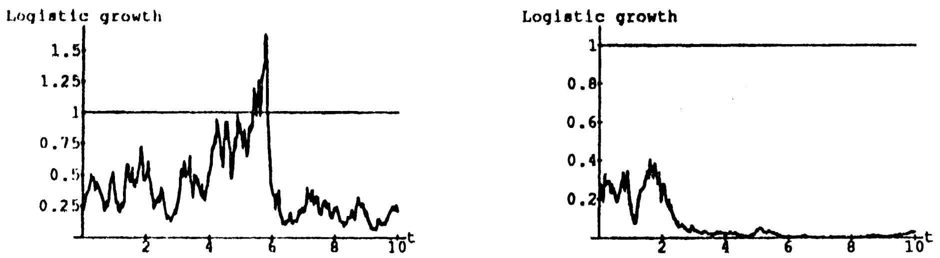
Different sample paths with $r = 1/5$, $\alpha = 1/2$. Starting point: 0.6



Different sample paths with $r = 1/5$, $\alpha = 1/2$. Starting point: 0.25



Different sample paths with $r = 1/5$, $\alpha = 1$. Starting point: 0.25



The non-Markovian nature of the solutions reflects the fact that the value of the Wick product $X_t \diamond (1 - X_t)$ at a given $\omega_0 \in \mathcal{S}'$ is not a function of $X_t(\omega_0)$ alone, but depends on the whole set of values $\{X_t(\omega); \omega \in \mathcal{S}'\}$.

The solutions $X_t^{(1)}$, $X_t^{(2)}$ are illustrated on the figure, which shows computer similations for various choices of r, t and starting point x. In a sense the use of Wick products gives a model of a population with a "memory": If the population reaches the value 1 (the capacity of the environment) from a lower starting point, it has got a momentum which makes it possible to grow even further.

It would be interesting to compare the solutions (4.7), (4.12) to the solution of the "traditional" stochastic model (3.9). Unfortunately this comparison does not seem to be straightforward, because it appears to be difficult to solve (3.9) as explicitly as we have solved (4.1).

Acknowledgements

We are grateful to P. Malliavin, P.A. Meyer and J. Potthoff for useful discussions. This work is supported by VISTA, a research cooperation between The Norwegian Academy of Science and Letters and Den Norske Stats Oljeselskap A.S. (Statoil).

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