GEOMETRY OF THE FANO THREEFOLD OF DEGREE 10 OF THE FIRST TYPE

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References.
§ 1. Preliminaries.

(1.1) DEFINITION. The smooth irreducible projective variety $X$ of dimension three over the field of complex numbers is called a Fano threefold, if the anticanonical sheaf $-K_X$ (resp. the anticanonical divisor $-K_X$) is ample.

Let $X$ be a Fano threefold and let $K_X$ be the canonical divisor of $X$. From the Riemann-Roch formula and from the Kodaira vanishing theorem it follows, that:

(a) $h^i(X, O_X(-mK_X)) = 0$, for $i=1,2$ and for any $m$ - integer; or, for $i > 0$, $m \geq 0$; or, for $i < 3$, $m < 0$;
(b) $h^0(X, O_X(-mK_X)) = (m(m+1)(2m+1)(K_X)^3)/12 + 2m + 1$.

In particular, $h^i(X, O_X) = 0$ for $i > 0$, and $h^0(X, -K_X) = \dim |-K_X| + 1 = -(K_X)^3/2 + 1$.

(1.2) DEFINITION. The integer $g = g(X) = -(K_X)^3/2 + 1$ is called a genus of the Fano threefold $X$.

The Fano variety $X$ is called a Fano variety of the main series, if the anticanonical divisor $-K_X$ is very ample. Therefore the linear system $|-K_X|$ gives an embedding of $X$ in $\mathbb{P}^{g+1}$ as a subvariety of degree $2g - 2$, $(g \geq 3)$. The intersections of $X$ with codimension $2$ subspaces $\mathbb{P}^{g-1} \subset \mathbb{P}^{g+1}$ are canonical curves of degree $2g - 2$ and of arithmetical genus $p_a = g(X)$.

The Fano variety $X$ is called a (Fano) variety of the 1-st kind, if $\text{Pic}(X) \cong \mathbb{Z}$, where $\text{Pic}(X)$ is the Picard group of $X$. The Fano varieties of the 1-st kind (i.e., with $\text{Pic} \cong \mathbb{Z}$) are classified by V.A.Iskovskikh ([1], [2], [4]) after G.Fano ([1], [2]) and L.Roth ([3]).
In the papers [MM1], [MM2] of S. Nori and S. Mukai is given a list of the Fano threefolds of rank $\text{Pic} \geq 2$ (i.e., the Fano 3-folds of the 2-nd kind). From a birational point of view both kinds of Fano threefolds are interesting, but non-trivial questions, connected with the problem of non-rationality, arise from the Fano 3-folds with $\text{Pic} \not\cong \mathbb{Z}$.

Such non-trivial examples are the cubic hypersurface in $\mathbb{P}^4$ ([CG], [B1]), the quartic double solid ([W], [C], [T1], [T2], [V]), the complete intersection of three quadrics in $\mathbb{P}^6$ ([B], [TN], [D]), the two-sheeted covering of the cone over the Veronese surface ([T3], [H]), the quartic hypersurface in $\mathbb{P}^4$ ([IM], [I3], [L]), etc.

(1.3) The Fano threefolds of the main series and of the 1-st kind have genus $g \leq 12$. The 3-folds of $g = 7$ and of $g \geq 9$ are rational. The variety of $g = 8$ is interpreted as an intersection of the Grassmannian of the lines in $\mathbb{P}^5$, embedded by Plücker in $\mathbb{P}^{14}$, with a codimension 5 subspace $\mathbb{P}^9 \subset \mathbb{P}^{14}$; the last is birational to a smooth cubic hypersurface in $\mathbb{P}^4$. The variety $X_8 \subset \mathbb{P}^6$ of genus 5 is a complete intersection of three quadrics in $\mathbb{P}^6$, etc. (see [I1], [B2]). In particular, the Fano varieties $X_{10} \subset \mathbb{P}^7$ of genus 6 are the last non-trivial examples of Fano threefolds for which $\text{Pic} \cong \mathbb{Z}$ and which have very ample anticanonical class. They are divided into two types (see [G]):

1-st type. An intersection of the Grassmannian of the lines in $\mathbb{P}^4$, embedded by Plücker in $\mathbb{P}^9$, with a codimension 2 subspace $\mathbb{P}^7 \subset \mathbb{P}^9$ and with a quadric;
2-nd type. An intersection of a cone (in $\mathbb{P}^7$) over the threefold $X_5 \subset \mathbb{P}^6$ with a quadric, where $X_5 \subset \mathbb{P}^6$ is an intersection of the Grassmannian of the lines in $\mathbb{P}^4$, embedded by Plücker in $\mathbb{P}^9$, with a codimension 3 subspace $\mathbb{P}^6 \subset \mathbb{P}^9$.

It is known, that the two types of Fano threefolds of genus $g = 6$ belong to one type of deformations ([G], [I4]).

The Fano threefold of genus $g = 6$ (resp. of degree $d = 2g - 2 = 10$) of the first of the abovementioned types is studied in the present paper.

(1.4) The Fano threefold of degree 10 of the 1-st type is represented as a 3-dimensional family of lines, lying in $\mathbb{P}^4$ (see for ex. §(1.5) and § 2). In § 2 are derived some elementary incidences, which are fulfilled for the lines $l \subset \mathbb{P}^4$, belonging to $X = X_{10}$ (see Propositions (2.2.2), (2.2.3), (2.2.4)). It is shown also the existence of a special hypersurface $R = R_6 \subset \mathbb{P}^4$ such that the lines $l \subset \mathbb{P}^4$, which belong to $X$, are at least three-tangents to $R_6$ (see (2.2.5) and (2.2.6)). In § 3 is considered a two-dimensional analogue of the threefold $X_{10}$. In fact, this is the Del Pezzo surface $S = S_{2.2}$ in $\mathbb{P}^4$, represented as an intersection of the Grassmannian $G(2,4) = G(1;\mathbb{P}^3)$, embedded as a Plücker quadric in $\mathbb{P}^5$, with a hyperplane $\mathbb{P}^4 \subset \mathbb{P}^5$ and with a quadric (see Example 3.2.). In this case, the lines $l \subset \mathbb{P}^3$, which belong to $S$, are at least bi-tangents to a Kummer surface in $\mathbb{P}^3$ (see (3.2.7)).
In the rest of § 3 it is shown, that the propositions from § 2 can be extended in an appropriate way (see Cor. (3.3.3) ).

§ 5, 6, 7 are devoted to a detailed study of the family \( C^3_e \) of rational normal cubic curves, lying on the threefold \( X = X_{10} \). First, in § 5 a geometrical investigation of the possible bundles of lines in \( P^4 \), which represent the cubics \( C \in C^3_e(X) \), is made (see § 5.2.). It is shown also the existence of a well-defined involution

\[
\sigma' : C^3_e \longrightarrow C^3_e
\]

inducing a natural embedding of the factorfamily \( C^3_{e,0} = C^3_e / \sigma' \) in the Grassmannian \( G = G(2,5) \) (see Cor. (5.3.2)). It is shown in § 6, that the factorfamily \( C^3_{e,0} \) can be embedded in a special projective bundle over the Grassmannian \( G = G(2,5) \) as a degeneration locus (more precisely, as a second determinantal) of an appropriate net of quadrics over the Grassmannian (see § 6.3., § 6.4. and Prop. (6.4.4)). The essential results in § 7 are the Corollaries (7.2.4) and (7.2.6), the Proposition (7.3.2) and their geometrical description in the Corollary (7.3.9). They are technical consequences from the results in § 6, but under some additional open conditions on the variety \( X_{10} \) (see (6.3.5) (i),(ii) and (7.1.2) (iii)). In particular, it is shown, that the family \( C^3_e \) of rational normal cubic curves on \( X_{10} \) can be embedded naturally in the projectivized bundle \( P_Z(\Sigma^*) = G_Z(4,\Sigma) \) as a zero-scheme of some section (see Prop. (7.3.2)); see
also (6.3.2) and (6.3.3) for the definitions of $Z$ and $L_c$.

The geometrical description of the embedding

$$\mathbb{C}^3 \rightarrow P_Z(\Sigma^*)$$

given in Corollary (7.3.3) is used in the formulation and in the proof of the Tangent Bundle Theorem (T.B.T.) for the family $\mathbb{C}^3$, stated and proven in § 8 (see Proposition (8.1.9) and Theorem (8.2.12)). The last describes the tangent bundle of the family $\mathbb{C}^3$ by means of some standard bundles, defined over $\mathbb{C}^3$ (see (8.2.12) and Remark (8.2.10)). In fact, the T.B.T., in its formulation presented in the paper, describes the disposition of the tangent spaces at the points of the Abel-Jacobi image of the family $\mathbb{C}^3$, inside the Intermediate Jacobian $J(X)$ of $X$.

In addition, some known results on the family of conics on $X_{10}$ are followed out in § 4 and in § 9. The original approach to the family of conics, which inspired the author to go on further and to apply an appropriate technique to the family of rational normal cubics on $X_{10}$, is due to D. Logachev (see [L1], [L2] and also (9.2.1), (9.2.5), (9.3.6), (9.3.8), (9.3.10), (9.3.11)).

The use and the possible applications of the T.B.T. to the study of the principally polarized Intermediate Jacobian of $X$ (see for ex. [CG]) are made in the author's comments in § 9.1.
Let $V = V_5$ be the five-dimensional space $\mathbb{C}^5$ and let $G = G(2,5) = G(2,V) = G(1:P(V))$ be the Grassmannian of the two-dimensional subspaces of $V$ (resp. of lines in $P(V) = P^4$). There is a standard Plücker embedding:

$$\text{Pl}: G = G(2,V) \hookrightarrow P^9 = P(\wedge^2 V),$$

which can be described as follows:

Let $0 \subset L \subset V$ be a 2-dimensional subspace of $V$ (i.e. $L \in G$), and let $L = \mathbb{C}\cdot u + \mathbb{C}\cdot v$ for some basis $\{u,v\}$ of $L$. Then, by definition, $\text{Pl}: L \longmapsto \text{the 1-dim. subspace } 0 \subset \mathbb{C}\cdot u \wedge v \subset \wedge^2 V$, i.e. $\text{Pl}(L) \in G(1, \wedge^2 V) = P(\wedge^2 V) = P^9$. It is easy to see that the definition of $\text{Pl}(L)$ is independent of the choice of the basis $\{u,v\}$ of $L$.

Let $P^4 = P(V)$ has homogeneous coordinates $x_0, x_1, \ldots, x_4$, and let $e_0, e_1, \ldots, e_4$ the corresponding dual basis of vectors in $V$. Let $e_{ij} = e_i \wedge e_j, 0 \leq i \leq j \leq 4$ and let $\{x_{ij}\}$ be the corresponding to $\{e_{ij}\}$ dual basis. The embedded Grassmannian $G = G(2,V) \subset P(\wedge^2 V)$ is described as the set of all classes of decomposable 2-forms (bi-vectors) in $\wedge^2 V$ modulo a multiplication with an element of $\mathbb{C}^*$. Therefore, the ideal $I(G)$ of $G \subset P^9$ is generated by the five Plücker quadrics $P_{1m} = x_{ij} x_{kl} - x_{ik} x_{jl} + x_{il} x_{jk}, 0 \leq m \leq 4, 0 \leq i < j < k < l \leq 4$ and $i,j,k,l \neq m$. The Plücker embedding represents the Grassmannian $G$ as a smooth subvariety $G \subset P^9$ of degree 5 and of dimension 6.

Let $P^7 \subset P^9$ be a codimension 2 subspace of $P^9 = P(\wedge^2 V)$, which is in general position with the embedded Grassmannian $G$, let $H_1$ and $H_2$ be two hyperplanes in $P^9$ such that $H_1 \cap H_2 = P^7$ and let $Q$ be a general (esp. smooth) quadric in $P^9$ (or, equivalently, in $P^7$). According to the
choice of the $P^7$ and $Q$, the variety $X = X_{10} = G \cap P^7 \cap Q = G \cap H_1 \cap H_2 \cap Q$ is smooth, and it is easy to see, that $X$ is a Fano variety of dimension $3$, of degree $10$ and of genus $6$; the anticanonical divisor $-K_X$ represents the hyperplane section of the embedded $X = X_{10} \subset P^7$.

§ 2. Some elementary incidences, connected

with the Schubert calculus on the

Grassmannian of the lines in $P^4$.

2.1. It is well known, that the integer homologies of the Grassmannian have no torsion and are generated freely by the homology classes of the Schubert cycles (see for ex. [GH] or [F]). In particular, the Schubert classes $G_{a,b}$, $3 \geq a \geq b \geq 0$ (of real codimension $2(a + b)$) of the Grassmannian $G = G(2,5) = G(2,V)$ are described as follows:

Let $P^4 = P(V)$ for $V = \mathbb{C}^5$ and let $P^0 \subset P^1 \subset P^2 \subset P^3 \subset P^4 = P(V)$ be a flag in $P(V)$. Using the natural isomorphism: $G(2,V) \cong G(1:P(V))$, which identify the two-dimensional subspace $0 \subset L \subset V$ and the projective line $1 = P(L) \subset P(V)$, we define $G'_{a,b} = \left\{ l : l \text{ is a line in } P^4 \text{ and } l \cap P^{3-a} \neq \emptyset , l \subset P^{4-b} \right\}$, where $3 \geq a \geq b \geq 0$.

The intersections of cycles in the Chow ring $A_*(G)$ can be described by the Pieri-formula (see for ex. [GH, ch.I, §5]):

$G'_{a,b} \cdot G'_{c,d} = \sum_{d+e=a+b+c, d \geq a \geq e \geq b} G'_{d,e}$. The intersection of general cycles $G'_{a,b} \cdot G'_{c,d}$ can be obtained from the Pieri-formula by the additive properties of the intersections of cycles (see for ex. the Giambelli formula in [GH, ch.I, §5]).
Let $E \in (H_1 \cap H_2) = G_0 \subseteq \mathbb{P}^7$, $Q$ and $X = X_{10} = G \cap \mathbb{P}^7 \cap Q = G \cap H_1 \cap H_2 \cap Q$ be as above.

Let $W = W_5 = G \cap \mathbb{P}^7 = G \cap H_1 \cap H_2$. The fourfold $W$ is represented by the homology class $G_{1,0} \cdot G_{1,0}$ in the Chow ring $A_*(G)$, i.e., $W = G_{1,0} \cdot G_{1,0} = G_{2,0} + G_{1,1}$ (here $=$ means the homology equivalence of cycles). In fact, the homology cycle of $G_{1,0}$ represents the hyperplane section according to the Plücker embedding, so the quadrics are represented by the cycle $2 \cdot G_{1,0}$.

Let $x$ be a general point of $\mathbb{P}^4$. Using the fact, that the set of the lines in $\mathbb{P}^4$, which pass through $x$, is represented by the cycle $G_{3,0}(x) = \{ l \in G : x \in l \}$ ($x$ is the $P^0$ in the flag, see above), and the homological equivalence $Q = 2 \cdot G_{1,0}$, we derive the following:

(2.2.2) Proposition.

(1) The set of the lines in $\mathbb{P}^4$, which belong to $W$ and which pass through a general point $x \in \mathbb{P}^4$ is homologous to the cycle $G_{3,2}$, which represents a plane bundle of lines with a center the point $x \in \mathbb{P}^4$ (i.e., $G_{3,2}$ is a Grassmann line in $G$);

(2) There are exactly two lines in $\mathbb{P}^4$, which belong to $X$ and which pass through the general point $x \in \mathbb{P}^4$.

Proof. It rests only to apply the formulae for the intersections of cycles on $G$.

Let now $\mathbb{P}^3 \subset \mathbb{P}^4$ be a general hyperplane. Using the fact, that the set (the Grassmannian) of all the lines in the
chosen $P^3$ is represented by the cycle $\mathcal{O}_{1,1}(P^3) = \{ l \in G : l \subset P^3 \}$ (if $P^3$ is the $P^3$ in the flag, see above), we obtain in a similar way:

(2.2.3) **PROPOSITION.**

(1) The set of the lines, which belong to $W$ and which lie in a fixed generally chosen hyperplane $P^3 \subset P^4$, is homologous to the cycle $\mathcal{O}_{3,1} + \mathcal{O}_{2,2}$ and represents a two-dimensional quadric $Q(P^3) = S(P^3)$, embedded in $G$;

(2) The set of the lines, which belong to $X$ and which lie in a fixed generally chosen $P^3 \subset P^4$ is homologous to the cycle $\mathcal{O}_{3,2}$ and represents an elliptic curve $C(P^3)$ of degree 4 in $G$. The curve $C(P^3)$ is an intersection of the quadric $Q(P^3)$ with the quadric $Q$ in the 3-dimensional projective space $Q(P^3) = \text{Span} Q(P^3)$.

From another point of view, the quadric surface $Q(P^3)$ is an intersection of the four-dimensional Plücker quadric $G(1:P^3)$, embedded in $G(1:P^4) \subset P^9$ by the natural embedding, corresponding to the embedding $P^3 \subset P^4$, with the subspace $P^7 \cap P^9$. From the last it follows immediately, that, if $P^3$ is chosen sufficiently general, then the quadric $Q(P^3)$ and the curve $C(P^3)$ are smooth subvarieties of $G$. As a set of lines in $P^3 \subset P^4$, the quadric $Q(P^3)$ is described as the set of all the lines in $P^3$, which intersect the both of a given pair of lines $l_1$ and $l_2$ in $P^3$, which does not intersect between them.

Let now $l \subset P^4$ be a generally chosen line, and let $\mathcal{O}_{2,0}(l) = \{ m \in G : m \cap l \neq \emptyset \}$ ($l$ is the $P^1$ in the flag, as above). In a similar way, we can prove the analogues
of the propositions above, namely:

(2.2.4) PROPOSITION. Let \( L \) be a general line in \( P^4 \). Then:

(1) The set of all the elements of \( W \), which intersect as lines in \( P^4 \) the line \( L \), is homologous to the Grassmann cycle \( 2 \cdot G_{3,1} + G_{2,2} \) and represents a linearly normal surface \( S_L \) of degree 3; more precisely, the surface \( S_L \) is embedded as a rational normal ruled surface \( S_L = \mathcal{F}_L \) in \( G \) by the linear system \( |C_0 + 2f| \), where \( C_0 \) is the \((-1)\)-section and \( f \) is the fiber of \( \mathcal{F}_L \);

(2) The set of the elements of \( X \), which intersect as lines in \( P^4 \) the line \( L \), is homologous to the cycle \( 6 \cdot G_{3,2} \) and represents a linearly normal curve \( C_L \) of degree 6 and of (arithmetical) genus 2.

(2.2.5) REMARKS.

(1) It is clear, that the curve \( C_L \) is an intersection of the surface \( S_L \) with the quadric \( Q \) in the four-dimensional projective space \( P^4(1) = \langle S_L \rangle \), The hyperelliptic linear system \( g_2 = \langle l'(x) + l''(x) \rangle \) maps \( C_L \) twice on the projective line \( l \subset P^1 \); the point \( x \in l \) has as preimages the two lines \( l'(x) \) and \( l''(x) \) as elements of \( C_L \).

(2) As in the previous case, it is easy to see that the curve \( C_L \) is non-singular under the general choice of the line \( L \subset P^4 \).
(3) Let \( l \) be chosen so that the curve \( C_1 \) is smooth. Then the hyperelliptic projection \( g_2^1 \) for \( C_1 \) has exactly 6 ramification points: \( x_1, x_2, \ldots, x_6 \); the last means, that the corresponding lines \( l'(x_i) \) and \( l''(x_i) \) (see (1)) coincide, \( i = 1, 2, \ldots, 6 \).

The arosed situation outlines the existence of a hypersurface \( R = R_6 \subset P^4 \) of degree 6, which is defined as the closed completion of the set of all the points \( x_i \), \( i = 1, 2, \ldots, 6 \) of ramification of the hyperelliptic systems \( g_2^1 = g_2^1(l) \) (see (3)) corresponding to the lines \( l \in X \), for which \( C_1 \) is smooth.

(2.2.6) DEFINITION - COROLLARY. The closed completion \( R = \overline{R^0} \) of the set \( R^0 = \{ x \in P^4 : \text{there is exactly one line (a "double" line), which pass through } x \text{ and which belongs to } X \text{ as an element of } G \} \) is a hypersurface of degree 6 in \( P^4 \). We call \( R = R_6 \) the ramification hypersurface for \( X \).

§ 3. The elements of \( X \) as three-tangents to the ramification hypersurface \( R \subset P^4 \).

3.1. Let \( l \subset P^4 \) be a line, which belongs, as a point of the Grassmannian, to \( X \). We shall show, that (when \( l \in X \) is choosen sufficiently general) the line \( l \) has three points of a simple contact with the ramification hypersurface \( R \) (see (2.2.6)). The point here is not in proving this fact (by the way proving the threetangency is not so difficult), but in the suggestion that the geometrical interpretation of the elements of the subvariety \( X \) of the Grassmann-
nian as lines, which are totally tangent to some hypersurface, is somewhat natural. The following example clarifies the situation in one more simple case.

3.2. Example.

Let \( P^3 = P(\mathbb{C}^4) \) be the projective 3-space, and let \( G(2,4) = G(2, \mathbb{C}^4) = G(1:P^3) \) be the Grassmannian of the lines in \( P^3 \), embedded as a Plücker quadric in the projective 5-space \( P^5 = P(\wedge^2 \mathbb{C}^4) \). Let \( S = G(1:P^3) \cap H \cap Q \) be a smooth intersection of the embedded Grassmannian with the hyperplane \( H = P^4 \subset P^5 \) and with the quadric \( Q \subset P^5 \).

Using, as in the case of \( X \), the Schubert representation of \( S \), we obtain easily:

(3.2.1) PROPOSITION.

(1) There are exactly two projective lines which pass through the general point \( x \in P^3 \) and which, as points of the Grassmannian \( G(1:P^3) \), belong to \( S \).

(2) There are exactly two lines, which lie in the generally chosen plane \( P^2 \subset P^3 \) and which belong to \( S \) as points of the \( G(1:P^3) \).

(3) Let \( l \subset P^3 \) be a general line. Then the set \( C_l = \{ m \subset P^3 - \text{a line} : m \in S \text{ and } m \cap l \neq \emptyset \} \), is an elliptic curve of degree 4 (the curve \( C_l \) is, in the general case, a non-singular, though special element of the anticanonical system \( |-K_S| \) )

(3.2.2) The four points of ramification \( x_1, x_2, x_3 \) and \( x_4 \) of the hyperelliptic linear system \( g_2^1 = |l'(x) + l''(x)| \) (here \( l'(x) \) and \( l''(x) \) are the two elements of \( S \), which pass through the point \( x \in l \), see Prop. (3.2.1) (1)).
define, just as in the case of X (see Remark (2.2.5) (3)), a surface \( R = R_4 \subset P^3 \) of degree 4 - the ramification (hyper)surface for \( S \).

(3.2.3) Let now \( P^2 \subset P^3 \) be a sufficiently general plane. Let \( l' \) and \( l'' \) be the two lines in \( P^2 \), which belong to \( S \) as Grassmann points (see Prop. (3.2.1) (2)). It is easy to see, that there exists a rational map \( \varphi : S \longrightarrow P^2 \) such that:

1. If \( x \in P^2 \setminus \{l' \cup l''\} \), and \( x \notin R \cap P^2 \), then
   \[ \varphi^{-1}(x) = \{l'(x), l''(x)\} \]
   where \( l'(x) \) and \( l''(x) \) are as above;

2. If \( x \in P^2 \cap R \) and \( x \notin l' \cup l'' \), then
   \[ \varphi^{-1}(x) = l(x) \]
   where \( l(x) = l'(x) = l''(x) \) is the unique element of \( S \), which passes through \( x \) as a line in \( P^3 \);

3. If \( x \in l' \) (or if \( x \in l'' \)), but \( x \neq P = l' \cap l'' \), then
   \[ \varphi^{-1}(x) = \{l', l''(x)\} \]
   (or resp. \( \varphi^{-1}(x) = \{l'', l'(x)\} \));

4. If \( x = P = l' \cap l'' \), then
   \[ \varphi^{-1}(x) = \{l', l''\} \]

(3.2.4) It is clear, that there exists a regularization \( \tilde{\varphi} \) of the rational map \( \varphi \), which makes the following diagram commutative:

\[ \begin{array}{ccc}
S & \xrightarrow{\varphi} & P^2 \\
\downarrow{\tilde{\varphi}} & & \\
\tilde{S} & \xrightarrow{\pi} & P^2
\end{array} \]

(3.2.5) In the diagram above the \( \pi \)-preimage of \( l' \cup l'' \) is an union of curves \( l'_1 + l'_2 + l''_1 + l''_2 \) such that:

(i) If \( \pi^{-1}(p) = \{p_1, p_2\} \), then \( l'_1 \cap l''_1 = p_1 \), \( l'_2 \cap l''_2 = p_2 \), \( l'_1 \cap l''_2 = \emptyset \), \( l'_1 \cap l''_1 = \emptyset \);

(ii) The map \( \sigma : \tilde{S} \longrightarrow S \) is a product of two \( \sigma \)-processes, which blow down the one of the two pairs of
non-intersecting lines (for ex. the lines \( l_1' \) and \( l_2'' \)) to a pair of non-singular points; moreover, the lines \( l_1' \) and \( l_2'' \) are, of course, \((-1)\)-curves on \( S \).

It is clear now, that the lines \( l' \) and \( l'' \) are bi-tangent to the ramification curve \( R \subset P^2 \) of \( \tilde{S} \).

(3.2.6) There are exactly 56 lines on the surface \( \tilde{S} \), which are \((-1)\)-curves and which are also the 56 preimages of the 28 bi-tangent lines to the plane quartic curve of ramification \( R \subset P^2 \); that is, the surface \( S \) is a Del Pezzo surface, which is obtained from the projective plane after blowing-up of 7 points in the general position with respect to the lines, conics and cubics on the plane. On the other hand, there are exactly 16 lines \((-1)\)-curves lying on the surface \( S \).

Let \( y_1, y_2, \ldots, y_7 \) be the seven points of the projective plane \( P = P^2 \), after blowing-up of which is obtained the surface \( \tilde{S} \) and let the lines \( l_1' \) and \( l_2'' \) (see above) be the preimages of the points \( y_6 \) and \( y_7 \). Then the 16 lines on \( S \) correspond to: (1) the conic, which pass through the points \( y_1, y_2, \ldots, y_5 \); (2) the exceptional divisors over the points \( y_1, y_2, \ldots, y_5 \); (3) the lines through the 10 pairs of points \((y_i, y_j)\), \(1 \leq i < j \leq 5\).

(3.2.7) The interpretation of the surface \( S \) as a subvariety of the Grassmannian \( G(1:P^3) \) permits to get some interesting conclusions about the ramification surface \( R \subset P^3 \).

(1) Let \( l_0 \) be a line on the surface \( S \). As a set of projective lines in \( P^3 \), the grassmann line \( l_0 \) is described by a plane pencil of lines through a fixed point \( x_0 = x(l_0) \). The plane of this pencil \( F_0^2 = P^2(l_0) = \)
stands in a special position with respect to the ramification surface $R \subset \mathbb{P}^3$. As the line $l_0 \subset S$ intersects five other lines $l_1, l_2, \ldots$, $l_5$ lying on the Del Pezzo surface $S = S_{2.2} = S_4 \subset \mathbb{P}^4$, so there are five points $x_1, x_2, \ldots, x_5$ in the plane $\mathbb{P}^2_0 = \mathbb{P}^2(l_0)$ each of them being the centre of the corresponding plane pencil of lines. In this case the analogue of the map $\mathcal{G}$ from above $\mathcal{G} : S \rightarrow \mathbb{P}^2_0$ is a birational morphism. More concretely, by definition, the formal preimage $\mathcal{G}^{-1}(x)$ of the point $x \in \mathbb{P}^2_0$ is defined to be the set of all the elements of $S$, which pass through $x$ as lines in $\mathbb{P}^3$. The regularization $\mathcal{K}$ of the rational map $\mathcal{G}$ can be given by the commutative diagram

$$
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\mathcal{K}} & \mathbb{P}^2_0 = \mathbb{P}^2(l_0) \\
\mathcal{G} \downarrow & & \downarrow \mathcal{G}
\end{array}
$$

where the regular map $\mathcal{K}$ is a composition of six blow-ups over the points $x_i$, $i = 0, 1, \ldots, 5$, and the map $\mathcal{G} : \tilde{S} \rightarrow S$ blows down the line $l \subset S$, which is the preimage of the conic $C$ in $\mathbb{P}^2_0$, passing through the five points $x_1, x_2, \ldots, x_5$.

(2) The plane $\mathbb{P}^2_0 = \mathbb{P}^2(l_0)$ is totally bi-tangent to the ramification surface $R$, along the conic $C$. It follows immediately from the description above that the projective lines, which are elements of $S \subset G(1: \mathbb{P}^3)$ are bi-tangents to the ramification surface $R$; but not every bi-tangent to $R$ belongs to $S$. For example, there are 28 bi-tangent
lines to $R$, which lie in the general plane $\mathbb{P}^2 \subset \mathbb{P}^3$, but only 2 of them belong to the surface $S \subset G(1; \mathbb{P}^3)$. There are 16 planes, corresponding to the 16 lines on $S$, each plane being totally tangent to the surface $R \subset \mathbb{P}^3$; moreover, the 16 points, which are centers of the corresponding pencils of lines in $\mathbb{P}^3$, are singular points for the quartic $R \subset \mathbb{P}^3$ and there are exactly 6 singular points of $R$, which lie on a fixed totally tangent plane, etc.

(3.2.8) The facts, stated till now, give us a sufficient reason to claim, that the surface $R$ is a Kummer surface in $\mathbb{P}^3$; for example, there are 16 singular points on the quartic $R$, but the quartic surface with a maximal number of singular points is a surface of Kummer and the 16 singular points are simple nodes (see [B3, ch.VIII, ex.]).

The family of the bi-tangent lines to a Kummer surface splits into 28 (rational) components; 16 of them are isomorphic to the projective plane, and they are the dual planes to the 16 totally tangent planes to $R$; the rest 12 components are isomorphic to Del Pezzo surfaces of degree 4 (as the surface $S$). The described set of 28 surfaces is a degeneration of the surface of the bi-tangents to a general quartic surface in $\mathbb{P}^3$, the last has been studied by Welters (see [W, part 1, §3]) and Tikhomirov (see [T1]).
3.3. The elements of $X$ as three-tangent lines to $R$.

(3.3.1) Let $l$ be a line in $P^4$, which belongs, as a point of the Grassmannian $G = G(1:P^4)$, to $X$, and let $P^2 \subset P^4$ be a sufficiently general plane through the line $l$. It follows from the adjunction formula, that the surface $S(P^2) = \{ m \subset P^4: m \cap P^2 \neq \emptyset \}$ is a K3-surface of degree 10, which represents a (special) hyperplane section of the threefold $X = X_{10}$. It is easy to see, that we can choose the line $l$ and then the plane $P^2$ through $l$ sufficiently general, such that the surface $S(P^2)$ be non-singular and moreover, such that the line $l$ be the unique element of $X$, which lies on the plane $P^2$. Let $\varphi: S = S(P^2) \longrightarrow P^2$ be, as in § 3.2., the natural rational map; that is, the formal preimage $\varphi^{-1}(x)$ of the point $x \in P^2$ is equal, by definition, to the set of all the lines in $P^4$, which pass through $x$ and which belong, as elements of the Grassmannian, to $X$. In the abovedescribed case the rational map $\varphi$ can be regularized by the commutative diagram

$$\begin{array}{ccc}
\tilde{S} & \xrightarrow{\tilde{\varphi}} & P^2 \\
\sigma \downarrow & & \\
S & \xrightarrow{\varphi} & P^2
\end{array}$$

where $\sigma$ is a blowing-down of one of the two preimages of the line $l$, to a non-singular point of $S$ and where $\tilde{\varphi}$ is a regular two sheeted covering with a ramification curve $R \cap P^2$ of degree 6. It is clear, that the two $\tilde{\varphi}$-preimages of the line $l$ are $(-1)$-curves on $\tilde{S}$ and hence, the line $l$ is a three-tangent to the ramification curve $R \cap P^2$; in particular, the line $l$ is a threetangent line to $R$. 
(3.3.2) Let $\mathbb{P}^5 \subset \mathbb{P}^7$ be any subspace of codimension 2; the intersection $\mathcal{C} = X \cap \mathbb{P}^5$ is a canonical curve of degree 10 and of arithmetical genus $p_a(\mathcal{C}) = 6$.

Let now $l \subset \mathbb{P}^3$ be a pair of a line $l$ and a hyperplane $\mathbb{P}^3$ through $l$. The curve $\mathcal{C} = X \cap \mathcal{O}_l(1) + X \cap \mathcal{O}_{l,1}(\mathbb{P}^3) = C_l + C(\mathbb{P}^3)$ (see § 2.2.) is an intersection of $X$ with a special codim. 2 subspace of $\mathbb{P}^7$. In particular, the curve $\mathcal{C}$ is a canonical curve of degree $\deg(\mathcal{C}) = 10$ and of arithmetical genus $p_a(\mathcal{C}) = 6$.

The Schubert calculus gives that $\deg(C_l) = 6$, $\deg(C(\mathbb{P}^3)) = 4$ and $(C_l, C(\mathbb{P}^3)) = 4$. As $p_a(C_m) = 2$ for the general line $m \subset \mathbb{P}^4$ (see Prop. (2.2.4) (2)) and $p_a(C(\mathbb{P}^3)) = 1$ for the general hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$ (see Prop. (2.2.3) (2)), then we derive from the numerical equalities above that

$p_a(C_l) = 2$ and that $p_a(C(\mathbb{P}^3)) = 1$ for any line $l \subset \mathbb{P}^4$ and for any $\mathbb{P}^3 \subset \mathbb{P}^4$. In fact, $p_a(C_l) \geq 2$ and $p_a(C(\mathbb{P}^3)) \geq 1$ follow from the corresponding equalities, mentioned just above. The equality $(C_l, C(\mathbb{P}^3)) = 4$ for $l \subset \mathbb{P}^3$ implies that $p_a(C) = p_a(C_l) + p_a(C(\mathbb{P}^3)) + 3$.

But $p_a(C) = 6$, since $\mathcal{C}$ is a canonical curve of degree 10. Therefore, the strong inequalities are impossible, that is $p_a(C_l) = 2$ and $p_a(C(\mathbb{P}^3)) = 1$ for any line $l \subset \mathbb{P}^4$ and for any hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$.

(3.3.3) COROLLARY. Let $l \subset \mathbb{P}^4$ be any line, and let $\mathbb{P}^3 \subset \mathbb{P}^4$ be any hyperplane. Let $C_l = \{ m \in X : m \cap l \neq \emptyset \}$, $C(\mathbb{P}^3) = \{ m \in X : m \subset \mathbb{P}^3 \}$ (see (2.2.3) (2) and (2.2.4) (2)). Then:

1. $C_l$ is a curve of degree 6 and of arith. genus 2;
2. $C(\mathbb{P}^3)$ is a curve of degree 4 and of arith. genus 1.

Moreover, if $l \subset \mathbb{P}^3$, then the curves $C_l$ and $C(\mathbb{P}^3)$...
intersect between themselves in four points (with multiplicities) and the curve \( C = C_1 + C(P^3) \) is a canonical curve of degree 10.

(3.3.4) In particular, let the line 1 be a three-tangent to the ramification hypersurface \( R = R_6 \subset P^4 \). Then the normalization of the curve \( C_1 \) is \( P^1 \) and the line 1 is a triple point of the curve \( C_1 \); moreover, the tangents to the three branches at the point \( 1 \in C_1 \) do not lie in one plane (see also the Corollary above).

§ 4. A description of the conics on \( X = X_{10} \).

4.1. The family of the conics on the Fano threefold \( X_{10} \) of the 1\textsuperscript{st} type (see (1.3)) is studied in the papers of P. Puts [P] and D. Logachev [L1], [L2]. In many respects the analysis of the family \( G^3 \) of the rational normal cubics in the present paper is similar to Logachev's study of the family of conics on \( X_{10} \); because of that we shall give a brief expose of some results about the family of conics on \( X = X_{10} \) "in the sense of Logachev".

4.2. Let \( X = G \cap H_1 \cap H_2 \cap Q = G \cap P^7 \cap Q \subset P^7 \subset P^9 = P(\wedge^2 V) \), \( V = V_5 \) be as before. The hypersurfaces \( H_1 \) and \( H_2 \), regarded as elements of \( P((\wedge^2 V)^*) = P(\wedge^2 V^*) \) (i.e., considered as one-dimensional subspaces of the vector space \( \wedge^2 V^* \)), determine a 2-dimensional subspace \( E \subset \wedge^2 V^* \). There exists a natural mapping \( S : E \otimes V \rightarrow V^* \) defined as follows:

Let \( H \in E \) and let \( v \in V \); then \( S(H \otimes v) = S_H(v) \in V^* \), where \( S_H : V \rightarrow V^* \) is the skew-symmetric linear mapping, corresponding to the element \( H \in \wedge^2 V^* \). The condition -
the variety \( W = G \cap H_1 \cap H_2 \) to be smooth - means that
the equality \( \text{rank } S_H = 4 \) is valid for any \( H \in E \).

Then the 2-dimensional subspace \( E \subset \wedge^2 V^* \) defines
the projective line \( P(E) \) and the embedding
\[
\psi^*_0 : P(E) \hookrightarrow P(V),
\]
where \( \psi^*_0 \) is defined by the rule:
\[
\psi^*_0 (H) = \text{Ker } S_{H}, \quad H \in E.
\]
In fact, \( \text{Ker } S_{H} \) is one-dimensional subspace of \( V \)
for any \( H \in E \), since \( W \) is smooth
(see above); hence, the definition of \( \psi^*_0 \) is correct.
The image \( C_0 = \psi^*_0 (P(E)) \) is a conic in \( P^4 = P(V) \).
The projective plane \( P^2 \subset P^4 \), spanned on the conic \( C_0 \)
corresponds to some fixed 3-dimensional subspace \( U \subset V \), such
that \( P^2_0 = P(U) \). The "dual" plane \( (P^2_0)^* = P(U^*) \) repre-
sents the set of the projective lines, lying in the plane \( P^2_0 \),
that is, \( (P^2_0)^* \) is a \( \sim \)-plane, lying in the fourfold
\( W \subset P^7 = H_1 \cap H_2 \); moreover, the \( (P^2_0)^* \) is the unique
\( \sim \)-plane in \( W \) (see [P] or [L1]).

Let \( l_0 = P((V/U)^*) \subset P(V^*) \) be the "ortogonal" line to
the plane \( P^2_0 = P(U) \subset P(V) \), that is,
\[
l_0 = \left\{ V_4 \subset V \subset \text{a four-dimensional subspace} : U \subset V_4 \subset V \right\}.
\]
There exists a natural isomorphism \( s : C_0 \rightarrow l_0 \), defined by the
rule: \( s(v) = S(E \otimes v) \subset P(V^*) \), for \( v \in l_0 \).

The element \( v \in C_0 \) defines the Schubert cycle \( \mathcal{C}^*(v) = \mathcal{C}^*_{3,1}(v, s(v)) = \left\{ l \in G = G(1; P^4) : v \in l \subset s(v) \right\} \).
The Schubert cycle \( \mathcal{C}^*(v) \) represents the set of all the
projective lines, passing through the point \( v \in C_0 \subset P(V) \)
and lying in the three-dimensional embedded subspace
\( P^3(v) = s(v) \subset P^4 = P(V) \), that is, \( \mathcal{C}^*(v) \) is a
\( \mathcal{C}^* \)-plane in the Grassmannian \( G \) (ibid.).
(4.2.1) CLAIM (Logachev, [L1]). The set \( \mathcal{C}(v) : v \in C_0 \)
describes all the \( \mathcal{C} \)-planes, lying in the fourfold \( W = G \cap H_1 \cap H_2 \). Moreover, for any \( v \in C_0 \), the set \( l(v) = \mathcal{C}(v) \cap (P_0^2)^* = \mathcal{C}(v) \cap (\text{Span } C_0)^* \) is a line, which
is tangent to the dual conic \( C_0^* \subset (P_0^2)^* \). For any \( V_4 \in l_0 \) the surface \( S(P(V_4)) = W \cap C_{1,1}(P(V_4)) \) is a
degenerated quadric, which splits into two planes: \( S(P(V_4)) = (P_0^2)^* \cup \mathcal{C}(s^{-1}(V_4)) \). The opposite is also true, namely:
If \( S(P(V_4)) \) splits into two planes, then \( V_4 \in l_0 \) (for the definition of \( S(P(V_4)) \) see (2.2.3)).

4.3. Let now \( q \subset X \) be a conic, lying on the Fano threefold \( X = G \cap H_1 \cap H_2 \cap Q \). As a subvariety of the Grassmannian
\( G = G(1:P(V)) \), the conic \( q \) represents an one-dimensional bundle of projective lines in \( F^4 = P(V) \), which is of
degree 2 with respect to the cycle \( C_{1,0} \) corresponding to the hyperplane section of the embedded by Plücker Grassmannian \( G \subset P^9 \). Consequently, the surface \( S_q = (\bigcup l, l - \text{a line in } F^4 \ & l \in q) \subset F^4 \) is one of the following:

1. \( S_q \subset \text{Span } S_q = P^3_q \subset F^4 \) is a non-singular quadric surface and \( q \) corresponds to one of the two bundles of
lines on the quadric \( S_q \), that is, \( q \) is a \( \mathcal{C} \)-conic on \( X \) (see [P]);

2. \( S_q \subset \text{Span } S_q = P^3_q \subset F^4 \) is a quadratic cone and \( q \) corresponds to the one-dimensional family of lines,
lying on the cone \( S_q \), that is, \( q \) is a \( \mathcal{C} \)-conic on \( X \) (ibid.);

3. \( \text{Span } S_q = P^2_q \subset F^4 \) is a plane and \( q \) corresponds to
the set of all the tangent lines to a fixed conic in \( P_2^3 \),
that is, \( q \) is a \( \mathcal{J} \)-conic on \( X \) (ibid.).

Since \( X = W \cap Q \) is a general intersection of \( W \)
with a quadric \( Q \subset P^7 = H_1 \cap H_2 \), we can suppose that
the above cases of possible conics, lying on \( X \), are realized
"in general position" that is, in particular, we can suppose,
that the \( \mathcal{J} \)-conics (resp., the \( \mathcal{G} \)-conics) on \( X = W \cap Q \)
are obtained as intersections of the \( \mathcal{J} \)-planes (resp., of
the \( \mathcal{G} \)-planes) on \( W \) with the quadric \( Q \). In the last
context, the opposite is also true (see \([P]\) or \([11]\) ).
In fact, the general conic on \( X \) is of type \( \mathcal{C} \); the
\( \mathcal{G} \)-conics on \( X \) correspond to the points of the curve \( C_0 \)
(see Claim \((4.2.1)\)) , and there is an unique \( \mathcal{J} \)-conic
on \( X \), which corresponds to the unique \( \mathcal{J} \)-plane \((P_0^3)^*\)
on \( W \).

4.4. Let now \( q \) be a \( \mathcal{C} \)-conic or a \( \mathcal{G} \)-conic on \( X \)
(that is, \( q \neq \) the unique \( \mathcal{J} \)-conic \( q_0 = (P_0^3)^* \cap Q \)
on \( X \) ); in particular, \( q \subset G(1 : P_3^3 = \text{Span } S_q) \).
As we
know (see \((2.2.3)\) and \((3.3.3)\) ), the surface \( S(P_3^3) = \)
\( W \cap \mathcal{G}'_1,1(P_3^3) \) is a quadric and the curve \( C(P_3^3) = \)
\( S(P_3^3) \cap Q \cap X \) is a space curve of degree 4 and of
arithmetical genus 1. As \( q \) is a component of the curve
\( C(P_3^3) \), there exists an additional conic \( \overline{q} \subset X \), such that
\( C(P_3^3) = q + \overline{q} \) and \((q, \overline{q}) = 2 \). In particular,
\( P_3^3 = P_3^3 = P_3^3(q, \overline{q}) \). The conic \( \overline{q} \) can be \( \mathcal{C} \)-conic,
\( \mathcal{G} \)-conic, or \( \mathcal{J} \)-conic. In fact, if \( \overline{q} = q_0 \), then the
symbol \( P_3^3 \) does not make sense. Then, under \( P_3^3 \) we mean
the 3-space \( P_3^3 \). The incidence \( \overline{q} = q_0 \) is possible
iff $q$ is a $C$-conic (see (4.2.1)). From the Claim (4.2.1) we derive also that in the case when $q$ is a $C$-conic, the quadric $S(P_q^3)$ splits into two planes: $S(P_q^3) = (P_q^3)^* + C(x_0(q))$ (here $x_0(q) \in C_0$ is the center of the cone $\bigcup l : l \subset P^4$ - a line & $l \subset q$), i.e. $\bar{q} = q_o$ is the unique $q$-conic on $X$. In the case, when $q$ is a $\mathcal{C}$-conic, $\bar{q}$ is also a $\mathcal{C}$-conic and the surface $S(P_q^3) = S(P_q^3)$ is a non-singular quadric surface in $W$.

(4.4.1) DEFINITION. The set of conics:
$$F_c = F_c(X) = \{q \text{ a conic} : q \subset X\}$$
is called a surface of the geometrical conics on $X$.

The adjective "geometrical" is necessary in the context of the following investigations; see, for example, the difficulties, arising from the attempts to define correctly the natural involution on the surface of the conics on $X$.

(4.4.2) DEFINITION. We call the set of pairs:
$$F = F(X) = \{(q, V_4) : q \text{ is a conic on } X, \text{ and } V_4 \text{ is a subspace of dim. } 4 \text{ of } V, \text{ such that } S_q \subset P(V_4)\}$$
an extended surface of conics on $X$ (or, simply, a surface of conics for $X$).

4.5. The mapping $q \mapsto \bar{q}, q \neq q_o$ defines a birational isomorphism of $F_c$, which can be completed to an involution $\iota$ on the surface $F = F(X)$ in view of the following considerations:

(4.5.1) It follows immediately from the definitions of $F$ and $F_c$ that the projection $(q, V_4) \mapsto q$ on the first factor defines naturally a morphism $r_F : F \longrightarrow F_c$, which is one-to-one over $F_c \setminus \{q_o\}$. The curve $r_F^{-1}(q_o) \subset F$.
is described by the set of all the four-dimensional subspaces \( V_4 \subset V \), for which \( \mathbb{P}(V_4) \) contains the projective plane \( \mathbb{P}^2_{q_0} = \mathbb{P}^2_0 \). Evidently, the curve \( r_F^{-1}(q_0) \) is naturally isomorphic to the curve of \( \mathcal{G} \)-conics 
\[
( = \{ q : q \text{ is a } \mathcal{G} \text{-conic on } X \} \subset F_c )
\]; moreover, the morphism \( r_F \) is a \( \mathcal{G} \)-process over the point \( q_0 \in F_c \). In this way the mapping \( \varphi : F_c \longrightarrow F_c \) defines correctly the involution \( i : F(X) \longrightarrow F(X) \).

(4.5.2) DEFINITION. We define \( F_0 = F_0(X) \) to be the factorsurface \( F/\iota \) of \( F \) under the involution \( \iota \); that is \( F_0 \) is identified with the set of all (non-ordered) pairs of involutive elements of \( F \).

4.6. The projection on the second factor \( (q, V_4) \longrightarrow V_4 \) defines a morphism \( \varphi : F \longrightarrow (\mathbb{P}^4)^* := \mathbb{P}(\mathbb{V}^*) \).

Let \( p_F : F \longrightarrow F_0 \) be the natural two-sheeted covering, defined by the involution \( \iota \). Clearly, there exists a commutative diagram:

\[
\begin{array}{ccc}
F & \xrightarrow{\varphi} & (\mathbb{P}^4)^* := \mathbb{P}(\mathbb{V}^*) \\
p_F \downarrow & & \downarrow \\
F_0 & \xrightarrow{\varphi_0} & \mathbb{P}(\mathbb{V}^*) \\
\end{array}
\]

because the involutive elements of \( F \) have same \( \varphi \)-images.

It is easy to see also that the mapping \( \varphi_0 \) provides an embedding of the factorsurface \( F_0 \) in the projective space \( \mathbb{P}(\mathbb{V}^*) \).
§ 5. The family $C^3$ of rational normal cubic curves on $X$.

5.1. PROPOSITION (see [G, §2. Prop. 2.1 and 2.2]).

($X = X_{10}$ is, as usual, a general Fano threefold of degree 10 and of the 1-st type, see (1.3))

(1) Let $C$ be a rational normal cubic curve on $X$. Then the normal sheaf $N_{C/X}$ is one of the following:

$$N_{C/X} = \begin{cases} O_C \oplus O_C & (1) \\ O_C(-1) \oplus O_C & (2) \\ O_C(-2) \oplus O_C & (3) \end{cases}$$

(2) There are rational normal cubic curves on $X$;

(3) Let $T$ be an irreducible reduced component of the family $C^3$ of all the rational normal cubic curves on $X$, and let $x$ be a sufficiently general point of $X$. Then:

(a) $\dim \{ C \in T : C \text{ passes through } x \} = 1$,

(b) The normal sheaf $N_{C/X}$ of the general $C \in T$ is isomorphic to $O_C \oplus O_C(1)$.

5.2. A geometrical representation of the rational normal cubics on $X$ as cubic scrolls.

(5.2.1) Let $C \subset X$ be a rational normal cubic, lying on the Fano threefold $X = G \cap H_1 \cap H_2 \cap Q$. The points of the curve $C$, regarded as lines in $P^4 = P(V)$, sweep out a surface $S = S_C$ of degree 3 in $P^4$. The ruled surface $S_C \subset P^4$ is one of the following:

Case 1. $S_C \subset \text{Span } S_C = P^4$, and:

(1.a) $S = \overline{F_4}$ is a rational normal cubic scroll embedded in $P^4$ by means of the linear system $|s + 2f|$, where $s$
is the \((-1)\)-section and \(f\) is the fiber of the ruled surface
\[
F_1 = \mathbb{P} ( \mathbb{P}^1 \oplus \mathbb{P}^1 ) (1)
\]

(1.b) \(S_0\) is a cone over a rational normal cubic curve \(Z\) of degree 3, with a center outside of the "ambient" space \(\text{Span } Z\);

Case 2. \(S_0 \subset \text{Span } S_0 = \mathbb{P}^3 \subset \mathbb{P}^4\), and:

(2.a) \(S_0\) is a projection of a cubic scroll \(F_1\) (see (1.a)) on \(\mathbb{P}^3 \subset \mathbb{P}^4\),

(2.b) \(S_0\) is a projection of a rational cubic cone (see (1.b)) on \(\mathbb{P}^3\).

(5.2.2) CLAIM. The cases (1.b) and (2.b) do not occur for the general \(X\).

Proof. We shall consider simultaneously the cases (1.b) and (2.b).

Let \(S = S_0\) be a rational cubic cone (in \(\mathbb{P}^4\), resp. in some \(\mathbb{P}^3 \subset \mathbb{P}^4\)), corresponding to some rational normal cubic \(C \subset X\). As we know (see 4.2) the set of all \(S\)-conics on \(X\) describes a rational curve on the surface of geometrical conics \(F_c\) (see Def. (4.4.1)). The lines, which are points of such a conic, sweep out a quadratic cone in \(\mathbb{P}^4\). The rational curve on \(F_c\), which points correspond to the \(S\)-conics on \(X\), is naturally isomorphic to the conic \(C_0 = \Psi_0(\mathbb{P}(E))\) (see Claim (4.2.1)). Therefore, the set of the centers of the corresponding quadratic cones describes some rational curve \(\hat{C}_0\) in \(\mathbb{P}^4\). In fact, the curve \(\hat{C}_0\) is a non-singular conic, lying on the plane.
it is not hard to see also that the curve \( \hat{C}_0 \) coincides with the conic \( C_0 \) (see Claim (4.2.1)), but we don't need such a precision.

Let \( x_0 \) be the center of the cubic cone, corresponding to the cubic \( C \subset X \) and let \( y \) be a center of some quadratic cone, corresponding to some \( \mathcal{G} \)-conic \( q \) on \( X \). Let \( l = \langle x_0, y \rangle \) be the line through the points \( x_0 \) and \( y \). As we know, the curve \( C_1 = \mathcal{G}_{2,0} (l) \cap X \) has a degree 6 and an arithmetical genus 2, independently of the choice of the line \( l \subset \mathbb{P}^4 \) (see Corollary (3.3.3) (1)). Consequently, the curve \( C_1 \) has an additional component \( L \) of degree 1, such that \( C_1 = C + q + L \), where \( C \) and \( q \) are the cubic and the conic from above (all the lines of \( C \) pass through \( x_0 \in l \), and all the lines of \( q \) pass through \( y \in l \)). As the curve \( C_1 \) is connected and all the projective lines, corresponding to the points of the "Grassmann" line \( L \), intersect the line \( l \subset \mathbb{P}^4 \), the center \( z \) of the plane pencil of lines, which describes \( L \), lies on the line \( l = \langle x_0, y \rangle \). Obviously, the point \( z \) is the unique point on the line \( \langle x_0, y \rangle \) with the property, that \( z \) is a center of some plane pencil of lines in \( \mathbb{P}^4 \), corresponding to some line, lying on \( X \).

The last considerations show, that the existence of a cubic \( C \subset X \) of the type (1.b) or (2.b) implies an existing of an isomorphism between the rational curve \( C_0 \) and some component of the curve \( \Gamma \) (see [P] or [M]), parametrizing the lines, lying on the Fano threefold \( X \). But, as it is known ([P, § 8]), if \( X \) is general, then
the family \( \Gamma \) has only one component; moreover, the geometrical genus of \( \Gamma \) is 71, which leads us into a contradiction. Therefore, the cases (1.b) and (2.b) do not occur, if \( X \) is sufficiently general.

(5.2.3) COROLLARY. Let \( C \) be a rational normal cubic curve, lying on \( X \). Then, the corresponding to \( C \) bundle of lines in \( \mathbb{P}^4 \) describes either the ruled surface \( \mathbb{F}_4^2 \), realized as a cubic scroll in \( \mathbb{P}^4 \), or some projection of a cubic scroll \( \mathbb{F}_4^2 \) on some \( \mathbb{P}^3 \subset \mathbb{P}^4 \).

(5.2.4) In both the cases (1.a) and (2.a) there exists an unique line \( l = l(C) \) in \( \mathbb{P}^4 \), which coincides geometrically with the (-1)-section of the corresponding \( \mathbb{F}_4^2 \) (or of its projection) and such that the cubic curve \( C \) is a component of the curve \( C_1 = \mathcal{C}_{2,0}(1) \cap X \).

(5.2.5) On the other hand, \( \deg C_1 = 6 \) and \( p_a(C_1) = 2 \). The last implies that there exists an additional rational normal cubic \( \overline{C} \) on \( X \), such that \( C_1 = C + \overline{C} \) and \( (C, \overline{C}) = 3 \). It is possible curve \( \overline{C} \) to be singular, which in our particular case means "a degenerated rational normal cubic"; because of this, as in the case of the family of conics, we introduce the following refinement of the definition of the family of rational normal cubics on \( X \):

(5.2.6) DEFINITION. We call the set \( \mathcal{C}^3 = \mathcal{C}^3(X) = \{ C \subset X : C \text{ is a rational normal (possibly, degenerate) cubic curve} \} \) a family of the rational normal cubic curves (or, simply r.n. cubics) on \( X \).
5.3. The rational mapping \( C \rightarrow \overline{C} \) defines an involution on the family of rational normal cubics on \( X \). To be precise, we need to consider separately the special cases of "degenerate" rat. normal cubics on \( X \).

Let \( C \in \mathcal{C}_g^3 \) be a degenerate r.n. cubic on \( X \). We can separate the following possible cases for \( C \):

(i) \( C = q + L \), where \( q \in \mathbb{P}_C(X) \) is a conic on \( X \) and \( L \subset X \) is a line, such that \( (q, L) = 1 \);

(ii) \( C = L_1 + L_2 + L_3 \), where \( L_1, L_2 \) and \( L_3 \) are lines on \( X \) and \( (L_1, L_2) = (L_2, L_3) = 1 \), \( (L_1, L_3) = 0 \).

We shall find the involutive curve \( \overline{C} \) in the cases (i) and (ii).

Case (i'). Let \( q \) be a \( \tau \)-conic. Then the surface \( S_q = \left( \bigcup l, l \subset \mathbb{P}^4 - \text{a line } \& l \in q \right) \) is a non-singular quadric and the set of lines \( \{ l : l \in q \} \) describes one of the two pencils of lines on \( S_q \). Let \( P_L^2 \) be the plane \( \left( \bigcup l, l \subset \mathbb{P}^4 - \text{a line } \& l \in L \right) \), and let \( y \in P_L^2 \) be the center of the plane pencil of lines \( \{ l : l \in L \} \). The equality \( (q, L) = 1 \) means that there exists a line \( l_0 \subset \mathbb{P}^4 \), which belongs to both of the pencils; evidently, \( y \in l_0 \). Let \( l' \subset S_q \) be the line from the second pencil of lines on \( S_q \), which passes through the point \( y \). Then the conic \( q \) and the line \( L \) are components of the curve \( C_{l'} = G_{2,0}(l') \cap X \).

Since \( \deg C_{l'} = 6 \) and \( p_a(C_{l'}) = 2 \), then the residue component \( \overline{C} = C_{l'} - q - L \) belongs to the family \( \mathcal{C}_g^3(X) \); moreover \( (q + L, \overline{C}) = 3 \), i.e. \( \overline{C} = q + L \) is the involutive of \( C = q + L \).
Case (i") Let $q$ be a $G'$-conic. Then the surface $S_q = \{ l : l \in P^4 \ - \text{a line & } l \in q \}$ is a quadratic cone and the equality $(q, L) = 1$ means that there is a line $l_o$, which is common for the pencils $\{ l : l \in q \}$ and $\{ l : l \in L \}$. Let $x_o$ be the singular point (the center) of the cone $S_q$ and let $y$ be the center of the plane pencil of lines, corresponding to $L$ (see (i')). Since $(q, L) = 1$, then the plane of the second pencil $P^2_L$ does not lie in the subspace $\text{Span } S_q \subset P^4$. In the opposite case the plane $P^2_L$ intersects the cone $S_q$ along a pair of lines $l_o$ and $l_o'$. Both $l_o$ and $l_o'$ are common members of the pencils $\{ l : l \in q \}$ and $\{ l : l \in L \}$, i.e. $(q, L) = 2$ -- a contradiction. Moreover, if the points $y$ and $x_o$ coincide, then the configuration $S_q + P^2_L$ corresponds to a degeneration of the case (1.b) (see (5.2.1)), which does not occur (see Claim (5.2.2)); the proof of the "degenerate" variant of the Claim (5.2.2) is, obviously, the same. Consequently, the points $x_o$ and $y$ determine the line $l_o = \langle x_o, y \rangle$. As in the case (i'),

$C_l = q + L + \overline{C}$, where $(q + L, \overline{C}) = 3$, and the involutive curve $\overline{C} = \overline{q + L}$ belongs to $P^3_2$.

Case (i'') Let $q = q_o$ be the unique $G$-conic on $X$. Then, the center $y$ of the bundle $\{ l : l \in L \}$ lies on the plane $P^2_o = \text{Span } C_o$ (see § 4.2). The points $v \in C_o$ are centers of the bundles of lines in $P^4$ corresponding to the Schubert cycles ($C$-planes) $C(v) =$

$= G_{3,1} (v, s(v))$ (see § 4.2) on the fourfold $W = G \cap H_1 \cap H_2 \subset H_1 \cap H_2 = P^7$. The "Grassmann"
quadric $Q$ intersects each $\mathcal{C}$-plane $\mathcal{C}(v)$ in the $\mathcal{C}$-conic $q(v) = \mathcal{C}(v) \cap Q$; the center of the corresponding cone $S_q(v)$ coincides with the point $v$, i.e. $x_0(v) = v$, $v \in C_0$. Let now $l_0 \subset P^2_0$ be a line through the center $y$ of the bundle $\{l : l \in L\}$. The line $l_0$ intersects the conic $C_0$ in two points: $v_1$ and $v_2$; the last are the centers of the cones, corresponding to the $\mathcal{C}$-conics $q(v_1)$ and $q(v_2)$. Therefore, the curve $C_{l_0}$ has as components the conics $q(v_1)$, $q(v_2)$, $q_0$ and the line $L$ (the centers $v_1, v_2$ and $y$ lie on $l_0$ and $l_0$ lies in the plane $P^2_0$ of the dual conic $q_0$), hence: $6 = \deg C_{l_0} \geq \deg q(v_1) + \deg q(v_2) + \deg q_0 + + \deg L = 7$ — a contradiction. Therefore, the case (i'') does not occur.

(5.3.1) REMARK. As $q_0 = (P^2_0)^* \cap Q$, the "Grassmann" conic $q_0$ is represented by the set of all lines in $P^2_0$ which are lines of intersection of the cones $S_q(v)$, $v \in C_0$ and the plane $P^2_0$. For every $v \in C_0$ the cone $S_q(v)$ intersects $P^2_0$ in a pair of lines through the point $v$; the set of all these lines coincides with the set of the tangent lines to some fixed conic $C'_0 \subset P^2_0$. In particular, every $\mathcal{C}$-conic $q(v)$, $v \in C_0$ intersects its "involutive" $\overline{q(v)} = q_0$ in a pair of points (or, in a double point, if $v$ is a point of an intersection of $C_0$ and $C'_0$), which correspond to the pair of tangents to the conic $C'_0$ through the point $v \in C_0$.

Case (ii) Let the degenerate rational normal cubic $C$ splits into a chain of three lines: $C = L_1 + L_2 + L_3$, 

\((L_1, L_2) = (L_2, L_3) = 1, (L_1, L_3) = 0\). Let \(P_1^2, P_2^2\)
and \(P_3^2\) be the planes, and \(y_1, y_2\) and \(y_3\) be the centers of the corresponding to the "Grassmann" lines \(L_1, L_2\) and \(L_3\) plane pencils of lines in \(P^4\). The conditions for the intersections give, that the points \(y_1\) and \(y_3\) lie in the plane \(P_2^2\). The case \(y_1 = y_2 = y_3\) represents a special degeneration of the case (1.b) or (2.b) (see (5.2.1)), which does not occur for the general \(X\) (see Claim (5.2.2)).

If \(y_1 \neq y_2\) or if \(y_1, y_2\) and \(y_3\) are collinear, then the line \(l = \langle y_1, y_2 \rangle = \langle y_3, y_2 \rangle\) is a common point of the "Grassmann" lines \(L_1\) and \(L_3\); but the last two do not intersect between them - a contradiction. Therefore, the line \(l_0 = \langle y_1, y_3 \rangle\) is correctly defined and moreover \(l_0\) does not pass through the point \(y_2\). Just as in the case (i), \(C_{l_0} = L_1 + L_2 + L_3 + \mathfrak{C}\), where the curve \(\mathfrak{C}\) belongs to the family \(\mathfrak{C}_3\), and

\((L_1 + L_2 + L_3, \mathfrak{C}) = 3\), i.e. \(\mathfrak{C} = \frac{L_1 + L_2 + L_3}{L_1 + L_2 + L_3}\) is the correctly defined involutive of \(C = L_1 + L_2 + L_3\).

(5.3.2) COROLLARY. Let \(\mathfrak{C}_3^3 = \mathfrak{C}_3^3(X)\) be the family of the rational normal cubics on \(X\). Then, there is a correctly defined involution \(\mathcal{G} : \mathfrak{C}_3^3 \longrightarrow \mathfrak{C}_3^3\), such that \(\mathcal{G}(C) = \mathfrak{C}\), \(C \in \mathfrak{C}_3^3\). The factorscheme \(\mathfrak{C}_3_{l_0}^3 = \mathfrak{C}_3^3 / \mathcal{G} = \{\text{the non-ordered pairs } (C, \mathfrak{C}) \text{ of involutive elements of } \mathfrak{C}_3^3\}\) is naturally embedded in the Grassmannian \(G = G(1 : P(V))\) by the rule:

\((C, \mathfrak{C}) \longrightarrow l\),

where the line \(l \subset P^4 = P(V)\) is defined as above (see
cases (5.2.1) (1.a), (2.a) and § 5.3 (i), (ii)); i.e. l is the unique line in \( P^4 = P(V) \), such that \( C_1 = C_{2,0} (1) \cap X = C + \bar{C} \). The involutive pair of rational normal cubics \( C \) and \( \bar{C} \) is defined equivalently by the numerical condition \( (C, \bar{C})_X = 3 \).

To prove the Corollary, it remains to note that the uniqueness of the line \( l \) is demonstrated separately at every of the cases above; from the last it follows immediately that \( \bar{C} = C \) for every \( C \in C^3(X) \).

5.4. COMMENTS.

(5.4.1) Let \((C, \bar{C})\) be an involutive pair of \( C^3_{2,0} \) and let \( C \cdot \bar{C} = t_1 + t_2 + t_3 \), where \( t_1, t_2 \) and \( t_3 \) are points of \( P^7 = H_1 \cap H_2 \), possibly with multiplicities. As \( C \) and \( \bar{C} \) are rational normal cubic curves, the intersection \( C \cdot \bar{C} \) determines correctly an unique plane (the plane, "spanned on the intersection points of \( C \) and \( \bar{C} \)).

Let \( l \subset P^4 \) be the line, such that \( C_1 = C + \bar{C} \), and let \( p_i = t_i \cap l \), \( i = 1, 2, 3 \) (the points \( t_1, t_2 \) and \( t_3 \) are regarded here as lines in \( P^4 \)). It is evident that the line \( l \) is a three-tangent to the ramification hypersurface \( R = R_6 \subset P^4 \) (see Def.-Cor. (2.2.6)), that is \( l \cdot R = 2p_1 + 2p_2 + 2p_3 \).

(5.4.2) The involutive pairs of the factorfamily \( C^3_{2,0} \) as sections of the rational cubic scrolls \( S_1 \subset W \) with the quadric \( Q \).

As above, let \( W = G \cap H_1 \cap H_2 \) and let \( l \) be a line in \( P^4 \). The intersection \( S_1 = C_{2,0} (1) \cdot W \) is homolo-
gous to the cycle $2 \cdot \mathcal{C}_{3,1} + \mathcal{C}_{2,2}$ and represents a surface of degree 3; moreover, for the general line $l \subset \mathbb{P}^4$, the surface $S_1$ coincides with a surface

$$\overline{F}_1 = \mathbb{P} \left( \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \right),$$

embedded in some four-dimensional subspace of $\mathbb{P}^7 = H_1 \cap H_2$ as a rational normal cubic scroll (see Proposition (2.2.4) (1)). As $X = W \cap Q$, where $Q$ is a (sufficiently general) quadric, the curve $C_1 = \mathcal{C}_{2,0}(1)$. $X$ is an intersection of the surface $S_1$ with $Q$; in the notations of Prop. (2.2.4), the curve $C_1$ belongs to the linear system $|2 \mathcal{C}_0 + 4.f|$

($\mathcal{C}_0$ is the $(-1)$-section, $f$ is the fiber of $\overline{F}_1$). The splitting $C_1 = C + \overline{C}$ means that the restriction of the quadric $Q$ on the surface $S_1$ ($= \overline{F}_1$, in the general case) can be represented as a sum of two hyperplane sections (elements of the linear system $|\mathcal{C}_0 + 2.f|$) of the cubic scroll $\overline{F}_1$. In the degenerate cases the surface $S_1$ is a degeneration of a family of $\overline{F}_1$'s; we need to be careful with the symbols, because the surfaces $S_0$ in the cases (1.a) and (2.a) (see § 5.2) and also the surfaces $S_q + \mathcal{P}_L^2$ and $\mathcal{P}_1^2 + \mathcal{P}_2^2 + \mathcal{P}_3^2$ (see § 5.3), which are cubic scrolls or their degenerations, represent cubic surfaces in $\mathbb{P}^4$, which are swept out by lines in $\mathbb{P}^4$ corresponding to the points of the curves $C \in \mathcal{C}_3$. We shall study the conditions for the splitting $C_1 = C + \overline{C}$ in detail in the next paragraph.
§ 6. The factorfamily $L^3_0$

as a determinantal variety.

6.1. Tautological sequences on $P(V^*)$ and on $G(2, V)$.

In the present section we need some details, concerning the well-known standard tautological sequences on the Grassmannians.

(6.1.1) We shall regard the "dual" projective space $(P^4)^* = P(V^*)$ as a Grassmannian: $P(V^*) = G(4, V) = \{ V_4 : V_4 - a four-dimensional subspace of V \}$.

In this interpretation the standard tautological sequence on $P(V^*)$:

$$0 \rightarrow \Omega_4, V \rightarrow P(V^*) \times V \rightarrow \Omega_1, V^* \rightarrow 0$$

parametrizes the family of embeddings

$$\{ 0 \rightarrow V_4 \rightarrow V : V_4 \in G(4, V) = P(V^*) \}.$$

Let $f = t(f_1, \ldots, f_4)$ and $g = t(g_1, \ldots, g_4)$ be the vector-columns of two given bases of the subspace $V_4 \subset V$ and let $f = A \cdot g$ be a change of the basis $f$ by the basis $g$. Then

$$\wedge^4 f = f_1 \wedge f_2 \wedge f_3 \wedge f_4 = \det A \cdot g_1 \wedge g_2 \wedge g_3 \wedge g_4 = \det A \cdot \wedge^4 g.$$ On the other hand, the Plucker embedding $\text{Pl}(\wedge^4) : G(4, V) \rightarrow P^4$ provides the natural isomorphism: $G(4, V) \cong P(V^*)$;

$\text{Pl}(\wedge^4) : V_4 \rightarrow \wedge^4 f$ (mod. proportionality by elements of $(\wedge^4)^*$, $V_4 \in G(4, V)$. From here we get immediately, that

$$\det \Omega_4, V = O_{G(4, V)} (-1),$$

according to the embedding above. Keeping in mind the last, we replace $G(4, V)$ with $P(V^*)$; in particular, we write

$$\det (\Omega^*_1, V^*) = \Omega^*_1, V^* = O_{G(4, V)} (1) = O_{P(V^*)} (1).$$
(6.1.2) Let now \( G = G(2, V) = \left\{ V_2 : V_2 \text{ is a} \right\} \) be the Grassmannian and let
\[
\mathcal{C} \rightarrow \mathcal{C}_2, V \rightarrow G(2, V) \times V \rightarrow \mathcal{C}_2^*, V^* \rightarrow 0
\]
be the standard tautological sequence on \( G \). Over the element \( V_2 \subset V \) of \( G(2, V) \) the embedding of the left side coincides with the natural embedding \( \mathcal{C} \rightarrow V_2 \rightarrow V \).

If we perform a base change \( f = A \cdot g \) in the fiber \( V_2 \) (\( f = t(f_1, f_2) \) and \( g = t(g_1, g_2) \) are two bases of \( V_2 \), as above), we obtain a change of the second exterior powers
\[
\wedge^2 f = \det A \cdot \wedge^2 g.
\]

On the other hand, the Plücker embedding
\[
\text{Pl}(\wedge^2) : G(2, V) \rightarrow \mathbb{P}(\wedge^2 V) = \mathbb{P}^9
\]
maps the element \( V_2 \in G \) to the class \( \wedge^2 f \) (mod. proportionality by elements of \( \mathcal{C}_2^* \)). In view of the last, the former change means that \( \det(\mathcal{C}_2, V) = 0_{G(2, V)}(-1) \), according to the Plücker embedding \( \text{Pl}(\wedge^2) \).

6.2. Pfaff ideals on \( P(V^*) \) and \( G(2, V) \).

(6.2.1) We shall define a subbundle \( \text{Pf} \) of the bundle of quadrics \( S^2 \wedge^2 \mathcal{C}_4, V \) over \( P(V^*) = G(4, V) \).

Let \( V_4 \subset V \) be an element of \( P(V^*) = G(4, V) \), let \( f = t(f_1, \ldots, f_4) \) and \( g = t(g_1, \ldots, g_4) \) be, as above, bases of \( V_4 \) and let \( y = (y_1, \ldots, y_4) \) and \( z = (z_1, \ldots, z_4) \) be the corresponding bases of coordinates on \( V_4 \). Assigned to the base \( f \), we define the fibre of \( \text{Pf} \) to be: \( \text{Pf}(V_4)_f = \) the Plücker quadric
\[
y_{12} \cdot y_{34} - y_{13} \cdot y_{24} + y_{14} \cdot y_{23}, \text{where } y_{ij},
\]
\( 1 \leq i < j \leq 4 \) are the coordinates, which correspond to the
basis \( f_{ij} = f_i \wedge f_j \), \( 1 \leq i < j \leq 4 \) of \( \Lambda^2 V_4 \) (the same for the coordinates \( z_{ij} \) and for the base vectors \( e_{ij} \) of \( \Lambda^2 V_4 \)). Let \( f = A \cdot g \) be a base change of \( V_4 \); the change of the coordinates is \( y \cdot A = z \), i.e. \( y = B \cdot z \), where \( B = A^{-1} \). The last allows to compute the change of \( \text{Pf} : \text{Pf}(V_4)_f = y_{12} \cdot y_{34} - y_{13} \cdot y_{24} + y_{14} \cdot y_{23} = \ldots = \det B \cdot (z_{12} \cdot z_{34} - z_{13} \cdot z_{24} + z_{14} \cdot z_{23}) = (\det A)^{-1} \cdot \text{Pf}(V_4)_g \); but \( A \) changes the bases of the bundle \( \mathcal{U}_{4,V} \) (see (6.1.1)), hence \( \text{Pf} = (\det \mathcal{U}_{4,V})^{-1} = \mathcal{O}_{P(V^*)}(1) \). Here we shall describe one geometrical interpretation of the bundle of quadrics \( \text{Pf} \).

Taking the second exterior power of the embedding
\[ 0 \rightarrow \mathcal{U}_{4,V} \rightarrow P(V^*) \times V \] from the tautological sequence, we obtain the embedding
\[ 0 \rightarrow \Lambda^2 \mathcal{U}_{4,V} \rightarrow P(V^*) \times \Lambda^2 V , \]
which parametrizes the family of embeddings \( \Lambda^2 V_4 \hookrightarrow \Lambda^2 V \), \( V_4 \in P(V^*) \). The fiber \( \text{Pf}(V_4) \) coincides with the one-dimensional vector space, spanned on the equation of the embedded Grassmannian \( G(2,V_4) = G_{1,1}(P(V_4)) \subset \text{Span} G_{1,1}(P(V_4)) = P(\Lambda^2 V_4) \) as a subvariety of \( G(2,V) \subset P(\Lambda^2 V) = P^9 \).

(6.2.2) Let we take the exterior product of the embedding
\[ 0 \rightarrow \mathcal{U}_{2,V} \rightarrow G(2,V) \times V \] with the constant bundle \( G(2,V) \times V \). The obtained embedding
\[ 0 \rightarrow \mathcal{U}_{2,V} \wedge V \rightarrow G(2,V) \times \Lambda^2 V \]
can be interpreted geometrically as follows:
Let $V_2 \subset V$ be an element of $G(2, V)$ and let $G_{2,0}(P(V_2))$ be the Schubert cycle \( \{ 1 \subset P(V) \text{ - a line :} \)

\[ 1 \cap P(V_2) \neq \emptyset \} = \{ L \subset V \text{ - a subspace of dim. } \geq 2 : \]

\[ \dim(L \cap V_2) \geq 1 \} \], embedded in $P(\wedge^2 V)$ as a subvariety of the embedded Grassmannian $G(2, V)$. We can check directly that $\text{Span } G_{2,0}(P(V_2)) = P(V_2 \wedge V)$.

Now we can define the "ideal" subbundle $\mathbb{I} \subset S^2(V_2 \wedge V)$, namely:

\[ \mathbb{I}(V_2) = \left[ \text{the set of all the quadrics in } P(V_2 \wedge V) \right] \text{ (regarded as elements of } S^2(V_2 \wedge V)^* \) which vanish on the subvariety $G_{2,0}(P(V_2)) = H^0(P(V_2 \wedge V), 0(2 - G_{2,0}(P(V_2)))) \subset S^2(V_2 \wedge V)^*$

As in (6.2.1) we compute the cocycle of the base changes of the bundle $\mathbb{I}$; as a result we obtain that $\mathbb{I}$ is isomorphic to the bundle $\mathcal{C}_{3,V}^*$.

(6.2.3) COROLLARY.

(i) Let $\mathcal{Pf} \subset S^2 \wedge^2 \mathcal{C}_{4,V}^*$ be the sheaf of quadrics over $P(V^*)$ with a fibre $\mathcal{Pf}(V_4) = H^0(P(\wedge^2 V_4), 0(2 - G_{1,1}(P(V_4))))$ over the element $V_4 \subset V$ of $P(V^*) = G(4, V)$. Then there is a natural isomorphism:

\[ \mathcal{Pf} \cong 0_{P(V^*)}(1) ; \]

(ii) Let $\mathbb{I} \subset S^2(V_2 \wedge V)$ be the bundle of quadrics over $G = G(2, V)$ with a fibre $\mathbb{I}(V_2) = H^0(P(V_2 \wedge V), 0(2 - G_{2,0}(P(V_2))))$.
over the element \( V_2 \subset V \) of \( G(2, V) \). Then there is a natural isomorphism:

\[
\mathcal{I} \cong \mathcal{I}_{3, V}^*,
\]

where \( \mathcal{I}_{3, V}^* \) is the tautological factorbundle over \( G(2, V) \).

(6. 1-2) COMMENTS. The bundles of quadrics \( Pf \rightarrow P(V^*) \) and \( \mathcal{I} \rightarrow G(2, V) \) are the components of degree 2 in the graded bundles of ideals of the families of the embedded Schubert cycles \( \{ G_{1,1}(P(V_4)) \subset P(\wedge^2 V_4) : V_4 \in P(V^*) \} \) and \( \{ G_{2,0}(P(V_2)) \subset P(V_2 \wedge V) : V_2 \in G(2, V) \} \) respectively. Obviously, the components \( Pf \) and \( \mathcal{I} \) generate the corresponding ideals.

6.3. The factorfamily \( \mathcal{G}_{2,0}^3 \) as a set of quadrics.

(6.3.1) Let \( Q \subset S^2(\mathcal{I}_{2, V} \wedge V)^* \) be the bundle of quadrics over \( G = G(2, V) \), which parametrizes the family of restrictions of the quadric \( Q \in S^2(\wedge^2 V)^* \) on the subspaces of the form \( V_2 \wedge V, V_2 \in G(2, V) \). As the quadric \( Q \) is sufficiently general, the corresponding bundle of restrictions \( Q(V_2) = [ \text{the } C^* \text{-class of the equation of the restriction of the quadric hypersurface } Q = 0 \text{ in } P(V_2 \wedge V) ] \) is correctly defined.

(*) NOTE. We use the same symbol \( Q \) for the quadric \( Q \in S^2 \wedge^2 V^* \), for the surface \( (Q = 0) \) in \( P(\wedge^2 V) \) and for the bundle "of quadrics" \( Q \) over \( G(2, V) \).

Obviously, the bundle \( Q \), defined above, is trivial, i.e. the corresponding sheaf \( Q \) is isomorphic to the structure sheaf \( O_{G(2, V)} \) over \( G = G(2, V) \).
(6.3.2) PROPOSITION. Let \( Z \subset P_G( S^2 ( \tau_2, V \wedge V)^* ) \) be the set:
\[
Z = \mathbb{P}\left\{ (V_2, q) : V_2 \in G, q \in H^0(\mathcal{P}(V_2 \wedge V) \circ (2 - \mathcal{G}^*_{2,0}(\mathcal{P}(V_2)).X) \right\}
\]
Then \( Z \) is naturally isomorphic to the projectivized bundle \( P_G( \tau^*_3, V^* \oplus Q \mathcal{G}_G ) \) over \( G = G(2,V) \).

Proof. The proposition follows immediately from the Corollary (6.2.3) (ii) and from (6.3.1). It remains to see that the fiber of the natural projection \( Z \twoheadrightarrow G(2,V) \) over the "point" \( V_2 \in G(2,V) \) coincides with the space
\[
\text{Span}\left\{ \bigvee (V_2) \cup Q(V_2) \right\} = \bigwedge (V_2) \oplus Q(V_2).
\]

(6.3.3) Let now \( X = G(2,V) \cap H_1 \cap H_2 \cap Q \) and let \( V_8 \subset V_{10} = \bigwedge^2 V \) be the subspace, such that \( P^7 = H_1 \cap H_2 = P(V_8) \). We define a bundle \( \Sigma \) over \( G(2,V) \) with fibers \( \Sigma(V_2) = (V_2 \wedge V) \cap V_8 \), \( V_2 \in G(2,V) \). We can suppose that the subspace \( P^7 = P(V_8) \) is choosen a "sufficiently general" in such a way, that all the intersections \( (V_2 \wedge V) \cap V_8 \) are transversal (and, hence, are vector spaces of dimension 5) for every \( V_2 \in G(2,V) \). Considerations on the level "intersection of Schubert cycles" give that the last requirement is fulfilled for the elements \( V_8 \) of an open subset of the Grassmannian \( G(8, V_{10}) \).

(6.3.4) Let \( V_2 \in G(2,V) \). As we know, \( S_P(V_2) = P_2,0(P(V_2)) \cap P(V_8) = \{
\}
\[ l \subset P^4 - \text{a line: } l \in W, l \cap P(V_2) \neq \emptyset \] (see Proposition (2.2.4) (1) and (5.4.2)). Let \( V_8 \subset V_{10} = \)
Let $V_8 \subset V_{10} = \wedge^2 V$ and $Q$ are chosen sufficiently general, such that:

(i) The variety $X = G \cap Q \cap P(V_8)$ is smooth;
(ii) $\dim (V_8 \cap (V_2 \wedge V)) = 5$, for every $V_2 \in G = G(2,V)$.

Let $\mathbb{I} \rightarrow G$ and $Q \rightarrow G$ be the bundles, as in (6.2.3) (ii) and (6.3.1). Then we can consider that $\mathbb{I}$ and $Q$ are embedded in $S^2 \Sigma^* X$ in such a way that

$$\mathbb{I}(V_2) = H^0(P(\Sigma(V_2)) \otimes 0)$$(6.3.5) COROLLARY. Let $V_8 \subset V_{10} = \wedge^2 V$ and $Q$ are chosen sufficiently general, such that:

(i) The variety $X = G \cap Q \cap P(V_8)$ is smooth;
(ii) $\dim (V_8 \cap (V_2 \wedge V)) = 5$, for every $V_2 \in G = G(2,V)$.

Let $\mathbb{I} \rightarrow G$ and $Q \rightarrow G$ be the bundles, as in (6.2.3) (ii) and (6.3.1). Then we can consider that $\mathbb{I}$ and $Q$ are embedded in $S^2 \Sigma^* X$ in such a way that

$$\mathbb{I}(V_2) = H^0(P(\Sigma(V_2)) \otimes 0)$$(6.3.5) COROLLARY. Let $V_8 \subset V_{10} = \wedge^2 V$ and $Q$ are chosen sufficiently general, such that:

(i) The variety $X = G \cap Q \cap P(V_8)$ is smooth;
(ii) $\dim (V_8 \cap (V_2 \wedge V)) = 5$, for every $V_2 \in G = G(2,V)$.

Let $\mathbb{I} \rightarrow G$ and $Q \rightarrow G$ be the bundles, as in (6.2.3) (ii) and (6.3.1). Then we can consider that $\mathbb{I}$ and $Q$ are embedded in $S^2 \Sigma^* X$ in such a way that

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(i) The variety $X = G \cap Q \cap P(V_8)$ is smooth;
(ii) $\dim (V_8 \cap (V_2 \wedge V)) = 5$, for every $V_2 \in G = G(2,V)$.

Let $\mathbb{I} \rightarrow G$ and $Q \rightarrow G$ be the bundles, as in (6.2.3) (ii) and (6.3.1). Then we can consider that $\mathbb{I}$ and $Q$ are embedded in $S^2 \Sigma^* X$ in such a way that

$$\mathbb{I}(V_2) = H^0(P(\Sigma(V_2)) \otimes 0)$$(6.3.5) COROLLARY. Let $V_8 \subset V_{10} = \wedge^2 V$ and $Q$ are chosen sufficiently general, such that:

(i) The variety $X = G \cap Q \cap P(V_8)$ is smooth;
(ii) $\dim (V_8 \cap (V_2 \wedge V)) = 5$, for every $V_2 \in G = G(2,V)$.

Let $\mathbb{I} \rightarrow G$ and $Q \rightarrow G$ be the bundles, as in (6.2.3) (ii) and (6.3.1). Then we can consider that $\mathbb{I}$ and $Q$ are embedded in $S^2 \Sigma^* X$ in such a way that

$$\mathbb{I}(V_2) = H^0(P(\Sigma(V_2)) \otimes 0)$$

(6.3.6) Let now $C \subset X$ be a rational normal cubic curve. As we know (see Corollary (5.3.2)), there exists a line $l \subset P^4 = P(V)$ (resp., a subspace $V_2 \subset V$, such that $P(V_2) = l$), such that $C_1 = C_{2,0}(l)$. $X = C + \bar{C}$ ($\bar{C}$ is the involutive of $C$). But $C_1 = S_1 \cdot Q$, where $S_1$, $l = P(V_2)$, is as in the Corollary above. The last means that there exists a quadric $Q$, which is a point of the space $\mathbb{I}(V_2) \oplus Q(V_2) \subset S^2 \Sigma^*(V_2)$ and which splits, as a subvariety of $P(\Sigma(V_2))$, into
two hyperplanes: \((q = 0) = \langle c \rangle + \langle \overline{c} \rangle \subset P(\Sigma(V_2))\).

The last is equivalent, in the interpretation of the Corollary (6.3.5), to the existence of a quadric of rank \(\leq 2\) in the vector-space "of quadrics" \((\mathbb{I} \oplus Q)(V_2)\). Obviously, the opposite is also true: the existence of a quadric \(q \in (\mathbb{I} \oplus Q)(V_2)\) of rank \(\leq 2\), \(q = \overline{H} \cdot \overline{H}\) means that the curves \(C = S_P(V_2) \cdot P(\overline{H})\) and \(\overline{C} = S_{P}(V_2) \cdot P(H)\) are an involutive pair of rational normal cubics such that
\[
C_P(V_2) = C + \overline{C}.
\]

(6.3.7) COROLLARY. The factorfamily \(\mathcal{L}_\circ = \mathcal{L}_3 / \mathcal{O}\) (see (5.3.2)) is embedded naturally in the projectivized vector bundle \(P_G(\mathbb{I} \oplus Q) \subset G(2,V) \times P(S^2\Sigma^*)\) as the set:
\[
D_2 = \{ (V_2, q) : V_2 \in G, q \in (\mathbb{I} \oplus Q)(V_2) \subset S^2\Sigma^*(V_2), \text{rank}(q) \leq 2 \}.
\]

Postponing the comments of the Corollary till later on, we shall explain in brief some facts about the degeneration loci (determinantals) (see [F, ch. 14]).

6.4. The factorfamily \(\mathcal{L}_3\) as a determinantal.

(6.4.1) DEFINITION.

(i) Let \(\varphi : E \rightarrow F\) be a homomorphism of vector bundles (a vector bundle map) over the variety \(Y\); let \(\text{rank } E = e\), \(\text{rank } F = f\). Let \(k\) be a non-negative integer, such that \(k \leq \min(\ e, \ f\)\). The locus
\[
D_k = D_k(\varphi) = \{ y \in Y : \text{rank } \varphi(y) \leq k \} \subset Y
\]
is called the \(k\)-th degeneration locus (the \(k\)-th determinantal) of \(\varphi\).
(ii) In particular, let \( F = E^* \otimes L \) for some invertible sheaf \( L \) on \( Y \). Multiplying by \( L^{-1} \), we obtain the map \( \gamma \otimes L^{-1} : E \otimes L^{-1} \rightarrow E^* \).

Taking the dual \( (\gamma \otimes L^{-1})^* : E^{**} = E \rightarrow E^* \otimes L \), we obtain another vector bundle map from \( E \) to \( F = E^* \otimes L \). The vector bundle map \( \gamma : E \rightarrow E^* \otimes L \) is called symmetric, if \( \gamma \) is "selfdual", i.e. if

\[
\gamma = (\gamma \otimes L^{-1})^* .
\]

Using the natural correspondence between so defined symmetric maps and the "nets of quadrics" \( \hat{\gamma} : L^{-1} \rightarrow S^2 E^* \), we define the \( k \)-th degeneration locus (the \( k \)-th determinantal) of the net of quadrics \( \hat{\gamma} : L^{-1} \rightarrow S^2 E^* \) to be the \( k \)-th determinantal of the corresponding (symmetric) vector bundle map \( \gamma : E \rightarrow E^* \otimes L \).

(6.4.2) As we know, \( \overline{1} \oplus Q \cong \ell_{3,V}^* \oplus O_G(2,V) \), that is \( Z = \mathbb{P} G(2,V) (\ell_{3,V}^* \oplus O_G(2,V)) \cong \mathbb{P} G(2,V) (\overline{1} \oplus Q) \) (see Corollary (6.2.3)).

Let \( \pi : Z = \mathbb{P} G(\ell_{3,V}^* \oplus O_G) \rightarrow G = G(2,V) \) be the natural projection and let

\[
\ell_{3,V}^* \oplus O_G(2,V) \cong \overline{1} \oplus Q \hookrightarrow S^2 \Sigma^*
\]

be the abovedescribed embedding of vector bundles. Taking the tautological sequence for the projectivized vector bundle \( \bar{\pi} : Z = \mathbb{P} G(\ell_{3,V}^* \oplus O_G) \rightarrow G(2,V) : \)

\[
0 \rightarrow O_{\bar{\pi}}(-1) \xrightarrow{j} \bar{\pi}^*(\ell_{3,V}^* \oplus O_G) \rightarrow R \rightarrow 0,
\]

we obtain the composition of natural embeddings:
of vector bundles over $\mathbb{Z}$.

(6.4.3) By the construction of the tautological sequences, the projectivization of the embedding of the sheaf $0_{\pi}(-1)$ (the relative $0(-1)$, or the tautological subbundle): 

$$P(0_{\pi}(-1)) \hookrightarrow P(\mathcal{K}^*(\mathcal{Z}_{3, V^*}^* \oplus O_{G}))$$

represents the embedding of the points of the fibers of $P_G(\mathcal{Z}_{3, V^*}^* \oplus O_{G})$, regarded, in view of the isomorphism

$$\mathcal{Z}_{3, V^*}^* \oplus O_{G} \simeq \mathbb{I} \oplus \mathbb{Q},$$

as quadratic hypersurfaces in the corresponding projective spaces $P(\sum(V_2))$, $V_2 \in G(2,V)$, lifted to $\mathbb{Z}$. From the last and from the Corollary (6.3.7) we obtain:

(6.4.4) PROPOSITION. The factorfamily $\mathcal{L}_0^3 = \mathcal{L}_0^3 / \sigma$ is embedded naturally in $Z = P_{G(2,V)}(\mathcal{Z}^*_{3, V^*} \oplus O_{G(2,V)})$ as a second determinantal of the net of quadrics:

$$0 \rightarrow 0_{\pi}(-1) \xrightarrow{j} \mathcal{K}^*(\mathcal{Z}_{3, V^*}^* \oplus O_{G(2,V)}) \xrightarrow{i} \mathcal{K}^*(S^2 \Sigma^*)$$

i.e. $\mathcal{L}_0^3 \simeq D_2(\hat{\varphi})$, where $\hat{\varphi} = i \cdot j$ (see (6.4.2)).
§ 7. The families $\mathcal{C}_0^3$ and $\mathcal{T}_0^3$ as zero-schemes.

7.1. We shall use the same symbol $q$ for the quadratic form $q \in S^2 \Sigma^*(V_2)$ and for the corresponding symmetric operator $q : \Sigma(V_2) \longrightarrow \Sigma^*(V_2)$. Then Proposition (6.4.4) means that $\mathcal{C}_0^3 = \{(V_2, q) \in \mathbb{Z} : \text{rank}(q) \leq 2\} = D_2(\mathfrak{F})$, (the $q$ in the pair is regarded as a form, the $q$ after the colon is regarded as an operator). The condition $\text{rank}(q) = k$ means that $\dim \text{Im}(q) = k$ ($\text{Im}(q)$ is a subspace of $\Sigma^*(V_2)$).

Let $0 \leq k \leq 5 = \dim \Sigma(V_2)$ and let

$$\tilde{D}_k = D_k(\mathfrak{F}) = \{(z; U_k) \in \mathcal{G} \Sigma(k, \Sigma^*) : \text{Im}(q) \subseteq U_k\}, \text{where} \quad Z \ni z = (V_2, q).$$

The natural projection $\mathcal{K}_k : (z; U_k) \longrightarrow z$ provides a map: $\mathcal{K}_k : \tilde{D}_k \longrightarrow D_k$, which is an isomorphism outside the locus $D_{k-1} \subset D_k$, $1 \leq k \leq 5 = \dim (\text{fibers of } \Sigma)$. The natural embedding

$$\tilde{D}_k \longrightarrow \mathcal{G} \Sigma\left(k, \Sigma^*\right)$$

can be included in the commutative diagram:

$$\begin{array}{ccc}
\tilde{D}_k & \longrightarrow & \mathcal{G} \Sigma\left(k, \Sigma^*\right) \\
\mathcal{K}_k \downarrow & & \downarrow \circ \\
D_k & \longrightarrow & \mathbb{Z}
\end{array}$$

where $\circ : \mathcal{G} \Sigma\left(k, \Sigma^*\right) \longrightarrow \mathbb{Z}$ is the natural projection.

(7.1.2) REMARK. Let $Q \in S^2 V_8^*$ (resp. ($Q = 0$) $\subset \mathbb{P}(V_8)$) be a quadric. It is not hard to check that the set of all the quadrics $Q$ in $\mathbb{P}(V_8)$ (resp. $Q \in S^2 V_8^*$),
such that some of the spaces \( \Pi(V_2) \oplus Q(V_2) \) contains a quadric of rank 1 (a double hyperplane in the corresponding \( P(\sum V_2) \)) (resp., of \( S^2 \sum^*(V_2) \)), has a codimension > 1 inside the space of quadrics in \( P(V_8) \) (resp., of \( S^2 V_8^* \)). Therefore, for the general quadric \( Q \) in \( V_8 \) the first determinantal \( D_1(\hat{\phi}) \) vanishes. Because of that we can suppose in addition that the quadric \( Q \) is chosen such that the open condition

(iii) \( D_1(\hat{\phi}) = \emptyset \)

is fulfilled (see also the conditions (i) and (ii) in the Corollary (6.3.5)).

Taking into account (iii) from the Remark, we obtain that the projection \( \tilde{\kappa}_2 : \tilde{D}_2 \longrightarrow D_2 \) is an isomorphism.

7.2. (7.2.1) Let

\[
0 \longrightarrow \tau_{2, \Sigma^*} \longrightarrow G_Z(2, \Sigma^*) \times \sum^* \longrightarrow \tau_{3, \Sigma} \longrightarrow 0
\]

be the standard tautological sequence on the grassmannization \( \varphi : G_Z(2, \Sigma^*) \longrightarrow Z \). The embedding \( \alpha : \tau_{2, \Sigma^*} \hookrightarrow G_Z(2, \Sigma^*) \times \sum^* \) defines an embedding \( S^2 \alpha : S^2 \tau_{2, \Sigma^*} \hookrightarrow G_Z(2, \Sigma^*) \times S^2 \Sigma^* \). The embedding \( S^2 \alpha \) can be included in the exact sequence:

\[
0 \longrightarrow S^2 \tau_{2, \Sigma^*} \longrightarrow S^2 G_Z(2, \Sigma^*) \times \sum^* \longrightarrow \text{Coker}(2) \longrightarrow 0 ,
\]

where \( \text{Coker}(2) \) is the factor-bundle \( S^2 \Sigma^* / S^2 \alpha (S^2 \tau_{2, \Sigma^*}) \) on \( G_Z(2, \Sigma^*) \).

(7.2.2) On the other hand, the net of quadrics \( \hat{\phi} \) de-
defines the embedding $0 \to O_{\mathcal{X}}(-1) \xrightarrow{\hat{\varphi}} S^2 \Sigma^* \to$ of bundles over $\mathcal{Z}$. Taking into account the projection $\hat{\varphi} : G_{\mathcal{Z}}(2, \Sigma^*) \to \mathcal{Z}$ we can look at the sheaves $O_{\mathcal{X}}(-1)$ and $\Sigma$ as sheaves on $G_{\mathcal{Z}}(2, \Sigma^*)$ (here we omit, for simplicity, the symbol $\hat{\varphi}^*$).

Composing the morphisms in the sequences above, we obtain the sequence:

$$0 \to O_{\mathcal{X}}(-1) \xrightarrow{\hat{\varphi}} G_{\mathcal{Z}}(2, \Sigma^*) \times S^2 \Sigma^* \xrightarrow{\varphi} \text{Coker } (2)$$

of natural maps of sheaves (resp., of bundles) over $G_{\mathcal{Z}}(2, \Sigma^*)$. Multiplying by the sheaf $(O_{\mathcal{X}}(-1))^* = \text{O}_{\mathcal{X}}(1)$, we obtain the composition:

$$0 \to O_G(2, \Sigma^*) \xrightarrow{\hat{\varphi} \otimes O_{\mathcal{X}}(1)} O_{\mathcal{X}}(1) \otimes S^2 \Sigma^* \xrightarrow{\delta^* \otimes O_{\mathcal{X}}(1)} O_{\mathcal{X}}(1) \otimes \text{Coker } (2);$$

the last defines a section $s_0$ of the sheaf $O_{\mathcal{X}}(1) \otimes \text{Coker } (2)$ over the variety $G_{\mathcal{Z}}(2, \Sigma^*)$,

$s_0 = (\hat{\varphi} \otimes O_{\mathcal{X}}(1))^* \cdot (\hat{\varphi} \otimes O_{\mathcal{X}}(1)).$

(7.2.3) By definition the zero scheme $Z(s_0) \subset G_{\mathcal{Z}}(2, \Sigma^*)$ of the section $s_0 \in H^0(G_{\mathcal{Z}}(2, \Sigma^*), O_{\mathcal{X}}(1) \otimes \text{Coker } (2))$ coincides with the $0$-th determinantal $D_0(s_0)$ of the section $s_0$, regarded as a vector bundle map

$s_0 : O_G(2, \Sigma^*) \to O_{\mathcal{X}}(1) \otimes \text{Coker } (2)$. Consequently,

$Z(s_0) = D_0(s_0) = \{(z; U_2) \in Z \times G(2, \Sigma^*(V_2)) : z = (V_2, q) \& s_0(z; U_2) = 0\}$

So we have:

(1) $\hat{\varphi} = i \cdot j$, where the embedding $j : O_{\mathcal{X}}(-1) \to$
\[ \mathcal{R}^* ( \mathcal{L}^*_{3, \mathcal{V}^*} \oplus O_{G(2, \mathcal{V})} ) \] represents the closed embedding of the "point" \( q \) in the fiber \( \mathcal{R}^{-1}(V) \subset \mathcal{Z} \) and the embedding \( \iota : \mathcal{R}^* ( \mathcal{L}^*_{3, \mathcal{V}^*} \oplus O_{G(2, \mathcal{V})} ) \hookrightarrow S^2 \sum^* \) corresponds to the natural representation of the points in the fibers of \( \mathcal{L}^*_{3, \mathcal{V}^*} \oplus O_{G(2, \mathcal{V})} \) as quadrics in the fibers \( \sum (V) \) of \( \sum \) (here we use the isomorphism \( \mathcal{L}^*_{3, \mathcal{V}^*} \oplus O_{G(2, \mathcal{V})} \sim \prod \oplus \mathcal{Q} \), see (6.2.3) (ii) and (6.3.1)).

(2) The maps \( \mathcal{Y}^* (z) = \mathcal{Y}^* (V_2, q) : S^2 \sum^*(V_2, q ; U_2) \longrightarrow \text{Coker} (2) \), as factor-maps, send the quadrics \( q \in S^2 \sum^*(V_2) \) to the corresponding classes \( q \mod S^2 U_2 \), over the elements \( (z ; U_2) = (V_2, q ; U_2) \in G_Z (2, \sum^*) \).

From (1) and (2) we derive, that
\[ Z (s_0) = \left\{ (V_2, q ; U_2) \in G_Z (2, \sum^*) : q \mod S^2 U_2 = 0 \right\} \]
\[ = \left\{ (V_2, q ; U_2) \in G_Z (2, \sum^*) : U_2 \in \text{Image} (q) \subset \sum^*(V_2) \right\} \]
S \( q \) regarded as an operator \( q \in \text{Sym} (\sum(V_2), \sum^*(V_2)) \).

Consequently, \( Z (s_0) = \hat{D}_2 (\hat{\mathcal{Q}}) \subset G_Z (2, \sum^*) \), (see 7.1 and (7.1.1)).

But, by the condition (iii) (see Remark (7.1.2)), \( \hat{D}_1 (\hat{\mathcal{Q}}) = \emptyset \), that is, there are no double hyperplanes in the fibers \( \mathcal{H}^0 (\mathcal{F}(\sum(V_2)), O(2 - \sigma_{2,0}(\mathcal{F}(V_2)), X_{10} )) \),
\( V_2 \in G(2, \mathcal{V}) \) of the bundle "of quadrics" \( \prod \oplus \mathcal{Q} \).
Consequently, the natural projection \( X : \tilde{D} ( \hat{\mathbf{y}} ) \rightarrow D ( \hat{\mathbf{y}} ) \) (cf. (7.1.2)) is an isomorphism.

Summing up the obtained incidences, we derive:

(7.2.4) COROLLARY. Let \( V_8 \subset V_{10} = \wedge^2 V \) and \( Q \) fulfill the open conditions (i), (ii) from the Corollary (6.3.5) and (iii) from the Remark (7.1.2). Let \( G^3 = G^3 / \sigma \) be the factorfamily of the family of rational normal cubics on \( X = X_{10} = G(2,V) \cap P(V_8) \cap Q \) and let \( s_o \subset H^0(G^3(2,\Sigma^*), O_{\pi} (1) \otimes \text{Coker} (2) ) \) be the section, described in (7.2.2). Then \( G^3 \) embeds naturally in \( G^3(2,\Sigma^*) \) as a zero scheme of the section \( s_o \), that is:

\[
G^3 \overset{\sim}{\longrightarrow} Z(s_o) \subset G^3(2,\Sigma^*).
\]

(7.2.5) NOTE. By convention, the symbol \( \beta^* \), where \( \beta : G^3(2,\Sigma^*) \rightarrow Z \), is dropped, i.e. with some abuse of the notations we write \( \beta^* O_{\pi} (1) \otimes \text{Coker} (2) = O_{\pi} (1) \otimes \text{Coker} (2) \).

There exists a natural isomorphism of duality:

\[
\delta : G^3(2,\Sigma^*) \overset{\sim}{\longrightarrow} G^3(3,\Sigma),
\]

induced by the natural duality of the fibers

\[
\delta(V_2) : G(2,\Sigma^*(V_2)) \overset{\sim}{\longrightarrow} G(3,\Sigma(V_2)), V_2 \in G(2,V).
\]

The composition \( s_o^* = \delta \cdot s_o \) defines a section of (the isomorphic \( \delta \)-preimage of) the sheaf \( O_{\pi} (1) \otimes \text{Coker} (2) \) on \( G^3(3,\Sigma) \). So we obtained:

(7.2.6) COROLLARY. In the conditions of the Corollary (7.2.4) there is a natural embedding \( G^3 \overset{\sim}{\longrightarrow} G^3(3,\Sigma) \), such that:
(1) \( \mathfrak{a}^3 \bigotimes \mathfrak{s}^* \xrightarrow{\sim} Z(\mathfrak{s}_0^*) \subset G_Z(3, \Sigma) \)

where \( \mathfrak{s}_0^* = \mathfrak{f} \cdot \mathfrak{s}_0 \) is the section, defined just above;

(2) Let \((\mathcal{C}, \overline{\mathcal{C}})\) be an element of \( \mathfrak{a}^3 \) (i.e. \((\mathcal{C}, \overline{\mathcal{C}})\)) is a pair of involutive rational normal cubics on \( X = X_{10} \).

Let \( V_2 \subset V \) be the subspace (of dim. = 2), such that

\[ C_P(V_2) = \sigma^*_{2,0}(P(V_2)).X_{10} = C + \overline{C} \]

(see Corollary (5.3.2)) and let \( H_C \) and \( H_{\overline{C}} \in \Sigma^*(V_2) \) be the hyperplanes in \( \Sigma(V_2) \) such that \( \langle \mathcal{C} \rangle = \text{Span}(\mathcal{C}) = P(H_C) \)
and \( \langle \overline{\mathcal{C}} \rangle = \text{Span}(\overline{\mathcal{C}}) = P(H_{\overline{C}}) \). Then the embedding

\[ \mathfrak{a}^3 \bigotimes \rightarrow G_Z(3, \Sigma) \]

sends the pair \((\mathcal{C}, \overline{\mathcal{C}})\) to the triple \((V_2, q = H_C \cdot H_{\overline{C}} ; H_C \cap H_{\overline{C}}) \in (V_2, q = H_C \cdot H_{\overline{C}}) \times G(3, \Sigma(V_2)) \subset G(3, \Sigma(V_2)) \).

Proof. Point (1) follows immediately from the definition of the section \( \mathfrak{s}_0^* = \mathfrak{f} \cdot \mathfrak{s}_0 \) and from the previous Corollary.

To prove (2) it is sufficient to see that the third member \( H_C \cap H_{\overline{C}} \) of the triple corresponds, by the duality \( \mathfrak{f} \), to the subspace \( U_2 := \text{Image}(q) \subset \Sigma^*(V_2) \), where

\[ q = H_C \cdot H_{\overline{C}} \]

is regarded as an operator from \( \text{Sym}(\Sigma(V_2), \Sigma^*(V_2)) \). Moreover, \( \text{dim}(\text{Image}(q = H_C \cdot H_{\overline{C}})) = 2 \), because of the fact, that \( D_1(\mathfrak{f}) = \emptyset \). In particular, \( H_C \) and \( H_{\overline{C}} \) are not proportional, that is \( U_2 = \text{Image}(q = H_C \cdot H_{\overline{C}}) \) belongs to the Grassmannian \( G(2, \Sigma(V_2)) \).
7.3.
(7.3.1) Let
\[ 0 \rightarrow \mathcal{T}_{4, \Sigma} \rightarrow P_Z(\Sigma^*) \times \Sigma \rightarrow \mathcal{T}_{1, \Sigma}^* \rightarrow 0 \]
be the standard tautological sequence on the projectivization \( P_Z(\Sigma^*) = G_Z(4, \Sigma) \). Taking the dual, we obtain the natural surjection \( P_Z(\Sigma^*) \times \Sigma^* \rightarrow \mathcal{T}_{4, \Sigma}^* \rightarrow 0 \), hence, the surjection
\[ \mathcal{E} : P_Z(\Sigma^*) \times S^2 \Sigma^* \rightarrow S^2 \mathcal{T}_{4, \Sigma}^* \rightarrow 0. \]

Let now \( \mathcal{T} : P_Z(\Sigma^*) \rightarrow \mathbb{Z} \) be the natural projection. Taking the \( \mathcal{T} \)-preimages of the sheaves, we lift the map \( \hat{\varphi} \) onto \( P_z(\Sigma^*) \):
\[ 0 \rightarrow O_X((-1)) \rightarrow P_z(\Sigma^*) \times S^2 \Sigma^*; \]
as usual the symbol \( \mathcal{T}^* \) is dropped.

Just as in (7.2.2), after multiplying by \( O_X(1) \), we obtain the composite map:
\[ 0 \rightarrow O_{P_Z}(\Sigma^*) \rightarrow P_z(\Sigma^*) \times O_X(1) \otimes S^2 \Sigma^* \rightarrow \]
\[ \mathcal{E} \otimes O_X(1) \rightarrow O_X(1) \otimes S^2 \mathcal{T}_{4, \Sigma}^*, \]
hence, we obtain a well-defined section \( s \) of the sheaf
\[ O_X(1) \otimes S^2 \mathcal{T}_{4, \Sigma}^* \text{ over } P_z(\Sigma^*), s = \]
\[ (\mathcal{E} \otimes O_X(1)) \circ (\hat{\varphi} \otimes O_X(1)). \]

As in § 7.2 we shall prove the following

(7.3.2) PROPOSITION. In the conditions of the Corollary (7.2.4), the family \( C^3 \) of the rational normal cubics on \( X \) is embedded naturally in the projectivized bundle \( P_z(\Sigma^*) \) as a zero scheme of the section \( s \), that is:
\[ C^3 \sim \rightarrow \mathbb{Z}(s) \subset P_z(\Sigma^*). \]
Proof. (for more details see the proof of the Corollary (7.2.4)). As in (7.2.3) by the definition of the tautological sequence the map \( \mathcal{E} : P^2(\Sigma^*) \times S^2 \Sigma^* \rightarrow S^2 \tau^* \) sends the element \((V_2, q ; Y_4)\) of the fibre \(\tau^{-1}(V_2, q)\) to the element \((V_2, q ; q|_{Y_4}) \in S^2 \tau^*_4, \Sigma\) (here \(q|_{Y_4}\) is the restriction of the quadric \(q \in S^2 \Sigma^*(V_2)\) to the subspace \(Y_4 \subset \Sigma(V_2), \dim(Y_4) = 4\)). Consequently, the elements of the zero scheme of \(s\) are defined by the condition: \(Y_4 \subseteq \ker(q); \) here the quadric \(q\) is regarded as an operator from \(\text{Sym}(\Sigma(V_2), \Sigma^*(V_2))\). In other words, the zero scheme \(Z(s) = \{(V_2, q ; Y_4) : (V_2, q) \in Z, Y_4 \subset \Sigma(V_2)\}\) is an element of \(G(4, \Sigma(V_2))\) such that \(Y_4 \subseteq \ker(q)\).

But the symmetric operator \(q : \Sigma(V_2) \rightarrow \Sigma^*(V_2)\), \(\dim \Sigma(V_2) = 5\), contains four-dimensional subspaces in its kernel, if and only if \(\text{rank}(q) \leq 2\). By the way, there are no quadrics \(q\) of rank \(= 1\), since \(D_1(\hat{q}) = \emptyset\) (see for example (7.2.3) - the proof of the Corollary (7.2.4)). Because of that \(\text{rank}(q) \leq 2\) if \(\text{rank}(q) = 2\), that is \(q = H \cdot \bar{H}\), for some non-proportional linear functions (non-coincident hyperplanes) \(H\) and \(\bar{H}\) of \(\Sigma^*(V_2)\). But the projective hyperplanes \(P(H)\) and \(P(\bar{H})\) cut out on \(X\) (in the described situation) a pair of involutive rational normal cubics \(C = X \cap P(H)\) and \(\overline{C} = X \cap P(\bar{H})\). Because of that the zero scheme \(Z(s)\)
describes the set of the components of the quadrics $q$ of rank $= 2$, $q$ being the second element of the pair $(V_2, q) \in D_2(\hat{\mathcal{G}})$ (see Proposition (6.4.4), § 7.1 and § 7.2). Equivalently, the zero scheme $Z(s)$ describes the set of the rational normal cubics (the intersections of $X = X_{10}$ with the corresponding components $P(H)$ and $P(\overline{H})$ of the quadrics $q = H.H$ of rank $= 2$, as above). The Proposition is proved.

From the proof of the Proposition (7.3.2), just as for the Corollary (7.2.6) (2), we obtain the following geometrical description of the embedding $\mathbb{A}_3 \hookrightarrow P_Z(\Sigma^*) = G_Z (4, \Sigma)$:

(7.3.3) COROLLARY. Let $C \in \mathbb{A}_3$ be a rational normal cubic curve on $X = X_{10}$. Let $V_2 \subset V$ be the subspace (of dim. $= 2$), such that $G_P(V_2) = G_{2,0}(P(V_2)) \cdot X = C + \overline{C}$, where $\overline{C}$ is the involutive of $C$ (see Corollary (5.3.2)) and let $H_C$ and $H_{\overline{C}} \in \Sigma^*(V_2)$ be the hyperplanes in $\Sigma(V_2)$, such that $\langle C \rangle = \text{Span}(C) = P(H_C)$ and $\langle \overline{C} \rangle = \text{Span}(\overline{C}) = P(H_{\overline{C}})$. Then the embedding $\mathbb{A}_3 \hookrightarrow P_Z(\Sigma^*) = G_Z (4, \Sigma)$ sends the element $C \in \mathbb{A}_3$ to the triple $(V_2, q = H_C.H_{\overline{C}}; H_C) \in (V_2, q = H_C.H_{\overline{C}}) \times G(4, \Sigma(V_2)) \subset G_Z (4, \Sigma) = P_Z(\Sigma^*)$. 
§ 8. Tangent Bundle Theorem for the family of rational normal cubics $c^3$.

8.1. Before formulating and proving the Tangent Bundle Theorem (T.B.T.), we shall carry out some constructions, which clarify the local situation. From now on we shall suppose that the curve $C \subseteq c^3$ is chosen sufficiently general, in particular, we can suppose that the curve $C$ and its involutive $\tilde{C}$ are smooth rational normal cubics with normal bundles $N_C/X$ and $N_{\tilde{C}}/X$, isomorphic to $O_{P^1} \oplus O_{P^1}(1)$ (see Proposition 5.1), etc.

(8.1.1) Let $C$ and $\tilde{C}$ be a pair of involutive rational normal cubics on $X$, let $V_2 \subseteq V$ be the corresponding subspace of dim. = 2, such that $C = C + \tilde{C}$, where $l = P(V_2)$ (see Corollary (5.3.2)). Let $W = G \cdot P(V_8)$, $X = X_{10} = W$, $Q = G \cdot P(V_8)$, $Q$ and $S_1 = G_{2,0}(1) \cdot W$ are defined as before.

Examine the pairs of successive embeddings:
$C \subseteq X \subseteq W$ and $C \subseteq S_1 \subseteq W$
and the corresponding exact sequences of normal sheaves:

(1) $0 \rightarrow N_C/X \rightarrow N_C/W \rightarrow N_{X/W} \otimes O_C \rightarrow 0$ and

(2) $0 \rightarrow N_C/S_1 \rightarrow N_C/W \rightarrow N_{S_1/W} \otimes O_C \rightarrow 0$.

Multiplying the sequences (1) and (2) by the sheaf $O(-1) = O_{P(V_8)}(-1)$ and identifying the middle members, we obtain the diagonal map:

(3) $\phi_C : N_C/S_1 \otimes O(-1) \rightarrow N_C/W \otimes O(-1) \rightarrow N_{X/W} \otimes O(-1) \otimes O_C$

Since $\deg (C) = 3$, then $O(-1) \otimes O_C \simeq O_C(-3)$,
$C \simeq P^1$. 
From $X = W \cdot Q$, where $Q$ is a quadric in $\mathbb{P}^7 = \mathbb{P}(V_2')$, it follows that $N_{X/W} = O_X(2)$. Therefore,

$N_{X/W} \otimes O(-1) \otimes O_c \cong O(1) \otimes O_c \cong O_c(3)$. \hfill (8.1.2)

On the other hand, $N_{C/S_1} = 0(1) \otimes O_c$, since $C$ is a hyperplane section of the rat. norm. cubic scroll $S_1 \subset \text{Span}(S_1) = P(\Sigma(V_2))$, $l = P(V_2)$; (see (5.4.2)). Therefore:

(3'). $\gamma_C : O_C \longrightarrow O(1) \otimes O_c = O_c(3)$.

Taking into account the identifications above, the sequence $(2) \otimes O(-1)$ takes the form:

$(2') : 0 \longrightarrow O_C \longrightarrow N_{C/W} \otimes O(-1) \longrightarrow N_{S_1/W} \otimes O(-1) \otimes O_c \longrightarrow 0$ \hfill (8.1.3)

**LEMMA.**

Let $C \in \mathbb{C}_i^3(X)$ be a general r. n. cubic. Then

$N_{S_1/W} \otimes O_c \cong O_c(-1) \oplus O_c(-1)$.

The proof of the lemma will be given later on (see (8.1.10)).

(8.1.4) From (8.1.3) we derive the identification:

(4) $N_{S_1/W} \otimes O(-1) \otimes O_c \cong O_c(-1) \oplus O_c(-1)$,

therefore the long exact sequence of cohomologies, associated to the sequence $(2')$, defines the natural isomorphisms:

(5) $H^i(C, O_c) \longrightarrow H^i(C, N_{C/W} \otimes O(-1))$, $i = 0, 1, \ldots$

In fact, all the cohomology groups of the sheaf $N_{S_1/W} \otimes O(-1) \otimes O_c$ vanish.

(8.1.5) Let now look at the sequence $(1) \otimes O(-1)$. Since $N_{C/X} \otimes O(-1) \cong O_c(-3) \oplus O_c(-2)$, then $H^0(C, N_{C/X} \otimes O(-1)) = 0$. 
From $H^1(\mathcal{C}, N_{\mathcal{C}/X}) = H^1(\mathcal{C}, O_\mathcal{C} \oplus O_\mathcal{C}(1)) = 0$, we obtain that the tangent space $T_{\mathcal{C}_3}(O)$ is isomorphic to $H^0(\mathcal{C}, N_{\mathcal{C}/X})$.

Let $\mathcal{R}_{\mathcal{C}_3} = T_{\mathcal{C}_3}^*$ be the cotangent sheaf of and let $\omega_X$ be the sheaf of higher differential forms on $X$ (the canonical sheaf of $X$). Using the Serre duality, we can write the following sequence of identities:

$$
\mathcal{R}_{\mathcal{C}_3}(C) = T_{\mathcal{C}_3}(C) = H^0(\mathcal{C}, N_{\mathcal{C}/X})^* =
= H^1(\mathcal{C}, N_{\mathcal{C}/X}^* \otimes \omega_\mathcal{C}) = H^1(\mathcal{C}, N_{\mathcal{C}/X} \otimes \det N_{\mathcal{C}/X}^* \otimes \omega_\mathcal{C}) =
= H^1(\mathcal{C}, N_{\mathcal{C}/X} \otimes O_C(-1) \otimes \omega_\mathcal{C}) = H^1(\mathcal{C}, N_{\mathcal{C}/X} \otimes 0(-1)) =
= H^1(\mathcal{C}, N_{\mathcal{C}/X} \otimes \omega_X), \text{ since } \omega_X = O(-1) \otimes O_X.
$$

From the last we obtain that the long exact cohomology sequence, associated to the exact sequence $(1) \otimes 0(-1)$ takes the form:

$$(1') \quad 0 \rightarrow (H^0(\mathcal{C}, N_{\mathcal{C}/X} \otimes \omega_X) = 0) \rightarrow H^0(\mathcal{C}, N_{\mathcal{C}/W} \otimes O(-1)) \rightarrow
\rightarrow H^0(\mathcal{C}, O(1) \otimes O_\mathcal{C}) \rightarrow (H^1(\mathcal{C}, N_{\mathcal{C}/X} \otimes \omega_X) = \mathcal{R}_{\mathcal{C}_3}(C)) \rightarrow
\rightarrow H^1(\mathcal{C}, N_{\mathcal{C}/W} \otimes 0(-1)) \rightarrow \ldots \ .
$$

Using (5) we obtain the sequence (see also (8.1.2) (3')):

$$(1'') \quad 0 \rightarrow H^0(\mathcal{C}, O_\mathcal{C}) \xrightarrow{H^0(\mathcal{C})} H^0(\mathcal{C}, O(1) \otimes O_\mathcal{C}) \rightarrow
\rightarrow \mathcal{R}_{\mathcal{C}_3}(C) \rightarrow (H^1(\mathcal{C}, O_\mathcal{C}) = 0) \ .
$$

(8.1.6) If we suppose for a while that we have performed a globalization of the constructions above, simultaneously for all the elements $\mathcal{C}$ of the family of rational normal cubics $\mathcal{C}_3 = \mathcal{C}_3(X)$, then the members of the sequence
(1') will be induced by restriction on the fibers from sheaves, defined globally on the family $\mathcal{C}^3$ (as the fiber $\mathcal{O}_{\mathcal{C}}(\mathcal{C})$).

(6) Remark. In fact, the diagonal map $\mathcal{Y}_0: \mathcal{O}_C \to \mathcal{O}(1) \otimes \mathcal{O}_C$ (see (3')) corresponds to the choice of the divisor $C \cap \mathcal{O}$ on $\mathcal{O}$; the last divisor represents the hyperplane section of $C$ in the projective space $\langle C \rangle = \text{Span}(C) \subset P^3$ (see (5.4.1) and (5.4.2)).

(8.1.7) On the other hand, as we know, the family $\mathcal{C}^3$ is embedded in $P_Z((\Sigma^*)) = G_Z(4, \Sigma)$ as a zero scheme of the section $s$, and the factorfamily $\mathcal{C}^3_0 = \mathcal{C}^3/\sigma$ is embedded in $G_Z(3, \Sigma)$ as a zero scheme of the section $s_0^*$ (see Cor. (7.2.6), Prop. (7.3.2) and Cor. (7.3.3)).

From the last we conclude, that the tautological sheaves $\mathcal{L}_3, \Sigma$ and $\mathcal{L}^*_2, \Sigma^*$ from the standard tautological sequence:

(a) $0 \to \mathcal{L}_3, \Sigma \to G_Z(3, \Sigma) \times \Sigma \to \mathcal{L}^*_2, \Sigma^* \to 0$

are well-defined on the family $\mathcal{C}^3_0$, and the tautological sheaves $\mathcal{L}_4, \Sigma$ and $\mathcal{L}^*_1, \Sigma^*$ from the sequence:

(b) $0 \to \mathcal{L}_4, \Sigma \to P_Z((\Sigma^*)) \times \Sigma \to \mathcal{L}^*_1, \Sigma \to 0$

are well-defined on the family $\mathcal{C}^3$.

But the existence of the natural 2-sheeted covering $\mathcal{K}: \mathcal{C}^3 \to \mathcal{C}^3_0$, induced from the involution $\sigma': \mathcal{C}^3 \to \mathcal{C}^3$ ($\sigma'(C) = \overline{C}$, $\sigma'^2 = \text{id}$)

does that the ($\mathcal{K}$-preimages of the) sheaves $\mathcal{L}_3, \Sigma$ and $\mathcal{L}^*_2, \Sigma^*$ are defined also on the family $\mathcal{C}^3$. 
as usual we write $\mathcal{E}_3, \Sigma$ and $\mathcal{E}^*, \Sigma^*$ instead of $\mathcal{R}^* \mathcal{E}_3, \Sigma$ and $\mathcal{R}^* \mathcal{E}^*, \Sigma^*$. Then, the (lifted by $\mathcal{R}$) exact sequence (a) on $\mathcal{E}$, together with the exact sequence (b), define an embedding $j: \mathcal{E}_3, \Sigma \hookrightarrow \mathcal{E}_4, \Sigma$ of sheaves on $\mathcal{E}^3$ (see also (7.2.6), (7.3.2) and (7.2.6)).

Let $R = \mathcal{E}_4, \Sigma / \mathcal{E}_3, \Sigma$ be the factor-sheaf of the embedding $j$. Hence, we obtain the exact sequence:

(c) $0 \rightarrow \mathcal{E}_3, \Sigma \rightarrow j \mathcal{E}_4, \Sigma \rightarrow R \rightarrow 0$

of sheaves on the family $\mathcal{E}^3$.

(8.1.8) From the cited corollaries and proposition we derive also that the embedding $j: \mathcal{E}_3, \Sigma \hookrightarrow \mathcal{E}_4, \Sigma$ corresponds to the embeddings $H_0 \cap H_0 \hookrightarrow H_0$ of subspaces of $\sum(V_2)$; remember that $P(H_0) = \text{Span}(C) \subset C \subset P(\sum(V_2))$, $P(H_0) = \text{Span}(\overline{C}) \subset P(\sum(V_2))$, and $C + \overline{C} = C_1 = 6 \cdot 2, 0 (1) \cdot X_{10} \subset \text{Span}(C_1) = P(\sum(V_2))$, where $1 = P(V_2)$ (ibid.).

But the last means exactly that the epimorphism $\beta: \mathcal{E}_4, \Sigma \rightarrow R = \mathcal{E}_4, \Sigma / \mathcal{E}_3, \Sigma$ coincides with the dual of the map $H^0(\mathcal{E}_4): H^0(C, O_C) \rightarrow H^0(C, 0 (1) \otimes O_C) \cong H^0(C, 0 P(\sum(V_2))(1) \otimes O_C)$

(see the sequence (1") and the description of the map $\mathcal{E}_4$ in the beginning of the Remark). In fact, the description of the epimorphism $\beta: \mathcal{E}_4, \Sigma \rightarrow R$ corresponds to the description of the global sections of
the linear system \( \left| P(H_C \cap H_{\overline{C}}) \right| \) on \( C \), defined by the map \( \psi_C \). Therefore, the exact sequence \( (1'') \) takes the form:

\[
(1'') \quad 0 \rightarrow (\zeta_4, \sum_1) / \zeta_3, \sum_1 (C) \rightarrow H^0(\psi_C) = \beta^* \rightarrow \zeta_4, \sum_1 (C) \rightarrow 0,
\]

therefore \( \mathcal{R}_C^{\mathcal{G}^3}(C) \cong \zeta_3, \sum_1^*(C) \).

So, we formulate the following

\((8.1.9)\) PROPOSITION (The open Tangent Bundle Theorem for the family \( C \) of rational normal cubic curves on \( X \)).

Let \( U = \{ C \in C^3 : C \) and \( \overline{C} \) are smooth rational normal cubic curves on \( X = X_{10} = G \cap F(V_8) \cap Q \), such that \( N_{C/X} \) and \( N_{\overline{C}/X} \) are isomorphic to \( 0 \oplus 0 \), and also, such that the surfaces \( S_1 \) and \( S_0 \) are rational normal cubic scrolls (see \((2.2.4)(1)\) and \( \S 5.2 \)) and \( N_{S_1/X} \otimes 0_C = N_{S_1/X} \otimes 0_{\overline{C}} \cong 0 \oplus 0 \).

Then, there exists a naturally defined isomorphism

\[
\zeta_3, \sum_1 | U \rightarrow \mathcal{R}_C^{\mathcal{G}^3} | U.
\]

To prove the Tangent Bundle Theorem \((8.1.9)\) we need:

(1) to prove Lemma \((8.1.3)\), and (2) to globalize the local constructions, leading to the sequence \((1'''\)).
Proof of Lemma (8.1.3).

We shall suppose in addition that the surface

\[ S_C = \bigcup l \colon 1 - \text{a line in } P^4 \text{ & } l \in C \]

is a rational normal cubic scroll (see (5.2.1)); by the way, the surface \( S_C \) cannot be a cone (see Corollary (5.2.3)). Though this superfluous restriction will be not used in the proof, the author supposes that the normal sheaves \( N_{C/Y} \) for the curves \( C \) of the type (5.2.1) (2.a) (see also Corollary (5.2.3)), where \( Y = Y_5 \) is the threefold defined below (see (1), (5) and (7)), are isomorphic to \( O_{P^1}(1) \oplus O_{P^1}(3) \).

In any case, the last does not prevent us from proving (8.1.3).

(1) It is well known (see \([F, (B.5.8)]\)) that the tangent bundle \( T(G(2,V)) \) is isomorphic to the bundle of homomorphisms \( \text{Hom}(\mathcal{E}_{2,V}, \mathcal{E}_{3,V}^*) = \mathcal{E}_{2,V}^* \otimes \mathcal{E}_{3,V}^* \).

Let \( P^3 = P(V_4) \), where \( 0 \subset V_4 \subset V \). Then, the embedded cycle \( \sigma_{1,1}(P^3) \) carries out, in practice, an embedding of the Grassmannian \( G(2,V_4) \) into \( G(2,V) \); in particular, we have the exact sequence:

\[
0 \rightarrow \text{Hom}(\mathcal{E}_{2,V_4}, \mathcal{E}_{2,V_4}^*) \rightarrow \text{Hom}(\mathcal{E}_{2,V}, \mathcal{E}_{3,V}^*) \otimes O_{\sigma_{1,1}(P^3)} \rightarrow N(\sigma_{1,1}(P^3)), G) \rightarrow 0.
\]

We can derive from here that the normal sheaf \( N(\sigma_{1,1}(P^3)), G) \) equals to the restriction \( \mathcal{E}_{2,V}^* \otimes O_{\sigma_{1,1}(P^3)} \).
Let now \( Y = Y_5 = G \cap P^6 \), and let \( P^3 \subset P^4 \) is chosen in such a way that \( G'_{1,1}(P^3) \) intersects \( Y \) transversely; we shall suppose also that \( Y \) is chosen to be a smooth intersection. The intersection \( q = G'_{1,1}(P^3) \cap Y \) is a conic; we shall suppose in addition that the surface \( S_q = \bigcup \{ l : l - \text{a line in } P^4 \land l \in q \} \) is a smooth quadric (see § 4.3 (1)). Then we shall have:

\[
N_{q/Y} = N(q, Y) = N(\ G'_{1,1}(P^3) \cap P^6 \ , \ G \cap P^6 \) =
\quad N(\ G'_{1,1}(P^3), G \ ) \otimes q = \tau_{2,1} \otimes q = \delta^{(1)} \oplus 0_{q(1)}.
\]

(2) Let now \( l \) be a line lying on \( Y = Y_5 \). It is easy to see that \( N_{1/Y} \) is one of the sheaves \( 0_{1} \oplus 0_{1} \) or \( 0_{1}(-1) \oplus 0_{1}(1) \). In [I1, (5.2)] (see also [I4, (7.2)]) was proven that \( N_{1/Y} = 0_{1} \oplus 0_{1} \) for the general line \( l \subset Y_5 \) (see also [FN]).

(3) Let now \( q \) and \( l \) be a conic and a line, lying on \( Y = Y_5 \) and let \( q \) and \( l \) intersect in a point. Let \( H = P^5 \subset P^6 \) be a sufficiently general hyperplane through \( q \cup l \), such that the surface \( S = S_5 = Y \cap H \) is smooth. The surface \( S \) is a Del Pezzo surface of degree 5 in \( P^5 = H \subset P^6 \). The surface \( S = S_5 \) is obtained from the projective plane \( P^2 \) after blowing-up of four points \( x_1, x_2, x_3 \) and \( x_4 \in P^2 \). As \( S_5 \) is chosen to be smooth, then no three of the points \( x_1, x_2, x_3 \) and \( x_4 \) are collinear.

(4) Let now \( z \in P^2 \) be a point, which does not lie on the union of the lines \( l_{ij} = \langle x_i, x_j \rangle \), \( 1 \leq i < j \leq 4 \).
Let $G = \left\{ x_k : 1 \leq k \leq 4 \right\}$ be the map described in (3). The inverse of $G$ is the rational map given by the non-complete linear system

$$\left| 0_{P^2} (3 - x_1 - x_2 - x_3 - x_4) \right| : P^2 \longrightarrow P^5.$$

Let now $Q$ be a conic in $P^2$ through the points $x_1, x_2, x_3$ and $z$. It is easy to see that the proper $G$-preimage $C \subset S = S_5$ of the conic $(Q = 0) \subset P^2$, $Q \in H^0 (0_{P^2} (2 - x_1 - x_2 - x_3 - z))$ is a rational normal cubic curve.

The line $l$ (see (3)) is one of the ten lines, which lie on the Del Pezzo surface $S = S_5 \subset P^5$. Obviously, we can suppose that the morphism $G$ is chosen in such a way that $l = G^{-1}(x_4) \subset S_5$.

Let $(t) = (t_1 : t_2)$ be the homogeneous coordinates of $P^1 = \left| 0_{P^2} (2 - x_1 - x_2 - x_3 - z) \right|$. Evidently, the proper preimage $C(t)$ of the conic $Q(t) \in P^1 = P^1(t)$ splits iff $Q(t)$ passes through the point $x_4$.

Let $(1) = (1 : 1)$, and let $x_4 \in Q(1)$. Then $C(1) = q \cup l$, where $q$ is the conic and $l$ is the line $l = G^{-1}(x_4)$; obviously, $(q, l)_S = 1$.

(5) Let now $\beta(t) : Y(t) \longrightarrow Y = Y_5$, $(t) \neq (1)$, be the blowing-up of the curve $C(t) \subset S_5 \subset Y = Y_5$; let also $S(t) = \beta^{-1}(C(t))$ be the ruled surface over the curve $C(t)$, $(t) \neq (1)$. It is easy to see that the normal sequence for $C(t) \subset S_5 \subset Y$ is:
\[ 0 \rightarrow O_C(t) (1) \rightarrow N_C(t)/Y \rightarrow O_C(t) (3) \rightarrow 0. \]

Therefore, \( N_C(t)/Y = O_C(t) (a) \oplus O_C(t) (b) \), where \( a + b = 4 \) and \( 1 \leq a \leq b \leq 3 \).

Let \( S = (t) \neq (1) \cup Y(t) \subset Y \).

Obviously, \( S \) (resp. \( Y \)) is a bundle of rational surfaces (resp. of rational threefolds) over the projective line \( P^1 = P^1(t) \).

(6) Let the general rational normal cubic \( C \subset Y_5 \) has a normal bundle \( O_C(1) \oplus O_C(3) \). Then we can choose the hyperplane \( H \) (and, hence, the surface \( S_5 = Y_5 \cap H \)) through the given degenerate rational normal cubic \( C(1) = q \cup l \) (see (3)), and the point \( z \in P^2 \) (see (4)) in such a way that \( N_C(t)/Y = O_C(t) (1) \oplus O_C(t) (3) \) for any \( (t) \in P^1(t) \) in some neighbourhood of \( (t) = (1) \). We can also suppose that the normal sheaves of \( q \) and \( l \) are as in (1) and (2), i.e. \( N_q/Y = O_q(1) \oplus O_q(1) \) and \( N_l/Y = O_l \oplus O_l \).

The exceptional surfaces \( S(t) = Y(t) \) are isomorphic to the ruled surface \( \mathcal{F}_2 = P(N_C(t)/Y) = P(O(1) \oplus O(3)) \), for all \( (t) \neq (1), \) around \( (1) \). The surface \( S(1) = S \setminus \left[ (t) \neq (1) \cup S(t) \right] \) is a degeneration of \( \mathcal{F}_2 \); on the other hand, \( S(1) \) is an union of the two quadrics - the quadric \( A \) over \( q \) and the quadric \( B \) over \( l \).
The smooth quadrics $A$ and $B$ intersect along the common generator $A \cap B$, because $(q, l)_S = 1$. But then the union $A \cup B$ cannot be a degeneration of the family of $\Gamma_2$'s, described above. Consequently, the general rational normal cubic curve $C \subset Y = Y_5$ has a normal bundle $N_{C/Y} = \mathcal{O}_C(2) \oplus \mathcal{O}_C(2)$ (see (5)).

(7) Let now $C \subset X = X_{10}$ be a general rational normal cubic curve. In particular, $C$ is embedded in the rational normal cubic scroll $S_1 \subset W$ as a hyperplane section (see Prop. (2.2.4) (1) and Cor. (5.3.2)).

Let $H = P^6 \subset P^7$ be a sufficiently general hyperplane through the curve $C$, such that the intersection $Y = Y_5 = W \cap H$ is smooth. In particular, let $H$ intersects $S_1$ transversely along the curve $C$. Then

$$N_{S_1/W} \otimes \mathcal{O}_C = N(S_1 \cap H)/(W \cap H) \otimes \mathcal{O}_C = N_{C/Y}.$$ 

Clearly, we can also suppose that the curve $C$ is sufficiently general in $Y = Y_5$, i.e. $N_{C/Y} = \mathcal{O}_C(2) \oplus \mathcal{O}_C(2)$. But then $N_{S_1/W} \otimes \mathcal{O}_C = N_{C/Y} = \mathcal{O}_C(2) \oplus \mathcal{O}_C(2)$. Lemma (8.1.3) is proved.
8.2. Proof of the Tangent Bundle Theorem (Prop. (8.1.9))

(8.2.1) The proof of the T.B.T. consists in the repeating of the local constructions from the beginning globally over the whole family \( \mathcal{C}_t^3 = \mathcal{C}_t^3(X) \). For this purpose we introduce the following subvarieties of the projectivized bundle \( P\mathcal{G}_t^3(\Sigma) \) on the places of the curve \( C \subset X = X_{10} \) and of the surface \( S_1 \subset W : \)

\[
D_{\mathcal{C}_t^3} = \left\{ (C, x) \in \mathcal{C}_t^3 \times X : x \in C \right\}
\]

and

\[
D_{\mathcal{C}_t^3}(G) = \left\{ (C, x) \in \mathcal{C}_t^3 \times W : x \in S_1 \right\}
\]

remind that \( C + G = C_1 = S_1 \). \( Q \subset S_1 \subset P(\Sigma(V_2)) \)

where \( l = P(V_2) \).

(8.2.2) The pairs of successive embeddings

\[
D_{\mathcal{C}_t^3} \subset \mathcal{C}_t^3 \times X \subset \mathcal{C}_t^3 \times W
\]

and

\[
D_{\mathcal{C}_t^3} \subset D_{\mathcal{C}_t^3}(G) \subset \mathcal{C}_t^3 \times W
\]

globalize respectively the pairs of embeddings \( C \subset X \subset W \) and \( C \subset S_1 \subset W \). Correspondingly to the normal sequences (8.1.1) (1) and (8.1.1) (2) they induce the sequences of normal sheaves :

\[
(1, \mathcal{G}) : 0 \to N(D_{\mathcal{C}_t^3}, \mathcal{C}_t^3 \times X) \to N(D_{\mathcal{C}_t^3}, \mathcal{C}_t^3 \times W) \to
\]

\[
\to N(\mathcal{C}_t^3 \times X, \mathcal{C}_t^3 \times W) \otimes O_{D_{\mathcal{C}_t^3}} \to 0
\]

and

\[
(2, \mathcal{G}) : 0 \to N(D_{\mathcal{C}_t^3}, D_{\mathcal{C}_t^3}(G)) \to N(D_{\mathcal{C}_t^3}, \mathcal{C}_t^3 \times W) \to
\]

\[
\to N(D_{\mathcal{C}_t^3}(G), \mathcal{C}_t^3 \times W) \otimes O_{D_{\mathcal{C}_t^3}} \to 0
\].

(8.2.3) Let \( O(1) \) be the antiauxthological sheaf on the projective bundle \( P_{G(2,V)}(\Sigma) \to G = G(2,V) \).
According to the definition of the bundle \( \Sigma \) (see (6.3.3)), the sheaf \( \Sigma \) on \( G \) is embedded in a natural way in the sheaf \( V_8 \otimes O_G \) (corresponding to the constant bundle \( G \times V_8 \) over \( G = G(2, V) \)). In fact, for every \( V_2 \in G = G(2, V) \), the embedding of the corresponding fibers

\[
\Sigma (V_2) \subset (V_8 \otimes O_G) (V_2)
\]

coincides with the natural embedding \( (V_2 \wedge V) \cap V_8 \subset V_8 \). Therefore, the sheaf \( O(1) \) coincides with the restriction of the sheaf

\[
0_P(V_8)(1) = 0_P(\wedge^2 V) \otimes 0_P(V_8)
\]
on the subvariety

\[
P_G(\Sigma) \ni P_G(V_8 \otimes O_G) = G \times P(V_8), \quad (\text{the "old" } \quad O(1), \text{ see above}).
\]

(8.2.4) As in the local case we multiply the sequence

\[
(2', \xi_2) \quad 0 \longrightarrow N(D_{\xi^3}(G)) \otimes 0(-1) \longrightarrow
\]

\[
\longrightarrow N(D_{\xi^3}, \xi^3 \times W) \otimes 0(-1) \longrightarrow N(D_{\xi^3}(G), \xi^3 \times W) \otimes O_{\xi^3} \otimes 0(-1) \longrightarrow 0
\]

Similarly, we obtain the sequence \((1', \xi_2) \otimes 0(-1)\) :

\[
(1', \xi_2) \quad 0 \longrightarrow N(D_{\xi^3}, \xi^3 \times X) \otimes 0(-1) \longrightarrow
\]

\[
\longrightarrow N(D_{\xi^3}, \xi^3 \times W) \otimes 0(-1) \longrightarrow 0(1) \longrightarrow 0,
\]
since \( N(\xi^3 \times X, \xi^3 \times W) = 0(2) \).

(8.2.5) Let, for the brevity, denote by \( B \) the sheaf \( N(D_{\xi^3}, \xi^3 \times W) \otimes 0(-1) \) and let \( \lambda \) and \( \beta \) are the natural projections for the Fano family \( D_{\xi^3} \).
As before, using the equality $R^1_{\beta} N(D_{\mathfrak{a}}^3, \mathfrak{c}^3 \times X) = 0$ which can be obtained on the level of fibers from the local equality $H^1(C, N_{C/X}) = 0$, and also the relative duality of Serre, we obtain:

$$\bigoplus_{i} \mathfrak{c}_i^3 = R^1_{\beta} \left( N(D_{\mathfrak{a}}^3, \mathfrak{c}^3 \times X) \otimes O(-1) \right)$$

(here $R^i_{\beta}$ is the operation of taking the $i$-th direct image).

(8.2.6) In a similar way, using the local equalities:

$$H^i(N_{S_1/W} \otimes O_C \otimes O(-1)) = 0, \ i = 1, 2, \ldots$$

we derive the identities:

$$(5, \mathfrak{c}_i) \quad R^i_{\beta} \left( N(D_{\mathfrak{a}}^3(G), \mathfrak{c}_i^3 \times W) \right) \otimes O_{D_{\mathfrak{a}}^3} \otimes O(-1) = 0, \ i = 1, 2, \ldots$$

(8.2.7) As is already clear, $N(\mathfrak{c}_3^3 \times X, \mathfrak{c}_3^3 \times W) = O(2)$.

Using the obtained just above and taking the direct $\beta$-images of the sequences $(1', \mathfrak{c}_i)$ and $(2', \mathfrak{c}_i)$, we obtain:

$$(2', \mathfrak{c}_i) \quad 0 \to R^1_{\beta} \left( N(D_{\mathfrak{a}}^3 \times D_{\mathfrak{a}}^3(G)) \otimes O(-1) \right) \to R^1_{\beta} B \to 0$$

and:

$$(1', \mathfrak{c}_i) \quad 0 \to R^1_{\beta} \left( N(D_{\mathfrak{a}}^3 \times X) \otimes O(-1) \right) \to R^1_{\beta} B \to 0$$

and:

$$0 \to \beta^*(N(D_{\mathfrak{a}}^3 \times \mathfrak{c}_i^3 \times X) \otimes O(-1)) \to \beta^* B \to 0$$

and:

$$\beta^* 0(1) \to \mathfrak{c}_i^3 \to R^1_{\beta} B \to R^1_{\beta} 0(1) \to 0,$$

where $\beta^* = R^0_{\beta}$.
(8.2.8) Now, we introduce the variety

$$D_{\mathbb{A}^3}(\langle \cdot \rangle) = \left\{ (C, x) : C \in \mathbb{C}^3, x \in \langle C \rangle = \text{Span}(C) \right\}. $$

As $\text{Span}(C) \subset P(\sum(C)) = P(\sum(V_2))$, then

$$D_{\mathbb{A}^3}(\langle \cdot \rangle)$$

is embedded naturally in the projectivized bundle $P_{\mathbb{A}^3}(\Sigma)$. In fact, the bundle $\Sigma$ is defined, originally, on the Grassmannian $G = G(2, V)$, but it is defined also on the subsets of $G(2, V)$ and on the preimages of these subsets under morphisms; as usual, we drop the symbols of the maps. From the Remark (6) (see (8.1.6)), (8.1.7) and (8.1.8), we conclude that

$$D_{\mathbb{A}^3}(\langle \cdot \rangle) = P_{\mathbb{A}^3}(\mathcal{C}_4, \Sigma),$$

(see also the Corollary (7.3.3)).

The Fano family $D_{\mathbb{A}^3}$ is embedded identically in the projectivized bundle $D_{\mathbb{A}^3}(\langle \cdot \rangle) = P_{\mathbb{A}^3}(\mathcal{C}_4, \Sigma)$; in particular, the projection $\beta : D_{\mathbb{A}^3}(\langle \cdot \rangle) \longrightarrow \mathbb{A}^3$ is a restriction of the projection $\lambda : D_{\mathbb{A}^3}(\langle \cdot \rangle) \longrightarrow \mathbb{A}^3$. From the properties of the antitauathological sheaf $O(1)$ on $P_{\mathbb{A}^3}(\mathcal{C}_4, \Sigma)$ we have:

$$\beta_* O(1) = \mathcal{C}_4^*, \Sigma$$

and $R^1_\beta O(1) = 0$.

(8.2.9) Let we denote the sheaf $N(D_{\mathbb{A}^3}, D_{\mathbb{A}^3}(G)) \otimes O(-1)$ by $A$. Taking into account the identity

$$\beta_* (N(D_{\mathbb{A}^3}, \mathbb{A}^3 \times X) \otimes O(-1)) = 0,$$

which follows from the corresponding local equality, we conclude, just as in the local case, that the exact sequence
can be written in the form:

\[(1'', \xi) : 0 \rightarrow \mathcal{F}_* \mathcal{A} \rightarrow \mathcal{V}_4, \Sigma \rightarrow \mathcal{P}_3 \rightarrow \mathcal{R}_1 \mathcal{B} \mathcal{A} \rightarrow 0,\]

(see the "local" sequences \((8.1.5) (1'')\) and \((8.1.6) (1'')\)).

According to the comments in the Remark \((8.1.6) (6)\), the sheaf \(\mathcal{F}_* \mathcal{A}\) coincides with the factorsheaf \(\mathcal{R} = \mathcal{V}_4, \Sigma / \mathcal{V}_3, \Sigma\). If, moreover, the curve \(C\) (\(C \in \mathcal{C}_3^3\)) is general (in the sense of the conditions of Proposition \((8.1.9)\)), then the first direct image \(\mathcal{R}_1 \mathcal{B} \mathcal{A}\) vanishes over \(C\), since the fiber of \(\mathcal{R}_1 \mathcal{B} \mathcal{A}\) over \(C \in U \subset \mathcal{C}_3^3\) is \(H^1(\mathcal{N}_C/\mathcal{S}_1 \otimes O(-1)) = H^1(C, O_C) = 0\) \((C \cong \mathbb{P}^1)\). The open T.B.T. (Proposition \((8.1.9)\)) is proved.

\((8.2.10)\) REMARK.

The correct formulation of the T.B.T. must take into account a joint geometrical interpretation of the sequences \((1, \xi)\) and \((2, \xi)\), taking into consideration also the disposition of the locus \(\mathcal{C}_3^3 \subset \mathcal{C}_3\), where the local constructions fail; it is already clear that \(\mathcal{C}_3^3 \subset \mathcal{C}_3 \setminus U\) (see \((8.1.4) (5)\) and \((8.1.3)\)).

\((8.2.11)\) The geometry of the Fano threefold \(X = X_{10}\) can provide an additional useful information by means of the geometrical analogue of the just proven open T.B.T. First, the exact sequence \((1'', \xi)\) induces a natural isomorphism

\[\Psi : \mathcal{V}_3^3, \Sigma | U \rightarrow \mathcal{P}_3 \mathcal{A}^3 | U\]

over the open subset \(U \subset \mathcal{C}_3^3\). The definition of \(H^0(\Psi_C)\) in \((8.1.5) (1'')\) shows that the fibre \(\mathcal{V}_3^3, \Sigma (C)\) coincides with the
embedded plane \( H_C \cap H_C' \subset \Sigma (C) ( = \Sigma (V_2) = \Sigma (C) ) \); see also the Corollary (7.2.6) (2). Here, as usual, \( C + C = C_P(V_2) \), \( P(H_C) = \langle C \rangle \), \( P(H_C') = \langle C' \rangle \).

The dual map \( \psi^* \) defines the isomorphism:

\[
\psi^* : \bigoplus c^3 |_U \longrightarrow \mathcal{T}_3, \Sigma |_U \quad \text{over } U \subset \mathcal{C}^3
\]

( here \( \bigoplus c^3 = \bigoplus c^3 \) is the tangent sheaf of the family \( \mathcal{C}^3 \)).

So, we can formulate the following:

(8.2.12) **THEOREM** (Geometric (open) T.B.T. for \( \mathcal{C}^3 \)).

There is a naturally defined isomorphism (see (8.1.9)):

\[
\psi^* : \bigoplus c^3 |_U \longrightarrow \mathcal{T}_3, \Sigma |_U
\]

such that \( \psi^* \) coincides with the natural map

\[
\psi^*(C) : \bigoplus c^3 (C) \cong H_C \cap H_C' \subset \Sigma (V_2) \subset V_8
\]

on the fiber \( \bigoplus c^3 (C) \); (see Corollary (5.3.2) and Corollary (7.2.6) for the definitions of \( V_2, C, H_C \) and \( H_C' \)).

§ 9. **Tangent Bundle Theorem for the family of conics on the Fano threefold** \( X = X_{10} \).

9.1. In the present section we shall follow some results of the Thesis of D. Logachev [L1] (see also [L2] and [P]) on the family of conics on \( X = X_{10} = G \cap P(V_8) \cap Q \).

The original idea of representing the family of conics on \( X_{10} \) as a degeneration locus, resp. as a zero scheme of a section of some vector bundle, and the corresponding statement of the tangent bundle theorem, belongs to Logachev.
The main observations in the present work are:

(1). The use of the simple fact that the conics and the rational normal cubics on $X_{10}$ correspond to the standard splittings of the codim. 2-cycles on $X_{10}$ of the types

$\mathcal{C}_{1,1} \cap X_{10}$ and $\mathcal{C}_{2,0} \cap X_{10}$ respectively; the last generate the subring of the ring of cycles of codimension 2 on $X = X_{10}$, which are induced from cycles on the Grassmannian $G(2, \nu)$;

(2). The possibility to perform the constructions of Logachev also to the family of the rational normal cubic curves on $X_{10}$.

The two marked families are, probably, the only families of curves on $X_{10}$ for which the mentioned methods and representations are adequate.

In particular, (1), together with the geometrical interpretations of the tangent bundle theorems for both families (see the T.B.T. (8.2.12) and the T.B.T.-s (9.4.1,3,5) below), grounds us to carry out a geometrical approach to the Abel-Jacobi maps for some families of curves, naturally arising on $X$ (see for ex. [CG]). Let, for example:

$\mathcal{C}_5^1 = \{ C \text{ - a curve} : C \subseteq X, p_a(C) = 1, \deg(C) = 5 \}$

be the family of the linearly normal elliptic quintics on $X$; the last corresponds to the set of components of the degenerate canonical curves of degree 10 on $X_{10}$ into pairs of elliptic curves. The T.B.T.-s for the families $F = \{ \text{conics on } X_{10} \}$ and $\mathcal{C}_3^2 = \{ \text{rat.norm. cubics on } X_{10} \}$, enables us to perform a geometrical approach to the Abel-Jacobi map of the family $W = \mathcal{C}_3^2 + F =$
Let \( \mathcal{C}_q = \mathcal{C}_1^5 \cap W = \{ c + q : c \in \mathbb{P}^3, q \in F, (c, q) = 2 \} \)
be the threedimensional family, the elements of which are, by definition, degenerate elliptic quintics on \( X \), which split into a sum of a rat.norm. cubic and a conic, intersecting between them in two points. The results in the paper \([V]\) of Claire Voisin on the Quartic Double Solids give us a hope to suppose that the general points \( z \) of the two-dimensional subvariety \( Z \) of the Intermediate Jacobian \( J(X_{10}) \) of \( X_{10} \), which is the Abel-Jacobi image of the "diagonal" family \( \mathcal{C}_q \), are double points of the theta-divisor \( (\mathcal{W})(X_{10}) \subset J(X_{10}) \). The author's suggest is that the tangent cones \( TC_z (\mathcal{W}) \) to the theta divisor \( \mathcal{W} \) at the singular points \( z \) of \( \mathcal{W} \), are quadrics of rank = 6. The last will provide us with a geometrical proof of the non-rationality of the Fano threefold \( X = \bigwedge^2 V \) (see \([V, \S 4]\)).

9.2. In (6.2.1) was defined the bundle \( Pf \rightarrow P(V^*) \); the fiber \( Pf(V_4) \) over the element \( V_4 \subset P(V^*) \) was defined to be the Plücker quadric, corresponding to the 4-space \( V_4 \) (for more details, see \( \S 6.2 \)). It was proven that the corresponding invertible sheaf \( Pf \) over \( P(V^*) \) is isomorphic to the sheaf \( (\det \zeta_{4,V})^{-1} = \mathcal{O}_{P(V^*)}(1) \).

Let now \( V_8 \), \( V_8 \subset V_{10} = \bigwedge^2 V \) be choosen sufficiently general, such that:
(i) the variety $X = G \cap P(V_8) \cap Q$ is smooth;
(ii) $\dim (V_8 \cap (\wedge^2 V_4)) = 4$, for every $V_4 \in P(V^*) = G(4,V)$.

Then we can define correctly a bundle $M$ over $P(V)$ with 4-dimensional fibers $M(V_4)$ such that
$M(V_4) = \wedge^2 V_4 \cap V_8$, $V_4 \in P(V^*) = G(4,V)$.

On the other hand, the restriction of the Plücker bundle $Pf \longrightarrow P(V^*)$ (regarded as a bundle of quadrics on $\wedge^2 \mathcal{C}_{4,V} \longrightarrow P(V^*)$) to the subbundle $M = \wedge^2 \mathcal{C}_{4,V} \cap V_8 \subset \wedge^2 \mathcal{C}_{4,V}$, represents the bundle $Pf$ as a subbundle of $S^2 M^* \longrightarrow P(V^*)$;
moreover, $Pf$ has fibers
$Pf(V_4) = H^0(P(\wedge^2 V_4 \cap V_8), 0(2 - (\mathcal{C}_{1,1}(P(V_4)) \cap P(V_8))))$
(see Corollary (6.2.3)).

We can also define an analogue to the bundle $Q \rightarrow G(2,V)$ which was regarded as a subbundle of $S^2 \Sigma^* \longrightarrow G(2,V)$, namely:

Let $X = G \cap P(V_8) \cap Q$, where $Q$ is a quadric. We define the bundle (the invertible sheaf) $Q$ over $P(V)$ such that:

1. $Q$ is a subbundle of $S^2 (\wedge^2 \mathcal{C}_{4,V})^*$;
2. $Q(V_4) = \mathcal{C} \cdot Q |_{P(\wedge^2 V_4) \subset H^0(P(\wedge^2 V_4), 0(2))}$, i.e., $Q(V_4)$ is the restriction of the equation of the quadric $Q$, etc.
From the conditions (i) and (ii) we derive that we can look at the bundle \( Q \longrightarrow \mathbb{P}(V^*) \) also as a bundle of quadrics (a "quadric") on the bundle \( M \). Moreover, it is evident that the corresponding sheaf \( Q \) is isomorphic to the structure sheaf \( \mathcal{O}_{\mathbb{P}(V^*)} \). Just as in \( \S \ 6.3 \) (see Prop. (6.3.2)) we can prove the following:

(9.2.1) PROPOSITION (Logachev, \([L1]\)).

Let \( Z \subset \mathbb{P}(V^*) \left( S^2 (\wedge^2 \mathcal{O}_{4,V})^* \right) \) be the set:

\[
Z = \mathbb{P} \left\{ (V_4, q) : V_4 \in \mathbb{P}(V^*), q \in H^0(\mathbb{P}(\wedge^2 V_4 \cap V_8), 0(2 - Q \cap \mathcal{O}_{1,1}(\mathbb{P}(V_4))) \right\}
\]

Then \( Z \) is naturally isomorphic to the projectivized bundle \( \mathbb{P}(V^*) \left( \mathcal{O}_{\mathbb{P}(V^*)}(1) \oplus \mathcal{O}_{\mathbb{P}(V^*)} \right) \).

Moreover, from (i) and (ii) (see above), follows, that:

\[
Z = \mathbb{P} \left\{ (V_4, q) : V_4 \in \mathbb{P}(V^*), q \in H^0(\mathbb{P}(\mathcal{O}(V_4)), 0(2 - \mathcal{O}_{1,1}(\mathbb{P}(V_4)) \cap X)) \right\}
\]

\[
\subset \mathbb{P}(V^*) \left( S^2 \mathcal{O}^* \right) \quad \text{(see Proposition (6.3.2))}.
\]

(9.2.2) From the definition of the variety \( Z \) follows that the fiber \( \gamma^{-1}(V_4) = Z(V_4) \) of the projective bundle \( \gamma : Z \longrightarrow \mathbb{P}(V^*) \) is isomorphic to the projective line \( \langle \text{Pf}(V_4), Q(V_4) \rangle = \text{Span} \left\{ \text{Pf}(V_4) \cup Q(V_4) \right\} \)

in the projective space \( \mathbb{P}(S^2 \mathcal{O}^*(V_4)) \) of quadrics in \( \mathbb{P}(\mathcal{O}(V_4)) \).
The surface $F_c$ of the geometrical conics on $X = X_{10}$, the surface $F$ (of "conics on $X$") and the factorsurface $F_0 = F/\iota = \{(q, \overline{q}; V_4) : q \in F_c, (q, V_4) = \iota(q, V_4)^2\}$ were defined in § 4.4 and in § 4.5 (here $\iota$ is the involution, which is correctly defined on $F$). The non-correctness of the defining the rational map $(-)^{(-)}: F_c \rightarrow F_c$ arises from the fact that the linear Span of the set $S_{q_0} = \left( \bigcup l : l \subset P(V) \text{ - a line, } l \in q_c \right)$ where $q_0$ is the unique $\mathbb{P}$-conic on $X$, is contained in an one-dimensional family of $P^3$'s; the last coincides with the line $P((V/U)^*) = \{P^3 : P^3 \subset P^4 = P(V) \text{ & (the plane) } P(U) = S_{q_0} \subset P^3 \}$ (see § 4.2 and § 4.3).

(9.2.3) Let now $V_4 \subset P(V^*)$ and let $W$ denotes, as usual, the intersection $W = G \cap P(V_8) \subset P(V_8) \subset P(\wedge^2 V)$. As we know (see Proposition (2.2.3) (2)) the cycle $Q(P(V_4)) = \sigma_{1,1}(P(V_4)) \cap P(V_8) \subset W$ is a two-dimensional quadric. It follows from the definition of the bundle $M \rightarrow P(V^*)$ (see the introduction of § 9.2) that $\langle Q(P(V_4)) \rangle = \text{Span} (Q(P(V_4))) = P(M(V_4))$. By definition, $W \cap Q = X$. Hence, the elliptic quartic $C(P(V_4))$ (see Corollary (3.3.3)), which is, by definition, an intersection of the cycle $\sigma_{1,1}(P(V_4))$ with $X$, coincides with the intersection of the quadrics $Q(P(V_4))$ and $Q$.
in the space \( P(\mathcal{M}(V_4)) = P(\wedge^2 V_4 \wedge V_8) \).

The quadric \( Q(P(V^4)) \) defines a section of the Pfaff bundle \( Pf \rightarrow P(V^*) \); the restrictions of the quadric \( Q \) on the fibers of the bundle \( M \rightarrow P(V^*) \) define a section of the bundle \( Q \rightarrow P(V) \), defined above. Therefore, the condition of splitting of the curve \( C(P(V^4)) \) into a pair of involutive conics is equivalent to the existence of a quadric of rank \( \leq 2 \) on the line \( \langle Pf(V^4) \cup Q(V^4) \rangle \) in the space of quadrics in \( P(M(V^4)) \). The last arguments are correct on the subset \( F_c \setminus \{ q_0 \} \subset F_c \) ( \( q_0 \) is the unique \( \emptyset \)-conic on \( X \) ), but it is not hard to see that the "exceptional" quadrics of \( F \), which lie on the exceptional line \( r_{F}^{-1}(q_0) \) (see (4.5.1)) correspond to degenerate quadrics in the bundle \( Pf \oplus Q \) (see for ex. § 4.3, § 4.4, Claim (4.2.1) and (4.5.1)). The involution \( i \) on \( F \) identifies the elements of the exceptional curve \( r_{F}^{-1}(q_0) \) with the elements of the curve of the \( \emptyset \)-conics on \( X \) (ibid.).

(9.2.4) The tautological sequence:

\[
0 \rightarrow 0 \gamma(-1) \rightarrow \gamma^* \left( O_{P(V^*)}(1) \oplus O_{P(V^*)} \right) \rightarrow O_Z(1) \rightarrow 0
\]

on the projective bundle

\[
\gamma : Z = P \left( O_{P(V^*)}(1) \oplus O_{P(V^*)} \right) \rightarrow P(V^*)
\]

defines, just as in (6.4.4) (see also (6.4.2)), a net of quadrics on \( Z \). It follows from the properties of the tautological sequence that the projectivization of the embedding
0 → 0 \gamma (-1) \overset{j}{\longrightarrow} \gamma^*(O_P(V^*) (1) \oplus O_P(V^*))

represents the "embedded" points of the fibers of the projective bundle \( Z \); these points are interpreted also as quadrics in the corresponding spaces \( P(M(V^4)) \).

Because of that, as in § 6, Proposition (6.4.4), we can conclude the following:

(9.2.5) CLAIM (Logachev, [L1]). The factorfamily (of pairs of involutive conics) \( F_0 = F / i \) is embedded naturally in the projective bundle

\[ Z = P_P(V^*) (O_P(V^*) (1) \oplus O_P(V^*)) \]
as a second determinantal of the net of quadrics:

\[ 0 → 0 \gamma (-1) \overset{j}{\longrightarrow} \gamma^*(O_P(V^*) (1) \oplus O_P(V^*)) \overset{i}{\longrightarrow} \gamma^*(S^2 M^*) \]
i.e. \( F_0 = D_2 (\hat{\varphi}) \), where \( \hat{\varphi} = i \cdot j \).

9.3. The remaining constructions and results are carried out parallel to the corresponding ones for the family of the rational normal cubics on \( X \) (see § 7).

(9.3.1) Let \( Z \) and \( \hat{\varphi} : 0 \gamma (-1) \longrightarrow \gamma^*(S^2 M^*) \)

be as above. We define:

\[ \tilde{D}_k = \tilde{D}_k (\hat{\varphi}) = \{ (z; U_k) = (V_4, q; U_k) \in G_Z(k, M^*) : \text{Im}(q) \subseteq U_k \} \subseteq G_Z(k, M^*) , \ 0 \leq k \leq 4 . \]

(9.3.2) It is not hard to prove that the first determinantal \( D_1 (\hat{\varphi}) \subset Z \) vanishes for the general intersection \( X = G \cap P(V_8) \cap Q \), hence, the natural projection

\[ \pi_2 : \tilde{D}_2 (\hat{\varphi}) \longrightarrow D_2 (\varphi) \] (see (7.1.1)) is an isomorphism for the general \( X \) (see Remark (7.1.2)).
(9.3.3) From Corollary (3.3.3) (2) we have that the curve $C(P(V_4)) \subset X$ is a linearly normal curve of arithmetical genus 1 and of degree 4; the linear normality follows from the fact that the curve $C(T(V_4))$ is a component of the canonical curve $C(P(V_4)) + C_1$, $l = P(V_2)$ (see (3.3.2) and Cor. (3.3.3)), where $V_2 \in G(2, V)$ is chosen in such a way that $V_2 \subset V_4$.

In particular, in the case, when $C(P(V_4))$ splits into a sum of two plane sections: $C(P(V_4)) = q + \overline{q}$, we obtain that $(q, \overline{q}) = 2$, hence, the intersection $\langle q \cap \overline{q} \rangle$ of the spaces $\langle q \rangle = \text{Span}(q)$ and $\langle \overline{q} \rangle = \text{Span}(\overline{q})$ is a line in $P(V_8)$. In the case when the two points of intersection of $q$ and $\overline{q}$ coincide, the line $\langle q \cap \overline{q} \rangle$ is equal, by definition, to their common tangent line (see also (5.4.1) and (5.4.2))

(9.3.4) The tautological sequence on $G_Z(2, M)$, together with the injection $\hat{\varphi}: O_\gamma(-1) \to \gamma^* S^2 M^*$, define, as in (7.2.2), a section $s_0$ of the sheaf $O_\gamma(1) \otimes \text{Coker}(2)$ over $G_Z(2, M)$, where $\text{Coker}(2) = \text{Coker}(S^2 \otimes_{2, M^*} \gamma^* S^2 M^*)$.

(9.3.5) The same arguments, as in Corollary (7.2.4), give us a ground to impose the following open conditions on the subspace $V_8 \subset V_{10} = \wedge^2 V$ (dim$(V_8) = 8$) and on the quadric $Q$, where $X = G \cap P(V_8) \cap Q$ (see also Corollary (6.3.5) and Remark (7.1.2)).
(i) \( X = G \cap P(V_8) \cap Q \) is smooth;

(ii) \( \dim (V_8 \cap (\wedge^2 V_4)) = 4 \), for every \( V_4 \in P(V^*) = G(4, V) \);

(iii) \( D_1(\hat{\varphi}) = \emptyset \).

(9.3.6) CLAIM (Logachev, [L1]). Let \( X = G \cap P(V_8) \cap ( \) where the subspace \( V_8 \subset \wedge^2 V \) and the quadric \( Q \) fulfill
the open conditions (i), (ii) and (iii). Let \( Z, M, \hat{\varphi}, s_0 \) and \( F_0 = F/1 \), where \( F \) is the surface
of conics on \( X \) (see Definition (4.4.2)), be as above.

Then \( F_0 \) is embedded in \( G_Z(2, M^*) \) as a zero scheme
of the section \( s_0 \), i.e.
\[
F_0 \overset{\sim}{\to} Z(s_0) \subset G_Z(2, M^*).
\]

(9.3.7) The section \( s_0 \), together with the duality
\( G_Z(2, M^*) \overset{\sim}{\to} G_Z(2, M) \) define a section \( s_0^* \) with
zeros on \( G_Z(2, M) \) (see (7.2.5)). So we can formulate

(9.3.8) COROLLARY. In the conditions of the Claim (9.3.6)
the"dual" natural embedding \( F_0 \hookrightarrow G_Z(2, M) \) repre-
sents the isomorphic image of \( F_0 \) in \( G_Z(2, M) \) as a
zero scheme of the section \( s_0^* \), i.e.
\[
F_0 \overset{\sim}{\to} Z(s_0^*) \subset G_Z(2, M).
\]

Moreover, this embedding can be described as follows:

Let \( (q, V_4) \) and \( (\overline{q}, V_4) = i(q, V_4) \) be a pair
of involutive elements (conics) of \( F \). Let \( H_q \) and
\( H_{\overline{q}} \in M^*(V_4) \) be the hyperplanes in \( M(V_4) \), such that
\( \langle q \rangle = \text{Span}(q) = P(H_q) \) and \( \langle \overline{q} \rangle = \text{Span}(\overline{q}) = P(H_{\overline{q}}) \).
Then the embedding $F_0 \hookrightarrow G_Z(2, M)$ sends the pair
$((q, V_4), (\overline{q}, V_4))$ to the triple
$(V_4, H_q \cdot \overline{H_q}; H_q \cap \overline{H_q}) \in G_Z(2, M)$.

(9.3.9) REMARK. Note that $H_q \cap \overline{H_q}$ is always a plane
since $P(H_q \cap \overline{H_q})$ coincides with the projective line
$q \cap \overline{q}$ (see (9.3.3)).

Similarly, as in the "rational normal cubic" case
(see (7.3.1)), the tautological sequence on $P_Z(M^*)$
together with the injection $\varphi$ define a section $s$ of
the sheaf $\mathcal{O}_{\nu}(1) \otimes S^2 \varphi^* \gamma_{4,M}$ over $P_Z(M^*)$. The same
arguments, as in (7.3.1) and (7.3.2), give

(9.3.10) CLAIM (Logachev, [L1]). In the conditions
of the Claim (9.3.6), the family $F$ of the conics for
the Fano threefold $X = X_{10} = G(2, V) \cap P(V_8) \cap \overline{Q}$ is
embedded naturally in the projective bundle $P_Z(M^*)$
as a zero scheme of the section $s$, defined just above, that is:

$$F \sim Z(s) \subset P_Z(M^*)$$

Using the duality $P_Z(M^*) \sim G_Z(3, M)$, we can
formulate the following:

(9.3.11) COROLLARY. The embedding $F \hookrightarrow P_Z(M^*) \sim G_Z(3, M)$
sends the element $((q, V_4)) \in F$ to the triple
$(V_4, H_q \cdot \overline{H_q}; H_q) \in G_Z(3, M)$ (the hyperplanes
$H_q$ and $\overline{H_q}$ are defined as in the Corollary (9.3.8)).
9.4. Let now \( p_F : F \rightarrow F_0 = F/\iota \) be the two-sheeted covering, which corresponds to the involution \( \iota : F \rightarrow F \) (see (4.5.1) and (4.5.2)). The embeddings \( F_0 \hookrightarrow G_Z(2, M) \) and \( F \hookrightarrow P_Z(M^*) = G_Z(3, M) \), together with \( p_F \) define natural maps \( F \rightarrow F_0 \rightarrow G_Z(2, M) \) and \( F \rightarrow G_Z(3, M) \). Therefore, the standard tautological bundles \( \varpi_{2,M} \) on \( G_Z(2, M) \) and \( \varpi_{3,M} \) on \( G_Z(3, M) = P_Z(M^*) \) can be lifted on the family \( F \).

Just as in the "rational normal cubic" case, there is a representation of the tangent bundle \( T_F \) by means of some bundle on \( F \), which can be described in geometrical terms.

(9.4.1) CLAIM (Logachev) The open Tangent Bundle Theorem for the family \( F \) of "conics" for \( X \); see [L1]).

Let \( l_\sigma = \{ (q, V_4) : q \text{ is a } \sigma \text{-conic on } X \} \subset F \) be the smooth rational curve of the \( \sigma \)-conics in the family of conics \( F \) (see § 4.2); let also \( Z, M, G_Z(2, M) \), \( P_Z(M^*) \) and the bundles \( \varpi_{2,M} \) and \( \varpi_{3,M} \) be as above. Let \( \varpi_F \) be the cotangent sheaf (resp., the cotangent bundle) of \( F \).

Let \( U = F \setminus (l_\sigma \cup \iota(l_\sigma)) \subset F \). Then there exists a naturally defined isomorphism of sheaves over the open subset \( U \subset F : \)

\[
\varpi^*_2 : \varpi^*_{2,M} \mid_U \xrightarrow{\sim} \varpi^*_F \mid_U
\]

(9.4.2) REMARK. The curve \( \iota(l_\sigma) \) coincides with the
exceptional curve $r_F^{-1}(q_0)$ (the curve of the exceptional "conics" of $F$ (see § 4.2, or the comments (9.2.2)).
The curves $l_\sigma$ and $i(l_\sigma)$ are mutually disjoint (ibid.)
Moreover, there is an exact global version of the T.B.T.
for the family of conics, proven by Logachev:

(9.4.3) **THEOREM** (Logachev, [11]). There is a naturally defined exact sequence of sheaves over the surface of conics $F$ for $X$:

$$
0 \rightarrow \mathcal{C}^*_{2,M} \rightarrow r^* \mathcal{M}_F \rightarrow 0_{l_\sigma} \vee 0_{i(l_\sigma)} \rightarrow 0
$$

where $r: F \rightarrow F_m$ is the morphism onto the minimal model $F_m$ of $F$. In fact, the morphism $r$ is a composition of $r_F: F \rightarrow F_c$ (which blows-down the curve $r_F^{-1}(q_0)$ of the exceptional conics in $F$), and of the blowing-down of the $r_F$-image of its involutive curve (the curve of the $\sigma$-conics on $X$ in $F_c$). The both contractions blow-down $(-1)$-curves to (non-singular) points of the smooth minimal surface $F_m$ (see also [P]).

(9.4.4) **COMMENTS.**

(1) Note that the statement of the global T.B.T. (9.4.3) is, in fact, a correct treatment of the analogue of the sequence $(1'', \mathcal{C}_\sigma)$ for the family $F$ (see (8.2.9), seq. (1'', $\mathcal{C}_\sigma$) and Remark (8.2.10)).

(2) The factor-sheaf $0_{l_\sigma} \vee 0_{i(l_\sigma)}$ has a support on the closet subset $l_\sigma \cup i(l_\sigma)$ of codim. = 1 in $F$.

(9.4.5) **COROLLARY** (see (8.2.12)).

Let $\psi^*: T_F | U \rightarrow \mathcal{C}^*_{2,M} | U$ be the
dual of $\psi$ (see Claim (9.4.1)). Then $\psi^*$ coincides with the natural map

$$\psi^*(q, V_4) : T_p(q, V_4) \to H_q \cap H_{q'} \subset M(V_4) \subset V_8$$

on the fiber $T_p(q, V_4)$, where $(q, V_4) \in U$ and the hyperplanes $H_q$ and $H_{q'}$ are defined as in Corollary (9.3.8).

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