Stochastic boundary value problems. A white noise functional approach.

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Abstract

We give a program for solving stochastic boundary value problems involving functionals of (multiparameter) white noise. As an example we solve the stochastic Schrödinger equation

$$\Delta u + V \cdot u = -f \qquad \text{in } D \subset \mathbf{R}^d, \quad u\Big|_{\partial D} = 0$$

where V is a positive, noisy potential. We represent the potential V by a white noise f unctional and interpret the product of the two distribution valued processes V and u as a Wick product $V \diamond u$. Such an interpretation is in accordance with the usual interpretation of a white noise product in ordinary stochastic differential equations. The solution u will not be a generalized white noise functional but can be represented as an L^1 functional process.

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1 Introduction

The general motivation to study stochastic differential equations from the point of view of applications is the need to understand systems where there are random fluctuations or noise, or where the available information is incomplete.

The theory for ordinary differential equations with the randomness represented by e.g. white noise is now well understood (see e.g. [Ø]). The theory for stochastic partial differential equations (SPDE) is less developed. See, however, [W]. We here present a framework for treating certain SPDE's, generalizing the approach in [LØU1]. We advocate these techniques on the equations

$$\Delta u + V \cdot u = -f \qquad \text{in } D \subset \mathbf{R}^d, \quad u\Big|_{\partial D} = 0 \tag{1.1}$$

where D is a given bounded domain, f a given deterministic function and V is a positive, noisy potential.

In this article we represent the positive noise V by the *Wick exponential* $\mathrm{Exp}(W)$ of d-parameter white noise W. The product of the two distribution valued processes V and u is then interpreted as a *Wick product* $V \diamond u$. This is related to the Wick product used in quantum field theory. This interpretation can be regarded as a natural extention of the Ito integral interpretation of ordinary stochastic differential equations with white noise.

It turns out that there does not exist a generalized white noise functional u which solves (1.1) (see e.g. [HKPS] for definition). However, the solution can be represented as an L^1 *functional process* $u(\varphi, x, \omega)$. Heuristically $u(\varphi, x, \omega)$ is the value of u when the test function ("window") φ is used, shifted to the point x and in the experiment ω.

The paper is organized as follows: In Section 2 we recall the definition of the white noise probability space $(\mathcal{S}'(\mathbf{R}^d), \mathcal{B}, \mu)$, the d-parameter white noise W and the d-parameter Brownian motion B_x. In Section 3 we define the Hermite transform $\mathcal{H}$, which transforms elements of $L^2(\mu)$ into analytic functions of infinitely many complex variables $z_1, z_2, \ldots$ In Section 4 we describe the inverse $\mathcal{H}^{-1}$ in terms of an explicit integral operator. The Wick product on $L^2(\mu)$ is defined in Section 5. Then in Section 6 we introduce the L^2 functional processes $\mathcal{L}^2$ and we illustrate how to solve the Walsh equation

$$\begin{cases} \Delta u = W & \text{in} \quad D \subset \mathbf{R}^d, \\ u = 0 & \text{on} \quad \partial D \end{cases}$$

in terms of $u \in \mathcal{L}^2$. The basic method is to apply the Hermite transform and its inverse.

We then turn to the concept of positive noise V in Section 7. We point out that for equation (1.1) there does not exist an L^2 functional process solution u. In fact, (1.1) does not even have a solution in $(\mathcal{S}^*)$, the space of generalized white noise functionals

(Section 8). However, we show that if we extend the Wick product to L^1 (Section 9) and use this L^1 interpretation in (1.1), there exists an L^1 functional process u solving the equation (Section 10).

After this paper was written we learned that ideas similar to ours based on the $\mathcal{S}$-transform (which is related to our Hermite transform, see [LØU1, §5]) and the Wick product have earlier been adopted by Kuo and Potthoff [KP] to solve certain SDEs. However, their method seems insufficient here because, as just mentioned, the solution of the equation we consider is not a generalized white noise functional. In fact, we think that in general the natural framework for solutions of stochastic partial differential equations is not $(\mathcal{S}^*)$, but the space of L^1 functional process.

2 The white noise probability space

In this section we introduce the basic probability space that we will use in the rest of the paper. Here we only state the main results. For more details we refer to [HKPS].

In the following d will denote a fixed positive integer, interpreted as the time-, space- or time-space dimension of the system we consider. More generally we will call d the *parameter dimension.* Let $\mathcal{S} = \mathcal{S}(\mathbf{R}^d)$ be the Schwartz space of rapidly decreasing smooth (C^∞) functions on $\mathbf{R}^d$. $\mathcal{S}$ is a Frechet space under the family of seminorms

$$\|f\|_{k,\alpha} := \sup_{x\in\mathbf{R}^d}(1+|x|^k)|\partial^\alpha f(x)|$$

where $k \geq 0$ is an integer and $\alpha = (\alpha_1, \dots, \alpha_m)$ is a multi-index of non-negative integers α_j and

$$\partial^\alpha f = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}x_1\dots\partial^{\alpha_m}x_m} f \qquad \text{where} \quad |\alpha| = \alpha_1 + \dots + \alpha_m$$

The dual $\mathcal{S}' = \mathcal{S}'(\mathbf{R}^d)$ of $\mathcal{S}$, equipped with the weak star topology, is the space of *tempered distributions.*

Let $\mathcal{B} = \mathcal{B}(\mathcal{S}')$ be the family of Borel subsets of $\mathcal{S}'$. We now consider the probability measure μ on $\mathcal{B}$ which is characterized by the following property:

$$E[e^{i\langle\cdot,\varphi\rangle}] := \int_{\mathcal{S}'} e^{i\langle\omega,\varphi\rangle}d\mu(\omega) = e^{-\frac{1}{2}\|\varphi\|^2} \qquad \text{for all} \quad \varphi\in\mathcal{S}, \tag{2.1}$$

where $\|\varphi\|^2 = \|\varphi\|^2_{L^2(\mathbf{R}^d)}$, $\langle\omega,\varphi\rangle = \omega(\varphi)$ for $\omega \in \mathcal{S}'$ and $E = E^\mu$ denotes expectation with respect to μ. The existence of such a measure μ is given by the *Bochner-Minlos theorem,* which can be found in [GV]. We call $(\mathcal{S}', \mathcal{B}, \mu)$ the *white noise probability space.* The justification for this name is the following:

From (2.1) it follows that

$$E[\langle\cdot,\varphi\rangle^2] = \|\varphi\|^2; \qquad \varphi\in\mathcal{S} \tag{2.2}$$

(see e.g. [LØU1, §2]) and using this isometry we can define

$$\langle\omega,\varphi\rangle := \lim_{k\to\infty}\langle\omega,\varphi_k\rangle \qquad \text{for all} \quad \varphi\in L^2(\mathbf{R}^d) \tag{2.3}$$

(the limit taken in $L^2(\mu)$) where φ_k is any sequence in $\mathcal{S}$ such that $\varphi_k\to\varphi$ in $L^2(\mathbf{R}^d)$. In particular, this allows us to define

$$\tilde{B}_t := \tilde{B}_{t_1,\ldots,t_d}(\omega) := \langle\omega,\chi_{[0,t_1]\times\cdots[0,t_d]}\rangle \qquad \text{for} \quad t_k\geq 0. \tag{2.4}$$

Then $\tilde{B}_t$ is (essentially) a *d-parameter Brownian motion*, in the sense that there exists a t-continuous version B_t of $\tilde{B}_t$ such that B_t is a d-parameter Brownian motion (sometimes also called d-parameter Brownian sheet).

The *(d-parameter) Ito integral* of $\varphi\in L^2(\mathbf{R}^d)$ is now defined by

$$\int_{\mathbf{R}^d}\varphi(t_1,\ldots,t_d)dB_{t_1,\ldots,t_d}(\omega) = \langle\omega,\varphi\rangle. \tag{2.5}$$

Note that if $d=1$ and supp$\varphi\subset[0,\infty)$ this coincides with the classical Wiener-Ito integral.

We define the *(d-parameter) white noise process* W_φ as follows:

$$W_\varphi(\omega) = \langle\omega,\varphi\rangle \qquad \text{for} \quad \varphi\in L^2(\mathbf{R}^d),\ \omega\in\mathcal{S}'. \tag{2.6}$$

In other words, we have

$$W_\varphi(\omega) = \int_{\mathbf{R}^d}\varphi(t)dB_t(\omega); \quad t=(t_1,\ldots,t_d) \tag{2.7}$$

By integration by parts for Wiener-Ito integrals we have

$$\int_{\mathbf{R}^d}\varphi(t)dB_t(\omega) = (-1)^d\cdot\int_{\mathbf{R}^d}\frac{\partial^d\varphi}{\partial t_1,\ldots\partial t_d}(t)\cdot B_t(\omega)dt \tag{2.8}$$

Therefore

$$W_\varphi(\omega) = \int_{\mathbf{R}^d}\varphi(t)dB_t(\omega) = \left((-1)^d\frac{\partial^d\varphi}{\partial t_1\ldots\partial t_d},B_t\right)_{\mathbf{R}^d} = \left(\varphi,\frac{\partial^d B}{\partial t_1\ldots\partial t_d}\right)_{\mathbf{R}^d}, \tag{2.9}$$

where $(\cdot,\cdot)_{\mathbf{R}^d}$ denotes the usual inner product in $L^2(\mathbf{R}^d)$. In other words, in the sense of distributions we have

$$W = \frac{\partial^d B_t}{\partial t_1\ldots\partial t_d} \tag{2.10}$$

The space $L^2(\mu)$ can be given a useful representation, which we now describe:

For $n = 0, 1, 2, \ldots$ and $x \in \mathbf{R}$ let

$$h_n(x) := (-1)^n e^{\frac{x^2}{2}} \cdot \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}) \tag{2.11}$$

be the *Hermite polynomials.* If $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a multi-index of non-negative integers we define

$$h_\alpha(u_1, \ldots, u_m) := h_{\alpha_1}(u_1) h_{\alpha_2}(u_2) \ldots h_{\alpha_m}(u_m). \tag{2.12}$$

Fix an orthonormal basis $\{e_k\}_{k=1}^\infty$ for $L^2(\mathbf{R}^d)$ and define

$$H_\alpha(\omega) = h_\alpha(\theta_1(\omega), \ldots, \theta_m(\omega)); \qquad \omega \in \mathcal{S}'(\mathbf{R}^d) \tag{2.13}$$

where

$$\theta_k(\omega) = \int_{\mathbf{R}^d} e_k(t) dB_t(\omega) \tag{2.14}$$

The following fundamental result can be regarded as a version of the celebrated Wiener-Ito chaos theorem:

Theorem 2.1 ([HKPS, Lemma 2.3]) The collection $\{H_\alpha(\cdot);\ \alpha \in \mathbf{N}^m;\ m = 0, 1, \ldots\}$ forms an orthogonal basis of $L^2(\mu)$. Moreover,

$$E[H_\alpha^2] := \|H_\alpha\|_{L^2(\mu)}^2 = \alpha! \tag{2.15}$$

where $\alpha! = \prod_{j=1}^m \alpha_j!$ when $\alpha = (\alpha_1, \ldots, \alpha_m)$.

The theorem implies that any $f \in L^2(\mu)$ has the unique representation

$$f(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \tag{2.16}$$

where $c_\alpha \in \mathbf{R}$ for each multi-index α and

$$\|f\|_{L^2(\mu)}^2 = \sum_\alpha \alpha! c_\alpha^2 \tag{2.17}$$

An alternative description of $L^2(\mu)$ is the following:

In any dimension there is an expansion of every $f \in L^2(\mathcal{S}'(\mathbf{R}^d), \mu)$ in terms of *multiple Ito integrals* (see [I]):

$$f(\omega) = \sum_{n=0}^\infty \int_{(\mathbf{R}^d)^n} f_n(u_1, \ldots, u_n) dB_u^{\otimes n} \tag{2.18}$$

where $f_n \in \hat{L}^2((\mathbf{R}^d)^n)$ (i.e. $f_n \in L^2((\mathbf{R}^d)^n)$ and f_n is symmetric) and

$$E[f^2] = \sum_{n=0}^\infty n! \left\| f_n \right\|_{L^2((\mathbf{R}^d)^n)}^2 . \tag{2.19}$$

Here $dB_u^{\otimes n} = dB_{u_1} \cdots dB_{u_n}$ denotes Ito's n-multiple differential on $(\mathbf{R}^d)^n$, as defined in [I].

Moreover, we have

$$\int_{(\mathbf{R}^d)^n} e_1^{\otimes\alpha_1} \otimes e_2^{\otimes\alpha_2} \otimes \cdots \otimes e_m^{\otimes\alpha_m} dB^{\otimes n} = \prod_{j=1}^m h_{\alpha_j}(\theta_j) \tag{2.20}$$

if $\alpha = (\alpha_1, \ldots, \alpha_m)$, $n = |\alpha| = \alpha_1 + \cdots + \alpha_m$ and θ_j is given by (2.14). Here – and later – $\otimes$ denotes the *symmetrized* tensor product, so that, e.g., $f \otimes g(x_1, x_2) = \frac{1}{2}[f(x_1)g(x_2) + f(x_2)g(x_1)]$ if $x_i \in \mathbf{R}$, and similarly for more than two variables. By (2.13) we can rewrite (2.20) as follows:

$$\int_{(\mathbf{R}^d)^{|\alpha|}} e^{\otimes\alpha} dB^{\otimes|\alpha|} = H_\alpha(\omega) \tag{2.21}$$

which shows the connection between the two representations (2.16), (2.18).

3 The Hermite transform on $L^2(\mu)$

We now describe a useful transformation $\mathcal{H}$ (*the Hermite transform*) which transforms a given $F \in L^2(\mu)$ into a complex valued function $\mathcal{H}(F) = \tilde{F}$ which is an analytic function of infinitely many complex variables $z_1, z_2, \ldots$ For $d=1$ this transform was introduced in [LØU1]. In the following $L^2(\mu)$ means $L^2(\mathcal{S}'(\mathbf{R}^d), \mu)$ (for general d) unless otherwise stated.

Definition 3.1 Let $F(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \in L^2(\mu)$. Then the *Hermite transform* $\mathcal{H}(F) = \tilde{F}$ of F is the formal power series in infinitely many complex variables $z_1, z_2, \ldots$ defined by

$$\mathcal{H}(F)(z) := \tilde{F}(z) := \sum_\alpha c_\alpha z^\alpha \tag{3.1}$$

where $z = (z_1, z_2, \ldots)$ and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_m^{\alpha_m}$ if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$.

Many of the results about Hermite transforms that were proved in [LØU1] and [LØU2] for $d = 1$ carry over to general d with minor modifications. We state these results, referring to where the proofs for $d = 1$ can be found:

Theorem 3.2 [LØU1, Lemma 5.3] If $X = \sum_\alpha c_\alpha H_\alpha \in L^2(\mu)$ then for each n the series

$$\tilde{X}(z) = \sum_\alpha c_\alpha z^\alpha \tag{3.2}$$

converges absolutely for $z = (z_1, \ldots, z_n, 0, 0, \ldots)$ with $|z_k| \leq M$ for all k. Therefore, for each n the function

$$\tilde{X}^{(n)}(z_1, \ldots, z_n) := \tilde{X}(z_1, \ldots, z_n, 0, 0, \ldots) \tag{3.3}$$

is *analytic* on $\mathbf{C}^n$.

Example 3.1

White noise W_φ was introduced in section 2. Since

$$W_\varphi=\int_{\mathbf{R}^d}\varphi(t)dB_t=\sum_k(\varphi,e_k)\int_{\mathbf{R}^d}e_k(t)dB_t=\sum_k(\varphi,e_k)h_1(\theta_k),$$

where $(\cdot,\cdot)$ denotes inner product in $L^2(\mathbf{R}^d)$, the Hermite transform of W_φ is given by

$$\tilde{W}_\varphi(z)=\sum_{k=1}^{\infty}(\varphi,e_k)z_k \tag{3.4}$$

4 The inverse Hermite transform

The usefulness of the Hermite transform stems partly from (3.4) and partly from the existence of an explicit inverse:

Let $\mathbf{N}$ denote the natural numbers. Define the measure λ on the product σ-algebra on $\mathbf{R}^{\mathbf{N}}$ by

$$d\lambda(\eta)=e^{-\frac{1}{2}\eta_1^2}\frac{d\eta_1}{\sqrt{2\pi}}\cdot e^{-\frac{1}{2}\eta_2^2}\frac{d\eta_2}{\sqrt{2\pi}}\cdots;\qquad \eta=(\eta_1,\eta_2,\ldots) \tag{4.1}$$

If n,k are positive integers put $J_{n,k}=\{\alpha=(\alpha_1,\ldots,\alpha_m);\ |\alpha|\le n \text{ and } \alpha_j=0 \text{ for } j>k\}$. If $\tilde{X}=\sum c_\alpha z^\alpha$ is the Hermite transform of $X\in L^2(\mu)$ we define

$$\tilde{X}^{(n,k)}=\sum_{\alpha\in J_{n,k}}c_\alpha z^\alpha \tag{4.2}$$

We remark that in [LØU1] and [LØU2] the 1-dimensional versions of the following results are slightly incorrectly stated, in the sense that either stronger conditions must be imposed on X or $\tilde{X}$ or the $d\lambda$-integrals of $\tilde{X}$ must be interpreted as limits of the $d\lambda$-integrals of the truncated $\tilde{X}^{(n,k)}$, as stated below:

Theorem 4.1 [LØU1, formula (5.11)] If $\tilde{X}$ is the Hermite transform of $X\in L^2(\mu)$ then

$$X(\omega)=\mathcal{H}^{-1}(\tilde{X}):=\lim_{n,k\to\infty}\int\tilde{X}^{(n,k)}(\theta+i\eta)d\lambda(\eta)\qquad(\text{limit in } L^2(\mu)) \tag{4.3}$$

where $\theta=(\theta_1,\theta_2,\ldots)$, with $\theta_k=\int_{\mathbf{R}^d}e_k dB$ as before.

Theorem 4.2 [LØU1, Corollary 6.2] Let $X\in L^2(\mu)$ and let $g:\mathbf{R}\to\mathbf{R}$ be continuous and bounded. Then

$$E[g(X)]=\lim_{n,k\to\infty}\int g\Big(\int\tilde{X}^{(n,k)}(\xi+i\eta)d\lambda(\eta)\Big)d\lambda(\xi) \tag{4.4}$$

It follows that

$$E[|X|^p]\le\liminf_{n,k\to\infty}\iint\left|\tilde{X}^{(n,k)}(\xi+i\eta)\right|^p d\lambda(\xi)d\lambda(\eta)\qquad\text{for all } p\ge1. \tag{4.5}$$

Corollary 4.3 [LØU2, §3] Suppose

$$f(z) = \sum_{\alpha} c_\alpha z^\alpha \qquad z = (z_1, z_2, \ldots)$$

is a formal power series in $z_1, z_2, \ldots$ with $c_\alpha \in \mathbf{R}$ for all α. Moreover, assume that, using the notation from (4.2),

$$\int \Big| \int \left(f^{(n,k)}(\xi + i\eta) - f^{(m,j)}(\xi + i\eta) \right) d\lambda(\eta) \Big|^2 d\lambda(\eta) \to 0 \qquad \text{as } n, k, m, j \to \infty \tag{4.6}$$

Then

$$F(\omega) := \sum_{\alpha} c_\alpha H_\alpha(\omega) \in L^2(\mathcal{S}'(\mathbf{R}^d), \mu).$$

5 The Wick product in $L^2(\mu)$

In quantum statistics there is a special product of random variables based on renormalization principles. For example, if

$$X = \int_{\mathbf{R}^d} \varphi dB \qquad \text{and} \qquad Y = \int_{\mathbf{R}^d} \psi dB$$

with $\varphi, \psi \in \mathcal{S}$, then this product of X and Y, called *the Wick product* and denoted by $X \diamond Y$, is given by

$$X \diamond Y = \int_{(\mathbf{R}^d)^2} \varphi \otimes \psi dB^{\otimes 2} \tag{5.1}$$

We extend this definition to $L^2(\mu)$ in the natural way by using the expansion (2.18):

Definition 5.1 Suppose

$$X = \sum_{n=0}^{\infty} \int_{(\mathbf{R}^d)^n} f_n dB \qquad \text{and} \qquad Y = \sum_{m=0}^{\infty} \int_{(\mathbf{R}^d)^m} g_m dB$$

are two elements of $L^2(\mu)$. Then we define

$$X \diamond Y = \sum_{n,m=0}^{\infty} \int_{(\mathbf{R}^d)^{n+m}} f_n \otimes g_m dB^{\otimes(n+m)} \tag{5.2}$$

provides the sum on the right converges in $L^1(\mu)$.

Alternatively, using (2.16) we see that this can also be formulated as follows:

If $X = \sum_\alpha a_\alpha H_\alpha$ and $Y = \sum_\beta b_\beta H_\beta$ are in $L^2(\mu)$ then

$$X \diamond Y = \sum_{\alpha,\beta} a_\alpha b_\beta H_{\alpha+\beta} \tag{5.3}$$

whenever this sum converges in $L^1(\mu)$.

The equivalence of (5.2) and (5.3) follows from (2.20), which gives the identity

$$H_\alpha \diamond H_\beta = \int_{(\mathbf{R}^d)^{(\alpha+\beta)}} e^{\otimes\alpha} \otimes e^{\otimes\beta} dB^{\otimes(\alpha+\beta)} = \int_{(\mathbf{R}^d)^{(\alpha+\beta)}} e^{\otimes(\alpha+\beta)} dB^{(\alpha+\beta)} = H_{\alpha+\beta}\,. \tag{5.4}$$

Note that the definition in (5.3) apparently depends on the choice of the base $\{e_k\}_{k=1}^\infty$ of $L^2(\mathbf{R}^d)$. However, the equivalence with (5.2) shows that this is not the case. In the Appendix we give an alternative, direct proof of independent interest that $X \diamond Y$ defined by (5.3) is independent of $\{e_k\}$.

Remark Our definition of Wick product $X \diamond Y$ coincides with the classical one, often denoted by $: XY :$, in cases like (5.1) but not in general. See the discussion in [LØU1, Remark to Th. 5.1].

As a motivation for the use of Wick products in stochastic differential equation we mention that if Y_t is an adapted and, for example, bounded stochastic process then

$$\int_0^T Y_t(\omega) dB_t(\omega) = \int_0^T Y_t \diamond W_t dt$$

in a sense that is made precise in [LØU2, Theorem 3.3]. Thus by representing multiplication by white noise in stochastic differential equations by Ito integrals one is really interpreting the product by white noise as a Wick product. In that sense the use of Wick products instead of ordinary products in stochastic partial differential equations is a natural extension of this principle. We also point out that Wick multiplication reduces to ordinary multiplication if one of the factors is deterministic (corresponding to all the f_n's being zero for $n > 0$ in the expansion (2.18)).

The following connection between Wick products and the Hermite transform $\mathcal{H}$ is crucial:

Theorem 5.2 [LØU1, Th. 5.5] Let $X, Y \in L^2(\mu)$ such that $X \diamond Y \in L^2(\mu)$. Then

$$\mathcal{H}(X \diamond Y) = \mathcal{H}(X) \cdot \mathcal{H}(Y), \tag{5.5}$$

where the product on the right is the usual complex product.

Example 5.3 *The square of white noise, $W_\varphi^{\diamond 2}$,* has the Hermite transform

$$\mathcal{H}(W_\varphi^{\diamond 2}) = \sum_{k,l=1}^{\infty} (\varphi, e_k)(\varphi, e_l) z_k z_l$$

6 Functional processes. The stochastic Poisson equation

We now introduce the generalized stochastic processes that we will work with. First we consider the L^2 case. If $X \in L^2(\mu)$ we can write, using (2.16),

$$X(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega)$$

By allowing each c_α to depend on test functions φ on $\mathbf{R}^d$ and on $x \in \mathbf{R}^d$ we get the following concept:

Definition 6.1 A *d-parameter generalized (white noise) L^2 functional process* is a sum of the form

$$X(\varphi, x, \omega) = \sum_\alpha c_\alpha(\varphi, x) H_\alpha(\omega); \qquad \varphi \in \mathcal{S}(\mathbf{R}^d),\ x \in \mathbf{R}^d,\ \omega \in \mathcal{S}' \tag{6.1}$$

where

$$c_\alpha(\cdot,\cdot) : \mathcal{S}(\mathbf{R}^d) \times \mathbf{R}^d \to \mathbf{R} \qquad (\text{for } |\alpha| \geq 1)$$

and

$$x \to c_\alpha(\varphi, x) \quad \text{is (Borel) measurable on } \mathbf{R}^d, \text{ for each } \varphi \in \mathcal{S}(\mathbf{R}^d).$$

If $\alpha = 0$ then $c_\alpha(\cdot)$ is just a measurable function on $\mathbf{R}^d$ (independent of φ).

We also require that

$$E[X(\varphi, x, \cdot)^2] = \sum_\alpha c_\alpha^2(\varphi, x)\alpha! < \infty \tag{6.2}$$

for all $\varphi \in \mathcal{S}(\mathbf{R}^d)$ and all $x \in \mathbf{R}^d$.

Remark Suppose that for each multi-index α we are given an element a_α such that

$$a_\alpha(\cdot) \in H^{-s}(\mathbf{R}^{d\cdot|\alpha|}) \qquad \text{for some} \quad s < \infty,$$

where in general H^r denotes the Sobolev space of order $r \in \mathbf{R}$. Then we can associate a function $c_\alpha(\cdot,\cdot)$ as above by putting

$$c_\alpha(\varphi, x) = a_\alpha(\varphi_x^{\otimes|\alpha|}) \qquad \text{where} \quad \varphi_x(u) = \varphi(u - x) \tag{6.3}$$

In other words, $c_\alpha(\varphi, x)$ is the result of applying a_α to (the tensor product of) the shifted test function φ. Put $M = |\alpha|d$, $a_\alpha = a$ and let D_k denote the distributional derivative of a in the k'th direction. Then for $\Phi, \Psi \in \mathcal{S}(\mathbf{R}^M)$ and $1 \leq k \leq M$ we have, putting $a(\Phi) = \langle a, \Phi \rangle$,

$$\begin{aligned}\int_{\mathbf{R}^M} \frac{\partial}{\partial y_k} \langle a, \Phi_y \rangle \Psi(y) dy &= -\int_{\mathbf{R}^M} \langle a, \Phi_y \rangle \frac{\partial \Psi}{\partial y_k} dy \\ &= -\langle a, \int_{\mathbf{R}^M} \Phi(u-y) \cdot \frac{\partial \Psi}{\partial y_k}(y) dy \rangle = -\langle a, \int_{\mathbf{R}^M} D_k \Phi(u-y) \cdot \Psi(y) dy \rangle \\ &= -\int_{\mathbf{R}^M} \langle a, D_k \Phi(u-y) \rangle \Psi(y) dy = \int_{\mathbf{R}^M} \langle D_k a, \Phi_y \rangle \Psi(y) dy.\end{aligned}$$

Therefore

$$\frac{\partial}{\partial y_k} \langle a, \Phi_y \rangle = \langle D_k a, \Phi_y \rangle. \tag{6.4}$$

So the distributional derivative of a evaluated at the y-shifted Φ coincides with the y-derivative of $\langle a, \Phi_y \rangle$. Returning to (6.3) we obtain

$$\frac{\partial}{\partial x_j} c_\alpha(\varphi, x) = \sum_{k=0}^{n-1} D_{j+kd} a_\alpha(\varphi_x^{\otimes n}) \tag{6.5}$$

where $n = |\alpha|$. Therefore processes of the form (6.1) are more general than processes of the form

$$Y_\varphi(\omega) = \sum_\alpha a_\alpha(\varphi^{\otimes n}) H_\alpha(\omega) \tag{6.6}$$

where $a_\alpha(\cdot) \in H^{-s}(\mathbf{R}^{nd})$ for some s and

$$E[Y_\varphi(\cdot)^2] = \sum_\alpha a_\alpha^2(\varphi^{\otimes n}) \alpha! < \infty$$

Such processes are called *functional processes.* They were introduced in [LØU1] in the case where the parameter dimension $d = 1$, and for such processes the derivatives were taken in distribution sense for each α. By (6.5) we see that this is (essentially) the same as taking the x-derivatives of

$$X(\varphi, x, \omega) = \sum_\alpha a_\alpha(\varphi_x^{\otimes |\alpha|}) H_\alpha(\omega).$$

It turns out that the L^2 requirement is too strong for many important applications. Therefore we extend the definition to L^p, $p > 0$ as follows:

Definition 6.2 Let $p>0$. A (d-parameter generalized) L^p *functional process* is a function

$$X(\varphi, x, \omega) : \mathcal{S} \times \mathbf{R}^d \times \mathcal{S}' \to \mathbf{R}$$

such that

(i) the map $x \to X(\varphi, x, \omega)$ is (Borel) measurable for all $\varphi \in \mathcal{S}$ and a.a. $\omega \in \mathcal{S}'$

and

(ii) the map $\omega \to X(\varphi, x, \omega)$ is in $L^p(\mu)$ for all $\varphi \in \mathcal{S}$ and all $x \in \mathbf{R}^d$.

The family of all L^p generalized functional processes is denoted by $\mathcal{L}^p$.

Conclusion

For a generalized functional process X we interpret $X(\varphi, x, \omega)$ as the measurement of X obtained by using the "window" φ shifted to the point x. And in stochastic differential equations involving $X(\varphi, x, \omega)$ the derivatives are taken in the x-variable, for each fixed window φ. In this sense our concept has similarities to Colombeau distributions [C]. For a more thorough motivation of our approach see [LØU3].

Generalized functional processes are both more general, mathematically simpler and easier to interpret than functional processes, so we will only consider this more general concept from now on. For simplicity we drop the word "generalized", so that "functional processes" really means "generalized functional processes" from now on.

The main idea in the solution of stochastic partial differential equations can now be summarized as follows:

Interpret the products involved as Wick products and look for solutions in the form of functional processes. By taking Hermite transform the equation is transformed to a (complex) deterministic partial differential equation involving usual products. If this equation can be solved, the inverse Hermite transform will give the solution of the original stochastic equation.

The stochastic Poisson equation

To illustrate the method we consider the equation

$$\begin{cases} \Delta u = W & \text{in } D \subset \mathbf{R}^d \\ u = 0 & \text{on } \partial D \end{cases} \tag{6.7}$$

where D is a given bounded open set in $\mathbf{R}^d$ and we use the notation

$$\Delta = \frac{1}{2}\sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2} \qquad \text{for the Laplacian operator } \Delta$$

This problem has been solved by Walsh [W]. The solution is given by the distribution valued process

$$u_O(\phi) = \int_D \Big[\int_D G(r,s)\phi(r)dr\Big] dB_s(\omega) \tag{6.8}$$

where $G(r,s)$ is the Green function corresponding to the deterministic equation

$$\Delta u = f \ \text{ in } D; \qquad u\Big|_{\partial D} = 0 \tag{6.9}$$

Based on our approach it is natural to make the following definition:

Definition 6.3 A functional process $u{=}u(\phi, x, \omega)$ is a *solution* of the stochastic Dirichlet problem

$$\Delta u = W \ \text{ in } D \subset \mathbf{R}^n, \qquad u\Big|_{\partial D} = 0$$

if for each $\phi \in \mathcal{S}$ there is a set $H_\phi \subset \mathcal{S}'$ with $\mu(H_\phi) = 1$ such that the function $x \to u(\phi, x, \omega)$ satisfies (in the classical, weak sense) the boundary value problem

$$\begin{cases} \Delta_x u(\phi, x, \omega) = W_{\varphi_x} & \text{for } x \in D \\ u(\phi, x, \omega) = 0 & \text{for } x \in \partial D \end{cases} \tag{6.10}$$

for all $\omega \in H_\phi$.

(Here, and in the following, Δ_x means that the Laplacian operator is taken with respect to the variable x.)

To illustrate our method, we now want to solve (6.7) using this definition. Taking $\mathcal{H}$-transforms on both sides in (6.10), we get as we expand $\widetilde{W}_\phi$ along a base $\{e_k\}_{k=1}^\infty$

$$\Delta \tilde{u}^{(n)}(x) = \sum_{k=1}^{n} (\phi_x, e_k) z_k \ \text{ for } x \in D; \qquad \tilde{u}^{(n)}\Big|_{\partial D} = 0 \tag{6.11}$$

for $n = 1, 2, 3, \ldots$

The solution formula of (6.9) extends by linearity to the case where f is $\mathbf{C}$-valued. Hence

$$\tilde{u}^{(n)}(x) = \tilde{u}^{(n)}(\phi, x, z) = \int_D G(x,r) \sum_{k=1}^{n} (\phi_r, e_k) z_k dr \tag{6.12}$$

Taking inverse transforms we get, with $z_k = \theta_k + i\eta_k$, θ_k as in (2.14),

$$\begin{aligned} u^{(n)}(\phi, x) &= \int_D \int_{\mathbf{R}^\mathbf{N}} G(x,r) \sum_{k=1}^{n} (\phi_r, e_k) z_k dr\, d\lambda(\eta) \\ &= \int_D G(x,r) \sum_{k=1}^{n} (\phi_r, e_k) \int_{\mathbf{R}^d} e_k(s) dB_s dr \\ &= \int_{\mathbf{R}^d} \int_D G(x,r) \sum_{k=1}^{n} (\phi_r, e_k) e_k(s) dr\, dB_s \end{aligned} \tag{6.13}$$

With this expression, it is easy to see that

$$u^{(n)}(\phi, x) \to u(\phi, x) = \int_{\mathbf{R}^d} \Big[\int_D G(x,r)\phi_r(s) dr \Big] dB_s \tag{6.14}$$

This is our solution to problem (6.7).

To compare this solution with the Walsh solution u_O in (6.8) we interpret u_O as a functional process $u_O(\phi, x)$ by defining

$$u_O(\phi, x) := u_O(\phi_x) = \int_D \int_D G(r,s)\phi_x(r) dr\, dB_s \tag{6.15}$$

Comparing with (6.13) we see that $u_O(\phi, x) = u(\phi, x)$ if, for all $x, s \in D$,

$$\int G(r,s)\phi(r-x) dr = \int G(x,r)\phi(s-r) dr \tag{6.16}$$

(Defining $G(r,s) = 0$ if $r \notin D$ or $s \notin D$ we may regard both integrals as integrals over $\mathbf{R}^d$.) Changing variables we see that (6.16) holds iff, for all $x, s \in D$,

$$\int G(x+y, s)\phi(y) dy = \int G(x, s-y)\phi(y) dy \tag{6.17}$$

If supp $\phi \subset (D - \{x\}) \cap (\{s\} - D)$ (where, in general, $A - B = \{a-b; a \in A, b \in B\}$) then (6.17) is valid because then, if we write $\phi = \Delta f$ with $f \in C_0^\infty(D)$ (the C^∞ functions with compact support in D) we get

$$\begin{aligned} &\int G(x+y,s)\Delta f(y) dy = \int G(y,s)\Delta f(y-x) dy = \int G(y,s)\Delta f_x(y) dy \\ &\quad = f_x(s) = f(s-x) = \int G(x,y)\Delta f(s-y) dy = \int G(x, s-y)\Delta f(y) dy\,. \end{aligned}$$

Thus if we define

$$\phi^{(\varepsilon)}(y) = \frac{1}{\varepsilon}\phi\Big(\frac{y}{\varepsilon}\Big) \qquad \text{for } \varepsilon > 0$$

we have

$$\lim_{\varepsilon \to 0} \int u_O(\phi^{(\varepsilon)}, x)\psi(x) dx = \lim_{\varepsilon \to 0} \int u(\phi^{(\varepsilon)}, x)\psi(x) dx \tag{6.18}$$

for all $\psi \in C_0^\infty(D)$. Using the terminology from Colombeau distributions ([C]) we conclude that we have *associated equality* between u_O and u in D. It follows that u_O and u both satisfy the same equation (in the weak sense)

$$\Delta u = \dot{W} \quad \text{in } D$$

This can also be seen directly by considering the Hermite transform $\tilde{u}_O$ of u_O, which is easily seen to be

$$\tilde{u}_O(\phi_x) = \sum_k \Big[\int_D \Big(\int_D G(s,r)\phi_x(r)dr\Big)e_k(s)ds\Big] z_k .$$

From this it follows that

$$\begin{aligned}
\Delta_x \tilde{u}_O(\phi_x) &= \sum_k \Big[\int \Big(\int G(s,r)\Delta_x\phi(r-x)dr\Big)e_k(s)ds\Big] z_k \\
&= \sum_k \Big[\int \Big(\int G(s,r)\Delta_r\phi(r-x)dr\Big)e_k(s)ds\Big] z_k \\
&= \sum_k \Big[\int \phi(s-x)e_k(s)dr\Big] z_k = \sum_k (\phi_x, e_k) z_k = \widetilde{W}_{\phi_x} ,
\end{aligned}$$

as claimed.

In spite of this u_O and u are not equal, and the explanation is that they do not satisfy the boundary requirement

$$u\Big|_{\partial D} = 0$$

in the same sense.

7 Positive noise

In many applications the noise that occurs is not white. The following example illustrates this:

If we consider fluid flow in a porous rock we often lack exact information about the *permeability* of the rock at each point. The lack of information makes it natural to model the permeability as a (multiparameter) noise. This noise will of course not be white but *positive* since permeability is always a nonnegative quantity. If we try to model permeability as a functional process we are therefore led to the following definition:

Definition 7.1 A functional process $X(\varphi, x, \omega)$ is called *positive* or *a positive noise* if

$$X(\varphi, x, \omega) \geq 0 \qquad \text{for a.a. } \omega \in \Omega$$

for all $\varphi \in \mathcal{S}$, $x \in \mathbf{R}^d$.

We now have the following useful characterization of positivity:

Theorem 7.2 [LØU1, Theorem 7.4] Let $X \in L^2(\mu)$. Then $X \geq 0$ a.s. if and only if

$$g_n(y) := \tilde{X}^{(n)}(iy)e^{-\frac{1}{2}y^2}; \qquad y \in \mathbf{R}^n \tag{7.1}$$

is positive definite for all n. $\tilde{X}^{(n)}(z)$ is defined by (3.3).

Theorem 7.3 [LØU1, Corollary 7.5] Let $X = X(\varphi, x, \omega)$ and $Y = Y(\varphi, x, \omega)$ be positive L^2 functional processes of the form (6.6). Then $X \diamond Y$ is also positive (when defined).

Theorem 7.4 [LØU1, Example 7.3] The Wick-exponential

$$\mathrm{Exp}[W_\varphi] := \sum_{n=0}^{\infty} \frac{1}{n!} W_\varphi^{\diamond n}$$

is a positive noise and can be computed by the relation

$$\mathrm{Exp}[W_\varphi] = \exp\Big(\int \varphi dB - \frac{1}{2}\|\varphi\|^2_{L^2}\Big)$$

8 An SDE with no solution in $L^2(\mu)$

We have already mentioned that it is sometimes necessary to consider L^p functional process for $p < 2$. In fact, equation (1.1) is an example of an equation whose solution is in $L^1(\mu)$ but not in $L^p(\mu)$ for $p > 1$. We now explain this in more detail.

As a model for the potential V we use the positive noise $\varepsilon \,\mathrm{Exp}\, W$ (where $\varepsilon > 0$) and we interpret the multiplication in the Wick sense. This gives the stochastic boundary value problem

$$\begin{cases} \Delta u = -\varepsilon \mathrm{Exp} W \diamond u - f & \text{in } D \subset \mathbf{R}^d \\ u = 0 & \text{on } \partial D \end{cases} \tag{8.1}$$

Definition 8.1 We say that an L^2 functional process $u = u(\varphi, x, \omega)$ is a *solution* of (8.1) if for all $\varphi \in \mathcal{S}$ and all $z = (z_1, z_2, \ldots, z_n, 0, 0, \ldots) \in \mathbf{C}^{\mathbf{N}}$ the function $x \to \tilde{u}(\varphi, x, \omega, z)$ solves the (deterministic) boundary value problem

$$\begin{cases} \Delta_x \tilde{u}(\varphi, x, z) = -\varepsilon \exp \tilde{W}_{\varphi_x}(z) \cdot \tilde{u}(\varphi, x, z) - (f * \varphi)(x) & \text{for all } x \in D \\ \lim\limits_{\substack{x \to y \\ x \in D}} \tilde{u}(\varphi, x, z) = 0 & \text{for all } y \in \partial_R D \end{cases} \tag{8.2}$$

where $(f*\varphi)(x)=\int\limits_{\mathbf{R}^n} f(y)\varphi(y-x)dy$ and $\partial_R D$ denotes the boundary points of D which are regular for the classical Dirichlet problem. (If necessary the first part of (8.2) is interpreted in the weak (distribution) sense.)

We will return to this equation in full generality in Section 10. Let us here consider the special case

$$d=1,\ \varepsilon=\frac{1}{2} \qquad \text{and} \qquad D=(0,\pi)\subset\mathbf{R}$$

Then, writing $u(x)=u(\varphi,x,\omega)$, (8.2) gets the form

$$\begin{cases} u''(x)=-\operatorname{Exp} W_{\varphi_x}\diamond u(x)-2(f*\varphi)(x);\ \ 0<x<1 \\ u(0)=u(\pi)=0 \end{cases} \tag{8.3}$$

Suppose u is an L^2 functional process solving (8.3). Then taking the $\mathcal{H}$-transform of (8.3) gives

$$\begin{cases} \tilde{u}''(x)=-\exp\tilde{W}_{\varphi_x}(z)\cdot\tilde{u}(x)-2(f*\varphi)(x);\ \ 0<x<1 \\ \tilde{u}(0)=\tilde{u}(\pi)=0 \end{cases} \tag{8.4}$$

where $\tilde{u}(x)=\tilde{u}(x;z_1,z_2,\dots)$ is the $\mathcal{H}$-transform of u. Now

$$\exp\tilde{W}_{\varphi_x}(z)=\exp\Big(\sum_j(\varphi_x,e_j)z_j\Big)$$

so it is clearly possible to choose e_1,φ and f such that

$$(f*\varphi)(x)=1 \qquad \text{for } x\in(0,1)$$

and

$$(e_1,\varphi_x)=\pi \qquad \text{for } x\in(0,1)$$

For this choice of e_1,φ and f and with $z=(2i,0,\dots)$ (8.4) gets the form

$$\begin{cases} \tilde{u}''(x)=-\tilde{u}(x)-2 \qquad \text{in } (0,1) \\ \tilde{u}(0)=\tilde{u}(\pi)=0 \end{cases} \tag{8.5}$$

The general solution of the first equation in (8.5) is

$$\tilde{u}(x)=-2+A\cos x+B\sin x \qquad (A,B \text{ constants})$$

but this $\tilde{u}$ has the boundary values

$$\tilde{u}(0)=-2+A, \qquad \tilde{u}(\pi)=-2-A$$

so a solution of (8.5) does not exist for $z=(2i,0,\dots)$. We conclude that it is not possible to find a z-analytic (entire) function $\tilde{u}(x;z)$ which solves (8.4). Hence $u(\varphi,x,\cdot)$ cannot be in $L^2(\mu)$. In fact, we shall see in Section 10 that u cannot be in $L^p(\mu)$ for any $p>1$. However, it is possible to find a solution in $L^1(\mu)$.

9 The Wick product in $L^1(\mu)$

Motivated by the example in Section 8 we now turn to L^1 functional processes. In order to define what it means that an L^1 functional process u solves (8.1) it is necessary to extend the definition of Wick product from $L^2(\mu)$ to $L^1(\mu)$. The most natural way of doing this is as follows:

Definition 9.1 Let $X, Y \in L^1(\mu)$. Suppose there exist $X_n, Y_n \in L^2(\mu)$ such that

$$X_n \to X \text{ in } L^1(\mu) \qquad \text{and } Y_n \to Y \qquad \text{in } L^1(\mu) \text{ as } n \to \infty$$

and

$$Z := \lim_{n\to\infty} X_n \diamond Y_n \quad \text{exists in } L^1(\mu)$$

Then we define

$$X \diamond Y = Z$$

It is necessary to show that this definition of $X \diamond Y$ does not depend on the actual sequences $\{X_n\}, \{Y_n\}$. This is done in the next Lemma:

Lemma 9.2 Let X_n, Y_n be as in Definition 9.1 and assume that $X_n', Y_n' \in L^2(\mu)$ also satisfy

$$X_n' \to X \quad \text{in } L^1(\mu), \qquad Y_n' \to Y \quad \text{in } L^1(\mu)$$

and

$$Z' := \lim_{n\to\infty} X_n' \diamond Y' \quad \text{exists in } L^1(\mu)$$

Then $Z' = Z$.

Moreover,

$$\mathcal{F}[X \diamond Y](\varphi) = e^{-\frac{1}{2}\|\varphi\|^2} \mathcal{F}[X](\varphi) \cdot \mathcal{F}[Y](\varphi) \tag{9.1}$$

where in general

$$\mathcal{F}[g](\varphi) = \int_{S'} e^{i\langle\varphi,\omega\rangle} g(\omega) d\mu(\omega), \quad \varphi \in \mathcal{S} \tag{9.2}$$

denotes the Fourier transform of $g \in L^1(\mu)$.

Proof Suppose that $X_n, Y_n \in L^2(\mu)$ satisfy the conditions of Definition 9.1 and put $Z = \lim X_n \diamond Y_n$. It suffices to prove that

$$\mathcal{F}[Z](\varphi) = e^{-\frac{1}{2}\|\varphi\|^2}\mathcal{F}[X](\varphi)\cdot\mathcal{F}[Y](\varphi)$$

To this end, recall that the connection between the Fourier transform and the $\mathcal{S}$-transform is given by

$$\mathcal{F}[g](\varphi) = e^{\frac{1}{2}\|\varphi\|^2}\mathcal{S}g(i\varphi) \tag{9.3}$$

(see [HKPS, formula (2.10)].) Moreover, since

$$\mathcal{H}g(z_1, z_2, \ldots) = \mathcal{S}g(z_1 e_1 + z_2 e_2 + \cdots) \tag{9.4}$$

(see [LØU1, The. 5.7]) we know by Theorem 3.3 that

$$\mathcal{S}(g_1 \diamond g_2)(\varphi) = \mathcal{S}g_1(\varphi)\cdot\mathcal{S}g_2(\varphi) \tag{9.5}$$

for $g_1, g_2 \in L^2(\mu)$, $\varphi \in \mathcal{S}$. This gives

$$\begin{aligned}
\mathcal{F}[X\diamond Y](\varphi) &= \lim_{n\to\infty}\mathcal{F}[X_n\diamond Y_n](\varphi)\\
&= \lim_{n\to\infty} e^{\frac{1}{2}\|\varphi\|^2}\mathcal{S}(X_n\diamond Y_n)(i\varphi)\\
&= \lim_{n\to\infty} e^{\frac{1}{2}\|\varphi\|^2}\mathcal{S}X_n(i\varphi)\cdot\mathcal{S}Y_n(i\varphi)\\
&= \lim_{n\to\infty} e^{-\frac{1}{2}\|\varphi\|^2}\mathcal{F}[X_n](\varphi)\cdot\mathcal{F}[Y_n](\varphi)\\
&= e^{-\frac{1}{2}\|\varphi\|^2}\mathcal{F}[X](\varphi)\cdot\mathcal{F}[Y](\varphi).
\end{aligned}$$

Corollary 9.3 Let $X, Y \in L^1$ and assume that $X \diamond Y$ exists. Then

$$E[X\diamond Y] = E[X]\cdot E[Y] \tag{9.6}$$

Proof Choose $\varphi = 0$ in (9.1).

Corollary 9.4 Suppose $u(\varphi, x, \omega)$ is an L^1 functional process which solves equation (8.1). Then $E[u(\varphi, x, \cdot)]$ coincides with the solution $v(\varphi, x)$ of the corresponding no-noise equation

$$\begin{cases} \Delta_x v(\varphi,x) = -\varepsilon v(\varphi,x) - (f*\varphi)(x) & \text{in } D\\ v(\varphi,x) = 0 & \text{on } \partial D\end{cases} \tag{9.7}$$

It is well-known that the solution v of (9.7) can be expressed as follows:

Let $\{b_t\}_{t\geq 0}$ be a Brownian motion on $\mathbf{R}^d$ (independent of B_x) with law $\hat{P}^x$ ($\hat{P}^x(b_0 = x) = 1$) and put

$$\tau_D = \inf\{t>0; b_t \notin D\}$$

Then by the Feynman-Kac formula we have

Corollary 9.5 Suppose $u(\varphi,x,\omega)$ is an L^1 functional process which solves (8.1). Then

$$E[u(\varphi,x,\cdot)] = \hat{E}^x\Big[\int_0^{\tau_D} e^{\varepsilon t}\cdot(f*\varphi)(b_t)dt\Big] \tag{9.8}$$

where $\hat{E}^x$ denotes expectation with respect to $\hat{P}^x$.

10 The stochastic Schrödinger equation

We now have the sufficient background for discussing the general stochastic equation (8.1). Modified to the $L^1(\mu)$ setting our definition of a solution of (8.1) becomes the following:

Definition 10.1 We say that an L^1 functional process $u = u(\varphi,x,\omega)$ is a solution of (8.1) if for all $\varphi\in\mathcal{S}$ there is a set $H_\varphi\subset\mathcal{S}'$ with $\mu(H_\varphi)=1$ such that for all $\omega\in H_\varphi$ the function $x\longrightarrow u(\varphi,x,\omega)$ solves the (deterministic) boundary value problem

$$\Delta_x u(\varphi,x,\omega) = -\varepsilon(\operatorname{Exp} W_{\varphi_x}\diamond u)(\omega) - (f*\varphi)(x) \quad \text{in } D \tag{10.1}$$

and

$$\lim_{\substack{x\to y\\ x\in D}} u(\varphi,x,\omega) = 0 \quad \text{for all } y\in\partial_R D \tag{10.2}$$

Remarks

1) It is a part of the definition (assumption on u) that

$$\operatorname{Exp} W_{\varphi_x}\diamond u$$

exists in $L^1(\mu)$, for all $\varphi\in\mathcal{S}$ and a.a. $x\in D$.

2) Again we interpret the first part, (10.1), in the weak (distribution) sense, i.e. for all $\omega\in H_\varphi$ the functions $x\longrightarrow u(\varphi,x,\omega)$ and $x\longrightarrow \operatorname{Exp} W_{\varphi_x}\diamond u(\varphi,x,\cdot)$ belong to $L^1_{\text{loc}}(D)$ and

$$(u,\Delta\psi) = -\varepsilon(\operatorname{Exp} W_{\varphi_x}\diamond u,\psi) - (f*\varphi,\psi) \tag{10.3}$$

for all $\psi\in C_0^\infty(D)$ (the smooth functions with compact support in D), where (,) denotes the usual inner product in $L^2(\mathbf{R}^d)$.

Theorem 10.2 Let D be a bounded domain in $\mathbf{R}^d$ and let f be a bounded continuous function on D. Let $\varepsilon(D)>0$ be the lowest positive eigenvalue of the operator $-\Delta$ in D. Then for $0<\varepsilon<\varepsilon(D)$ the function

$$u(\varphi,x,\omega)=\hat{E}^x\Big[\int_0^{\tau_D}\operatorname{Exp}\Big[\varepsilon\int_0^t \operatorname{Exp}(W_{\varphi_{b_s}})ds\Big](f*\varphi)(b_t)dt\Big] \tag{10.4}$$

is an L^1 functional process which solves (8.1). Here $b_t,\hat{E}^x,\tau_D$ is as in Corollary 9.5.

The proof will be split into several lemmas:

Put

$$u_n(x)=u_n(\varphi,x,\omega)=\sum_{k=0}^{n}\frac{\varepsilon^k}{k!}\hat{E}^x\Big[\int_0^{\tau_D}\Big(\int_0^s \operatorname{Exp} W_{\varphi_{b_r}}dr\Big)^{\diamond k}F(b_s)ds\Big] \tag{10.5}$$

for $n=0,1,2,\ldots$, where $F=f*\varphi$.

Lemma 10.3

$$u_n(\varphi,x,\cdot)\in L^2(\mu) \text{ for all } \varphi,x \text{ and all } n. \tag{10.6}$$

Proof Define

$$v_k(\varphi,x,\cdot)=\hat{E}^x\Big[\int_0^{\tau_D}\Big(\int_0^s \operatorname{Exp} W_{\varphi_{b_r}}dr\Big)^{\diamond k}F(b_s)ds\Big];\quad k=1,2,\ldots \tag{10.7}$$

Then, with $c=\sup|F|$, $\Phi_j=\varphi_{b_{r_j}}=\varphi(x-b_{r_j})$ we have

$$\begin{aligned}
v_k^2(\varphi,x,\cdot)&\le c^2\Big(\hat{E}^x\Big[\int_0^{\tau_D}\Big(\int_0^s\cdots\int_0^s \operatorname{Exp}\sum_{j=1}^k W_{\Phi_j}dr_1\ldots dr_k\Big)ds\Big]\Big)^2\\
&\le c^2\hat{E}^x\Big[\tau_D\cdot\int_0^{\tau_D}\Big(\int_0^s\cdots\int_0^s \exp\Big(W_{\sum_{j=1}^k\Phi_j}-\frac{1}{2}\Big\|\sum_{j=1}^k\Phi_j\Big\|^2\Big)dr_1\ldots dr_k\Big)^2ds\Big]\\
&\le c^2\hat{E}^x\Big[\tau_D\cdot\int_0^{\tau_D}s^k\Big(\int_0^s\cdots\int_0^s \exp\Big(2W_{\sum_{j=1}^k\Phi_j}-\Big\|\sum_{j=1}^k\Phi_j\Big\|^2\Big)dr_1\ldots dr_k\Big)ds\Big]\\
&= c^2\hat{E}^x\Big[\tau_D\cdot\int_0^{\tau_D}s^k\Big(\int_0^s\cdots\int_0^s \operatorname{Exp}\Big(W_{2\sum_{j=1}^k\Phi_j}\Big)\exp\Big(\Big\|\sum_{j=1}^k\Phi_j\Big\|^2\Big)dr_1\ldots dr_k d\Big)ds\Big]\\
&\le c^2\exp(k^2\|\varphi\|^2)\hat{E}^x\Big[\tau_D\cdot\int_0^{\tau_D}s^k\Big(\int_0^s\cdots\int_0^s \operatorname{Exp}\Big(W_{2\sum_{j=1}^k\Phi_j}\Big)dr_1\ldots dr_k\Big)ds\Big]
\end{aligned} \tag{10.8}$$

This gives

$$
\begin{aligned}
E^{\mu}[v_k^2(\varphi,x,\cdot)] &\le c^2\exp(k^2\|\varphi\|^2)\hat{E}^x\left[\tau_D\cdot\int_0^{\tau_D}s^k\Big(\int_0^s\cdots\int_0^s 1\cdot dr_1\ldots dr_k\Big)ds\right]\\
&= c^2\exp(k^2\|\varphi\|^2)\hat{E}^x\left[\tau_D\cdot\frac{1}{2k+1}\cdot\tau_D^{2k+1}\right]\\
&= \frac{c^2}{2k+1}\cdot\exp(k^2\|\varphi\|^2)\hat{E}^x\Big[\tau_D^{2k+2}\Big]<\infty\,,
\end{aligned}
\tag{10.9}
$$

since $\hat{E}^x[\exp(\varepsilon\tau_D)]<\infty$ for all $\varepsilon<\varepsilon(D)$ (see e.g. [D; Section 8.8]). This proves Lemma 10.3.

Lemma 10.4 For $n\ge 1$ we have

$$
\Delta_x u_n = -\varepsilon\,\mathrm{Exp}\,W_{\varphi_x}\diamond u_{n-1} - F\quad\text{for all } x\in D \tag{10.10}
$$

where Δ is interpreted in distribution sense.

Proof The Hermite transform of v_k is given by

$$
\tilde{v}_k(x)=\tilde{v}_k(x,z)=\hat{E}^x\left[\int_0^{\tau_D}\Big(\int_0^s\exp\widetilde{W}_{\varphi_{b_r}}(z)dr\Big)^kF(b_s)ds\right]\qquad k=0,1,2,\ldots
$$

Fix $t>0$. For bounded measurable g we define

$$
(P_tg)(x)=\hat{E}^x[g(b_t)]
$$

By the Markov property we have

$$
\begin{aligned}
P_t\tilde{v}_k(x) &:= \hat{E}^x[\tilde{v}_k(b_t),t<\tau_D]=\hat{E}^x\left[\hat{E}^{b_t}\Big[\int_0^{\tau_D}\Big(\int_0^s\exp\widetilde{W}_{\varphi_{b_r}}dr\Big)^kF(b_s)ds\Big],t<\tau_D\right]\\
&= \hat{E}^x\left[t<\tau_D,\hat{E}^x\Big[\int_0^{\tau_D\circ\theta_t}\Big(\int_0^s\exp\widetilde{W}_{\varphi_{b_r\circ\theta_t}}dr\Big)^kF(b_s\circ\theta_t)ds\Big|\mathcal{F}_t\Big]\right],
\end{aligned}
$$

where θ_t denotes the shift operator $b_r\circ\theta_t=b_{r+t}$. Since $\tau_D=t+\tau_D\circ\theta_t$ on $\{t<\tau_D\}$ we get

$$
\begin{aligned}
P_t\tilde{v}_k(x) &= \hat{E}^x\left[t<\tau_D,\hat{E}^x\Big[\int_t^{\tau_D}\Big(\int_t^s\exp\widetilde{W}_{\varphi_{b_r}}dr\Big)^kF(b_s)ds\Big|\mathcal{F}_t\Big]\right]\\
&= \hat{E}^x\Big[t<\tau_D,\int_t^{\tau_D}\Big(\int_t^s\exp\widetilde{W}_{\varphi_{b_r}}dr\Big)^kF(b_s)ds\Big]
\end{aligned}
\tag{10.11}
$$

This gives

$$
\begin{aligned}
&\frac{1}{t}\{P_t\tilde{v}_k(x)-\tilde{v}_k(x)\}\\
&\quad=-\frac{1}{t}\Big\{\hat{E}^x\Big[t\geq\tau_D,\int_0^{\tau_D}\Big(\int_0^s\exp\widetilde{W}_{\varphi_{b_r}}dr\Big)^kF(b_s)ds\Big]\\
&\qquad+\hat{E}^x\Big[t<\tau_D,\int_0^{\tau_D}\Big(\Big(\int_0^s\exp\widetilde{W}_{\varphi_{b_r}}dr\Big)^k-\int_t^{\tau_D}\Big(\int_t^s\exp\widetilde{W}_{\varphi_{b_r}}dr\Big)^k\Big)F(b_s)ds\Big]\Big\}\\
&\quad=-\frac{1}{t}\hat{E}^x\Big[t\geq\tau_D,\int_0^{\tau_D}\Big(\int_0^s\exp\widetilde{W}_{\varphi_{b_r}}dr\Big)^kF(b_s)ds\Big]\\
&\qquad-\frac{1}{t}\hat{E}^x\Big[t<\tau_D,\int_0^{t}\Big(\int_0^s\exp\widetilde{W}_{\varphi_{b_r}}dr\Big)^kF(b_s)ds\Big]\\
&\qquad-\frac{1}{t}\hat{E}^x\Big[t<\tau_D,\int_t^{\tau_D}\Big\{\Big(\int_0^s\exp\widetilde{W}_{\varphi_{b_r}}dr\Big)^k-\Big(\int_t^s\exp\widetilde{W}_{\varphi_{b_r}}dr\Big)^k\Big\}F(b_s)ds\Big]\\
&\quad=I_1(t)+I_2(t)+I_3(t)
\end{aligned}
\tag{10.12}
$$

We deal with these 3 expressions separately. First note that, with $M=\sup_x|\widetilde{W}_x|$ and $c=\sup_x|F(x)|$,

$$
\begin{aligned}
\lim_{t\to 0}|I_1(t)| &\leq \lim_{t\to 0}\frac{c}{t}\exp(Mk)\cdot\hat{E}^x\Big[t\geq\tau_D,\int_0^{\tau_D}s^kds\Big]\\
&\leq \frac{c\exp(Mk)}{k+1}\lim_{t\to 0}\frac{1}{t}\hat{E}^x\Big[t\geq\tau_D,\tau_D^{k+1}\Big]\\
&\leq \frac{c\exp(Mk)}{k+1}\cdot\lim_{t\to 0}\frac{1}{t}\cdot t^{k+1}=0\,.
\end{aligned}
\tag{10.13}
$$

For the second term we get similarly

$$
\begin{aligned}
\lim_{t\to 0}|I_2(t)| &\leq \frac{c\exp(Mk)}{t}\cdot\hat{E}^x\Big[t<\tau_D,\int_0^t s^kds\Big]\\
&\leq \frac{c\exp(Mk)}{k+1}\cdot\frac{t^{k+1}}{t}\longrightarrow 0\quad\text{as } t\to 0\,.
\end{aligned}
\tag{10.14}
$$

Using the mean value theorem we see that

$$
I_3(t)=-\frac{1}{t}\hat{E}^x\Big[t<\tau_D,\int_t^{\tau_D}k\cdot\Big(\int_\alpha^s\exp\widetilde{W}_{\varphi_{b_r}}dr\Big)^{k-1}\exp\widetilde{W}_{\varphi_{b_\alpha}}\cdot t\cdot F(b_s)ds\Big]
$$

where $0<\alpha<t$. This gives

$$
\lim_{t\to 0}I_3(t)=-k\cdot\hat{E}^x\Big[\int_0^{\tau_D}\Big(\int_0^s\exp\widetilde{W}_{\varphi_{b_r}}dr\Big)^{k-1}F(b_s)ds\Big]\cdot\exp\widetilde{W}_{\varphi_x}
\tag{10.15}
$$

From (10.12)–(10.15) we conclude that

$$
\lim_{t\to 0}\frac{1}{t}\Big(P_t\tilde{v}_k(x)-\tilde{v}_k(x)\Big)=-k\cdot\tilde{v}_{k-1}(x)\cdot\exp\widetilde{W}_{\varphi_x}\quad\text{for } k\geq 1\,.
\tag{10.16}
$$

For $\tilde{v}_0(x)$ it is well-known that

$$\lim_{t\to 0}\frac{1}{t}\Big(P_t\tilde{v}_0(x)-\tilde{v}_0(x)\Big)=-F(x) \tag{10.17}$$

From this it follows that in distribution sense we have

$$\Delta v_0(x)=-F(x) \tag{10.18}$$

and

$$\Delta v_k(x)=-k\operatorname{Exp}W_{\varphi_x}\diamond v_{k-1}(x) \qquad \text{for } k\geq 1 \tag{10.19}$$

Since

$$u_n=\sum_{k=0}^{n}\frac{\varepsilon^k}{k!}v_k$$

we get from (10.18)–(10.19) that

$$\begin{aligned}\Delta u_n &= \sum_{k=1}^{n}\frac{\varepsilon^k}{k!}\Delta v_k - F = -\varepsilon\sum_{k=1}^{n}\frac{\varepsilon^{k-1}}{(k-1)!}v_{k-1}\diamond\operatorname{Exp}W_{\varphi_x}-F\\ &= -\varepsilon u_{n-1}\diamond\operatorname{Exp}W_{\varphi_x}-F\,,\end{aligned}$$

which proves Lemma 10.4.

Lemma 10.5

$$\sup_x\Big\|u_n\diamond\operatorname{Exp}W_{\varphi_x}-u_m\diamond\operatorname{Exp}W_{\varphi_x}\Big\|_{L^1(\mu)}\to 0 \qquad \text{as } m,n\to\infty$$

Proof We may assume $m<n$. Then

$$\begin{aligned}w_{n,m}&:=u_n\diamond\operatorname{Exp}W_{\varphi_x}-u_m\diamond\operatorname{Exp}W_{\varphi_x}=\\ &=\sum_{m+1}^{n}\frac{\varepsilon^k}{k!}\hat{E}^x\Big[\int_0^{\tau_D}\Big(\int_0^s\operatorname{Exp}W_{\varphi_{b_r}}dr\Big)^{\diamond k}\diamond\operatorname{Exp}W_{\varphi_x}\cdot F(b_s)ds\Big]\,.\end{aligned}$$

So, putting $\Phi_0=\varphi_x$ and using the notation from the proof of Lemma 10.3,

$$\begin{aligned}E^\mu[|w_{n,m}|]&\leq cE^\mu\left[\sum_{m+1}^{n}\frac{\varepsilon^k}{k!}\hat{E}^x\Big[\int_0^{\tau_D}\Big(\int_0^s\cdots\int_0^s\exp\Big(W_{\sum_{j=0}^{n}\Phi_j}-\frac{1}{2}\Big\|\sum_{j=0}^{k}\Phi_j\Big\|^2\Big)dr_1\ldots dr_k ds\Big]\right]\\ &=c\cdot\sum_{m+1}^{n}\frac{\varepsilon^k}{k!}\hat{E}^x\Big[\int_0^{\tau_D}s^k ds\Big]=c\sum_{m+1}^{n}\frac{\varepsilon^k}{(k+1)!}\hat{E}^x\Big[\tau_D^{k+1}\Big]\end{aligned}$$

$\to 0$ as $m,n\to\infty$, uniformly in x, since

$$\sup_x\hat{E}^x\Big[\exp(\varepsilon\tau_D)\Big]<\infty$$

by our choice of ε.

Remark Lemma 10.5 implies that $\operatorname{Exp} W_{\varphi_x} \diamond u \in L^1(\mu)$ exists and

$$\operatorname{Exp} W_{\varphi_x} \diamond u = \lim_{n\to\infty} \operatorname{Exp} W_{\varphi_x} \diamond u_n \qquad \text{in } L^1(\mu) \tag{10.20}$$

Lemma 10.6

$$\sup_x \|u_n - u\|_{L^1(\mu)} \qquad \text{as } n \to \infty$$

Proof This is proved in the same way as Lemma 10.5. We omit the details.

Proof of Theorem 10.2: From Lemma 10.4 we have

$$\Delta u_n = -\varepsilon \operatorname{Exp} W_{\varphi_x} \diamond u_{n-1} - f \qquad \text{for all } x \in D$$

Therefore, if $\psi \in C_0^\infty(D)$ and (,) denotes inner product in $L^2(D)$, we have for all $n \geq 1$

$$(u_n, \Delta\psi) = (-\varepsilon \operatorname{Exp} W_{\varphi_x} \diamond u_{n-1}, \psi) - (F, \psi) \tag{10.21}$$

As $n \to \infty$ this converges to, by Lemma 10.5 (10.20) and Lemma 10.6,

$$(u, \Delta\psi) = (-\varepsilon \operatorname{Exp} W_{\varphi_x} \diamond u, \psi) - (F, \psi)\,,$$

which shows that (10.1) holds.

The proof that (10.2) holds follows the usual argument for the stochastic solution of the Poisson problem (see e.g. [Ø, Ch. IX]) and is omitted.

Appendix: Basis-invariance of the Wick product

Apparently the alternative definition (5.3) of the Wick product depends on the choice of basis elements $\{e_k\}_{k=1}^\infty$ for $L^2(\mathbf{R}^d)$. In this appendix we will prove directly that this is not the case.

First we establish some properties of Hermite polynomials. Recall that the Hermite polynomials are defined by the relation

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right)$$

We adopt the convention that $h_{-1}(x) = 0$. Then for $n = 0, 1, 2 \ldots$ we have

$$h_{n+1}(x) = h_1(x)h_n(x) - nh_{n-1}(x) \tag{A.1}$$

If $x = (x_1, x_2, \ldots, x_N)$ is a vector, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a multi-index, we define

$$h_\alpha(x) = h_{\alpha_1}(x_1)h_{\alpha_2}(x_2)\ldots h_{\alpha_N}(x_N)$$

Formulated in this language, (A1) takes the form

$$h_{\alpha+\beta}(x) = h_\alpha(x)h_\beta(x) - \frac{\alpha!}{(\alpha-\beta)!}h_{\alpha-\beta}(x) \qquad \text{if } |\beta| = 1 \tag{A.2}$$

Lemma A.1 If $a^2+b^2=1$, then for all $n=0,1,2\dots$ we have

$$h_n(ax+by)=\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}h_k(x)h_{n-k}(y)$$

Proof By induction. The cases $n=0,1$ are trivial. We use (A1) to get

$$\begin{aligned}
h_{n+1}(ax+by) &= (ax+by)h_n(ax+by)-nh_{n-1}(ax+by)\\
&= (ax+by)\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}h_k(x)h_{n-k}(y)\\
&\quad -n\sum_{k=0}^{n-1}\binom{n-1}{k}a^kb^{n-1-k}h_k(x)h_{n-1-k}(y)\\
&= \sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k}xh_k(x)h_{n-k}(y)\\
&\quad +\sum_{k=0}^{n}\binom{n}{k}a^kb^{n+1-k}h_k(x)yh_{n-k}(y)\\
&\quad -\sum_{k=0}^{n-1}\binom{n-1}{k}na^kb^{n-1-k}h_k(x)h_{n-1-k}(y)
\end{aligned}$$

Using the equation (A1) backwards, we have the following

$$\begin{aligned}
&=\sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k}h_{k+1}(y)\\
&+\sum_{k=0}^{n-1}\binom{n}{k}a^kb^{n+1-k}h_k(x)h_{n+1-k}(y)\\
&+\sum_{k=1}^{n}\binom{n}{k}a^{k+1}b^{n-k}h_{k-1}(x)h_{n-k}(y)\\
&+\sum_{k=0}^{n-1}\binom{n}{k}a^kb^{n+1-k}h_k(x)yh_{n-1-k}(y)\\
&-\sum_{k=0}^{n-1}\binom{n-1}{k}na^kb^{n-1-k}h_k(x)h_{n-1-k}(y)
\end{aligned}$$

The sum of the two first terms gives the required expression. As for the three last terms, we have when we let S denote the sum of these

$$S=\sum_{k=0}^{n-1}\left[\binom{n}{k+1}(k+1)a^2+\binom{n}{k}(n-k)b^2-\binom{n-1}{k}n\right]a^kb^{n-1-k}h_k(x)h_{n-1-k}(y)$$

Here

$$\binom{n}{k+1}(k+1)a^2+\binom{n}{k}(n-k)b^2-\binom{n-1}{k}n=(a^2+b^2-1)\binom{n-1}{k}n=0$$

and this proves the lemma. □

Proposision A.2 If $a = (a_1, a_2, \ldots, a_N)$ with $\sum_{i=1}^{N} a_i^2 = 1$, we have

$$h_n\left(\sum_{i=1}^{N} a_i x_i\right) = \sum_{\substack{\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_N)\\ |\alpha|=n}} \frac{n!}{\alpha!} a^\alpha h_\alpha(x)$$

Proof Trivial if $N = 1$. If $N \geq 1$, we put

$$\sum_{i=1}^{N+1} a_i x_i = a_1 x_1 + \sqrt{\sum_{i=2}^{N+1} a_i^2} \sum_{i=2}^{N+1} \frac{a_i}{\sqrt{\sum_{i=2}^{N+1} a_i}} x_i =: a_1 x_1 + by$$

i.e.

$$b = \sqrt{\sum_{i=2}^{N+1} a_i^2} \qquad y = \sum_{i=2}^{N+1} \left(\frac{a_i}{b}\right) x_i$$

Then $a_1^2 + b^2 = 1$, so by Lemma A1

$$h_n\left(\sum_{i=1}^{N+1} a_i x_i\right) = \sum_{k=0}^{n} \binom{n}{k} a_1^k b^{n-k} h_k(x_1) h_{n-k}(y)$$

The induction hypothesis applies to $h_{n-k}(y)$. Hence

$$h_n\left(\sum_{i=1}^{N+1} a_i x_i\right) = \sum_{k=0}^{n} \binom{n}{k} a_1^k b^{n-k} h_k(x_1) \sum_{\substack{\alpha=(0,\alpha_2,\ldots,\alpha_{N+1})\\ |\alpha|=n-k}} \frac{(n-k)!}{\alpha!} \frac{a^\alpha}{b^{|\alpha|}} h_\alpha(x_2, x_3, \ldots, x_{N+1})$$

$$= \sum_{k=0}^{n} \sum_{\substack{\alpha=(0,\alpha_2,\ldots,\alpha_{N+1})\\ |\alpha|=n-k}} \frac{n!}{k!\alpha!} a_1^k a_2^{\alpha_2} \ldots a_{N+1}^{\alpha_{N+1}} h_k(x_1) h_\alpha(x_2, \ldots, x_{N+1})$$

□

Proposition A.3 Let $a, b_1, b_2, \ldots, b_N, x$ be vectors in $\mathbf{R}^M$. If all the inner products $\langle a, b_i \rangle = 0$ $i = 1, 2, \ldots, N$, then

$$\sum_{\substack{|\alpha|=n\\ |\beta_1|=1,\ldots,|\beta_N|=1}} \frac{n!}{\alpha!} a^\alpha b_1^{\beta_1} \ldots b_N^{\beta_N} h_{\alpha+\beta_1+\cdots+\beta_N}(x)$$

$$= \sum_{\substack{|\alpha|=n\\ |\beta_1|=1,\ldots,|\beta_N|=1}} \frac{n!}{\alpha!} a^\alpha b_1^{\beta_1} \ldots b_N^{\beta_N} h_\alpha(x) h_{\beta_1+\cdots+\beta_N}(x)$$

Proof By induction again. This time we use induction on the number of β-s. We first consider the case with one β. We use (A2) to get

$$\sum_{\substack{|\alpha|=n\\|\beta|=1}} \frac{n!}{\alpha!} a^\alpha b^\beta h_{\alpha+\beta}(x) = \sum_{\substack{|\alpha|=n\\|\beta|=1}} \frac{n!}{\alpha!} a^\alpha b^\beta h_\alpha(x) h_\beta(x) - \sum_{\substack{|\alpha|=n\\|\beta|=1}} \frac{n!}{(\alpha-\beta)!} a^\alpha b^\beta h_{\alpha-\beta}(x)$$

$$= \sum_{\substack{|\alpha|=n\\|\beta|=1}} \frac{n!}{\alpha!} a^\alpha b^\beta h_\alpha(x) h_\beta(x) - \sum_{|\beta|=1} a^\beta b^\beta \left(\sum_{|\alpha|=n-1} \frac{n!}{\alpha!} a^\alpha h_\alpha(x) \right)$$

If $\langle a, b\rangle = 0$, the last term vanishes. This proves the case with one β. By induction we will assume that the statement is true on all levels up to N. We use (A2) again to get

$$\sum_{\substack{|\alpha|=n\\|\beta_1|=1,\ldots,|\beta_{N+1}|=1}} \frac{n!}{\alpha!} a^\alpha b_1^{\beta_1} \ldots b_{N+1}^{\beta_{N+1}} h_{\alpha+\beta_1+\cdots+\beta_N+\beta_{N+1}}(x)$$

$$= \sum_{\substack{|\alpha|=n\\|\beta_1|=1,\ldots,|\beta_{N+1}|=1}} \frac{n!}{\alpha!} a^\alpha b_1^{\beta_1} \ldots b_{N+1}^{\beta_{N+1}} h_{\alpha+\beta_1+\cdots+\beta_N}(x) h_{\beta_{N+1}}(x)$$

$$- \sum_{\substack{|\alpha|=n\\|\beta_1|=1,\ldots,|\beta_{N+1}|=1}} \frac{n!}{\alpha!} a^\alpha b_1^{\beta_1} \ldots b_{N+1}^{\beta_{N+1}} \frac{(\alpha+\beta_1+\cdots+\beta_N)!}{(\alpha+\beta_1+\cdots+\beta_N-\beta_{N+1})!} h_{\alpha+\beta_1+\cdots+\beta_N-\beta_{N+1}}(x)$$

We now use the induction hypothesis on the first term.

$$= \sum_{\substack{|\alpha|=n\\|\beta_1|=1,\ldots,|\beta_{N+1}|=1}} \frac{n!}{\alpha!} a^\alpha b_1^{\beta_1} \ldots b_{N+1}^{\beta_{N+1}} h_\alpha(x) h_{\beta_1+\cdots+\beta_N}(x) h_{\beta_{N+1}}(x)$$

$$- \sum_{\substack{|\alpha|=n\\|\beta_1|=1,\ldots,|\beta_{N+1}|=1}} \frac{n!}{\alpha!} a^\alpha b_1^{\beta_1} \ldots b_{N+1}^{\beta_{N+1}} \frac{(\alpha+\beta_1+\cdots+\beta_N)!}{(\alpha+\beta_1+\cdots+\beta_N-\beta_{N+1})!} h_{\alpha+\beta_1+\cdots+\beta_N-\beta_{N+1}}(x)$$

Then we use (A2) backwards in the first expression.

$$= \sum_{\substack{|\alpha|=n\\|\beta_1|=1,\ldots,|\beta_{N+1}|=1}} \frac{n!}{\alpha!} a^\alpha b_1^{\beta_1} \ldots b_{N+1}^{\beta_{N+1}} h_\alpha(x) h_{\beta_1+\cdots+\beta_N}(x)$$

$$+ \sum_{\substack{|\alpha|=n\\|\beta_1|=1,\ldots,|\beta_{N+1}|=1}} \frac{n!}{\alpha!} a^\alpha b_1^{\beta_1} \ldots b_{N+1}^{\beta_{N+1}} \frac{(\beta_1+\cdots+\beta_N)!}{(\beta_1+\cdots+\beta_N-\beta_{N+1})!} h_\alpha(x) h_{\beta_1+\cdots+\beta_N-\beta_{N+1}}(x)$$

$$- \sum_{\substack{|\alpha|=n\\|\beta_1|=1,\ldots,|\beta_{N+1}|=1}} \frac{n!}{\alpha!} a^\alpha b_1^{\beta_1} \ldots b_{N+1}^{\beta_{N+1}} \frac{(\alpha+\beta_1+\cdots+\beta_N)!}{(\alpha+\beta_1+\cdots+\beta_N-\beta_{N+1})!} h_{\alpha+\beta_1+\cdots+\beta_N-\beta_{N+1}}(x)$$

$$=: I + II - III$$

Now observe that

$$\frac{(\beta_1+\cdots+\beta_N)!}{(\beta_1+\cdots+\beta_N-\beta_{N+1})!} = \#\{\beta_i = \beta_{N+1}, i \le N\}$$

$$\frac{(\alpha+\beta_1+\cdots+\beta_N)!}{(\alpha+\beta_1+\cdots+\beta_N-\beta_{N+1})!}=\frac{\alpha!}{(\alpha-\beta_{N+1})!}+\#\{\beta_i=\beta_{N+1}, i\le N\}$$

We consider the second term II, and have

$$\begin{aligned}
II = & \sum_{\substack{|\alpha|=n \\ |\beta_1|=1,\dots,|\beta_{N+1}|=1}} \frac{n!}{\alpha!}a^\alpha b_1^{\beta_1}\dots b_{N+1}^{\beta_{N+1}} h_\alpha(x) h_{\beta_1+\dots+\beta_N-\beta_{N+1}}(x)\cdot\#\{\beta_1=\beta_{N+1}, i\le N\} \\
= & \sum_{|\beta_{N+1}|=1} b_{N+1}^{\beta_{N+1}} b_1^{\beta_{N+1}} \sum_{\substack{|\alpha|=n \\ |\beta_2|=1,\dots,|\beta_N|=1}} \frac{n!}{\alpha!}a^\alpha b_2^{\beta_2}\dots b_N^{\beta_N} h_\alpha(x) h_{\beta_2+\dots+\beta_N}(x) \\
+ & \sum_{|\beta_{N+1}|=1} b_{N+1}^{\beta_{N+1}} b_2^{\beta_{N+1}} \sum_{\substack{|\alpha|=n \\ |\beta_1|=1,|\beta_3|=1,\dots,|\beta_N|=1}} \frac{n!}{\alpha!}a^\alpha b_1^{\beta_1} b_3^{\beta_3}\dots b_N^{\beta_N} h_\alpha(x) h_{\beta_1+\beta_3+\dots+\beta_N}(x) \\
+ & \\
\vdots & \\
+ & \sum_{|\beta_{N+1}|=1} b_{N+1}^{\beta_{N+1}} b_N^{\beta_{N+1}} \sum_{\substack{|\alpha|=n \\ |\beta_1|=1,\dots,|\beta_{N-1}|=1}} \frac{n!}{\alpha!}a^\alpha b_1^{\beta_1}\dots b_{N-1}^{\beta_{N-1}} h_\alpha(x) h_{\beta_1+\dots+\beta_{N-1}}(x)
\end{aligned}$$

The induction hypothesis applies (backwards) to all of these, so we get that the second term II is equal to the expression

$$\sum_{\substack{|\alpha|=n \\ |\beta_1|=1,\dots,|\beta_{N+1}|=1}} \frac{n!}{\alpha!}a^\alpha b_1^{\beta_1}\dots b_{N+1}^{\beta_{N+1}} h_{\alpha+\beta_1+\dots+\beta_N-\beta_{N+1}}(x)\#\{\beta_i=\beta_{N+1}, i\le N\}$$

Now we can finally subtract the third term III from the second II, and get

$$\begin{aligned}
II-III = & -\sum_{\substack{|\alpha|=n \\ |\beta_1|=1,\dots,|\beta_{N+1}|=1}} \frac{n!}{\alpha!}a^\alpha b_1^{\beta_1}\dots b_{N+1}^{\beta_{N+1}} \frac{\alpha!}{(\alpha-\beta_{N+1})!} h_{\alpha+\beta_1+\dots\beta_N-\beta_{N+1}}(x) \\
= & -\sum_{|\beta_{N+1}|=1} a^{\beta_{N+1}} b_{N+1}^{\beta_{N+1}} \left\{ \sum_{\substack{|\alpha|=n \\ |\beta_1|=1,\dots,|\beta_N|=1}} \frac{n!}{\alpha!}a^\alpha b_1^{\beta_1}\dots b_{N+1}^{\beta_{N+1}} h_{\alpha+\beta_1+\dots+\beta_N}(x) \right\} = 0
\end{aligned}$$

and this proves the proposition. □

We now proceed to show basis-invariance. So we consider two bases $\{e_k\}_{k=1}^\infty$ and $\{\hat{e}_k\}_{k=1}^\infty$ for $L^2(\mathbb{R}^d)$. We let $\theta_k=\int e_k dB$ and $\hat{\theta}_k=\int \hat{e}_k dB$ denote the corresponding first order integrals and we let $\diamond$ and $\hat{\diamond}$ denote the Wick products that arise from the two bases. To prove that $\diamond=\hat{\diamond}$ we proceed as follows:

Lemma A.4 For each pair of integers n and k

$$\theta_k^{\diamond n}=\theta_k^{\hat{\diamond} n}=h_n(\theta_k)$$

Proof Since $\|\theta_k\|_{L^2(\Omega)}=1$, it can be approximated in L^2 by a sum $\sum_{i=1}^{N} a_i\hat{\theta}_i$ where $\sum_{i=1}^{N} a_i^2=1$. Then by definition of the $\hat{\diamond}$ product

$$\begin{aligned}\theta_k^{\hat{\diamond} n} &\approx \left(\sum_{i=1}^{N} a_i\hat{\theta}_i\right)^{\hat{\diamond} n} = \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_N)\\|\alpha|=n}} \frac{n!}{\alpha!}a^{\alpha}h_{\alpha}(\hat{\theta}_1,\hat{\theta}_2,\dots,\hat{\theta}_N)\\ &= h_n\left(\sum_{i=1}^{N} a_i\hat{\theta}_i\right) \approx h_n(\theta_k) = \theta_k^{\hat{\diamond} n}\end{aligned}$$

In the third equality we used proposition A.2. We now let $N \to \infty$, and this proves the lemma. □

Corollary A.5 If n, m and k are non-negative integers

$$h_n(\theta_k)\,\hat{\diamond}\,h_m(\theta_k) = h_n(\theta_k)\diamond h_m(\theta_k) = h_{n+m}(\theta_k)$$

Proof

$$h_n(\theta_k)\,\hat{\diamond}\,h_m(\theta_k) = \theta_k^{\hat{\diamond} n}\,\hat{\diamond}\,\theta_k^{\hat{\diamond} m} = \theta_k^{\hat{\diamond} n+m} = h_{n+m}(\theta_k)$$

by Lemma A.4. □

Proposition A.6 For all finite length multi-indices α and β

$$H_{\alpha}(\omega)\,\hat{\diamond}\,H_{\beta}(\omega) = H_{\alpha}(\omega)\diamond H_{\beta}(\omega) = H_{\alpha+\beta}(\omega)$$

Proof Because of Corollary A.5 it suffices to prove that for all $n_1, n_2, \dots, n_M$

$$h_{n_1}(\theta_1)\hat{\diamond}h_{n_2}(\theta_2)\hat{\diamond}\cdots\hat{\diamond}h_{n_M}(\theta_M) = h_{n_1}(\theta_1)\cdot h_{n_2}(\theta_2)\cdots h_{n_M}(\theta_M)$$

As in the proof of Lemma A.4, we may just as well assume that $\theta_1, \theta_2, \dots, \theta_M$ is in some finite dimensional subspace generated by the $\hat{\theta}_k$-s. I.e. we may assume that

$$\theta_1 = \sum_{i=1}^{N} a_i\hat{\theta}_i \qquad \theta_2 = \sum_{i=1}^{N} b_i^{(2)}\hat{\theta}_i \qquad \cdots \qquad \theta_M = \sum_{i=1}^{N} b_i^{(M)}\hat{\theta}_i$$

where in particular $a = (a_1, a_2, \dots, a_N)$ is orthogonal to all the $b^{(i)}$-s. By propositions A2 and A3, we get

$$
\begin{aligned}
&h_{n_1}(\theta_1)\hat{\diamond}h_{n_2}(\theta_2)\hat{\diamond}\cdots\hat{\diamond}h_{n_M}(\theta_M) \\
&= h_{n_1}\left(\sum_{i=1}^{N} a_i\hat{\theta}_i\right)\hat{\diamond}\underbrace{\left(\sum_{i=1}^{N} b_i^{(2)}\hat{\theta}_i\right)\hat{\diamond}\cdots\hat{\diamond}\left(\sum_{i=1}^{N} b_i^{(2)}\hat{\theta}_i\right)}_{n_2-\text{times}}\hat{\diamond}\underbrace{\left(\sum_{i=1}^{N} b_i^{(3)}\hat{\theta}_i\right)\hat{\diamond}\cdots}_{n_3-\text{times}} \text{ etc.} \\
&= \sum_{\substack{|\alpha|=n_1 \\ |\beta_1|=1,\dots,|\beta_{n_2+\cdots+n_M}|=1}} \frac{n!}{\alpha!}a^\alpha b_1^{\beta_1}b_2^{\beta_2}\cdots b_{n_2+\cdots+n_M}^{\beta_{n_2+\cdots+n_M}} h_{\alpha+\beta_1+\cdots+\beta_{n_2+\cdots+n_M}}(\hat{\theta}_1,\dots,\hat{\theta}_N) \\
&= \sum_{\substack{|\alpha|=n_1 \\ |\beta_1|=1,\dots,|\beta_{n_2+\cdots+n_M}|=1}} \frac{n!}{\alpha!}a^\alpha b_1^{\beta_1}\cdots b_{n_2+\cdots+n_M}^{\beta_{n_2+\cdots+n_M}} h_\alpha(\hat{\theta}_1,\dots,\hat{\theta}_N)h_{\beta_1+\cdots+\beta_{n_2+\cdots+n_M}}(\hat{\theta}_1,\dots,\hat{\theta}_N) \\
&= h_{n_1}(\theta_1)\cdot\{h_{n_2}(\theta_2)\hat{\diamond}\cdots\hat{\diamond}h_{n_M}(\theta_M)\}
\end{aligned}
$$

and the claim follows by repeated use of this argument. □

Proposition A.6 says that the Wick product is invariantly defined on all elements in a base for $L^2(\Omega)$. The same then applies to all finite dimensional linear spans of such elements and hence to limits of such spans. We have thus proved the following

Theorem A.7 Whenever defined, the Wick-product $X\diamond Y$ of two elements $X, Y \in L^2(\Omega)$ is invariant under base changes on $L^2(\mathbf{R}^d)$.

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