ON THE C^*–ALGEBRA GENERATED BY THE LEFT REGULAR REPRESENTATION OF A LOCALLY COMPACT GROUP.

by

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Abstract: Let λ denote the left regular representation of a locally compact group G on $L^2(G)$ and $C^*(\lambda(G))$ the C^*–algebra generated by $\lambda(G)$. We show that the amenability of G and the amenability of G considered as a discrete group may both be characterized in terms of $C^*(\lambda(G))$.

1 Introduction.

We first fix some notation. Throughout this note we let G denote a locally compact (Hausdorff topological) group equipped with a fixed left Haar measure μ, and G_d denote the group G considered as a discrete group. As usual, $L^1(G)$, $L^2(G)$ and $L^\infty(G)$ are defined with respect to μ. The left regular representation of G on $L^2(G)$, defined by

$$\lambda(g)\xi)(h) = \xi(g^{-1}h) \ , \ \xi \in L^2(G) \ , \ g,h \in G,$$

is well known to be a (strongly) continuous unitary representation of G. We shall denote by λ_d the left regular representation of G_d on $l^2(G_d)$. All undefined terminology in this paper is explained in at least one of the following references: [2], [5], [7], [11], [13], [14].

Much attention has been devoted to the study of the following operator algebras associated with G: the full group C^*–algebra $C^*(G)$, the reduced group C^*–algebra $C^*_r(G)$ and the group von Neuman algebra $vN(G)$. We recall that $C^*(G)$ is defined as the enveloping C^*–algebra of $L^1(G)$ considered as an involutive Banach algebra with an approximate identity. If $\mathcal{B}(L^2(G))$ denotes the bounded linear operators on $L^2(G)$, then $C^*_r(G)$ is the C^*–subalgebra of $\mathcal{B}(L^2(G))$ generated by the convolution operators T_f, $f \in L^1(G)$, where $T_f(\xi) = f \star \xi$, $\xi \in L^2(G)$. At last, $vN(G)$ is the von Neumann subalgebra of $\mathcal{B}(L^2(G))$ generated by $\lambda(G) = \{\lambda(g), \ g \in G\}$, or equivalently $vN(G) = \lambda(G)'' = C^*_r(G)''$, where $''$ denotes the double commutant (in $\mathcal{B}(L^2(G))$). The purpose of this note is to draw the attention to $C^\star(\lambda(G))$, the $C^\star$–subalgebra of $\mathcal{B}(L^2(G))$ generated by $\lambda(G)$. Of course, when G is discrete, we have $C^\star(\lambda(G)) = C^\star_r(G)$, and we will therefore mainly be interested in the non-discrete case. In this case, it is known that $C^\star_r(G)$ and $C^\star(G)$ are non-unital ([10; Cor. 1 and 2]), while $C^\star(\lambda(G))$ is always unital.

The only paper we are aware of which explicitely deals with $C^\star(\lambda(G))$ in the non-discrete case is [8], where Kodaira and Kakutani essentially show that when G is abelian, then $C^\star(\lambda(G))$ is $\star$–isomorphic to $\mathcal{C}(\widehat{G_d})$, the continuous complex functions on the dual group of G_d. This result is nicely exposed by Arveson in [1], where he generalizes it to other $C^\star$–algebras generated by abelian unitary groups. Further, when G is abelian, it is well known that $C^\star(G) \simeq C^\star_r(G) \simeq \mathcal{C}_o(\widehat{G})$, the continuous complex functions on the dual group of G which vanish at infinity. Thus, $C^\star(\lambda(G))$ on one hand and $C^\star(G) \simeq C^\star_r(G)$ on the other hand contain rather different information in the abelian case. However, still in this case, we also have $C^\star_r(G_d) \simeq \mathcal{C}(\widehat{G_d})$, hence $C^\star(\lambda(G)) \simeq C^\star_r(G_d)$, which shows that the topology of G is not reflected in $C^\star(\lambda(G))$. One may therefore wonder whether all the topological flavour of G does disappear in $C^\star(\lambda(G))$ in the non-abelian case too.

We shall show that this suggestion is not generally true. Our approach relies heavily on the now well-developped theory of amenability ([13], [14]). We recall that G is called amenable whenever there exists a left invariant mean on $L^\infty(G)$, i.e. a state on $L^\infty(G)$ which is invariant under left translations. A deep $C^\star$–algebraic characterization of the amenability of G is that $C^\star(G)$

and $C_r^\star(G)$ are canonically $\star$-isomorphic. ([12; Theorem 4.21] or [13; Theorem 8.9]). Another characterization via $C^\star(\lambda(G))$ is possible: our first result (Theorem 1) is that G is amenable if and only if there exists a non zero multiplicative linear functional on $C^\star(\lambda(G))$. We notice that the "only if" part is known in the discrete case ([3; Theorem 2], [12; Proof of prop. 1.6]). This result provides a natural $C^\star$–explanation to the fact that an abelian group G is amenable: $C^\star(\lambda(G))$ is then an abelian $C^\star$–algebra and therefore possess a non-zero multiplicative linear functional by Gelfand's theory. Of course, this is not the most efficient way to prove this fact which is an easy consequence of the Markov–Kakutani fixed point theorem (cf. [13; Proposition 0.15]).

By combining a remark of Arveson in [1] and some arguments of Figà–Talamanca in [6], one obtains that if G_d is amenable, then $C^\star(\lambda(G)) \simeq C_r^\star(G_d)$. With the help of Theorem 1, we can conclude that G_d is amenable if and only if G is amenable and $C^\star(\lambda(G)) \simeq C_r^\star(G_d)$. (Theorem 2). Hence, if G is an amenable group such that G_d is not amenable (f.ex. $G = SO(3)$), then $C^\star(\lambda(G))$ is not $\star$–isomorphic to $C_r^\star(G_d)$.

At last, we characterize the nuclearity of $C^\star(\lambda(G))$. We recall that a $C^\star$–algebra is called nuclear if there is a unique way of forming its tensor product with any other $C^\star$–algebra. For some equivalent definitions, the reader may consult [9], [13] or [15] where further references are given. As a sample of the work of many hands, we quote the following from [13; 1.31 and 2.35]:

G is amenable if and only if G is inner amenable and $C_r^\star(G)$ is nuclear,

if and only if G is inner amenable and $vN(G)$ is injective.

Inner amenability of G means here that there exists a state on $L^\infty(G)$ invariant under the action on $L^\infty(G)$ by inner automorphisms of G, while $vN(G)$ is injective whenever there exists a norm one projection from $\mathcal{B}(L^2(G))$ onto $vN(G)$. We also recall that there exist non-amenable groups G such that $C_r^\star(G)$ is nuclear and $vN(G)$ is injective. Now, since any discrete group is inner amenable in the above sense, we have G_d is amenable if and only if $C_r^\star(G_d)$ is nuclear, a result proved by Lance in [9; Theorem 4.2]. We shall use this to conclude that G_d is amenable if and only if $C^\star(\lambda(G))$ is nuclear (Theorem 2). Especially, we get that if G is amenable but G_d is not, then $C^\star(\lambda(G))$ is non-nuclear while $C_r^\star(G)$ is nuclear and $vN(G)$ is injective.

2 The results.

We begin with a lemma which is surely known to specialists, but for the convenience of the reader we sketch the proof.

Lemma A: Let $\mathcal{A}$ denote a unital $C^\star$–algebra, $\mathcal{U}(\mathcal{A})$ its unitary group and φ a state on $\mathcal{A}$. Let $x \in \mathcal{A}$ and $u \in \mathcal{U}(\mathcal{A})$. Then

a) $\varphi(xa) = \varphi(x)\varphi(a)$ for all a in $\mathcal{A}$ if and only if $\varphi(xx^\star) = |\varphi(x)|^2$.

b) $\varphi(ax) = \varphi(a)\varphi(x)$ for all a in $\mathcal{A}$ if and only if $\varphi(x^\star x) = |\varphi(x)|^2$.

c) $\varphi(ua) = \varphi(au) = \varphi(u)\varphi(a)$ for all a in $\mathcal{A}$ if and only if $|\varphi(u)| = 1$.

d) If $\mathcal{V}$ is a subgroup of $\mathcal{U}(\mathcal{A})$ which generates $\mathcal{A}$ as a $C^\star$–algebra, then φ is multiplicative if and only if $|\varphi(v)| = 1$ for all v in $\mathcal{V}$.

Proof:

a) Suppose $\varphi(xx^\star) = |\varphi(x)|^2$ and let $a \in \mathcal{A}$. Then, by the Cauchy–Schwartz inequality, we get

$$\begin{aligned} |\varphi(xa) - \varphi(x)\varphi(a)|^2 &= |\varphi((x - \varphi(x))a)|^2 \\ &\leq \varphi(a^\star a)\varphi((x - \varphi(x))(x - \varphi(x))^\star) \\ &= \varphi(a^\star a)(\varphi(xx^\star) - |\varphi(x)|^2) \\ &= 0. \end{aligned}$$

Hence $\varphi(xa) = \varphi(x)\varphi(a)$ as desired.

The only if part is trivial.

b) may be deduced from a) or proved similarly.

c) follows from a) and b).

d) follows from c) and an easy density argument.

□

Theorem 1: G is amenable if and only if there exists a non-zero multplicative linear functional on $C^\star(\lambda(G))$.

Proof: Suppose G is amenable. Then there exists a net $\{\xi_i\}$ in $\{\xi \in L^2(G) | \parallel \xi \parallel_2 = 1\}$ such that

$$\parallel \lambda(g)\xi_i - \xi_i \parallel_2 \to 0 \text{ for all } g \text{ in } G.$$

(cf. [13; Theorem 4.4] or [14; Corollary 6.15]). For each i, define φ_i on $C^*(\lambda(G))$ by

$$\varphi_i(x) = < x\xi_i, \xi_i >, \quad x \in C^*(\lambda(G)).$$

Then $\{\varphi_i\}$ is a net in the state space of $C^*(\lambda(G))$ which (by Banach–Alaoglu's theorem) is weak*–compact. Hence we may pick a weak*–limit point of this net, say φ, which is a state on $C^*(\lambda(G))$. Now, since

$$\begin{aligned} |\varphi_i(\lambda(g)) - 1|^2 &= | < (\lambda(g)\xi_i - \xi_i), \xi_i > |^2 \\ &\leq \| \lambda(g)\xi_i - \xi_i \|_2 \to 0 \text{ for all } g \text{ in } G, \end{aligned}$$

we clearly have $\varphi(\lambda(g)) = 1$ for all g in G. As $\lambda(G)$ generates $C^*(\lambda(G))$ by definition, it follows from lemma A d) that φ is a non-zero multiplicative linear functional on $C^*(\lambda(G))$.

Conversly, suppose φ is such a functional on $C^*(\lambda(G))$. Then, as φ preserves adjoints ([11; Prop. 2.1.9]), φ is a state on $C^*(\lambda(G))$ such that $|\varphi(\lambda(g))| = 1$ for all g in G. By the Hahn-Banach theorem for states ([2; Prop. 2.3.24]), we may extend φ to a state $\tilde{\varphi}$ on $\mathcal{B}(L^2(G))$ which satisfies

$$|\tilde{\varphi}(\lambda(g))| = 1 \text{ for all } g \text{ in } G.$$

As a consequence of lemma A c), we then have

$$\begin{aligned} \tilde{\varphi}(\lambda(g)x\lambda(g^{-1})) &= \tilde{\varphi}(\lambda(g))\tilde{\varphi}(x\lambda(g^{-1})) \\ &= \tilde{\varphi}(\lambda(g))\tilde{\varphi}(x)\tilde{\varphi}(\lambda(g^{-1})) \\ &= |\tilde{\varphi}(\lambda(g))|^2\tilde{\varphi}(x) \\ &= \tilde{\varphi}(x) \end{aligned}$$

for all g in G and x in $\mathcal{B}(L^2(G))$.

The amenability of G follows readily from this in a quite standard way. If M_f denotes the multiplication operator on $L^2(G)$ by $f \in L^\infty(G)$, then one obtains a left invariant mean m on $L^\infty(G)$ by defining $m(f) = \tilde{\varphi}(M_f)$, $f \in L^\infty(G)$, and using that $M_{f_g} = \lambda(g)M_f\lambda(g^{-1})$ for all f in $L^\infty(G)$ and g in G, where $f_g(h) = f(g^{-1}h)$, $h \in G$.

□

When U is a continuous unitary representation of G on a Hilbert space $\mathcal{H}$, we denote by π_U the canonically associated $\star$–representation of $C^\star(G)$ in $\mathcal{B}(\mathcal{H})$. We recall that if V is such another representation of G, then U is said to be weakly contained in V (resp. equivalent to V) whenever $ker\ \pi_V \subseteq ker\ \pi_U$ (resp. $ker\ \pi_V = ker\ \pi_U$). We shall also need the fact that $\pi_U(C^\star(G))$ is the closure (in the uniform topology) of $\pi_U(L^1(G))$ in $\mathcal{B}(\mathcal{H})$. We refer to [5] for more information on this matter.

By regarding G as a discrete group, we may consider λ as a representation of G_d in $L^2(G)$. To avoid confusion, we shall denote this representation by λ°. For each $g \in G$, we let δ_g denote the characteristic function of $\{g\}$ in G.

Lemma B: $C^\star(\lambda(G)) = \pi_{\lambda^\circ}(C^\star(G_d))$.

Proof: Let $\xi, \eta \in L^2(G)$. Then for all g in G we have

$$\begin{aligned} < \pi_{\lambda^\circ}(\delta_g)\xi, \eta > &= \sum_{h \in G} \delta_g(h) < \lambda^\circ(h)\xi, \eta > = < \lambda^\circ(g)\xi, \eta > \\ &= < \lambda(g)\xi, \eta > . \end{aligned}$$

Hence $\pi_{\lambda^\circ}(\delta_g) = \lambda(g)$, $g \in G$. This clearly implies that $C^*(\lambda(G)) \subseteq \pi_{\lambda^\circ}(C^*(G_d))$. To prove the converse inclusion, let $f \in l^1(G_d)$. Then choose a sequence of complex functions f_n with finite support such that $f_n \to f$ in l^1-norm. From the above, we have $\pi_{\lambda^\circ}(f_n) \in C^*(\lambda(G))$ for all n. Since

$$\begin{aligned} \| \pi_{\lambda^\circ}(f_n) - \pi_{\lambda^\circ}(f) \| &= \| \pi_{\lambda^\circ}(f_n - f) \| \\ &\leq \| f_n - f \|_1 \to 0 \end{aligned}$$

we get $\pi_{\lambda^\circ}(f) \in C^*(\lambda(G))$.

Thus $\pi_{\lambda^\circ}(l^1(G_d)) \subseteq C^*(\lambda(G))$, so

$$\pi_{\lambda^\circ}(C^*(G_d)) = \overline{\pi_{\lambda^\circ}(l^1(G_d))}^{\|\cdot\|} \subseteq C^*(\lambda(G)).$$

□

The next lemma is a corollary of [1] and [6], but for the sake of completeness, we sketch the proof.

Lemma C: λ_d is weakly contained in λ°. Further, if G_d is amenable, then λ_d is weakly equivalent to λ° and $C^*(\lambda(G))$ is $\star$–isomorphic to $C^*_r(G_d)$.

Proof: For each finite subset F of G, there exists a ξ_F in $L^2(G)$ such that $\| \xi_F \|_2 = 1$ and $< \lambda(g)\xi_F, \xi_F >= 0$ for all g in F, $g \neq e$ (the identity of G). This follows from the easily verified fact that there exists a Borel subset $W = W(F)$ of G such that $0 < \mu(W) < \infty$ and $\mu(gW \cap W) = 0$ for all g in F, $g \neq e$, and then by setting $\mu(W)^{1/2} \cdot \xi_F = \chi_W$ (the characteristic function of W).

Define so $\varphi_F(g) = < \lambda(g)\xi_F, \xi_F > = < \lambda^\circ(g)\xi_F, \xi_F >$ for each g in G. Then φ_F a positive definite function on G_d associated to λ°. Further, if we regard $\{F \subseteq G,\ F \text{ finite}\}$ as a directed set ordered by inclusion, then we clearly have

$$\varphi_F(g) \longrightarrow \delta_e(g) \text{ for all } g \text{ in } G.$$

Since $\delta_e(g) = < \lambda_d(g)\delta_e, \delta_e >$ for all g in G, δ_e is a positive definite function on G_d associated to λ_d. As δ_e is a cyclic vector for λ_d, we then get from [5; Prop. 18.1.4] that λ_d is weakly contained in λ° as desired.

Now, suppose G_d is amenable. Then ρ is weakly contained in λ_d for all unitary representations ρ of G_d (use [5; Prop. 18.3.5] together with [5; Prop. 18.3.6] or [14; Theorem 8.9]). Especially, λ° is then weakly contained in λ_d. Hence λ_d is weakly equivalent to λ°.

Since $C^*_r(G_d) = \pi_{\lambda_d}(C^*(G_d))$ and

$$C^*(\lambda(G)) = \pi_{\lambda^\circ}(C^*(G_d)) \text{ (by lemma } B\text{)},$$

this implies that $C^*(\lambda(G)) \simeq C^*_r(G_d)$.

□

Theorem 2: The following statements are equivalent:

(i) G_d is amenable.

(ii) G is amenable and $C^*(\lambda(G)) \simeq C^*_r(G_d)$.

(iii) $C^*(\lambda(G))$ is nuclear.

(iv) $C^*_r(G_d)$ is nuclear.

Proof: (i) $\Leftrightarrow$ (iv) is proved by Lance in [9; Theorem 4.2].

(i) $\Rightarrow$ (ii) Suppose G_d is amenable. Then G is amenable ([13; Problem 1.12] or [14; Prop. 4.21]) and $C^*(\lambda(G)) \simeq C_r^*(G_d)$ by lemma B.

(ii) $\Rightarrow$ (i) Suppose G is amenable and $C^*(\lambda(G)) \simeq C_r^*(G_d)$. From Theorem 1, we then know that $C^*(\lambda(G))$ possess a nonzero multiplicative linear functional, and therefore that $C_r^*(G_d)$ possess one too. Since $C_r^*(G_d) = C^*(\lambda_d(G_d))$, Theorem 1 now implies that G_d is amenable.

(iii) $\Rightarrow$ (iv) Suppose $C^*(\lambda(G))$ is nuclear. Since λ_d is weakly contained in λ° by lemma B, this implies that $\pi_{\lambda_d}(C^*(G_d)) = C_r^*(G_d)$ is a quotient C^*–algebra of $\pi_{\lambda^\circ}(C^*(G_d)) = C^*(\lambda(G))$. As it is known that a quotient C^*–algebra of a nuclear C^*–algebra is itself nuclear ([4; Corollary 4]), we obtain that $C_r^*(G_d)$ is nuclear.

(iv) $\Rightarrow$ (iii) Suppose $C_r^*(G_d)$ is nuclear. Since we now know that (iv) $\Rightarrow$ (ii), we have $C^*(\lambda(G)) \simeq C_r^*(G_d)$, so $C^*(\lambda(G))$ is nuclear too.

$\square$

We conclude this note with some remarks on

$$X(G) = \{\varphi : \ C^*(\lambda(G)) \to \mathbf{C} | \varphi \text{ is nonzero, linear and multiplicative}\}$$

which is a weak*–closed subset of the state space of $C^*(\lambda(G))$. Theorem 1 says that $X(G) \neq \phi$ if and only if G is amenable. When G is abelian, the result of Kodaira and Kakutani mentionned in the introduction may be interpreted as the fact that $X(G)$ is homeomorphic to $\widehat{G_d}$. In the non–abelian case, $X(G)$ is of course a rather primitive C^*–algebraic invariant for

$C^*(\lambda(G))$, but it has the advantage of being easily computed in some cases, as the following illustrates.

Let H denote a discrete group and CH its commutator subgroup. Then H/CH is abelian and it is not difficult to show, as it has been observed by Watatani in [16], that if H is amenable, then $X(H)$ is homeomorphic to $\widehat{H/CH}$. Hence, if G_d is amenable, we get via Theorem 2 that $X(G)$ is homeomorphic to $\widehat{G_d/CG_d}$. If G is amenable but G_d is not, one can show that $X(G)$ contains a copy of $\widehat{G/CG}$ and may itself be embedded in $\widehat{G_d/CG_d}$, but we don't know whether anything more general can be said here. If f.ex. $G = SO(3)$, then $CG_d = G_d$, so $X(G) = \{\hat{1}\}$ (where $\hat{1}$ denotes the state on $C^*(\lambda(G))$ determined by $\hat{1}(\lambda(g)) = 1$ for all g in G, cf. the proof of Theorem 1).

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