THE STOCHASTIC VOLTERRA EQUATION

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Abstract

We study the stochastic (Skorohod) integral equation of the Volterra type

$$X_t(\omega) = Y_t(\omega) + \int_0^t b(t,s)X_s(\omega)ds + \int_0^t \sigma(t,s)X_s(\omega)\delta B_s(\omega)$$

where Y, b and σ are given functions; b and σ are bounded, deterministic and Y_t is stochastic, not necessarily adapted. The stochastic integral (δB) is taken in the Skorohod sense.

In general there need not exist a classical stochastic process $X_t(\omega)$ satisfying this equation. However, we show that a unique solution exists in the following extended senses:

(I) As a functional process
(II) As a generalized white noise functional (Hida distribution).

Moreover, in both cases we find explicit solution formulas. The formulas are similar to the formulas in the deterministic case ($\sigma \equiv 0$), but with Wick products in stead of ordinary (pointwise) products.

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§1. Introduction

The classical (deterministic) Volterra equation of the second kind has the form

$$X_t = Y_t + \int_0^t \gamma(t,s) X_s ds \quad 0 \le t \le T \tag{1.1}$$

where $Y_t, \gamma(t,s)$ are given functions, $T > 0$ is a given constant. This equation occurs in many applications, some of which are described in [GLS]. See also [T] and Example 1.1 below. Suppose now that the system is randomly perturbed or that there is insufficient/noisy information about the function $\gamma(t,s)$. In both cases a possible mathematical formulation would be to put

$$\gamma(t,s) = b(t,s) + \sigma(t,s) \cdot W_s, \tag{1.2}$$

where $b(t,s)$ and $\sigma(t,s)$ are deterministic functions and $W_s = W_s(\omega); \omega \in \Omega$ (a probability space) denotes "white noise" (see definitions below). We also allow $Y_t = Y_t(\omega)$ to be random. This gives - formally - the equation

$$X_t(\omega) = Y_t(\omega) + \int_0^t b(t,s) X_s(\omega) ds + \text{``}\int_0^t \sigma(t,s) X_s(\omega) W_s(\omega) ds\text{''} \tag{1.3}$$

where the last term (in quotation marks) remains to be defined.

If Y_t is an *adapted* stochastic process, then it is natural to assume that a solution X_t of (1.3) must be adapted too, and this leads to the following interpretation of (1.3):

$$X_t(\omega) = Y_t(\omega) + \int_0^t b(t,s) X_s(\omega) ds + \int_0^t \sigma(t,s) X_s(\omega) dB_s(\omega) \tag{1.4}$$

where the last term denotes the usual Ito integral and $B_t(\omega)$ denotes Brownian motion whose t-derivative is $W_t(\omega)$ (in distribution sense).

In this paper we are mainly interested in the case when Y_t is *not* adapted. In this case we of course cannot expect X_t to be adapted and then the Ito integral in (1.4) is not defined. However, equation still makes sense if we replace the Ito integral by the more general *Skorohod integral*:

$$(1.5) \qquad X_t(\omega) = Y_t(\omega) + \int_0^t b(t,s)X_s(\omega)ds + \int_0^t \sigma(t,s)X_s(\omega)\delta B_s(\omega)$$

REMARK. The Skorohod integral

$$\int_0^T Z_s(\omega)\delta B_s(\omega)$$

is defined for all processes $Z_t(\omega)$ (adapted or not) such that

$$(1.6) \qquad \int_0^T E[Z_s^2]ds + \sum_{m=1}^{\infty}(m+1)!\|\tilde{f}_m\|^2 < \infty$$

Here $\tilde{f}_m(t_1,\cdots,t_m,t)$ is the symmetrization of $f_t^{(m)}(t_1,\cdots,t_m)$, where $f_t^{(m)}$ is the m'th order term in the Wiener-Ito chaos expansion of Z_t:

$$Z_t(\omega) = \sum_{m=0}^{\infty}\int_{\mathbf{R}^m} f_t^{(m)}(t_1,\cdots,t_m)dB_{t_1}\cdots dB_{t_m}$$

If $Z_s(\omega)$ is adapted, then the Skorohod integral coincides with the Ito integral [NZ].

We will use (1.5) as our mathematical model for a randomly perturbed Volterra integral equation, or a *stochastic Volterra integral equation* for short. The purpose of this paper is to study the existence and uniqueness of a solution of (1.5). Moreover, we will find an explicit solution formula. It turns out that in general (without strong conditions on Y_t, b and σ) there does *not* exist a (classical) stochastic process X_t satisfying (1.5). However, we will prove that a solution exists (and is unique) in the following extended senses:

(I) As a *functional process* (see §3)

(II) As a generalized white noise functional (or *Hida distribution*) (see §4).

Skorohod Volterra equations with anticipating kernel (but non-anticipating initial condition $X_0 \equiv Y_t(\forall t)$) have been studied in [PP], see also [BM] and the survey in [Pa]. In [O] the stochastic Volterra equation is studied in the setting of Ogawa-type integrals. To the best of our knowledge our paper is the first to discuss the Skorohod interpretation with anticipating initial conditions.

We now explain these two approaches in more detail:

(I) *The functional process approach.* ([LØU1],[LØU3],[HLØUZ]) (see §3 for details).

Here we regard the solution X as a generalized stochastic process of the form

$$(1.7) \qquad X = X_t^\phi = X(\phi,t,\omega)$$

where $\phi \in \mathcal{S}$ (the Schwartz space of rapidly decreasing functions on $\mathbf{R}$). Heuristically $X(\phi, t, \omega)$ can be regarded as the result of measuring X (at time t and in the experiment ω) through the averaging/test function or "window" ϕ.

White noise W may be regarded as such a functional process by the definition

$$W(\phi, t, \omega) := W_{\phi_t}(\omega) := \int \phi_t(s) dB_s, \tag{1.8}$$

where $\phi_t(s) = \phi(s - t)$ is the window ϕ shifted by the amount t. Note that for each fixed ϕ both $X(\phi, \cdot, \cdot)$ and $W(\phi, \cdot, \cdot)$ are continuous stochastic processes. There is a striking formula for the Skorohod integral in terms of *the Wick product* $\diamond$ as follows (see Lemma 2.1):

$$\int_{\mathbf{R}} (\phi * Y)_t \delta B_t = \int_{\mathbf{R}} Y_t \diamond W_{\phi_t} dt \quad \forall \phi \in \mathcal{S} \tag{1.9}$$

where $*$ denotes convolution with respect to t, i.e.

$$(\phi * Y)_t(\omega) = \int_{\mathbf{R}} \phi(t - s) Y_s(\omega) ds. \tag{1.10}$$

In view of this we say that a functional process $X_t = X_t^{\phi}(\omega) = X(\phi, t, \omega)$ is a *solution of (1.5)* if for all $\phi \in \mathcal{S}$ there exists a stochastic process $X_t = X_t^{\phi}$ such that

$$X_t^{\phi} = Y_t^{\phi} + \int_0^t b(t, s) X_s^{\phi} ds + \int_0^t \sigma(t, s) X_s^{\phi} \diamond W_{\phi_s} ds \ ; \ 0 \le t \le T \tag{1.11}$$

where we allow $Y_t = Y_t^{\phi}$ to be a functional process too. In §3 we show that a functional process solution of (1.5) exists under certain conditions on $Y_t^{\phi}, b(t, s)$ and $\sigma(t, s)$. Moreover, we give an explicit solution formula:

$$X_t = Y_t + \int_0^t H(t, s) \diamond Y_s ds, \tag{1.12}$$

where $H(t, s) = H(t, s, \omega)$ is a random kernel constructed from $K(t, s, \omega) := b(t, s) + \sigma(t, s) W_{\phi_s}(\omega)$.

(II) *The generalized white noise functional (Hida distribution) approach* [HKPS] (see §4)

In this setting we regard X_t and the other elements of equations (1.5) as elements of the space $(\mathcal{S})^*$ of *Hida distributions* (or *generalized white noise functionals*). The *pointwise* white noise W_t may be regarded as an element of $(\mathcal{S})^*$. By Corollary 3.4 in [LØU2] we have

$$\int_{\mathbf{R}} Z_t \delta B_t = \int_{\mathbf{R}} Z_t \diamond W_t dt \tag{1.13}$$

for all stochastic processes satisfying (1.6) (adapted or not). In view of this the natural interpretation of equation (1.5) in the Hida distribution setting is

$$X_t = Y_t + \int_0^t b(t,s)X_s ds + \int_0^t \sigma(t,s)X_s \diamond W_s ds \tag{1.14}$$

where Y_t now is regarded as an element of $(\mathcal{S})^*$.

Note that by (1.13) the relation between the pointwise white noise W_t and the functional process W_ψ is (choose $Z_t = \psi(t)$ in (1.13)):

$$W(\psi, 0, \omega) = \int_{\mathbf{R}} \psi(t) W_t dt, \tag{1.15}$$

i.e. $W(\psi)$ is the result of "smearing out" the singular noise W_t by the test function ψ.

In this $(\mathcal{S})^*$-setting we prove an existence and uniqueness result for (1.14) and, here too, we obtain a solution formula of the type (1.12). Moreover, we show that the solution of (1.14) is actually in $L^2(\mu)$ under some conditions.

EXAMPLE 1.1 A number of applications of Volterra equations can be found in [GLS, p. 4-13]. Here we present an economic example, with a structure related to the population dynamics example presented in Ex. 2.2 in [GLS]. Our example leads to a stochastic Volterra equation of the form considered in this paper:

An investment in an economic production, for example the purchase of new production equipment, will usually have effects over a long period of time. Let $X(t,u)$ denote the capital distribution at time t resulting from the investments which have *age* u (i.e. which were made u units of time ago). More precisely, let

$$\int_U X(t,u)du \quad \text{denote the total capital gained}$$

at time t from all investments with age $u \in U$. Assume that

$$\frac{\partial X(t,u)}{\partial t} + \frac{\partial X(t,u)}{\partial u} = -m(u)X(t,u), \tag{1.16}$$

where $m(u) \geq 0$ denotes the age-dependent "death" rate of the equipmentment/machines involved in the production. Moreover, assume that the amount of new capital $X(t,0)$ at time t is described by the equation

$$X(t,0) = \int_0^\infty X(t,u)p(u)du \tag{1.17}$$

where $p(u)$ is *the productivity* of the equipment with age u, i.e. $p(u)$ is the production at age u per capital unit. (In this model we only consider the part $X(t,u)$ of the produced capital that is reinvested into the production process.)

We assume that the initial capital distribution $X(0,u) = \phi(u)$ is known. Then the solution $X(t,u)$ of (1.16) is given by

$$X(t,u) = \begin{cases} \phi(u-t)\cdot\exp(-\int_0^t m(s+u-t)ds) & ; \quad 0 \le t < u \\ X(t-u,0)\cdot\exp(-\int_0^u m(s)ds) & ; \quad t \ge u \end{cases} \tag{1.18}$$

Substituting this in (1.17) we get the Volterra equation

$$X(t,0) = Y(t) + \int_0^t K(t-s)X(s,0)ds \tag{1.19}$$

where

$$Y(t) = \int_0^\infty \phi(s)\exp(-\int_0^t m(s+r)dr)p(t+s)ds \tag{1.20}$$

and

$$K(t) = p(t)\exp(-\int_0^t m(s)ds) \tag{1.21}$$

If the productivity function $p(u)$ is subject to random fluctuations we could model $p(u)$ by

$$p(u) = p_0(u) + \epsilon W_u \tag{1.22}$$

where $\epsilon > 0$ and W_u denotes white noise as before. This leads to a stochastic Volterra equation of the form (1.4) with $X_t = X(t,0)$,

$$b(t,s) = p_0(t-s)\exp(-\int_0^{t-s} m(r)dr) \quad ; \quad 0 \le s \le t \tag{1.23}$$

$$\sigma(t,s) = \epsilon\exp(-\int_0^{t-s} m(r)dr) \quad ; \quad 0 \le s \le t \tag{1.24}$$

and

$$\begin{aligned} Y_t = &\int_0^\infty \phi(s)\exp(-\int_0^t m(s+r)dr)p_0(t+s)ds \\ &+\epsilon\int_t^\infty \phi(v-t)\exp(-\int_0^t m(v-t+r)dr)dB_v \end{aligned} \tag{1.25}$$

Note that Y_t is not adapted in this case.

§2. Some mathematical preliminaries

Let $(\mathcal{S}', \mathcal{B}, \mu)$ denote *the white noise probability space*, i.e. μ is the probability measure on the Borel subsets $\mathcal{B}$ of the space $\mathcal{S}' = \mathcal{S}'(\mathbf{R})$ of tempered distributions on $\mathbf{R}$, with the property that

$$\int_{\mathcal{S}'} e^{i<\omega,\phi>} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2} \tag{2.1}$$

for all $\phi \in \mathcal{S}$, where $\|\phi\|^2 = \int_{\mathbf{R}} |\phi|^2 dx$ and $<\omega, \phi> = \omega(\phi)$ is the action of $\omega \in \mathcal{S}'$ (the dual of $\mathcal{S}$) on $\phi \in \mathcal{S}$. See [HKPS] for more information.

Recall that *the white noise process* W is the map

$$W : \mathcal{S} \times \mathcal{S}' \to \mathbf{R}$$

given by

$$W(\phi, \omega) = W_\phi(\omega) = <\omega, \phi>; \phi \in \mathcal{S}, \omega \in \mathcal{S}' \tag{2.2}$$

i.e.

$$W_\phi(\omega) = \int_{\mathbf{R}} \phi(t) dB_t$$

where the right hand side denotes the Wiener-Ito integral with respect to Brownian motion B_t.

There is also a *pointwise*, singular version W_t of white noise, which we describe below. Heuristically we may regard W_t as the limit of W_ϕ as $\phi \to \delta_t$, the point mass at t. This limit exists in the space $(\mathcal{S})^*$ of Hida distributions. For definition and properties of $(\mathcal{S})^*$ see [HKPS]. An alternative description of $(\mathcal{S})^*$ can be given as follows (see [Z]): Let

$$e_n(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{n-1}(\sqrt{2}x) \tag{2.3}$$

be the *Hermite function* of order $n \geq 1$, where

$$h_m(x) = (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m} e^{-\frac{x^2}{2}} \tag{2.4}$$

is the *Hermite polynomial* of order $m \geq 0$.

It is well known that $\{e_n\}_{n=1}^\infty$ forms an orthonormal base of $L^2 = L^2(\mathbf{R})$. Moreover, e_n is an eigenfunction for the operator

$$A = -\frac{d^2}{dx^2} + x^2 + 1$$

with eigenvalue $2n$, i.e.

$$Ae_n = 2ne_n \;,\; n = 1, 2, \cdots \tag{2.5}$$

Put $\theta_j(\omega) = \int_{\mathbf{R}} e_j(t)dB_t$ and define

$$H_\alpha(\omega) = \prod_{j=1}^{m} h_{\alpha_j}(\theta_j) \tag{2.6}$$

for all multi-indices $\alpha = (\alpha_1, \cdots, \alpha_m)$.

By the Wiener-Ito chaos theorem we have that each $f \in L^2(\mu)$ can be represented as a sum

$$f(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \tag{2.7}$$

where

$$\|f\|^2_{L^2(\mu)} = \sum_\alpha \alpha! c_\alpha^2, \quad \text{and } \alpha! = \prod_{i=1}^{m} \alpha_i! \tag{2.8}$$

We say that $f \in L^2(\mu)$ is a *Hida test function*, $f \in (\mathcal{S})$, if

$$A_f(k) := \sup_\alpha c_\alpha^2 \alpha! (2\mathrm{N})^{\alpha k} < \infty \quad \text{for all } k < \infty \tag{2.9}$$

where

$$(2\mathrm{N})^\alpha := \prod_{j=1}^{m} (2j)^{\alpha_j} \quad \text{if } \alpha = (\alpha_1, \cdots, \alpha_m). \tag{2.10}$$

The dual of $(\mathcal{S})$, denoted by $(\mathcal{S})^*$ (the space of Hida distributions) can be represented as the set of formal sums

$$F = \sum_\alpha b_\alpha H_\alpha \tag{2.11}$$

where

$$\sup_\alpha b_\alpha^2 \alpha! ((2\mathrm{N})^{-\alpha})^q < \infty \quad \text{for some } q < \infty. \tag{2.12}$$

The action of $F \in (\mathcal{S})^*$ (given by (2.11)) on the test function $f \in (\mathcal{S})$ (given by (2.7)) is

$$< F, f >= \sum_\alpha \alpha! b_\alpha c_\alpha \tag{2.13}$$

In general we have

$$(\mathcal{S}) \subset L^p(\mu) \subset (\mathcal{S})^* \quad \text{for all } p \in (1, \infty)$$

We can now define the *pointwise* white noise W_t in $(\mathcal{S})^*$ by

$$W_t(\omega) = \sum_{k=1}^{\infty} e_k(t) h(\theta_k) \tag{2.14}$$

$(= \sum_{n=1}^{\infty} b_n H_{\epsilon_n}(\omega)$ with $\epsilon_n = (0,0,\cdots,1)$ with 1 on the n'th place and $b_n = e_n(t))$

Then

$$\sup_\alpha b_\alpha^2 \alpha! (2\mathrm{N})^{-\alpha q} = \sup_n e_n^2(t) \cdot 1 \cdot (2n)^{-q} < \infty$$

for $q > -\frac{1}{12}$, since $\|e_n\|_\infty = 0(n^{-\frac{1}{12}})$ as $n \to \infty$ (See [HP, Formula (21.3.3)]). So $W_t \in (\mathcal{S})^*$ as claimed.

In the following we will use the convention that W_t means the pointwise white noise (in $(\mathcal{S})^*$) if $t \in \mathbf{R}$ while W_ψ means the white noise defined by (2.3) if $\psi \in \mathcal{S}$. In spite of the ambiguouity of this notation we think it will be clear from the context what we mean.

If $F = \sum_\alpha a_\alpha H_\alpha$ and $G = \sum_\beta b_\beta H_\beta$ are two elements of $(\mathcal{S})^*$ we define their *Wick product* $F \diamond G$ as the element of $(\mathcal{S})^*$ given by

$$F \diamond G = \sum_{\alpha,\beta} a_\alpha b_\beta H_{\alpha+\beta} = \sum_\gamma \Big(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \Big) H_\gamma \tag{2.15}$$

Using the characterization (2.9) one can prove that both $(\mathcal{S})^*$ and $(\mathcal{S})$ are closed under $\diamond$, i.e. $f,g \in (\mathcal{S})^* \Rightarrow f \diamond g \in (\mathcal{S})^*$ and similarly for $(\mathcal{S})$. (See the argument in [Z]).

There is an alternative to the representation (2.7): If $f \in L^2(\mu)$ there exist functions $f_n \in \hat{L}^2(\mathbf{R}^n)$ such that

$$f(\omega) = \sum_{n=0}^{\infty} \int_{\mathbf{R}^n} f_n(t_1,\cdots,t_n) dB_t^{\otimes n} \tag{2.16}$$

and

$$\|f\|_{L^2(\mu)}^2 = \sum_{n=0}^{\infty} n! \|f_n\|_{L^2(\mathbf{R}^n)}^2.$$

Here $\hat{L}^2(\mathbf{R}^n)$ denotes the space of *symmetric* L^2- functions on $\mathbf{R}^n$ and $dB_t^{\otimes n} = dB_{t_1} dB_{t_2} \cdots dB_{t_n}$ denotes the n'th iterated Ito integral. Using this representation the Wick product of two functions

$$f = \sum_n \int f_n dB^{\otimes n} \quad \text{and} \quad g = \sum_m \int g_m dB^{\otimes m}$$

in $L^2(\mu)$ can be expressed as (when convergent)

$$f \diamond g = \sum_{n,m} \int f_n \hat{\otimes} g_m dB^{\otimes(n+m)} \tag{2.17}$$

where $\hat{\otimes}$ denotes the symmetrized tensor product.

The Wick product plays a crucial role in our solution of the stochastic Volterra equation. Since $L^1(\mu)$ is *not* contained in $(\mathcal{S})^*$, an extra definition is needed for that case (see [HLØUZ]):

Suppose there exist $X_n, Y_n \in L^2(\mu)$ such that $X_n \to X$ in $L^1(\mu)$, $Y_n \to Y$ in $L^1(\mu)$, $X_n \diamond Y_n \in L^1(\mu)$ for all n and $Z := \lim X_n \diamond Y_n$ exists in $L^1(\mu)$.

Then we define

$$X \diamond Y = Z \tag{2.18}$$

This definition does not depend on the specific choice of $\{X_n\}, \{Y_n\}$. In fact, we have

$$\mathcal{F}[X \diamond Y](\phi) = e^{\frac{1}{2}\|\phi\|^2} \mathcal{F}[X](\phi) \cdot \mathcal{F}[Y](\phi) \quad \forall \phi \in \mathcal{S}, \tag{2.19}$$

where

$$\mathcal{F}[X](\phi) = \int_{\mathcal{S}'} e^{i<\omega,\phi>} X(\omega) d\mu(\omega) \quad , \phi \in \mathcal{S} \tag{2.20}$$

is the Fourier transform of X.

(For a proof, see [HLØUZ, Lemma 9.2]). A survey of the properties of the Wick product is given in [GHLØUZ].

Now let us recall two important transforms on $(\mathcal{S})^*$:

If $F \in (\mathcal{S})^*$ then *the $\mathcal{S}$-transform of F (first introduced in [KT]), $\mathcal{S}F$,* is a map from $\mathcal{S}$ into $\mathbf{C}$ defined by

$$\mathcal{S}F(\phi) = e^{-\frac{1}{2}\|\phi\|^2} < F, \exp < \cdot, \phi >> \tag{2.21}$$

(It can be proved that the function $\omega \to \exp < \omega, \phi >; \omega \in \mathcal{S}'$ belongs to $(\mathcal{S})$, so (2.21) is well-defined).

Note that if $F \in L^2(\mu)$ then

$$\mathcal{S}F(\phi) = e^{-\frac{1}{2}\|\phi\|^2} \int_{\mathcal{S}'} \exp(< \omega, \phi >) F(\omega) d\mu(\omega) \tag{2.22}$$

The Hermite transform (first introduced in [LØU1]) of F, $\mathcal{H}F$, is a map from the space $\mathbf{C}_0^{\mathbf{N}}$ of all finite sequences of complex numbers into $\mathbf{C}$ (the set of complex numbers) defined by

$$\mathcal{H}F(z_1, z_2, \cdots) := \tilde{F}(z_1, z_2, \cdots) = \mathcal{S}F(z_1 e_1 + z_2 e_2 + \cdots); (z_1, z_2, \cdots) \in \mathbf{C}_0^{\mathbf{N}} \tag{2.23}$$

Equivalently (see [LØU1, Th. 5.7]) if $F \in (\mathcal{S})^*$ has the expansion

$$F(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega)$$

then, using multi-index notation $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots$ if $z = (z_1, z_2, \cdots)$ and $\alpha = (\alpha_1, \alpha_2, \cdots)$ we have

$$\mathcal{H}F(z) = \sum_\alpha c_\alpha z^\alpha \quad ; \; z \in \mathbf{C}_0^{\mathbf{N}} \tag{2.24}$$

It can be shown that the series converges and represents an analytic function of $z \in \mathbf{C}_0^{\mathbf{N}}$, for all $F \in (\mathcal{S})^*$ (see [HKPS]). The characterizations of the Wick product in terms of the $\mathcal{S}$- and the $\mathcal{H}$- transform are the following:

If $F, G \in (\mathcal{S})^*$ then

(2.25) $$\mathcal{H}(F \diamond G)(z) = \mathcal{H}F(z) \cdot \mathcal{H}G(z); \quad z \in \mathbf{C}_0^{\mathbf{N}}$$

and

(2.26) $$\mathcal{S}(F \diamond G)(\phi) = \mathcal{S}F(\phi) \cdot \mathcal{S}G(\phi); \quad \phi \in \mathcal{S}$$

Finally we recall that there is an explicit *inverse of the Hermite transform* [LØU1]: Let λ be the probability measure on $\mathbf{R}^{\mathbf{N}}$ defined by

(2.27) $$\int_{\mathbf{R}^{\mathbf{N}}} f(y_1, \cdots, y_n) d\lambda(y) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} f(y) e^{-\frac{1}{2}|y|^2} dy$$

if f is a bounded measurable function of $y = (y_1, y_2, \cdots) \in \mathbf{R}^{\mathbf{N}}$ depending only on the first n coordinates $y_1, \cdots, y_n$. Let

$$X = \sum_\alpha c_\alpha H_\alpha(\omega) \quad \text{be in } L^2(\mu),$$

so that X has the Hermite transform

$$\tilde{X}(z) = \mathcal{H}X(z) = \sum_\alpha c_\alpha z^\alpha$$

Then we recover X from $\tilde{X}$ by

(2.28) $$X(\omega) := \mathcal{H}^{-1}\tilde{X} := \lim_{n,k\to\infty} \int_{\mathbf{R}^{\mathbf{N}}} \tilde{X}^{(n,k)}(\theta + iy) d\lambda(y) \quad (\text{limit in } L^2(\mu))$$

where $\theta + iy = (\theta_1 + iy_1, \theta_2 + iy_2, \cdots)$ with $\theta_k = \int e_k dB$ and

(2.29) $$\tilde{X}^{(n,k)}(z) = \sum_{\alpha \in J_{n,k}} c_\alpha z^\alpha; J_{n,k} = \{\alpha; |\alpha| \leq n \quad \text{and } \alpha_j = 0 \quad \text{for } j > k\}$$

is the doubly truncated series for $\tilde{X}$.

Moreover, we have the estimate [HLØUZ, Th. 4.2]:

(2.30) $$E[|X|^p] \leq \liminf_{n,k\to\infty} \int\int |\tilde{X}^{(n,k)}(\xi + i\eta)|^p d\lambda(\xi) d\lambda(\eta)$$

for all $p \in [1, \infty)$.

Before we finish this section, let us recall the main results in [PS] which will be used intensively in §4:

DEFINITION 2.1. Let F be a complex valued functional on $\mathcal{S}(\mathbf{R})$. We call F a *U-functional* if the following conditions are satisfied:

C.1 For all $\phi, \psi \in \mathcal{S}(\mathbf{R})$, the mapping $\lambda \to F(\psi + \lambda\phi), \lambda \in \mathbf{R}$, has an entire analytic extension, which will be denoted by $F(\psi + z\phi), z \in \mathbf{C}$

C.2 There exists a $p \in \mathbf{N}_0$ (the set of non-negative integers) and $C_1, C_2 > 0$ so that for all $z \in \mathbf{C}, \phi \in \mathcal{S}(\mathbf{R})$,

$$|F(z\phi)| \leq C_1 \exp(C_2|z|^2\|\phi\|_{2,p}^2) \tag{2.31}$$

where $\|\phi\|_{2,p} = \|A^p\phi\|_{L^2(\mathbf{R})}$.

Then we have following theorems from [PS]:

THEOREM 2.1 [PS]. If $\Phi \in (\mathcal{S})^*$ then $\mathcal{S}\Phi$ is a U-functional. Conversely, if F is a U-functional, then there is a unique Φ in $(\mathcal{S})^*$, so that $\mathcal{S}\Phi = F$.

THEOREM 2.2 [PS]. Assume that $F_n, n \in \mathbf{N}$ (the set of natural numbers), and F are U-functionals and let $\Phi_n, n \in \mathbf{N}$, and Φ, respectively, denote the associated Hida distributions in $(\mathcal{S})^*$. Then the following are equivalent:

(a) The sequence $(\Phi_n, n \in \mathbf{N})$ converges strongly to Φ
(b) The sequence $(F_n, n \in \mathbf{N})$ converges pointwise to F and for all large enough $n \in \mathbf{N}$, the estimate (2.31) holds for every F_n uniformly in n.

§3. The functional process approach

Functional processes were first introduced in [LØU1] in a study of certain stochastic differential equations involving functionals of white noise. An extended multiparameter version was used in [HLØUZ]. Functional processes may be regarded as a generalization of distribution valued processes.

DEFINITION 3.1 Let $p > 0$. A (one-parameter) L^p *functional process* is a function

$$X : \mathcal{S} \times \mathbf{R} \times \mathcal{S}' \to \mathbf{R}$$

such that

(i) the map $t \to X(\phi, t, \omega)$ is (Borel) measurable for each $\phi \in \mathcal{S}$ and a.a. $\omega \in \mathcal{S}'$ and

(ii) the map $\omega \to X(\phi, t, \omega)$ is in $L^p(\mu)$ for each $\phi \in \mathcal{S}$ and each $t \in \mathbf{R}$

EXAMPLE 3.2 We may regard the white noise process as a functional process $W(\phi, t, \omega)$ by defining

$$W(\phi, t, \omega) = W_{\phi_t}(\omega) = < \omega, \phi_t > \tag{3.1}$$

where

$$\phi_t(s) = \phi(s - t) \tag{3.2}$$

is the t-shift of the test function ϕ.

Note that if $F \in \mathcal{S}'$ and D denotes the differentiation operator we have

$$< DF, \phi_x(\cdot) > = - < F, D\phi_x(\cdot) > = - < F, \frac{d}{dy}\phi(y - x) > = < F, \frac{d}{dx}\phi_x(y) >, \tag{3.3}$$

so taking distributional derivatives of F and applying the result to ϕ_x is the same as applying F to the derivative of ϕ_x with respect to x. In view of this it is natural to interpret distributional differential equations with respect to ϕ involving functional processes $X(\phi, x, \omega)$ as ordinary differential equations in x for each ϕ.

The second observation which is relevant for the interpretation of (1.5) is the following:

LEMMA 3.3 Let $Y_t(\omega)$ be a stochastic process and let $\phi \in \mathcal{S}$ be such that

$$Z_t := (\phi * Y)_t$$

satisfies condition (1.6). Then

$$\int_{\mathbf{R}} (\phi * Y)_t \delta B_t = \int_{\mathbf{R}} Y_t \diamond W_{\phi_t} dt,$$

where $*$ denotes convolution, i.e.

$$(\phi * Y)_t(\omega) = \int_{\mathbf{R}} \phi(t - s) Y_s(\omega) ds$$

Proof. By Corollary 3.4 in [LØU2] we have

$$\int_{\mathbf{R}} Z_t \delta B_t = \int_{\mathbf{R}} Z_t \diamond W_t dt \tag{3.4}$$

for all processes Z_t satisfying (1.6), where the right hand side is regarded as an element in $(\mathcal{S})^*$.

Applying this to $Z_t = (\phi * Y)_t$ we get

$$\begin{aligned}
\int_{\mathbf{R}} (\phi * Y)_t \delta B_t &= \int_{\mathbf{R}} (\int_{\mathbf{R}} \phi(t-s) Y_s(\omega) ds) \diamond W_t dt \\
&= \int_{\mathbf{R}} Y_s(\omega) \diamond (\int_{\mathbf{R}} \phi(t-s) W_t dt) ds \\
&= \int_{\mathbf{R}} Y_s(\omega) \diamond W_{\phi_s} ds, \quad \text{as required.}
\end{aligned}$$

In view of Lemma 2.1 and (3.3) the following interpretation of (1.5) is natural:

We say that a functional process $X(\phi, t, \omega) = X_t^\phi$ is a solution of (1.5) if, for all $\phi \in \mathcal{S}$,

$$X_t^\phi = Y_t^\phi + \int_0^t b(t,s) X_s^\phi ds + \int_0^t \sigma(t,s) X_s^\phi \diamond W_{\phi_s} ds \quad t \geq 0 \tag{1.11}$$

where $Y_t^\phi = Y(\phi, t, \omega)$ is a given functional process.

LEMMA 3.4 Let $b(t,s)$ and $\sigma(t,s)$ be two bounded deterministic functions satisfying

$$b(t,s) = \sigma(t,s) = 0 \quad \text{if} \quad 0 \leq t < s \tag{3.5}$$

Fix $\phi \in \mathcal{S}$ and define, for $(t,s) \in [0,\infty) \times [0,\infty)$,

$$K_1(t,s) := K(t,s) := b(t,s) + \sigma(t,s) W_{\phi_s}(\omega) \tag{3.6}$$

and inductively

$$K_{n+1}(t,s) = \int_0^t K_n(t,u) \diamond K(u,s) du (= \int_s^t K_n(t,u) \diamond K(u,s) du) \quad ; n \geq 1. \tag{3.7}$$

Then for all (t,s)

$$\|K_n(t,s)\|_{L^2(\mu)} \leq \frac{M^n (1 + \|\phi\|)^n}{\sqrt{n!}} \quad ; \quad n = 1, 2, \cdots \tag{3.8}$$

and therefore the series

$$H(t,s,\omega) := \sum_{n=1}^\infty K_n(t,s,\omega) \tag{3.9}$$

converges in $L^2(\mu)$ uniformly for $(t,s) \in [0,\infty) \times [0,\infty)$.

Proof. First note that, if we put

$$\gamma(t,s) = \gamma(t,s,\omega) = \sigma(t,s)W_{\phi_s} \tag{3.10}$$

then

$$K_2(t,s) = \int_s^t K(t,u) \diamond K(u,s)du$$
$$= \int_s^t (b(t,u) + \gamma(t,u)) \diamond (b(u,s) + \gamma(u,s))du$$

and

$$K_3(t,s) = \int_s^t K_2(t,v) \diamond K(v,s)dv$$
$$= \int\int_{s\le v\le u\le t} [(b(t,u) + \gamma(t,u)) \diamond (b(u,v) + \gamma(u,v)) \diamond (b(v,s) + \gamma(v,s))]dudv$$

So by induction

$$K_n(t,s) = \int \cdots \int_{s\le u_{n-1}\le\cdots\le u_1\le t} [\prod_{0\le k\le n-1}^{\diamond} (b(u_k,u_{k+1}) + \gamma(u_k,u_{k+1}))]du_1\cdots du_{n-1}, \tag{3.11}$$

where $u_0 = t$ and $u_n = s$ and $\prod^{\diamond}$ indicates that the Wick product is used.

Now

$$\prod_{0\le k\le n-1}^{\diamond} (b(u_k,u_{k+1}) + \gamma(u_k,u_{k+1})) = \sum_{\alpha,\beta} b_\alpha(u)\gamma_\beta(u) \tag{3.12}$$

the sum being taken over all partitions $\{\alpha,\beta\}$ of $\{0,1,\cdots,n-1\}$ (i.e. $\alpha \cap \beta = \emptyset$ and $\alpha \cup \beta = \{0,1,\cdots,n-1\}$) and we have used the notation

$$b_\alpha(u) = b(u_{\alpha_1},u_{\alpha_1+1})\cdots b(u_{\alpha_j},u_{\alpha_j+1}) \quad \text{if} \quad \alpha = \{\alpha_1,\cdots,\alpha_j\} \tag{3.13}$$

and

$$\gamma_\beta(u) = \gamma(u_{\beta_1},u_{\beta_1+1}) \diamond\cdots\diamond \gamma(u_{\beta_k},u_{\beta_{k+1}}) \quad \text{if} \quad \beta = \{\beta_1,\cdots,\beta_k\} \tag{3.14}$$

Since

$$E[|W_{\phi_{u_1}} \diamond\cdots\diamond W_{\phi_{u_k}}|^2]^{1/2} = \sqrt{k!}\|\phi\|^k \tag{3.15}$$

we obtain from (3.12) that (with $|\beta|$ = the cardinality of β)

$$\|K_n(t,s)\|_{L^2(\mu)} \le \int\cdots\int_{s\le u_{n-1}\le\cdots\le u_1\le t} \sum_{\alpha,\beta} |b_\alpha(u)| \cdot |\sigma_\beta(u)|\sqrt{|\beta|!}\|\phi\|^{|\beta|}du_1\cdots du_{n-1} \tag{3.16}$$

Choose $M < \infty$ such that

$$|b(t,s)| \le M \quad \text{and} \quad |\sigma(t,s)| \le M \quad \text{for all} \quad t, s.$$

Then from (3.16) we get (putting $|\beta| = k$)

$$\|K_n(t,s)\|_{L^2(\mu)} \le M^n \cdot \frac{1}{n!} \cdot \sum_{k=0}^{n} \binom{n}{k} \sqrt{k!} \|\phi\|^k$$
$$\le \frac{M^n}{\sqrt{n!}} (1 + \|\phi\|)^n.$$

LEMMA 3.5. Let K, K_n be as in Lemma 3.4. Then

$$K_n(t,s) = \int_s^t K_{n-j}(t,u) \diamond K_j(u,s) du \quad \text{for} \quad j = 1, 2, \cdots, n-1. \tag{3.17}$$

Proof. We proceed by induction. By definition (3.17) holds for $j = 1$ for all n. Suppose (3.17) holds for $n = n_0$ and all $j \le n_0 - 1$ and also for $n = n_0 + 1$ if $j = j_0 \le n_0 - 1$. Then

$$\int_s^t K_{n_0+1-(j_0+1)}(t,u) \diamond K_{j_0+1}(u,s) du$$
$$= \int_s^t K_{n_0-j_0}(t,u) \diamond (\int_s^u K_{j_0}(u,v) \diamond K(v,s) dv) du$$
$$= \int_s^t K(v,s) \diamond (\int_v^t K_{n_0-j_0}(t,u) \diamond K_{j_0}(u,v) du) dv$$
$$= \int_s^t K(v,s) \diamond K_{n_0}(t,v) dv = K_{n_0+1}(t,s).$$

□

LEMMA 3.6 Let $K(t,s)$ and $H(t,s)$ be as in Lemma 3.4. Then

$$H(t,s) - K(t,s) = \int_s^t K(t,u) \diamond H(u,s) du$$

Proof. By (3.9) and (3.7) we have

$$\int_s^t K(t,u) \diamond H(u,s) du = \lim_{N \to \infty} \int_s^t (\sum_{n=1}^{N} K(t,u) \diamond K_n(u,s)) du$$
$$= \lim_{N \to \infty} \sum_{n=1}^{N} K_{n+1}(t,s) = H(t,s) - K(t,s).$$

□

We are now ready for the first main result of this section:

THEOREM 3.7 Let $b(t,s)$ and $\sigma(t,s)$ be bounded deterministic functions satisfying

$$b(t,s) = \sigma(t,s) = 0 \quad \text{if} \quad 0 \leq t < s \tag{3.5}$$

Let $Y_t = Y(\phi, t, \omega)$ be an L^1 functional process (not necessarily adapted) such that

$$\sum_{m=1}^{\infty} \|K_m(s,u) \diamond Y_u \diamond K_n(t,s)\|_{L^2(\mu)} \leq C < \infty \quad \text{for all } t, s, u, n \tag{3.18}$$

(with C independent of t, s, u and n) where K_n is defined by (3.7). Define

$$X_t := X(\phi, t, \omega) := Y_t + \int_0^t H(t,s) \diamond Y_s ds. \tag{3.19}$$

Then

$$X_s \diamond K_n(t,s) \in L^1(ds \times d\mu) \quad \text{for all} \quad t, n \tag{3.20}$$

and X_t is the unique L^1 functional process which satisfies (3.20) and solves the ((1.11) interpretation of the) stochastic Volterra equation

$$X_t = Y_t + \int_0^t b(t,s) X_s ds + \int_0^t \sigma(t,s) X_s \delta B_s \tag{1.5}$$

Proof. We first verify that X_t given by (3.19) satisfies (3.20): From (3.18) we have

$$\left(\int_0^s H(s,u) \diamond Y_u du\right) \diamond K_n(t,s) = \left(\sum_{m=1}^{\infty} \int_0^s K_m(s,u) \diamond Y_u du\right) \diamond K_n(t,s)$$
$$= \sum_{m=1}^{\infty} \int_0^s K_m(s,u) \diamond Y_u \diamond K_n(t,s) du$$

which converges absolutely in $L^2(ds \times d\mu)$

Next we verify that if X_t is defined by (3.19) then X_t satisfies (1.11), i.e.

$$X_t = Y_t + \int_0^t X_s \diamond K(t,s) ds \tag{3.21}$$

Substituting for X and using Lemma 3.5 we get

$$\int_0^t X_s \diamond K(t,s)ds = \int_0^t Y_s \diamond K(t,s)ds + \int_0^t (\int_0^s H(s,u) \diamond Y_u du) \diamond K(t,s)ds$$

$$= \int_0^t Y_s \diamond K(t,s)ds + \int_0^t (Y_u \diamond \int_u^t K(t,s) \diamond H(s,u)ds)du$$

$$= \int_0^t Y_s \diamond K(t,s)ds + \int_0^t Y_u \diamond (H(t,u) - K(t,u))du$$

$$= \int_0^t Y_u \diamond H(t,u)du = X_t - Y_t,$$

which proves that X_t defined by (3.19) satisfies (1.5).

It remains to prove uniqueness:

Suppose $X_t^{(1)}, X_t^{(2)}$ are two L^1 functional processes such that for all t

$$X_s^{(i)} \diamond K_n(t,s) \in L^1(d\mu \times ds) \quad \text{for} \quad i = 1,2.$$

Then since both processes satisfying (3.21) we get by subtraction that

$$Z_s := X_s^{(1)} - X_s^{(2)}$$

satisfies the equation

$$Z_t = \int_0^t K(t,s) \diamond Z_s ds$$

This gives

$$Z_t = \int_0^t K(t,s) \diamond (\int_0^s K(s,u) \diamond Z_u du)ds$$

$$= \int_0^t K_2(t,u) \diamond Z_u du$$

Proceeding by induction we see that

$$Z_t = \int_0^t K_n(t,u) \diamond Z_u du \quad \text{for all} \quad n. \tag{3.22}$$

Applying the $\mathcal{F}$-transform on both sides we get

$$\mathcal{F}Z_t(\phi) = e^{\frac{1}{2}\|\phi\|^2} \int_0^t (\mathcal{F}K_n(t,u))(\phi)(\mathcal{F}Z_u)(\phi)du \quad \phi \in \mathcal{S}.$$

Since $K_n(t,u) \to 0$ in $L^2(\mu)$ as $n \to \infty$, uniformly in (t,u), we see that

$$(\mathcal{F}K_n(t,u))(\phi) \to 0 \quad \text{as} \quad n \to \infty, \quad \text{uniformly in} \quad (t,u)$$

and we conclude from (3.22) that $\mathcal{F}Z_t(\phi) = 0$ for all ϕ. Therefore $Z_t = 0$, which proves uniqueness.

The following is an important special case of Theorem 3.7:

THEOREM 3.8 Let $b(t,s), \sigma(t,s)$ be as in Theorem 3.7. Suppose $Y_t = Y(\phi, t, \omega)$ satisfies (3.18) and in addition that Y_t is independent of t, i.e.

$$Y_t = Y_0 \quad \text{for all} \quad t \tag{3.23}$$

Then the unique L^1 functional process X_t which satisfies (3.20) and solves the Volterra equation

$$X_t = Y_0 + \int_0^t b(t,s) X_s ds + \int_0^t \sigma(t,s) X_s \delta B_s \tag{3.24}$$

is given by

$$X_t := X(\phi, t, \omega) = Y_0 \diamond (1 + \int_0^t H(t,s) ds) \tag{3.25}$$

REMARK. Note that the unique solution $x_t(\omega)$ of the *non-anticipating* Volterra equation

$$x_t^{(a)} = a + \int_0^t b(t,s) x_s^{(a)} ds + \int_0^t \sigma(t,s) x_s^{(a)} dB_s \quad (a \text{ constant}) \tag{3.26}$$

is (by Theorem 3.8)

$$x_t^{(a)} = a(1 + \int_0^t H(t,s) ds). \tag{3.27}$$

The connection between the solution $x_t^{(a)}$ of the non-anticipating equation (3.26) and the solution X_t of the anticipating equation (3.24) is therefore

$$X_t = Y_0 \diamond x_t^{(1)} \tag{3.28}$$

In particular, note that if Y_0 is anticipating then X_t does *not* coincide with $x_t^{(a)}$ with $a = Y_0$!

Condition (3.18) may be difficult to use in specific cases. In order to get a more tractable condition we establish the following result of independent interest:

LEMMA 3.9 Let b, σ be as in Theorem 3.7 and let $x_t = x_t^{(1)}$ be the (non-anticipating) solution of (3.26). Let $\mathcal{H}x_t = \tilde{x}_t$ be the Hermite transform of x_t. Then (with λ as in §2)

$$\int\int |\tilde{x}_t(x+iy)|^p d\lambda(x) d\lambda(y) < \infty \quad \text{for all} \quad p < \infty$$

Proof. Taking $\mathcal{H}$-transforms of (3.26) we get

$$\tilde{x}_t(z) = 1 + \int_0^t b(t,s)\tilde{x}_s(z)ds + \int_0^t \sigma(t,s)\tilde{x}_s(z)\tilde{W}_{\phi_s}(z)ds, \tag{3.29}$$

with $\tilde{W}_{\phi_s}(z) = \sum_n (\phi_s(\cdot), e_n)z_n$; $z = (z_1, \cdots, z_2, \cdots) \in \mathbf{C}_0^{\mathbf{N}}$, where $(\psi, e_n) = \int_{\mathbf{R}} \psi e_n dx$ denotes the inner product in $L^2(\mathbf{R})$.

If we choose M such that

$$|b(t,s)| \le M \quad \text{and} \quad |\sigma(t,s)| \le M \quad \text{for all} \quad t,s$$

then

$$|\tilde{x}_t(z)| \le 1 + M\int_0^t |\tilde{x}_s|ds + M \cdot \int_0^t |\tilde{x}_s| \cdot |\tilde{W}_{\phi_s}|ds \tag{3.30}$$

By the Gronwall inequality (see e.g. [EK, Appendix 5])

$$\begin{aligned}
|\tilde{x}_t(z)| &\le \exp(Mt + M\int_0^t |\tilde{W}_{\phi_s}|ds) \\
&\le \exp(Mt + M\int_0^t (\sum_n |(\phi_s, e_n)||x_n| + \sum_n |(\phi_s, e_n)| \cdot |y_n|)ds) \\
&\le \exp(Mt) \cdot \exp(M\sum_n (\int_0^t |(\phi_s, e_n)|ds) \cdot |x_n| + \sum_n (\int_0^t |(\phi_s, e_n)|ds) \cdot |y_n|)
\end{aligned}$$

Therefore, if we put $a_n = Mp \cdot \int_0^t |(\phi_s, e_n)|ds$,

$$\begin{aligned}
&\int\int |\tilde{x}_t(z)|^p d\lambda(x)d\lambda(y) \le \exp(pMt) \cdot [\int \exp(\sum_n a_n|x_n|)d\lambda(x)]^2 \\
&= \exp(pMt)\prod_n [\int_{\mathbf{R}} \exp(a_n|x_n|)e^{-\frac{|x_n|^2}{2}} \frac{dx_n}{\sqrt{2\pi}}]^2
\end{aligned} \tag{3.31}$$

If $a > 0$ we have

$$\begin{aligned}
&\int_{\mathbf{R}} \exp(a|t| - \frac{1}{2}t^2)\frac{dt}{\sqrt{2\pi}} \\
&= \exp(\frac{1}{2}a^2)[\int_{-\infty}^0 \exp(-\frac{1}{2}(t+a)^2)\frac{dt}{\sqrt{2\pi}} + \int_0^\infty \exp(-\frac{1}{2}(t-a)^2)\frac{dt}{\sqrt{2\pi}}] \\
&= \exp(\frac{1}{2}a^2)[\int_{-\infty}^a + \int_{-a}^\infty \exp(-\frac{1}{2}y^2)\frac{dy}{\sqrt{2\pi}}] \\
&= \exp(\frac{1}{2}a^2)[1 + 2\int_0^a \exp(-\frac{1}{2}y^2)\frac{dy}{\sqrt{2\pi}}] \le \exp(\frac{1}{2}a^2)[1+2a].
\end{aligned} \tag{3.32}$$

Next consider (with the operator A is in (2.5))

$$\sum_{n=1}^{\infty} a_n = pM \cdot \sum_{n=1}^{\infty} \int_0^t |(\phi_s, e_n)| ds = pM \int_0^t (\sum_{n=1}^{\infty} |(\phi_s, e_n)|) ds$$

$$= pM \int_0^t \sum_{n=1}^{\infty} |(A^{-2} e_n, A^2 \phi_s)| ds$$

$$(3.33) \qquad \leq pM \int_0^t \sum_{n=1}^{\infty} \|A^{-2} e_n\|_{L^2} \|A^2 \phi_s\|_{L^2} ds$$

$$\leq pM \int_0^t \sum_{n=1}^{\infty} (2n)^{-2} \|A^2 \phi_s\|_{L^2} ds$$

$$= pMt \|A^2 \phi\|_{L^2} \cdot \sum_{n=1}^{\infty} (2n)^{-2} < \infty,$$

since $\phi \in \mathcal{S}$ and $A^2 \phi_s(x) = (A^2 \phi)_s(x)$.

Combining (3.31) with (3.32) and (3.33) we get

$$\int\int |\tilde{x}_t(z)|^p d\lambda(x) d\lambda(y) \leq \exp(pMt) \prod_{n=1}^{\infty} \exp(a_n^2)[1 + 2a_n]^2$$

$$= \exp(pMt) \cdot \exp(\sum_{n=1}^{\infty} a_n^2 + 2\ln(1 + 2a_n))$$

$$\leq \exp(pMt) \cdot \exp(\sum_{n=1}^{\infty} a_n^2 + 4a_n) < \infty.$$

If we apply Lemma 3.9 in Theorem 3.8 we get the following:

THEOREM 3.10. Suppose $b(t,s), \sigma(t,s)$ are as in Theorem 3.7 and suppose $Y_0 = Y_0(\phi, \omega)$ satisfies

$$(3.34) \qquad \tilde{Y}_0 \in L^{1+\epsilon}(\lambda \times \lambda) \quad \text{for some} \quad \epsilon = \epsilon(\phi) > 0, \forall \phi \in \mathcal{S}.$$

Then

$$(3.25) \qquad X_t = Y_0 \diamond x_t^{(1)} = Y_0 \diamond (1 + \int_0^t H(t,s) ds)$$

is the unique L^1 functional process which satisfies (3.20) and solves the Volterra equation

$$(3.24) \qquad X_t = Y_0 + \int_0^t b(t,s) X_s ds + \int_0^t \sigma(t,s) X_s \delta B_s$$

§4. The generalized white noise functional approach

In this section, we consider the following equation (4.1) in the Hida distribution setting,

$$X_t = Y_t + \int_0^t b(t,s)X_s ds + \int_0^t \sigma(t,s)X_s \diamond W_s ds \tag{4.1}$$

where Y_t is regarded as a process in $(\mathcal{S})^*$.

Throughout this section, we assume that $b(t,s)$ and $\sigma(t,s)$ are bounded deterministic functions satisfying

$$b(t,s) = \sigma(t,s) = 0 \quad \text{if} \quad 0 \leq t < s. \tag{4.2}$$

Now let us state our first result:

THEOREM 4.1. Assume there exist constants $c_1, c_2 > 0, p \in \mathbf{N}_0$, independent of t, such that the estimate (2.31) holds for $F = \mathcal{S}Y_t$ for all t. Then the equation (4.1) has a unique solution X_t in $(\mathcal{S})^*$, which is given by

$$X_t = Y_t + \int_0^t H(t,s) \diamond Y_s ds \tag{4.3}$$

where

$$H(t,s) = \sum_{\nu=1}^{\infty} K_\nu(t,s) \tag{4.4}$$

$$\begin{aligned} K_1(t,s) &= b(t,s) + \sigma(t,s)W_s \\ K_{\nu+1}(t,s) &= \int_0^t K_1(t,u) \diamond K_\nu(u,s) du; \ \nu = 1, 2, \cdots \end{aligned} \tag{4.5}$$

The series (4.4) converges strongly in $(\mathcal{S})^*$.

Proof. It is enough to construct the solution on any fixed interval $[0,T]$. We divide the proof into several steps.

LEMMA 4.2. $K_\nu(t,s)$ is a well-defined generalized functional for all ν, s, t.

Proof. By proposition 2.6 in [PS] and Theorem 3.1 in [Po], it suffices to prove that $\mathcal{S}K_\nu(t,s)$ can be bounded in the sense of (2.31) uniformly in t, s. We see this by induction:

Since $b(t,s)$ and $\sigma(t,s)$ are bounded, it is clear that there exist constants $c_1 > 0, p_1 \in \mathbf{N}_0$, independent of t, s, such that for $\phi \in \mathcal{S}(\mathbf{R})$

$$|\mathcal{S}K_1(t,s)(z\phi)| \leq c_1 \exp(c_1|z|\|\phi\|_{2,p_1}) \tag{4.6}$$

Suppose there are constants $c_\nu, p_\nu \in \mathbf{N}_0$ such that

$$|\mathcal{S}K_\nu(t,s)(z\phi)| \le c_\nu \exp(c_\nu |z| \|\phi\|_{2,p_\nu}) \tag{4.7}$$

for $\phi \in \mathcal{S}, t, s \le T$.

By (4.5) we have

$$\begin{aligned}|\mathcal{S}K_{\nu+1}(t,s)(z\phi)| &\le \int_0^t |\mathcal{S}K_1(t,u)(z\phi)| \cdot |\mathcal{S}K_\nu(u,s)(z\phi)| du \\ &\le \int_0^t c_1 \exp(c_1|z|\|\phi\|_{2,p_1}) \cdot c_\nu \exp(c_\nu |z| \|\phi\|_{2,p_\nu}) du \\ &\le c_{\nu+1} \exp(c_{\nu+1}|z|\|\phi\|_{2,p_{\nu+1}}) \quad \text{for} \quad \phi \in \mathcal{S}(\mathbf{R}), t, s \le T.\end{aligned}$$

where $c_{\nu+1} = \max(Tc_1c_\nu, c_1 + c_\nu), p_{\nu+1} = p_1 \vee p_\nu$. This completes the proof of Lemma 4.2.

LEMMA 4.3. The series $H(t,s) = \sum_{\nu=1}^{\infty} K_\nu(t,s)$ converges strongly in $(\mathcal{S})^*$.

We first show that $\hat{H}(t,s)(\phi) := \sum_{\nu=1}^{\infty} \mathcal{S}K_\nu(t,s)(\phi)$ is a U-functional. By the definition, we need to verify the following two claims :

Claim 1: $\hat{H}(t,s)(\psi + z\phi)$ is an entire function of $z \in \mathbf{C}$ for any $\psi, \phi \in \mathcal{S}(\mathbf{R})$.

Proof of Claim 1. Since $\mathcal{S}K_\nu(t,s)(\psi + z\phi)$ is analytic, it is sufficient to show that $\sum_{\nu=1}^{\infty} \mathcal{S}K_\nu(t,s)(\psi + z\phi)$ converges uniformly on compact sets in $\mathbf{C}$.

In fact, for any $M > 0$, put $\hat{K}(t,s,z) = b(t,s) + \sigma(t,s)(\psi(s) + z\phi(s))$ and

$$\begin{aligned}&\lambda(t,s) = \sup_{|z|\le M} |\hat{K}(t,s,z)| \\ &A^2(t) = \int_0^T \lambda^2(t,u) du, B^2(s) = \int_0^T \lambda^2(u,s) du.\end{aligned} \tag{4.8}$$

Then

$$\begin{aligned}&\sup_{|z|\le M} |\mathcal{S}K_2(t,s)(\psi + z\phi)|^2 \\ &\le \sup_{|z|\le M} |\int_s^t \hat{K}(t,u,z)\hat{K}(u,s,z) du|^2 \\ &\le \sup_{|z|\le M} (\int_s^t |\hat{K}(t,u,z)|^2 du)(\int_s^t |\hat{K}(u,s,z)|^2 du) \\ &\le \int_0^T \lambda^2(t,u) du \int_0^T \lambda^2(u,s) du \\ &= A^2(t) B^2(s)\end{aligned}$$

Similarly

$$(4.9)\qquad \begin{aligned}&\sup_{|z|\le M}|\mathcal{S}K_3(t,s)(\psi+z\phi)|^2\\ &\le \sup_{|z|\le M}|\int_s^t \hat{K}(t,u,z)\mathcal{S}K_2(u,s)(\psi+z\phi)du|^2\\ &\le A^2(t)B^2(t)\int_s^t A^2(u)du\end{aligned}$$

Inductively we have

$$(4.10)\qquad \begin{aligned}&\sup_{|z|\le M}|\mathcal{S}K_{\nu+2}(t,s)(\psi+z\phi)|^2\\ &\le A^2(t)B^2(s)\frac{1}{\nu!}(\int_0^T A^2(u)du)^\nu\end{aligned}$$

This implies that $\sum_{\nu=1}^{\infty}\mathcal{S}K_\nu(t,s)(\psi+z\phi)$ converges uniformly on $|z|\le M$. Hence the claim follows.

Claim 2: There exist $c_1, c_2 > 0$ and $p\in\mathbf{N}_0$ such that

$$(4.11)\qquad |\hat{H}(t,s)(z\phi)|\le c_1\exp(c_2|z|^2\|\phi\|_{2,p}^2)$$

for $\phi\in\mathcal{S}(\mathbf{R}), t,s\le T$.

Proof of Claim 2. Set

$$\begin{aligned}\lambda(t,s,z)&=|b(t,s)+\sigma(t,s)z\phi(s)|\\ A^2(t,z)&=\int_0^T\lambda^2(t,u,z)du\\ B^2(s,z)&=\int_0^T\lambda^2(u,s,z)du\end{aligned}$$

As the proof of (4.10), we get that

$$(4.12)\qquad |\mathcal{S}K_{\nu+2}(t,s)(z\phi)|\le A(t,z)B(s,z)\frac{(\int_0^T A^2(u,z)du)^{\nu/2}}{\sqrt{\nu!}}$$

Thus

$$(4.13)\qquad \begin{aligned}|\hat{H}(t,s)(z\phi)|&\le\lambda(t,s,z)+\sum_{\nu=0}^{\infty}|\mathcal{S}K_{\nu+2}(t,s)(z\phi)|\\ &\le\lambda(t,s,z)+A(t,z)B(s,z)\sum_{\nu=0}^{\infty}\frac{1}{\sqrt{\nu!}}(\int_0^T A^2(u,z)du)^{\nu/2}\\ &\le\lambda(t,s,z)+A(t,z)B(s,z)\sqrt{2}\exp(2\int_0^T A^2(u,z)du)\end{aligned}$$

On the other hand, it is easy to see that there exist constants $c_3, c_4 > 0$ and $p_1 \in \mathbf{N}_0$ such that

$$\int_0^T A^2(u,z)du \le c_3 + c_4 \int_0^T \phi(u)^2 du$$

$$(4.14) \qquad |A(t,z)| \le c_3 + c_4|z|\Big(\int_0^T \phi(u)^2 du\Big)^{\frac{1}{2}}$$

$$|B(s,z)| \le c_3 + c_4|z|\|\phi\|_{2,p_1}$$

$$|\lambda(t,s,z)| \le c_3 + c_4|z|\|\phi\|_{2,p_1}$$

for all $\phi \in \mathcal{S}(\mathbf{R}), t, s \le T$.

Combining (4.14) with (4.13), we have (4.11).

Now let us complete the proof of Lemma 4.3. Let $H(t,s) \in (\mathcal{S})^*$ with $\mathcal{S}(H(t,s))(\phi) = \hat{H}(t,s)(\phi)$. Put $\Phi_n = \sum_{\nu=1}^{n} K_\nu(t,s)$. Then we see that

$$\mathcal{S}\Phi_n(\phi) \to \mathcal{S}(H(t,s))(\phi) \quad \text{for} \quad \phi \in \mathcal{S}(\mathbf{R})$$

and $|\Phi_n(z\phi)|$ is uniformly controlled by (4.11). Applying Theorem 2.2, we obtain Lemma 4.3.

LEMMA 4.4. X_t (given by (4.3)) is well-defined and satisfies (4.1).

Proof. By the assumption, there is some $p_1 \in \mathbf{N}_0$ so that

$$(4.15) \qquad |\mathcal{S}Y_t(z\phi)| \le c\exp(c|z|^2\|\phi\|^2_{2,p_1})$$

for $\phi \in \mathcal{S}(\mathbf{R})$, all $t > 0$.

Combining with (4.11), we obtain that for some $p \in \mathbf{N}_0, c_1 > 0$

$$(4.16) \qquad |\mathcal{S}Y_s(z\phi)\mathcal{S}H(t,s)(z\phi)| \le c_1\exp(c_1|z|\|\phi\|^2_{2,p})$$

Here the constant c_1 is independent of s, t, ϕ, z. Again from proposition 2.6 in [PS] and Theorem 3.1 in [Po] we conclude that $X_t = Y_t + \int_0^t H(t,s) \diamond Y_s$ is well-defined, with $\mathcal{S}$-transform:

$$\mathcal{S}X_t(\phi) = \mathcal{S}Y_t(\phi) + \int_0^t \hat{H}(t,s)(\phi)\mathcal{S}Y_s(\phi)ds.$$

Likewise,

$$Z_t := Y_t + \int_0^t b(t,s)X_s ds + \int_0^t \sigma(t,s)X_s \diamond W_s ds$$

is also a well defined process in $(\mathcal{S})^*$. From our construction and Theorem 1.5 in [T], we know that $\mathcal{S}X_t = \mathcal{S}Z_t$. This means that X_t satisfies (4.1). The uniqueness follows from the uniqueness of the solution of the following deterministic equation:

$$\hat{Z}_t(\phi) = \mathcal{S}Y_t(\phi) + \int_0^t b(t,s)\hat{Z}_s(\phi)ds + \int_0^t \sigma(t,s)\hat{Z}_s(\phi)\phi(s)ds.$$

This completes the proof of Theorem 4.1.

REMARKS.
(1): In particular, if $Y_t = Y_0 \in (\mathcal{S})^*$ in Theorem 4.1, then $X_t = Y_0 \diamond x_t^{(1)}$ where as before $x_t^{(1)}$ is the solution of the equation

$$x_t^{(1)} = 1 + \int_0^t b(t,s)x_s^{(1)}ds + \int_0^t \sigma(t,s)x_s^{(1)}dB_s$$

This formula is a natural extension of the non-anticipating case. See Theorem 3.8 and the Remark there.

(2): There is a close connection between the Hida distribution solution $X_t \in (\mathcal{S})^*$ of equation (4.1) and the functional process solution X_t^ϕ of equation (1.11) found in §3. To see this put

$$\phi(x) = \begin{cases} c\exp(\frac{1}{|x|^2-1}) & |x| \le 1 \\ 0 & |x| > 1 \end{cases},$$

with c chosen such that $\int_{\mathbf{R}} \phi(x)dx = 1$.

Set $\phi^{(\epsilon)}(x) = \frac{1}{\epsilon}\phi(\frac{x}{\epsilon})$ for $\epsilon > 0$. Then for any $\psi \in \mathcal{S}(\mathbf{R}), \phi^{(\epsilon)} * \psi \to \psi$. Let $X_t^{(\epsilon)}$ be the solution found in Section 3 of equation (1.11) with $\phi := \phi^{(\epsilon)}$. If $Y_t^{(\epsilon)} := Y_t^{\phi^{(\epsilon)}}$ is good enough then $\mathcal{S}X_t^{(\epsilon)}(\psi)$ satisfies

$$\begin{aligned} \mathcal{S}X_t^{(\epsilon)}(\psi) = \mathcal{S}Y_t^{(\epsilon)}(\psi) &+ \int_0^t b(t,s)\mathcal{S}X_s^{(\epsilon)}(\psi)ds \\ &+ \int_0^t \sigma(t,s)\mathcal{S}X_s^{(\epsilon)}(\psi)\phi^{(\epsilon)} * \psi(s)ds \quad \text{for all } \epsilon > 0. \end{aligned} \tag{4.17}$$

Thus if $Y_t^{(\epsilon)} \to Y_t$ in $(\mathcal{S})^*$, then by Theorem 2.2 one can see that $X_t^{(\epsilon)} \to X_t$ in $(\mathcal{S})^*$, where X_t is the solution of equation (4.1).

In the rest of this section, we treat a special case

$$\begin{aligned} &\sigma(t,s) = f(t)\sigma(s), \quad b(t,s) = f(t)b(s) \\ &Y_t \equiv Y_0 = \sum_{n=0}^{\infty} \int_{\mathbf{R}^n} F_n(t_1 \cdots t_n)dB_t^{\otimes n} \in L^2(\mu) \end{aligned} \tag{4.18}$$

We will prove that in this situation the solution of equation (4.1) is actually in $L^2(\mu)$:

THEOREM 4.5

(I) Assume $\delta \le f \le \frac{1}{\delta}$ (δ is a positive constant), $f' \in L^2_{\text{loc}}(\mathbf{R}_+)$, $\|A\sigma f\|_{L^2(\mathbf{R})} < +\infty$. Let $Y_t \equiv Y_0$ as in (4.18) and assume that $\sum_{n=0}^{\infty} n!\|A^{\otimes n}F_n\|^2_{L^2(\mathbf{R}^n)} < +\infty$. Then the solution X_t of (4.1) is in $L^2(\mu)$ and

$$X_t = -\int_0^t \frac{f(t)f'(s)}{f^2(s)}(Y_0 \diamond \exp(X_t^{(0)} - X_s^{(0)} - \frac{1}{2}(< X^{(0)} >_t - < X^{(0)} >_s))ds + \frac{f(t)}{f(0)}(Y_0 \diamond \exp(X_t^{(0)} - \frac{1}{2} < X^{(0)} >_t)) \tag{4.19}$$

where $X_t^{(0)} = \int_0^t \sigma(s)f(s)dB_s + \int_0^t b(s)f(s)ds$.

(II) Assume $f' \in L^4_{\text{loc}}(\mathbf{R}_+), \delta \le f \le \frac{1}{\delta}$. Let $Y_t \equiv Y_0 \in L^2(\mu)$ and assume that

$$\int\int |\mathcal{H}Y_0(\xi + i\eta)|^4 d\lambda(\xi)d\lambda(\eta) < +\infty \tag{4.20}$$

where $d\lambda$ is as in §2.

Then $X_t \in L^2(\mu)$ and (4.19) holds.

Proof. (I): In this case the solution $X_t \in (\mathcal{S})^*$ found in Theorem 4.1 satisfies

$$\mathcal{S}X_t(\phi) = \mathcal{S}Y_0(\phi) + \int_0^t f(t)b(s)\mathcal{S}X_s(\phi)ds + \int_0^t f(t)\sigma(s)\phi(s)\mathcal{S}X_s(\phi)ds \tag{4.21}$$

Set $\overline{g}_t(\phi) = \frac{\mathcal{S}X_t(\phi)}{f(t)}$, then

$$\overline{g}_t(\phi) = \frac{\mathcal{S}Y_0(\phi)}{f(t)} + \int_0^t b(s)f(s)\overline{g}_s(\phi)ds + \int_0^t \sigma(s)f(s)\overline{g}_s(\phi)\phi(s)ds \tag{4.22}$$

Therefore we have

$$\mathcal{S}X_t(\phi) = -\int_0^t \frac{f(t)f'(s)}{f^2(s)}\mathcal{S}Y_0(\phi)\exp(\int_s^t \sigma(u)f(u)\phi(u)du + \int_s^t b(u)f(u)du) + \frac{f(t)}{f(0)}\mathcal{S}Y_0(\phi)\exp(\int_0^t \sigma(s)f(s)\phi(s)ds + \int_0^t b(s)f(s)ds) \tag{4.23}$$

This gives the formula (4.19).

We denote by $X_t^{(1)}$ and $X_t^{(2)}$ the two parts of the R.H.S in (4.19). Then

$$\mathcal{S}X_t^{(1)}(\phi) = -\int_0^t \frac{f(t)f'(s)}{f^2(s)}\mathcal{S}Y_0(\phi)\exp(\int_s^t \sigma(u)f(u)\phi(u)du + \int_s^t b(u)f(u)du)ds \tag{4.24}$$

Now suppose $X_t^{(1)} \in (\mathcal{S})^*$ has the formal expansion

$$X_t^{(1)} = \sum_{n=0}^{\infty} \int_{\mathbf{R}^n} G_{n,t}(s_1, \cdots, s_n) dB_s^{\otimes n} \quad \text{(See [PS])}.$$

Since $\mathcal{S}X_t^{(1)}(\phi) = \sum_{n=0}^{\infty} < G_{n,t}, \phi^{\otimes n} >$ (see [PS]), (4.24) indicates that

$$G_{n,t} = -\int_0^t \frac{f(t)f'(s)}{f^2(s)} \exp(\int_s^t b(u)f(u)du) \sum_{m=0}^{n} F_{n-m} \hat{\otimes} \frac{(\overline{\sigma}_{s,t})^{\otimes m}}{m!} ds$$

where $\overline{\sigma}_{s,t}(u) = \sigma(u)f(u)1_{[s,t]}(u)$. Therefore,

$$\begin{aligned}
\|G_{n,t}\|^2_{L^2(\mathbf{R}^n)} &\leq (\int_0^t \frac{f^2(t)f'(s)^2}{f^4(s)} \exp(2\int_s^t b(u)f(u)du)ds) \int_0^t \|\sum_{m=0}^{n} F_{n-m}\hat{\otimes}\frac{(\overline{\sigma}_{s,t})^{\otimes m}}{m!}\|^2_{L^2(\mathbf{R}^n)} ds \\
&\leq C_t \int_0^t (n+1)\sum_{m=0}^{n} \|F_{n-m}\|^2_{L^2(\mathbf{R}^{n-m})} \frac{(\|\overline{\sigma}_{s,t}\|^2_{L^2(\mathbf{R})})^m}{(m!)^2} ds \\
&\leq C_t(n+1)t\sum_{m=0}^{n} \|F_{n-m}\|^2_{L^2(\mathbf{R}^{n-m})} \frac{(\|\overline{\sigma}\|^2_{L^2(\mathbf{R})})^m}{(m!)^2}
\end{aligned}$$

where $\overline{\sigma}(u) = \sigma(u)f(u)$ and C_t is a constant.

Thus

$$\begin{aligned}
&\sum_{n=0}^{\infty} n!\|G_{n,t}\|^2_{L^2(\mathbf{R}^n)} \leq tC_t\sum_{n=0}^{\infty}(n+1)\sum_{m=0}^{n}\binom{n}{m}(n-m)!\|F_{n-m}\|^2_{L^2(\mathbf{R}^{n-m})}\frac{(\|\overline{\sigma}\|^2_{L^2(\mathbf{R})})^m}{m!} \\
&\leq tC_t\sum_{n=0}^{\infty}\sum_{m=0}^{n}(n+1)\binom{n}{m}2^{-2n}(n-m)!\|A^{\otimes n-m}F_{n-m}\|^2_{L^2}\frac{(\|A\overline{\sigma}\|^2_{L^2})^m}{m!} \\
&\leq tC_t\sum_{n=0}^{\infty}\sum_{m=0}^{n}(n-m)!\|A^{\otimes n-m}F_{n-m}\|^2_{L^2}\frac{(\|A\overline{\sigma}\|^2_{L^2})^m}{m!} \\
&\leq tC_t(\sum_{n=0}^{\infty} n!\|A^{\otimes n}F_n\|^2_{L^2})\exp(\|A\overline{\sigma}\|^2_{L^2}) < +\infty.
\end{aligned}$$

This shows that $X_t^{(1)} \in L^2(\mu)$. Similarly, we have $X_t^{(2)} \in L^2(\mu)$.

(II): By the estimate (2.30), it is sufficient to prove

$$\int\int |\mathcal{S}X_t((\xi_1 + i\eta_1)e_1 + \cdots + (\xi_n + i\eta_n)e_n + \cdots)|^2 d\lambda(\xi)d\lambda(\eta) < +\infty \tag{4.25}$$

In fact, from (4.23) we know that

$$\lim_{N\to\infty}\int\int|SX_t((\xi_1+i\eta_1)e_1+\cdots+(\xi_N+i\eta_N)e_N)|^2 d\lambda(\xi)d\lambda(\eta)$$
$$\leq C_0(\int\int|\mathcal{H}Y_0(\xi+i\eta)|^4 d\lambda(\xi)d\lambda(\eta))\times$$
$$\lim_{N\to\infty}\int\int|\exp(\int_s^t\sigma(u)f(u)((\xi_1+i\eta_1)e_1+\cdots+(\xi_N+i\eta_N)e_N)du)|^4 d\lambda(\xi)d\lambda(\eta)$$
$$\leq C_1(\lim_{N\to\infty}\int\exp(4\sum_{i=1}^n\xi_i\int_0^t\sigma(u)f(u)e_i(u)du)d\lambda(\xi))$$
$$\leq C_1\exp(8\int_0^t\sigma^2(u)f^2(u)du)<+\infty.$$

where C_0 and C_1 are appropriate constants. This ends the proof of (II) and the proof of Theorem 4.5 is complete.

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