**Homogeneous spaces**
**B. Komrakov seminar**

# THREE-DIMENSIONAL ISOTROPICALLY–FAITHFUL HOMOGENEOUS SPACES

VOLUME III


B. Komrakov A. Tchourioumov

E. Belin
Z. Belyankina
O. Chizh
B. Doubrov
I. Goncharov
T. Gur'yanova
S. Kimbar
B. Komrakov, Jnr.
L. Levin
N. Mozhei
A. Palvinsky
D. Shabes



**International Sophus Lie Centre**

**ISLC** International Sophus Lie Centre
*Address:*
Belarus, 220123, Minsk, P.O. Box 70
*Current address:*
Centre for Advanced Study
at The Norwegian Academy of Science and Letters
P.O. Box 7606 Skillebekk
0205 Oslo
Norway
*e-mail:* islc@shs.no

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# CONTENTS

## Foreword

We consider classification of lower-dimensional homogeneous spaces an immediate continuation and global version of classification results obtained by Sophus Lie. Two-dimensional homogeneous spaces were classified locally by Sophus Lie [L1] and globally by G.D. Mostow [M]. (See also our preprint [KTD], where the complete classification of two-dimensional homogeneous spaces, both locally and globally, is presented.) S. Lie also obtained some results in classification of three-dimensional homogeneous spaces and described all subalgebras in the Lie algebra $\mathfrak{gl}(3, \mathbb{C})$. A detailed account of these classifications can be found in [L2].

The problem of finding the complete description of three- and four-dimensional homogeneous spaces as pairs, (group, subgroup) or even (algebra, subalgebra), is extremely important and rich in applications, but it is a very difficult one: "The description of arbitrary transitive actions on manifolds $M$, where $\dim M \geqslant 3$, presently seems to be unattainable."([GO], p. 232)

Minimal transitive actions, that is, those that have no proper transitive subgroups, on three-dimensional manifolds were classified in [G]. The problem of local classification of three- and four-dimensional homogeneous spaces was chosen by one of the authors, B. Komrakov, as the topic of Dr. Sci. thesis for A. Tchourioumov, the other author. (Some of the results can be found in [Tch].)

An important subclass in all homogeneous spaces is formed by isotropically-faithful spaces. In particular, it contains all homogeneous spaces that admit an invariant affine connection. The present preprint gives the local classification of three-dimensional isotropically-faithful homogeneous spaces.

In 1990, the International Sophus Lie Centre, jointly with the University of Belarus, organized an experimental group of 25 students majoring in mathematics and working in accordance with a special syllabus oriented to modern differential-geometric methods in the study of nonlinear differential equations. The following idea arose: to split up the classification problem mentioned above into smaller parts and give each part to a student; in the process of learning new material, the student will then try to apply his newly acquired knowledge to this problem as an illustration.

Suppose, for example, that the student is learning about differential equations; he then writes out trajectories of one-parameter subgroups on the specific manifold that he has been given. Studying differential geometry, he computes invariant affine connections, metrics, curvature tensors, geodesics, etc., with special emphasis on his example, and so on.

In their first year, the students all took an advanced course in Lie algebras and the main part of the work on all these "smaller parts" was completed by 12 students. We had no time to give our students an introductory course in cohomologies of Lie algebras, and although their computation constitutes a considerable part of the work, we do not use this language.

This work was started in Tartu University, Estonia (August 1991), continued at the Institute of Astrophysics and Atmosphere Physics in Tõravere, Estonia (December 1991 to March 1992), then at the "Bears' Lakes" Space Center of the Special Research Bureau of Moscow Power Engineering Institute (August 1993), and finished at the University of Oslo and the Center for Advanced Study (SHS) at the

Norwegian Academy of Science and Letters. (Naturally, most of the time from August 1991 to November 1993 was spent in Minsk, Belarus.) The story of this work was rich in experiences and events only indirectly connected with mathematics, something we will not here dwell on at length. We would, however, like to express our gratitude to those who directly or indirectly made it possible for us to complete this work.

In the future, we are going to proceed with the study of geometry of three-dimensional homogeneous spaces in the following directions:

- description of invariant affine connections on three-dimensional homogeneous spaces together with their curvature and torsion tensors, holonomy groups, geodesics, etc.;
- description of invariant tensor geometric structures and their properties;
- global classification of three-dimensional isotropically-faithful homogeneous spaces and description of inclusions among the corresponding transformation groups;
- description of differential invariants for the homogeneous spaces to be found and of the corresponding invariant differential equations;
- description of discrete subgroups in transformation groups together with description of the corresponding topological factor spaces.

## Introduction

It is known that the problem of classification of homogeneous spaces $(\overline{G}, M)$ is equivalent to the classification (up to equivalence) of pairs of Lie groups $(\overline{G}, G)$ such that $G \subset \overline{G}$. Two pairs $(\overline{G}_1, G_1)$ and $(\overline{G}_2, G_2)$ are said to be equivalent if there exists an isomorphism of Lie groups $\pi \colon \overline{G}_1 \to \overline{G}_2$ such that $\pi(G_1) = G_2$.

By linearization, the problem can be reduced to the problem of classification of pairs of Lie algebras $(\bar{\mathfrak{g}}, \mathfrak{g})$ viewed up to equivalence of pairs. The structure of all pairs of Lie groups $(\overline{G}, G)$ corresponding to a given pair of Lie algebras $(\bar{\mathfrak{g}}, \mathfrak{g})$ was described in [M]. In the study of homogeneous spaces it is important to consider not the group $\overline{G}$ itself, but its image in $\mathrm{Diff}(M)$. In other words, it is sufficient to consider only the effective action of the group $\overline{G}$ on the manifold M. In terms of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$, this condition is equivalent to the condition for $\mathfrak{g}$ to contain no proper ideals of $\bar{\mathfrak{g}}$. In this case we say that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is *effective.*

In the present work we classify all isotropically–faithful pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ of codimension 3.

**Definition.** A pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is said to be *isotropically-faithful* if the natural $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$ is faithful.

We say that a homogeneous space $(\overline{G}, M)$ is isotropically-faithful if so is the corresponding pair $(\bar{\mathfrak{g}}, \mathfrak{g})$. From geometrical point of view it means that the natural action of the stabilizer $\overline{G}_x$ of an arbitrary point $x \in M$ on $T_x M$ has discrete kernel.

We divide the solution of our problem into the following parts:

(1) We classify (up to isomorphism) all faithful three–dimensional $\mathfrak{g}$-modules $U$. This is equivalent to classifying all subalgebras of $\mathfrak{gl}(3, \mathbb{R})$ viewed up to conjugation.

(2) For each $\mathfrak{g}$-module $U$ obtained in (1) we classify (up to equivalence) all pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ such that the $\mathfrak{g}$-modules $\bar{\mathfrak{g}}/\mathfrak{g}$ and $U$ are isomorphic.

In Chapter I we give basic definitions and introduce the notation to be employed. Here we also solve part (1) of the problem by classifying subalgebras in $\mathfrak{gl}(3, \mathbb{R})$.

In Chapter II we develop methods for constructing pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ given a three-dimensional faithful $\mathfrak{g}$-module $U$. This involves computation of the first cohomology space of $\mathfrak{g}$ with values in the natural module $\mathcal{L}(U, \mathfrak{g})$. A series of techniques described in Chapter II allows, in some cases, to simplify the computation considerably.

Finally, Chapter III gives the classification of three-dimensional isotropically-faithful pairs itself.

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## 4. Four-dimensional case

**Proposition 4.1.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.1 is trivial.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $0$ | $0$ | $u_1$ | $u_2$ | $0$ |
| $e_2$ | $0$ | $0$ | $2e_3$ | $-2e_4$ | $u_1$ | $-u_2$ | $0$ |
| $e_3$ | $0$ | $-2e_3$ | $0$ | $e_2$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $0$ | $2e_4$ | $-e_2$ | $0$ | $u_2$ | $0$ | $0$ |
| $u_1$ | $-u_1$ | $-u_1$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ |
| $u_2$ | $-u_2$ | $u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by $e_1$ and $e_2$.

**Lemma.** *Any virtual structure $q$ on generalized module 4.1 is equivalent to one of the following:*

$$C(e_1) = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = C(e_3) = C(e_4) = 0.$$

*Proof.* Let $q$ be a virtual structure on generalized module 4.1. Note that $\mathfrak{a} = \mathbb{R}e_2 \oplus \mathbb{R}e_3 \oplus \mathbb{R}e_4$ is a semisimple subalgebra of the Lie algebra $\mathfrak{g}$. Without loss of generality it can be assumed that $q(\mathfrak{a}) = \{0\}$. Therefore

$$C(e_2) = C(e_3) = C(e_4) = 0, \ C(e_1) = (c^1_{ij})_{\substack{1 \leqslant i \leqslant 4 \\ 1 \leqslant j \leqslant 3}}.$$

Checking condition (6), Chapter II, we obtain:

$$C(e_1) = \begin{pmatrix} 0 & 0 & c_{13}^1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C(e_2) = C(e_3) = C(e_4) = 0.$$

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 4.1. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by a virtual structure determined in the Lemma.

Then

$$\begin{array}{llll} [e_1, e_2] = 0, & & & \\ [e_1, e_3] = 0, & [e_2, e_3] = 2e_3, & & \\ [e_1, e_4] = 0, & [e_2, e_4] = -2e_4, & [e_3, e_4] = e_2, & \\ [e_1, u_1] = u_1, & [e_2, u_1] = u_1, & [e_3, u_1] = 0, & [e_4, u_1] = u_2, \\ [e_1, u_2] = u_2, & [e_2, u_2] = -u_2, & [e_3, u_2] = u_1, & [e_4, u_2] = 0, \\ [e_1, u_3] = pe_1, & [e_2, u_3] = 0, & [e_3, u_3] = 0, & [e_4, u_3] = 0. \end{array}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 10, Chapter II). Thus

$$\begin{array}{ll} \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) \supset \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}u_3, & \bar{\mathfrak{g}}^{(0,2)}(\mathfrak{h}) \supset \mathbb{R}e_3, \\ \bar{\mathfrak{g}}^{(0,-2)}(\mathfrak{h}) \supset \mathbb{R}e_4, & \bar{\mathfrak{g}}^{(1,1)}(\mathfrak{h}) \supset \mathbb{R}u_1, \\ \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) \supset \mathbb{R}u_2, & \end{array}$$

$$\begin{aligned} [u_1, u_2] &\in \bar{\mathfrak{g}}^{(2,0)}(\mathfrak{h}), \\ [u_1, u_3] &\in \bar{\mathfrak{g}}^{(1,1)}(\mathfrak{h}), \\ [u_2, u_3] &\in \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}), \end{aligned}$$

and

$$\begin{aligned} [u_1, u_2] &= 0, \\ [u_1, u_3] &= \beta_1 u_1, \\ [u_2, u_3] &= \gamma_2 u_2. \end{aligned}$$

Using the Jacobi identity we see that $p = 0$, $\beta_1 = \gamma_2$ and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | 0 | 0 | $u_1$ | $u_2$ | 0 |
| $e_2$ | 0 | 0 | $2e_3$ | $-2e_4$ | $u_1$ | $-u_2$ | 0 |
| $e_3$ | 0 | $-2e_3$ | 0 | $e_2$ | 0 | $u_1$ | 0 |
| $e_4$ | 0 | $2e_4$ | $-e_2$ | 0 | $u_2$ | 0 | 0 |
| $u_1$ | $-u_1$ | $-u_1$ | 0 | $-u_2$ | 0 | 0 | $\beta_1 u_1$ |
| $u_2$ | $-u_2$ | $u_2$ | $-u_1$ | 0 | 0 | 0 | $\beta_1 u_2$ |
| $u_3$ | 0 | 0 | 0 | 0 | $-\beta_1 u_1$ | $-\beta_1 u_2$ | 0 |

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_i) = e_i, \ i = 1,2,3,4, \\ &\pi(u_1) = u_1, \\ &\pi(u_2) = u_2, \\ &\pi(u_3) = u_3 + \beta_1 e_1. \end{aligned}$$

The proof of the Proposition is complete.

**Proposition 4.2.** *Any pair $(\bar{\mathfrak{g}},\mathfrak{g})$ of type 4.2 is equivalent to one and only one of the following pairs:*

*1.*

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | 0 | 0 | $\lambda u_1$ | $\lambda u_2$ | $u_3$ |
| $e_2$ | 0 | 0 | $2e_3$ | $-2e_4$ | $u_1$ | $-u_2$ | 0 |
| $e_3$ | 0 | $-2e_3$ | 0 | $e_2$ | 0 | $u_1$ | 0 |
| $e_4$ | 0 | $2e_4$ | $-e_2$ | 0 | $u_2$ | 0 | 0 |
| $u_1$ | $-\lambda u_1$ | $-u_1$ | 0 | $-u_2$ | 0 | 0 | 0 |
| $u_2$ | $-\lambda u_2$ | $u_2$ | $-u_1$ | 0 | 0 | 0 | 0 |
| $u_3$ | $-u_3$ | 0 | 0 | 0 | 0 | 0 | 0 |

*2.* $\lambda = \frac{1}{2}$

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | 0 | 0 | $\frac{1}{2}u_1$ | $\frac{1}{2}u_2$ | $u_3$ |
| $e_2$ | 0 | 0 | $2e_3$ | $-2e_4$ | $u_1$ | $-u_2$ | 0 |
| $e_3$ | 0 | $-2e_3$ | 0 | $e_2$ | 0 | $u_1$ | 0 |
| $e_4$ | 0 | $2e_4$ | $-e_2$ | 0 | $u_2$ | 0 | 0 |
| $u_1$ | $-\frac{1}{2}u_1$ | $-u_1$ | 0 | $-u_2$ | 0 | $u_3$ | 0 |
| $u_2$ | $-\frac{1}{2}u_2$ | $u_2$ | $-u_1$ | 0 | $-u_3$ | 0 | 0 |
| $u_3$ | $-u_3$ | 0 | 0 | 0 | 0 | 0 | 0 |

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix},$$
$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors $e_1$ and $e_2$.

**Lemma.** *Any virtual structure $q$ on the generalized module 4.2 is trivial.*

*Proof.* Let $q$ be a virtual structure on the generalized module 4.2. Note that $\mathfrak{a} = \mathbb{R}e_2 \oplus \mathbb{R}e_3 \oplus \mathbb{R}e_4$ is a semisimple subalgebra of the Lie algebra $\mathfrak{g}$. Without loss of generality it can be assumed that $q(\mathfrak{a}) = \{0\}$. Therefore

$$C(e_2) = C(e_3) = C(e_4) = 0, \quad C(e_1) = (c^1_{ij})_{\substack{1 \leqslant i \leqslant 4 \\ 1 \leqslant j \leqslant 3}}.$$

Checking condition (6), Chapter II, we obtain:

$$C(e_1) = \begin{pmatrix} 0 & 0 & c^1_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = C(e_3) = C(e_4) = 0.$$

Put

$$H = \begin{pmatrix} 0 & 0 & c^1_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $C_1(x) = C(x) + A(x)H - HB(x)$. Then

$$C_1(e_i) = 0, \quad 1 \leqslant i \leqslant 4.$$

By corollary 2, Chapter II, the virtual structures $C$ and $C_1$ are equivalent.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 4.2. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the virtual structure determined in the Lemma. Then

$$\begin{array}{llll}
[e_1, e_2] = 0 & & & \\
[e_1, e_3] = 0 & [e_2, e_3] = 2e_3 & & \\
[e_1, e_4] = 0 & [e_2, e_4] = -2e_4 & [e_3, e_4] = e_2 & \\
[e_1, u_1] = \lambda u_1 & [e_2, u_1] = u_1 & [e_3, u_1] = 0 & [e_4, u_1] = u_2 \\
[e_1, u_2] = \lambda u_2 & [e_2, u_2] = -u_2 & [e_3, u_2] = u_1 & [e_4, u_2] = 0 \\
[e_1, u_3] = u_3 & [e_2, u_3] = 0 & [e_3, u_3] = 0 & [e_4, u_3] = 0
\end{array}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}}^\alpha(\mathfrak{h}) = \mathfrak{g}^\alpha(\mathfrak{h}) \times U^\alpha(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 10, Chapter II). Thus

$$\begin{array}{ll}
\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) \supset \mathbb{R}e_1 \oplus \mathbb{R}e_2, & \bar{\mathfrak{g}}^{(0,2)}(\mathfrak{h}) \supset \mathbb{R}e_3, \\
\bar{\mathfrak{g}}^{(0,-2)}(\mathfrak{h}) \supset \mathbb{R}e_4, & \bar{\mathfrak{g}}^{(\lambda,1)}(\mathfrak{h}) \supset \mathbb{R}u_1, \\
\bar{\mathfrak{g}}^{(\lambda,-1)}(\mathfrak{h}) \supset \mathbb{R}u_2, & \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) \supset \mathbb{R}u_3.
\end{array}$$

Therefore

$$[u_1, u_2] \in \bar{\mathfrak{g}}^{(2\lambda,0)}(\mathfrak{h}),$$
$$[u_1, u_3] \in \bar{\mathfrak{g}}^{(\lambda+1,1)}(\mathfrak{h}),$$
$$[u_2, u_3] \in \bar{\mathfrak{g}}^{(\lambda+1,-1)}(\mathfrak{h}),$$

and

$$[u_1, u_2] = a_1 e_1 + a_2 e_2 + \alpha_3 u_3,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = 0.$$

Using the Jacobi identity we see that the pair ($\bar{\mathfrak{g}}$,$\mathfrak{g}$) has the form:

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | 0 | 0 | $\lambda u_1$ | $\lambda u_2$ | $u_3$ |
| $e_2$ | 0 | 0 | $2e_3$ | $-2e_4$ | $u_1$ | $-u_2$ | 0 |
| $e_3$ | 0 | $-2e_3$ | 0 | $e_2$ | 0 | $u_1$ | 0 |
| $e_4$ | 0 | $2e_4$ | $-e_2$ | 0 | $u_2$ | 0 | 0 |
| $u_1$ | $-\lambda u_1$ | $-u_1$ | 0 | $-u_2$ | 0 | $\alpha_3 u_3$ | 0 |
| $u_2$ | $-\lambda u_2$ | $u_2$ | $-u_1$ | 0 | $-\alpha_3 u_3$ | 0 | 0 |
| $u_3$ | $-u_3$ | 0 | 0 | 0 | 0 | 0 | 0 |

where $(2\lambda - 1)\alpha_3 = 0$. Consider the following cases:

1°. $\lambda \neq \frac{1}{2}$.
Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.
2°. $\lambda = \frac{1}{2}$.
2.1°. $\alpha_3 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.
2.2°. $\alpha_3 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i, \ i = 1, \dots, 4,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = \alpha_3 u_3.$$

Since $\dim \mathcal{D}(\bar{\mathfrak{r}}(\mathcal{D}\bar{\mathfrak{g}}_1)) \neq \dim \mathcal{D}(\bar{\mathfrak{r}}(\mathcal{D}\bar{\mathfrak{g}}_2))$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

Thus the proof of the Proposition is complete.

**Proposition 4.3.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.3 is trivial.*

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | 0 | 0 | $u_1$ | $u_2$ | $u_3$ |
| $e_2$ | 0 | 0 | $e_3$ | $-e_4$ | $u_1$ | 0 | $-u_3$ |
| $e_3$ | 0 | $-e_3$ | 0 | $e_2$ | 0 | $u_1$ | $u_2$ |
| $e_4$ | 0 | $-e_4$ | $-e_2$ | 0 | $u_2$ | $u_3$ | 0 |
| $u_1$ | $-u_1$ | $-u_1$ | 0 | $-u_2$ | 0 | 0 | 0 |
| $u_2$ | $-u_2$ | 0 | $-u_1$ | $-u_3$ | 0 | 0 | 0 |
| $u_3$ | $-u_3$ | $u_3$ | $-u_2$ | 0 | 0 | 0 | 0 |

*Proof.* Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

**Proposition 4.4.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.4 is trivial.*

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | 0 | $e_4$ | $u_1$ | 0 | 0 |
| $e_2$ | 0 | 0 | 0 | 0 | 0 | $u_2$ | 0 |
| $e_3$ | 0 | 0 | 0 | $-e_4$ | 0 | 0 | $u_3$ |
| $e_4$ | $-e_4$ | 0 | $e_4$ | 0 | 0 | 0 | $u_1$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | 0 | $-u_2$ | 0 | 0 | 0 | 0 | 0 |
| $u_3$ | 0 | 0 | $-u_3$ | 0 | 0 | 0 | 0 |

*Proof.* Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

**Proposition 4.5.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.5 is trivial.*

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | 0 | 0 | $u_1$ | $u_2$ | $u_3$ |
| $e_2$ | 0 | 0 | $e_4$ | $-e_3$ | $-u_3$ | 0 | $u_1$ |
| $e_3$ | 0 | $-e_4$ | 0 | $e_2$ | $-u_2$ | $u_1$ | 0 |
| $e_4$ | 0 | $e_3$ | $-e_2$ | 0 | 0 | $-u_3$ | $u_2$ |
| $u_1$ | $-u_1$ | $u_3$ | $u_2$ | 0 | 0 | 0 | 0 |
| $u_2$ | $-u_2$ | 0 | $-u_1$ | $u_3$ | 0 | 0 | 0 |
| $u_3$ | $-u_3$ | $-u_1$ | 0 | $-u_2$ | 0 | 0 | 0 |

*Proof.* Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

**Proposition 4.6.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.6 is equivalent to one and only one of the following pairs:*

*1.*

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $\lambda e_3$ | $e_4$ | $u_1$ | $\lambda u_2$ | 0 |
| $e_2$ | 0 | 0 | $-e_3$ | $-e_4$ | 0 | 0 | $u_3$ |
| $e_3$ | $-\lambda e_3$ | $e_3$ | 0 | 0 | 0 | 0 | $u_2$ |
| $e_4$ | $-e_4$ | $e_4$ | 0 | 0 | 0 | 0 | $u_1$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | $-\lambda u_2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_3$ | 0 | $-u_3$ | $-u_2$ | $-u_1$ | 0 | 0 | 0 |

*2.* $\lambda = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $0$ | $e_4$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $-e_4$ | $0$ | $0$ | $u_3$ |
| $e_3$ | $0$ | $e_3$ | $0$ | $0$ | $e_4$ | $e_3$ | $e_2+u_2$ |
| $e_4$ | $-e_4$ | $e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $-e_4$ | $0$ | $0$ | $-u_1$ | $0$ |
| $u_2$ | $0$ | $0$ | $-e_3$ | $0$ | $u_1$ | $0$ | $u_3$ |
| $u_3$ | $0$ | $-u_3$ | $-e_2-u_2$ | $-u_1$ | $0$ | $-u_3$ | $0$ |

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by vectors $e_1$ and $e_2$.

**Lemma.** *Any virtual structure $q$ on generalized module 4.6 is equivalent to one of the following:*

*a)* $\lambda = 0$

$$C(e_1) = C(e_2) = C(e_4) = 0, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & q & 0 \\ p & 0 & 0 \end{pmatrix};$$

*b)* $\lambda = \frac{1}{2}$

$$C(e_1) = C(e_2) = C(e_4) = 0, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix};$$

*c)* $\lambda \notin \{0, \frac{1}{2}\}$

$$C(e_1) = C(e_2) = C(e_3) = C(e_4) = 0.$$

*Proof.* Let $q$ be a virtual structure on the generalized module 4.6. Without loss of generality it can be assumed that $q$ is primary. Since

$$\begin{aligned} &\mathfrak{g}^{(0,0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2, && U^{(1,0)}(\mathfrak{h}) \supseteq \mathbb{R}u_1, \\ &\mathfrak{g}^{(\lambda,-1)}(\mathfrak{h}) \supseteq \mathbb{R}e_3, && U^{(\lambda,0)}(\mathfrak{h}) \supseteq \mathbb{R}u_2, \\ &\mathfrak{g}^{(1,-1)}(\mathfrak{h}) \supseteq \mathbb{R}e_4, && U^{(0,1)}(\mathfrak{h}) = \mathbb{R}u_3 \end{aligned}$$

we have

$$C(e_1) = \begin{pmatrix} 0 & c_{12}^1 & 0 \\ 0 & c_{22}^1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & c_{12}^2 & 0 \\ 0 & c_{22}^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & c_{13}^3 \\ 0 & 0 & c_{23}^3 \\ 0 & c_{32}^3 & 0 \\ c_{41}^3 & c_{42}^3 & 0 \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{42}^4 & 0 \end{pmatrix}.$$

Consider the following cases:

1°. $\lambda = 0$. Then

$$C(e_1) = \begin{pmatrix} 0 & c_{12}^1 & 0 \\ 0 & c_{22}^1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & c_{12}^2 & 0 \\ 0 & c_{22}^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & c_{13}^3 \\ 0 & 0 & c_{23}^3 \\ 0 & c_{32}^3 & 0 \\ c_{41}^3 & 0 & 0 \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{42}^4 & 0 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & c_{13}^3 & 0 \\ 0 & -c_{42}^4 + c_{13}^3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $C_1(x) = C(x) + A(X)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = \begin{pmatrix} 0 & c_{12}^1 & 0 \\ 0 & c_{22}^1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_1(e_2) = \begin{pmatrix} 0 & c_{12}^2 & 0 \\ 0 & c_{22}^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_1(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{23}^3 \\ 0 & c_{32}^3 & 0 \\ c_{41}^3 & 0 & 0 \end{pmatrix}, \quad C_1(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since for any virtual structure $q$ condition (6), Chapter II, must be satisfied, after direct calculation we obtain:

$$C(e_1) = C(e_2) = C(e_4) = 0, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & q & 0 \\ p & 0 & 0 \end{pmatrix}.$$

2°. $\lambda = \frac{1}{2}$.

$$C(e_1) = C(e_2) = C(e_4) = 0, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c^3_{42} & 0 \end{pmatrix}.$$

Since for any virtual structure $q$ condition (6), Chapter II, must be satisfied, after direct calculation we obtain:

$$C(e_1) = C(e_2) = C(e_4) = 0, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix}, \quad p \in \mathbb{R}.$$

3°. $\lambda \notin \{0, \frac{1}{2}\}$. Then

$$C(e_1) = C(e_2) = C(e_3) = C(e_4) = 0.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 4.6. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 10, Chapter II). Consider the following cases:

1°. $\lambda = 0$. Then

$$\begin{array}{llll} [e_1, e_2] = 0, & & & \\ [e_1, e_3] = 0, & [e_2, e_3] = -e_3, & & \\ [e_1, e_4] = e_4, & [e_2, e_4] = -e_4, & [e_3, e_4] = 0, & \\ [e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = pe_4, & [e_4, u_1] = 0, \\ [e_1, u_2] = 0, & [e_2, u_2] = 0, & [e_3, u_2] = qe_3, & [e_4, u_2] = 0, \\ [e_1, u_3] = 0, & [e_2, u_3] = 0, & [e_3, u_3] = pe_2 + u_2, & [e_4, u_3] = u_1. \end{array}$$

Since

$$\begin{array}{ll} \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}u_2, & \\ \bar{\mathfrak{g}}^{(0,-1)}(\mathfrak{h}) = \mathbb{R}e_3, & \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) = \mathbb{R}e_4, \\ \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) = \mathbb{R}u_1, & \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}) = \mathbb{R}u_3, \end{array}$$

we have

$$[u_1, u_2] = \alpha_1 u_1,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = \gamma_3 u_3.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $0$ | $e_4$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $-e_4$ | $0$ | $0$ | $u_3$ |
| $e_3$ | $0$ | $e_3$ | $0$ | $0$ | $pe_4$ | $pe_3$ | $pe_2 + u_2$ |
| $e_4$ | $-e_4$ | $e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $-pe_4$ | $0$ | $0$ | $-pu_1$ | $0$ |
| $u_2$ | $0$ | $0$ | $-pe_3$ | $0$ | $pu_1$ | $0$ | $pu_3$ |
| $u_3$ | $0$ | $-u_3$ | $-pe_2 - u_2$ | $-u_1$ | $0$ | $-pu_3$ | $0$ |

Consider the following cases:

1.1°. $p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

1.2°. $p \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i, \quad i = 1, \dots, 4,$$
$$\pi(u_j) = \frac{1}{p} u_j, \quad j = 1, 2, 3.$$

2°. $\lambda = \frac{1}{2}$. Then

$$\begin{array}{llll}
[e_1, e_2] = 0, & & & \\
[e_1, e_3] = \frac{1}{2} e_3, & [e_2, e_3] = -e_3, & & \\
[e_1, e_4] = e_4, & [e_2, e_4] = -e_4, & [e_3, e_4] = 0, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = 0, \\
[e_1, u_2] = \frac{1}{2} u_2, & [e_2, u_2] = 0, & [e_3, u_2] = pe_4, & [e_4, u_2] = 0, \\
[e_1, u_3] = 0, & [e_2, u_3] = 0, & [e_3, u_3] = u_2, & [e_4, u_3] = u_1.
\end{array}$$

Since

$$\begin{array}{ll}
\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2, & \bar{\mathfrak{g}}^{(1/2,-1)}(\mathfrak{h}) = \mathbb{R}e_3, \\
\bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) = \mathbb{R}e_4, & \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) = \mathbb{R}u_1, \\
\bar{\mathfrak{g}}^{(1/2,0)}(\mathfrak{h}) = \mathbb{R}u_2, & \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}) = \mathbb{R}u_3,
\end{array}$$

We obtain

$$[u_1, u_2] = 0,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = 0.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

3°. $\lambda \notin \{0, \frac{1}{2}\}$.
Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

Since the Lie algebra $\bar{\mathfrak{g}}_2$ is nonsolvable, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

This completes the proof of the Proposition.

**Proposition 4.7.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.7 is trivial:*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $-e_3$ | $e_2$ | $0$ | $\lambda u_1 - u_2$ | $u_1 + \lambda u_2$ | $\lambda u_3$ |
| $e_2$ | $e_3$ | $0$ | $0$ | $e_2$ | $0$ | $0$ | $u_1$ |
| $e_3$ | $-e_2$ | $0$ | $0$ | $e_3$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $0$ | $-e_2$ | $-e_3$ | $0$ | $0$ | $0$ | $u_3$ |
| $u_1$ | $u_2 - \lambda u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-u_1 - \lambda u_2$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-\lambda u_3$ | $-u_1$ | $-u_2$ | $-u_3$ | $0$ | $0$ | $0$ |

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} \lambda & 1 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

**Lemma.** *Any virtual structure $C$ on generalized module 4.7 is trivial.*

*Proof.* Put $C(e_i) = (c^i_{jk})_{\substack{1 \leqslant j \leqslant 4,\\ 1 \leqslant k \leqslant 3}}$, $1 \leqslant i \leqslant 4$. Checking condition (6), Chapter II, we obtain:

$$C(e_1) = \begin{pmatrix} c^1_{11} & c^1_{12} & \lambda c^4_{13} \\ -c^4_{22} + \lambda c^4_{21} - c^4_{31} & c^4_{21} + \lambda c^4_{22} - c^4_{32} & \frac{\lambda c^4_{23} - c^4_{33}}{2} \\ c^4_{21} + \lambda c^4_{31} - c^4_{32} & c^4_{31} + \lambda c^4_{32} + c^4_{22} & \frac{\lambda c^4_{33} + c^4_{23}}{2} \\ c^1_{41} & c^1_{42} & \lambda c^4_{43} \end{pmatrix},$$

$$C(e_2)=\begin{pmatrix} 0 & 0 & \frac{c^1_{12}+\lambda c^1_{11}}{\lambda^2+1} \\ \frac{-\lambda c^1_{41}-c^1_{42}}{\lambda^2+1} & \frac{c^1_{41}-\lambda c^1_{42}}{\lambda^2+1} & c^4_{21}-c^4_{43} \\ \frac{-\lambda c^1_{11}-c^1_{12}}{\lambda^2+1} & \frac{c^1_{11}-\lambda c^1_{12}}{\lambda^2+1} & c^4_{31}-c^4_{13} \\ 0 & 0 & \frac{c^1_{42}+\lambda c^1_{41}}{\lambda^2+1} \end{pmatrix},$$

$$C(e_3)=\begin{pmatrix} 0 & 0 & \frac{\lambda c^1_{12}-c^1_{11}}{\lambda^2+1} \\ \frac{\lambda c^1_{11}+c^1_{12}}{\lambda^2+1} & \frac{\lambda c^1_{12}-c^1_{11}}{\lambda^2+1} & c^4_{22}+c^4_{13} \\ \frac{-\lambda c^1_{41}-c^1_{42}}{\lambda^2+1} & \frac{c^1_{41}-\lambda c^1_{42}}{\lambda^2+1} & c^4_{32}-c^4_{43} \\ 0 & 0 & \frac{\lambda c^1_{42}-c^1_{41}}{\lambda^2+1} \end{pmatrix},$$

$$C(e_4)=\begin{pmatrix} 0 & 0 & c^4_{13} \\ c^4_{21} & c^4_{22} & c^4_{23} \\ c^4_{31} & c^4_{32} & c^4_{33} \\ 0 & 0 & c^4_{43} \end{pmatrix}.$$

Put

$$H=\begin{pmatrix} \frac{c^1_{12}+\lambda c^1_{11}}{\lambda^2+1} & \frac{\lambda c^1_{12}-c^1_{11}}{\lambda^2+1} & c^4_{13} \\ c^4_{21} & c^4_{22} & \frac{c^4_{23}}{2} \\ c^4_{31} & c^4_{32} & \frac{c^4_{33}}{2} \\ \frac{c^1_{42}+\lambda c^1_{41}}{\lambda^2+1} & \frac{\lambda c^1_{42}-c^1_{41}}{\lambda^2+1} & c^4_{43} \end{pmatrix}$$

and $C_1(x)=C(x)+A(x)H-HB(x)$. Then $C_1(e_i)=0$, $1\leqslant i\leqslant 4$.

By corollary 2, Chapter II, the virtual structures $C$ and $C_1$ are equivalent.

Thus we have

$$\begin{array}{llll}
[e_1,e_2]=-e_3, & & & \\
[e_1,e_3]=e_2, & [e_2,e_3]=0, & & \\
[e_1,e_4]=0, & [e_2,e_4]=e_2, & [e_3,e_4]=e_3, & \\
[e_1,u_1]=\lambda u_1-u_2, & [e_2,u_1]=0, & [e_3,u_1]=0, & [e_4,u_1]=0, \\
[e_1,u_2]=u_1+\lambda u_2, & [e_2,u_2]=0, & [e_3,u_2]=0, & [e_4,u_2]=0, \\
[e_1,u_3]=\lambda u_3, & [e_2,u_3]=u_1, & [e_3,u_3]=u_2, & [e_4,u_3]=u_3.
\end{array}$$

Put

$$\begin{aligned}
[u_1,u_2]&=a_1e_1+a_2e_2+a_3e_3+a_4e_4+\alpha_1u_1+\alpha_2u_2+\alpha_3u_3,\\
[u_1,u_3]&=b_1e_1+b_2e_2+b_3e_3+b_4e_4+\beta_1u_1+\beta_2u_2+\beta_3u_3,\\
[u_2,u_3]&=c_1e_1+c_2e_2+c_3e_3+c_4e_4+\gamma_1u_1+\gamma_2u_2+\gamma_3u_3.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is trivial.

**Proposition 4.8.** *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 4.8 is equivalent to one and only one of the following pairs:*

*1.*

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $(1-\lambda)e_3$ | $-\lambda e_4$ | $u_1$ | $0$ | $\lambda u_3$ |
| $e_2$ | $0$ | $0$ | $-\mu e_3$ | $(1-\mu)e_4$ | $0$ | $u_2$ | $\mu u_3$ |
| $e_3$ | $(\lambda-1)e_3$ | $\mu e_3$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $\lambda e_4$ | $(\mu-1)e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-\lambda u_3$ | $-\mu u_3$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ |

*2.* $\lambda = 0,\ \mu = 0$

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $0$ | $e_4$ | $0$ | $u_2$ | $0$ |
| $e_3$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $0$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $e_4$ |
| $u_3$ | $0$ | $0$ | $-u_1$ | $-u_2$ | $0$ | $-e_4$ | $0$ |

*3.* $\lambda = 0,\ \mu = 0$

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $0$ | $e_4$ | $0$ | $u_2$ | $0$ |
| $e_3$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $0$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $-e_4$ |
| $u_3$ | $0$ | $0$ | $-u_1$ | $-u_2$ | $0$ | $e_4$ | $0$ |

*4.* $\lambda = 0,\ \mu = 0$

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $0$ | $e_4$ | $0$ | $u_2$ | $0$ |
| $e_3$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $0$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $e_3$ |
| $u_2$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $\alpha e_4$ |
| $u_3$ | $0$ | $0$ | $-u_1$ | $-u_2$ | $-e_3$ | $-\alpha e_4$ | $0$ |

$, 0 < \alpha \leqslant 1$

*5.* $\lambda = 0,\ \mu = 0$

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $0$ | $e_4$ | $0$ | $u_2$ | $0$ |
| $e_3$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $0$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $\alpha u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $e_3$ |
| $u_2$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $-\alpha e_4$ |
| $u_3$ | $0$ | $0$ | $-u_1$ | $-\alpha u_2$ | $-e_3$ | $\alpha e_4$ | $0$ |

$, \alpha > 0$

*6.* $\lambda = 0$, $\mu = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $0$ | $e_4$ | $0$ | $u_2$ | $0$ |
| $e_3$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $0$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $\alpha u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-e_3$ |
| $u_2$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $-\alpha e_4$ |
| $u_3$ | $0$ | $0$ | $-u_1$ | $-\alpha u_2$ | $e_3$ | $\alpha e_4$ | $0$ |

$, 0 < \alpha \leqslant 1$

*7.* $\lambda = 0$, $\mu = \frac{1}{2}$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ | $e_4$ |
| $e_2$ | $0$ | $0$ | $-\frac{1}{2}e_3$ | $\frac{1}{2}e_4$ | $0$ | $u_2$ | $\frac{1}{2}u_3$ |
| $e_3$ | $-e_3$ | $\frac{1}{2}e_3$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $0$ | $-\frac{1}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-e_4$ | $-\frac{1}{2}u_3$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ |

*8.* $\lambda = 0$, $\mu = \frac{1}{2}$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ | $\alpha e_4$ |
| $e_2$ | $0$ | $0$ | $-\frac{1}{2}e_3$ | $\frac{1}{2}e_4$ | $0$ | $u_2$ | $\frac{1}{2}u_3 + e_4$ |
| $e_3$ | $-e_3$ | $\frac{1}{2}e_3$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $0$ | $-\frac{1}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-\alpha e_4$ | $-\frac{1}{2}u_3 - e_4$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ |

*9.* $\lambda = -1$, $\mu = 1$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $2e_3$ | $e_4$ | $u_1$ | $0$ | $-u_3$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $0$ | $0$ | $u_2$ | $u_3$ |
| $e_3$ | $-2e_3$ | $e_3$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_2$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $u_3$ | $-u_3$ | $-u_1$ | $-u_2$ | $-u_2$ | $0$ | $0$ |

*10.* $\lambda = -\frac{1}{2}$, $\mu = \frac{1}{2}$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $\frac{3}{2}e_3$ | $\frac{1}{2}e_4$ | $u_1$ | $0$ | $-\frac{1}{2}u_3$ |
| $e_2$ | $0$ | $0$ | $-\frac{1}{2}e_3$ | $\frac{1}{2}e_4$ | $0$ | $u_2$ | $\frac{1}{2}u_3$ |
| $e_3$ | $-\frac{3}{2}e_3$ | $\frac{1}{2}e_3$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $-\frac{1}{2}e_4$ | $-\frac{1}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $e_4$ |
| $u_2$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $\frac{1}{2}u_3$ | $-\frac{1}{2}u_3$ | $-u_1$ | $-u_2$ | $-e_4$ | $0$ | $0$ |

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 1-\lambda & 0 & \\ 0 & 0 & 0 & 0 & -\lambda \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 1-\mu \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda - 1 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda & \mu - 1 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors $e_1$ and $e_2$.

**Lemma.** *Any virtual structure $q$ on generalized module 4.8 is equivalent to one of the following:*

*a)* $\lambda = -1, \ \mu = 1$

$$C(e_i) = 0, \ i = 1, 2, 4, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix};$$

*b)* $\lambda = 0, \ \mu = 2$

$$C(e_i) = 0, \ i = 1, 2, 3, \quad C(e_4) = \begin{pmatrix} 0 & 2p & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

*c)* $\lambda = 0, \ \mu = \frac{1}{2}$

$$C(e_i) = 0, \ i = 3, 4, \quad C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q \end{pmatrix};$$

*d)* $\lambda = 0, \ \mu = 0$

$$C(e_i) = 0, \ i = 3, 4, \quad C(e_1) = \begin{pmatrix} 0 & 0 & -p \\ 0 & 0 & -q \\ p & 0 & 0 \\ 0 & q & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & -r \\ 0 & 0 & -s \\ r & 0 & 0 \\ 0 & s & 0 \end{pmatrix};$$

e) $(\lambda,\mu)\notin\{(-1,1),(0,2),(0,\frac{1}{2}),(0,0)\}$

$$C(e_i)=0,\quad i=1,\dots,4.$$

*Proof.* Let $q$ be a virtual structure on generalized module 4.8. Without loss of generality it can be assumed that $q$ is primary. We have:

$$\begin{aligned}
\mathfrak{g}^{(0,0)}(\mathfrak{h}) &\supseteq \mathbb{R}e_1\oplus\mathbb{R}e_2, & U^{(1,0)}(\mathfrak{h}) &\supseteq \mathbb{R}u_1,\\
\mathfrak{g}^{(1-\lambda,-\mu)}(\mathfrak{h}) &\supseteq \mathbb{R}e_3, & U^{(0,1)}(\mathfrak{h}) &\supseteq \mathbb{R}u_2,\\
\mathfrak{g}^{(-\lambda,1-\mu)}(\mathfrak{h}) &\supseteq \mathbb{R}e_4, & U^{(\lambda,\mu)}(\mathfrak{h}) &\supseteq \mathbb{R}u_3.
\end{aligned}$$

Consider the following cases:

1°. $\mu=2,\ \lambda=0$. Then

$$C(e_i)=0,\ i=1,2,3,\quad C(e_4)=\begin{pmatrix}0&c_{12}^4&0\\0&c_{22}^4&0\\0&0&0\\0&0&0\end{pmatrix},$$

where $c_{12}^4,\ c_{22}^4\in\mathbb{R}$.

Since for any virtual structure $q$ condition (6), Chapter II, must be satisfied, after direct calculation we obtain

$$C(e_i)=0,\ i=1,2,3,\quad C(e_4)=\begin{pmatrix}0&2p&0\\0&p&0\\0&0&0\\0&0&0\end{pmatrix}.$$

2°. $\mu=\lambda=0$. Then

$$C(e_1)=\begin{pmatrix}0&0&c_{13}^1\\0&0&c_{23}^1\\c_{31}^1&0&0\\0&c_{42}^1&0\end{pmatrix},\quad C(e_2)=\begin{pmatrix}0&0&c_{13}^2\\0&0&c_{23}^2\\c_{31}^2&0&0\\0&c_{42}^2&0\end{pmatrix},$$

$$C(e_3)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&c_{33}^3\\0&0&0\end{pmatrix},\quad C(e_4)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&c_{43}^4\end{pmatrix}.$$

Put

$$H=\begin{pmatrix}0&0&0\\0&0&0\\c_{33}^3&0&0\\0&c_{43}^4&0\end{pmatrix}$$

and $C_1(x)=C(x)+A(x)H-HB(x)$ for $x\in\mathfrak{g}$. Then

$$C_1(e_1)=\begin{pmatrix}0&0&c_{13}^1\\0&0&c_{23}^1\\c_{31}^1&0&0\\0&c_{42}^1&0\end{pmatrix},\quad C_1(e_2)=\begin{pmatrix}0&0&c_{13}^2\\0&0&c_{23}^2\\c_{31}^2&0&0\\0&c_{42}^2&0\end{pmatrix},\quad C_1(e_3)=C_1(e_4)=0.$$

Since for any virtual structure $q$ condition (6), Chapter II, must be satisfied, after direct calculation we obtain

$$C_1(e_i)=0,\ i=3,4,\quad C_1(e_1)=\begin{pmatrix}0&0&-p\\0&0&-q\\p&0&0\\0&q&0\end{pmatrix},\quad C_1(e_2)=\begin{pmatrix}0&0&-r\\0&0&-s\\r&0&0\\0&s&0\end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures $C$ and $C_1$ are equivalent.

Similarly we obtain the other results of Lemma.

Let $(\bar{\mathfrak{g}},\mathfrak{g})$ be a pair of type 4.8. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Consider the following cases:

1°. $\lambda=-1,\ \mu=1$. Then

$$\begin{array}{llll}
[e_1,e_2]=0, & & & \\
[e_1,e_3]=2e_3, & [e_2,e_3]=-e_3, & & \\
[e_1,e_4]=e_4, & [e_2,e_4]=0, & [e_3,e_4]=0, & \\
[e_1,u_1]=u_1, & [e_2,u_1]=0, & [e_3,u_1]=0, & [e_4,u_1]=0,\\
[e_1,u_2]=0, & [e_2,u_2]=u_2, & [e_3,u_2]=0, & [e_4,u_2]=0,\\
[e_1,u_3]=-u_3, & [e_2,u_3]=u_3, & [e_3,u_3]=pe_4+u_1, & [e_4,u_3]=u_2.
\end{array}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}}^\alpha(\mathfrak{h})=\mathfrak{g}^\alpha(\mathfrak{h})\times U^\alpha(\mathfrak{h}) \text{ for all } \alpha\in\mathfrak{h}^*$$

(Proposition 10, Chapter II). Therefore

$$[u_1,u_2]=0,$$
$$[u_1,u_3]=\gamma_2u_2$$
$$[u_2,u_3]=0.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ has the form

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $2e_3$ | $e_4$ | $u_1$ | $0$ | $-u_3$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $0$ | $0$ | $u_2$ | $u_3$ |
| $e_3$ | $-2e_3$ | $e_3$ | $0$ | $0$ | $0$ | $0$ | $u_1+pe_4$ |
| $e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\gamma_2u_2$ |
| $u_2$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $u_3$ | $-u_3$ | $-u_1$ | $-u_2$ | $-\gamma_2u_2$ | $0$ | $0$ |

and is equivalent to the pair $(\bar{\mathfrak{g}}_9,\mathfrak{g}_9)$, whenever $\gamma_2+p\neq 0$, by means of the mapping $\pi:\bar{\mathfrak{g}}_9\to\bar{\mathfrak{g}}$:

$$\begin{array}{ll}
\pi(e_1)=e_1, & \pi(u_1)=p+\gamma_2u_1+pe_4,\\
\pi(e_2)=e_2, & \pi(u_2)=u_2,\\
\pi(e_3)=p+\gamma_2e_3, & \pi(u_3)=u_3.\\
\pi(e_4)=e_4, &
\end{array}$$

If $p+\gamma_2 = 0$, then the pair is equivalent to the trivial pair by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$:

$$\begin{aligned} &\pi(e_1) = e_1, && \pi(u_1) = p + \gamma_2 u_1 + pe_4, \\ &\pi(e_2) = e_2, && \pi(u_2) = u_2, \\ &\pi(e_3) = e_3, && \pi(u_3) = u_3. \\ &\pi(e_4) = e_4, \end{aligned}$$

2°. $\mu = \lambda = 0$. Then

$$\begin{array}{llll} [e_1, e_2] = 0, & & & \\ [e_1, e_3] = e_3, & [e_2, e_3] = 0, & & \\ [e_1, e_4] = 0, & [e_2, e_4] = e_4, & [e_3, e_4] = 0, & \\ [e_1, u_1] = u_1 + pe_3, & [e_2, u_1] = re_3, & [e_3, u_1] = 0, & [e_4, u_1] = 0, \\ [e_1, u_2] = qe_4, & [e_2, u_2] = u_2 + se_4, & [e_3, u_2] = 0, & [e_4, u_2] = 0, \\ [e_1, u_3] = -pe_1 - qe_2, & [e_2, u_3] = -re_1 - se_2, & [e_3, u_3] = u_1, & [e_4, u_3] = u_2. \end{array}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}),$$

where

$$\begin{aligned} \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) &= \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}u_3, \\ \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) &= \mathbb{R}e_3 \oplus \mathbb{R}u_1, \\ \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}) &= \mathbb{R}e_4 \oplus \mathbb{R}u_2. \end{aligned}$$

Therefore

$$\begin{aligned} [u_1, u_2] &= 0, \\ [u_1, u_3] &= b_3 e_3 + \beta_1 u_1, \\ [u_2, u_3] &= c_4 e_4 + \gamma_2 u_2. \end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $e_3$ | 0 | $u_1$ | 0 | 0 |
| $e_2$ | 0 | 0 | 0 | $e_4$ | 0 | $u_2$ | 0 |
| $e_3$ | $-e_3$ | 0 | 0 | 0 | 0 | 0 | $u_1$ |
| $e_4$ | 0 | $-e_4$ | 0 | 0 | 0 | 0 | $u_2$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | $b_3e_3+\beta_1u_1$ |
| $u_2$ | 0 | $-u_2$ | 0 | 0 | 0 | 0 | $c_4e_4+\gamma_2u_2$ |
| $u_3$ | 0 | 0 | $-u_1$ | $-u_2$ | $-b_3e_3-\beta_1u_1$ | $-c_4e_4-\gamma_2u_2$ | 0 |

Put

$$\alpha = \frac{1}{\sqrt{|b_3 + \frac{\beta_1^2}{4}|}}, \qquad \beta = \frac{1}{\sqrt{|c_4 + \frac{\gamma_2^2}{4}|}}.$$

Consider the following cases:

2.1°. $b_3 + \frac{\beta_1^2}{4} = 0$.

2.1.1°. $c_4 + \frac{\gamma_2^2}{4} = 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \mathfrak{g}$, where

$$\begin{aligned}
&\pi(e_i) = e_i, \;\; i = 1, \ldots, 4,\\
&\pi(u_1) = \frac{\beta_1}{2} e_3 + u_1,\\
&\pi(u_2) = \frac{\gamma_2}{2} e_4 + u_2,\\
&\pi(u_3) = \beta u_3 - \frac{\beta_1}{2} e_1 - \frac{\gamma_2}{2} e_2.
\end{aligned}$$

2.1.2°. $c_4 + \frac{\gamma_2^2}{4} > 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \mathfrak{g}$, where

$$\begin{aligned}
&\pi(e_i) = e_i, \;\; i = 1, 2, 3,\\
&\pi(e_4) = \beta e_4,\\
&\pi(u_1) = \frac{\beta_1}{2} e_3 + u_1,\\
&\pi(u_2) = \frac{\beta\gamma_2}{2} e_4 + u_2,\\
&\pi(u_3) = u_3 - \frac{\beta_1}{2} e_1 - \frac{\lambda\gamma_2}{2} e_2.
\end{aligned}$$

2.1.3°. $c_4 + \frac{\gamma_2^2}{4} < 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \mathfrak{g}$, where

$$\begin{aligned}
&\pi(e_i) = e_i, i = 1, 2, 3,\\
&\pi(e_4) = \beta e_4,\\
&\pi(u_1) = \frac{\beta_1}{2} e_3 + u_1,\\
&\pi(u_2) = \frac{\beta\gamma_2}{2} e_4 + u_2,\\
&\pi(u_3) = \beta u_3 - \frac{\beta_1}{2} e_1 - \frac{\beta\gamma_2}{2} e_2.
\end{aligned}$$

2.2°. $b_3 + \frac{\beta_1^2}{4} > 0$.

2.2.1°. $c_4 + \frac{\gamma_2^2}{4} > 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ by means of the mapping

$\pi : \bar{\mathfrak{g}}_4 \to \mathfrak{g}$, where

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = e_2, \quad \pi(u_1) = \frac{\alpha\beta_1}{2}e_3 + u_1, \\ &\pi(e_3) = \alpha e_3, \quad \pi(u_2) = \frac{\alpha(\gamma_2 - \sqrt{\gamma_2^2 + 4c_4})}{2}e_4 + u_2, \\ &\pi(e_4) = \alpha e_4, \quad \pi(u_3) = \alpha u_3 - \frac{\alpha\beta_1}{2}e_1 - \frac{\alpha(\gamma_2 - \sqrt{\gamma_2^2 + 4c_4})}{2}e_2. \end{aligned}$$

Note that two pairs $(\bar{\mathfrak{g}}_4', \mathfrak{g}_4')$ and $(\bar{\mathfrak{g}}_4'', \mathfrak{g}_4'')$ corresponding, respectively, to the values $\alpha$ and $1/\alpha$ of the parameter are equivalent by means of the following mapping $\pi : \bar{\mathfrak{g}}_4'' \to \bar{\mathfrak{g}}_4'$:

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = e_2, \quad \pi(u_1) = u_2, \\ &\pi(e_3) = \sqrt{\alpha}e_4, \pi(u_2) = u_1, \\ &\pi(e_4) = \sqrt{\alpha}e_3, \pi(u_3) = \frac{1}{\sqrt{\alpha}}u_3. \end{aligned}$$

So, we can assume that $0 < \alpha \leqslant 1$.

2.2.2°. $c_4 + \frac{\gamma_2^2}{4} < 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_5 \to \mathfrak{g}$, where

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = e_2, \quad \pi(u_1) = \frac{\alpha\beta_1}{2}e_3 + u_1, \\ &\pi(e_3) = \alpha e_3, \quad \pi(u_2) = \frac{\alpha\gamma_2}{2b}e_4 + u_2, \\ &\pi(e_4) = \beta e_4, \quad \pi(u_3) = \frac{1}{\alpha}u_3 - \frac{\beta_1}{2}e_1 - \frac{\gamma_2}{2}e_2. \end{aligned}$$

2.2.3°. $c_4 + \frac{\gamma_2^2}{4} = 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one of the pairs $(\bar{\mathfrak{g}}', \mathfrak{g})$, considered in case 2.1.2°, by means of the mapping $\pi : \bar{\mathfrak{g}}' \to \mathfrak{g}$, where

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = e_2, \quad \pi(u_1) = u_2, \\ &\pi(e_3) = e_4, \quad \pi(u_2) = u_1, \\ &\pi(e_4) = e_3, \quad \pi(u_3) = u_3. \end{aligned}$$

2.3°. $b_3 + \frac{\beta_1^2}{4} < 0$. 2.3.1°. $c_4 + \frac{\gamma_2^2}{4} < 0$. 2.3.1.1°. $\frac{\alpha}{\beta} \leqslant 1$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_6 \to \mathfrak{g}$, where

$$\begin{aligned}
&\pi(e_1) = e_1, \\
&\pi(e_2) = e_2, \quad \pi(u_1) = \frac{\alpha\beta_1}{2} e_3 + u_1, \\
&\pi(e_3) = \alpha e_3, \quad \pi(u_2) = \frac{\alpha\gamma_2}{2b} e_4 + u_2, \\
&\pi(e_4) = \beta e_4, \quad \pi(u_3) = \frac{1}{\alpha} u_3 - \frac{\beta_1}{2} e_1 - \frac{\gamma_2}{2} e_2.
\end{aligned}$$

2.3.1.2°. $\frac{\alpha}{\beta} > 1$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_6 \to \mathfrak{g}$, where

$$\begin{aligned}
&\pi(e_1) = e_1, \\
&\pi(e_2) = e_2, \quad \pi(u_1) = \frac{\alpha\gamma_2}{2b} e_4 + u_2, \\
&\pi(e_3) = \beta e_4, \quad \pi(u_2) = \frac{\alpha\beta_1}{2} e_3 + u_1, \\
&\pi(e_4) = \alpha e_3, \quad \pi(u_3) = \frac{\beta}{\alpha^2} u_3 - \frac{\beta_1\beta}{2\alpha} e_1 - \frac{\gamma_2\beta}{2\alpha} e_2.
\end{aligned}$$

2.3.2°. $c_4 + \frac{\gamma_2^2}{4} = 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one of the pairs $(\bar{\mathfrak{g}}', \mathfrak{g}')$, considered in case 2.1°, by means of the mapping $\pi : \bar{\mathfrak{g}}' \to \mathfrak{g}$, where

$$\begin{aligned}
&\pi(e_1) = e_1, \\
&\pi(e_2) = e_2, \quad \pi(u_1) = u_2, \\
&\pi(e_3) = e_4, \quad \pi(u_2) = u_1, \\
&\pi(e_4) = e_3, \quad \pi(u_3) = u_3.
\end{aligned}$$

2.3.3°. $c_4 + \frac{\gamma_2^2}{4} > 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to one of the pairs $(\bar{\mathfrak{g}}', \mathfrak{g}')$, considered in case 2.2°, by means of the mapping $\pi : \bar{\mathfrak{g}}' \to \mathfrak{g}$, where

$$\begin{aligned}
&\pi(e_1) = e_1, \\
&\pi(e_2) = e_2, \quad \pi(u_1) = u_2, \\
&\pi(e_3) = e_4, \quad \pi(u_2) = u_1, \\
&\pi(e_4) = e_3, \quad \pi(u_3) = u_3.
\end{aligned}$$

Consider the homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(4, \mathbb{R})$, $i = 1, \dots, 6$, where $f_i(x)$ is the matrix of the mapping $\mathrm{ad}_{\mathcal{D}\bar{\mathfrak{g}}_i} x$ in the basis $\{e_3, u_1, e_4, u_2\}$ of $\mathcal{D}\bar{\mathfrak{g}}_i$, $x \in \bar{\mathfrak{g}}_i$.

Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$, $i = 1, \dots, 6$, are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1, \dots, 6$, are not equivalent.

In a similar way we obtain the other results of the Proposition.

**Proposition 4.9.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.9 is equivalent to one and only one of the following pairs:*

*1.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $-\lambda e_3 + e_4$ | $-\lambda e_4 - e_3$ | $-u_2$ | $u_1$ | $\lambda u_3$ |
| $e_2$ | $0$ | $0$ | $(1-\mu)e_3$ | $(1-\mu)e_4$ | $u_1$ | $u_2$ | $\mu u_3$ |
| $e_3$ | $\lambda e_3 - e_4$ | $(\mu-1)e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $e_3 + \lambda e_4$ | $(\mu-1)e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-\lambda u_3$ | $-\mu u_3$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ |

*2.* $\lambda = 0,\ \mu = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_4$ | $-e_3$ | $-u_2$ | $u_1$ | $0$ |
| $e_2$ | $0$ | $0$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $0$ |
| $e_3$ | $-e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $e_4$ |
| $u_2$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $e_3$ |
| $u_3$ | $0$ | $0$ | $-u_2$ | $-u_1$ | $-e_4$ | $-e_3$ | $0$ |

*3.* $\lambda = 0,\ \mu = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_4$ | $-e_3$ | $-u_2$ | $u_1$ | $0$ |
| $e_2$ | $0$ | $0$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $0$ |
| $e_3$ | $-e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $-e_4$ |
| $u_2$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $-e_3$ |
| $u_3$ | $0$ | $0$ | $-u_2$ | $-u_1$ | $e_4$ | $e_3$ | $0$ |

*4.* $\lambda = 0,\ \mu = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_4$ | $-e_3$ | $-u_2$ | $u_1$ | $0$ |
| $e_2$ | $0$ | $0$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $0$ |
| $e_3$ | $-e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $\alpha e_4 + e_3$ |
| $u_2$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $\alpha e_3 - e_4$ |
| $u_3$ | $0$ | $0$ | $-u_2$ | $-u_1$ | $-\alpha e_4 - e_3$ | $-\alpha e_3 + e_4$ | $0$ |

$, \alpha \geqslant 0$

*5.* $\lambda = 0,\ \mu = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_4$ | $-e_3$ | $-u_2$ | $u_1$ | $0$ |
| $e_2$ | $0$ | $0$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $0$ |
| $e_3$ | $-e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $\alpha e_4 - e_3$ |
| $u_2$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $\alpha e_3 + e_4$ |
| $u_3$ | $0$ | $0$ | $-u_2$ | $-u_1$ | $-\alpha e_4 + e_3$ | $-\alpha e_3 - e_4$ | $0$ |

$, \alpha \geqslant 0$

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda & -1 \\ 0 & 0 & 1 & -\lambda \end{pmatrix}, \qquad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\mu & 0 \\ 0 & 0 & 0 & 1-\mu \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda & \mu-1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \lambda & \mu-1 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector $e_2$.

**Lemma.** *Any virtual structure $q$ on generalized module 4.9 is equivalent to one of the following:*

*a)* $\lambda^2 + \mu^2 \neq 0$

$$C(e_i) = 0, \quad i = 1, \dots, 4;$$

*b)* $\lambda = \mu = 0$

$$C(e_1) = \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & s \\ r & -s & 0 \\ -s & r & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & q \\ 0 & 0 & p \\ q & -p & 0 \\ -p & -q & 0 \end{pmatrix}, \quad C(e_3) = C(e_4) = 0.$$

*Proof.* Let $q$ be a virtual structure on the generalized module 4.9. Without loss of generality it can be assumed that $q$ is primary. Consider the following cases:

*a)* $\lambda^2 + \mu^2 \neq 0$. Since

$$\begin{aligned} &\mathfrak{g}^{(0)}(\mathfrak{g}) \supset \mathbb{R}e_1 \oplus \mathbb{R}e_2, \quad &&\mathfrak{g}^{(1-\mu)}(\mathfrak{g}) \supset \mathbb{R}e_3 \oplus \mathbb{R}e_4, \\ &U^{(1)}(\mathfrak{h}) \supset \mathbb{R}u_1 \oplus \mathbb{R}u_2, \quad &&U^{(\mu)}(\mathfrak{h}) \supset \mathbb{R}u_3, \end{aligned}$$

we have

$$C(e_1) = \begin{pmatrix} 0 & 0 & c^1_{13} \\ 0 & 0 & c^1_{23} \\ c^1_{31} & c^1_{32} & c^1_{33} \\ c^1_{41} & c^1_{42} & c^1_{43} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c^2_{13} \\ 0 & 0 & c^2_{23} \\ c^2_{31} & c^2_{32} & c^2_{33} \\ c^2_{41} & c^2_{42} & c^2_{43} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} c^3_{11} & c^3_{12} & 0 \\ c^3_{21} & c^3_{22} & 0 \\ 0 & 0 & c^3_{33} \\ 0 & 0 & c^3_{43} \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} c^4_{11} & c^4_{12} & 0 \\ c^4_{21} & c^4_{22} & 0 \\ 0 & 0 & c^4_{33} \\ 0 & 0 & c^4_{43} \end{pmatrix}.$$

To be definite, let $\lambda \neq 0$. Put

$$H_1 = \begin{pmatrix} 0 & 0 & h_{13} \\ 0 & 0 & h_{23} \\ h_{31} & h_{32} & h_{33} \\ h_{41} & h_{42} & h_{43} \end{pmatrix},$$

where the set of coefficients $h_{ij}$ $(1 \leqslant i \leqslant 4,\ 1 \leqslant j \leqslant 3)$ is a solution of the following system of linear equations:

$$\begin{cases} c^1_{33} = h_{43} + 2\lambda h_{33}, \\ c^1_{43} = h_{33} + 2\lambda h_{43}, \\ c^3_{43} = h_{42} + h_{13}, \\ c^4_{33} = h_{31} + h_{13}, \\ c^3_{33} = h_{32} - \lambda h_{13} - (\mu - 1)h_{23}, \\ c^4_{43} = h_{41} - \lambda h_{13} - (\mu - 1)h_{23}, \\ c^1_{13} = \lambda h_{13}, \\ c^1_{23} = \lambda h_{23}. \end{cases}$$

Since the matrix of the system is non-degenerate, we see that there exists a unique solution.

Now put $C_1(x) = C(x) + A(x)H_1 - H_1 B(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c^1_{31} & c^1_{32} & 0 \\ c^1_{41} & c^1_{42} & \end{pmatrix}, \quad C_1(e_2) = \begin{pmatrix} 0 & 0 & c^2_{13} \\ 0 & 0 & c^2_{23} \\ c^2_{31} & c^2_{32} & c^2_{33} \\ c^2_{41} & c^2_{42} & c^2_{43} \end{pmatrix},$$

$$C_1(e_3) = \begin{pmatrix} c^3_{11} & c^3_{12} & 0 \\ c^3_{21} & c^3_{22} & 0 \\ 0 & 0 & c^3_{33} \\ 0 & 0 & c^3_{43} \end{pmatrix}, \quad C_1(e_4) = \begin{pmatrix} c^4_{11} & c^4_{12} & 0 \\ c^4_{21} & c^4_{22} & 0 \\ 0 & 0 & c^4_{33} \\ 0 & 0 & c^4_{43} \end{pmatrix}.$$

By corollary 2, Charter II, the virtual structures $C$ and $C_1$ are equivalent.

Since for any virtual structure $q$ condition (6), Charter II, must be satisfied, after direct calculation we obtain:

$$C_1(e_i) = 0, \quad i = 1, \dots, 4.$$

If $\lambda = 0$ and $\mu \neq 0$, we can similarly show that the virtual structure $q$ is trivial.

*b*) $\lambda = \mu = 0$. Since

$$\mathfrak{g}^{(0)}(\mathfrak{g}) \supset \mathbb{R}e_1 \oplus \mathbb{R}e_2, \quad \mathfrak{g}^{(1)}(\mathfrak{g}) \supset \mathbb{R}e_3 \oplus \mathbb{R}e_4,$$
$$U^{(1)}(\mathfrak{h}) \supset \mathbb{R}u_1 \oplus \mathbb{R}u_2, \quad U^{(0)}(\mathfrak{h}) \supset \mathbb{R}u_3,$$

we have

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c^1_{31} & c^1_{32} & 0 \\ c^1_{41} & c^1_{42} & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c^2_{13} \\ 0 & 0 & c^2_{23} \\ c^2_{31} & c^2_{32} & c^2_{33} \\ c^2_{41} & c^2_{42} & c^2_{43} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^3_{33} \\ 0 & 0 & c^3_{43} \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^4_{33} \\ 0 & 0 & c^4_{43} \end{pmatrix}.$$

Put

$$H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c^4_{33} & c^3_{33} & 0 \\ c^4_{43} & c^3_{43} & 0 \end{pmatrix},$$

and $C_2(x) = C(x) + A(x)H_2 - H_2B(x)$ for $x \in \mathfrak{g}$. Then

$$C_2(e_1) = \begin{pmatrix} 0 & 0 & c^1_{13} \\ 0 & 0 & c^1_{23} \\ c^1_{31} & c^1_{32} & c^1_{33} \\ c^1_{41} & c^1_{42} & c^1_{43} \end{pmatrix}, \; C(e_2) = \begin{pmatrix} 0 & 0 & c^2_{13} \\ 0 & 0 & c^2_{23} \\ c^2_{31} & c^2_{32} & c^2_{33} \\ c^2_{41} & c^2_{42} & c^2_{43} \end{pmatrix}, \; C(e_3) = C(e_4) = 0.$$

Since for any virtual structure $q$ condition (6), Charter II, must be satisfied, after direct calculation we obtain:

$$\begin{cases} c^1_{13} = -c^1_{42} = c^1_{31}, \\ c^1_{41} = c^1_{32} = -c^1_{23}, \\ c^2_{13} = c^2_{31} = -c^2_{42}, \\ c^2_{41} = c^2_{32} = -c^2_{23}. \end{cases}$$

The proof of the Lemma is complete.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 4.9. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Consider the following cases:

1°. $\lambda^2 + \mu^2 \neq 0$. Then

$$\begin{array}{llll}
[e_1, e_2] = 0, & & & \\
[e_1, e_3] = e_4 - \lambda e_3, & [e_2, e_3] = (1-\mu)e_3, & & \\
[e_1, e_4] = -e_3 - \lambda e_4, & [e_2, e_4] = (1-\mu)e_4, & [e_3, e_4] = 0, & \\
[e_1, u_1] = -u_2, & [e_2, u_1] = u_1, & [e_3, u_1] = 0, & [e_4, u_1] = 0, \\
[e_1, u_2] = u_1, & [e_2, u_2] = u_2, & [e_3, u_2] = 0, & [e_4, u_2] = 0, \\
[e_1, u_3] = \lambda u_3, & [e_2, u_3] = \mu u_3, & [e_3, u_3] = u_2 & [e_4, u_3] = u_1.
\end{array}$$

Using the Jacobi identity we obtain

$$[u_1, u_2] = 0,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = 0,$$

and see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

2°. $\lambda = \mu = 0$. Then

$$\begin{array}{llll}
[e_1, e_2] = 0, & & & \\
[e_1, e_3] = e_4, & [e_2, e_3] = e_3, & & \\
[e_1, e_4] = -e_3, & [e_2, e_4] = e_4, & [e_3, e_4] = 0, & \\
[e_1, u_1] = re_3 - se_4 - u_2, & [e_2, u_1] = qe_3 - pe_4 + u_1, & [e_3, u_1] = 0, & [e_4, u_1] = 0, \\
[e_1, u_2] = -se_3 - re_4 + u_1, & [e_2, u_2] = -pe_3 - qe_4 + u_2, & [e_3, u_2] = 0, & [e_4, u_2] = 0, \\
[e_1, u_3] = re_1 + se_2, & [e_2, u_3] = qe_1 + pe_2, & [e_3, u_3] = u_2, & [e_4, u_3] = u_1.
\end{array}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 10, Charter II). Thus

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}u_3, \qquad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}e_3 \oplus \mathbb{R}e_4 \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_2.$$

Therefore

$$[u_1, u_2] \subset \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}),$$
$$[u_1, u_3] \subset \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}),$$
$$[u_2, u_3] \subset \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}),$$

and

$$[u_1, u_2] = 0,$$
$$[u_1, u_3] = b_3 e_3 + b_4 e_4 + \beta_1 u_1 + \beta_2 u_2,$$
$$[u_2, u_3] = c_3 e_3 + c_4 e_4 + \gamma_1 u_1 + \gamma_2 u_2.$$

Using the Jacobi identity we see that

$$p = 0, \quad q = 0, \quad r = 0, \quad s = 0,$$
$$c_3 = b_4, \quad c_4 = -b_3, \quad \gamma_1 = -\beta_2, \quad \gamma_2 = \beta_1.$$

Consider the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ of the following form:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_4$ | $-e_3$ | $-u_2$ | $u_1$ | $0$ |
| $e_2$ | $0$ | $0$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $0$ |
| $e_3$ | $-e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $xe_3 + ye_4$ |
| $u_2$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $ye_3 - xe_4$ |
| $u_3$ | $0$ | $0$ | $-u_2$ | $-u_1$ | $-xe_3 - ye_4$ | $xe_4 - ye_3$ | $0$ |

,

where

$$x = b_3 + \frac{\beta_1\beta_2}{2}, \qquad y = b_4 + \frac{(\beta_2^2 - \beta_1^2)}{4}.$$

The pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}', \mathfrak{g}')$ are equivalent by means of the mapping $\pi : \bar{\mathfrak{g}}' \to \mathfrak{g}$, where

$$\pi(e_i) = e_i, \ i = 1, \dots, 4,$$
$$\pi(u_1) = u_1 - \frac{\beta_2}{2} e_3 - \frac{\beta_1}{2} e_4,$$
$$\pi(u_2) = u_2 + \frac{\beta_2}{2} e_4 - \frac{\beta_1}{2} e_3,$$
$$\pi(u_3) = u_3 - \frac{\beta_2}{2} e_1 + \frac{\beta_1}{2} e_2.$$

Consider the following cases:

2.1°. $x = y = 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is trivial.

2.2°. $x = 0$ and $y > 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}'$, where

$$\pi(e_i) = e_i, \ i = 1, \dots, 4,$$
$$\pi(u_j) = \frac{1}{\sqrt{y}} u_j, \ j = 1, 2, 3.$$

2.3°. $x = 0$ and $y < 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}'$, where

$$\pi(e_i) = e_i, \ i = 1, \dots, 4,$$
$$\pi(u_j) = \frac{1}{\sqrt{-y}} u_j, \ j = 1, 2, 3.$$

2.4°. $x > 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}'$, where

$$\pi(e_i) = e_i, \ i = 1, \dots, 4,$$
$$\pi(u_j) = \frac{1}{\sqrt{x}} u_j, \ j = 1, 2, 3,$$

and $\alpha = y/x$.

2.5°. $x < 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}'$, where

$$\pi(e_i) = e_i, \ i = 1, \dots, 4,$$
$$\pi(u_j) = \frac{1}{\sqrt{-x}} u_j, \ j = 1, 2, 3,$$

and $\alpha = -y/x$.

It remains to show that the pairs determined in the Proposition are not equivalent to each other whenever $\lambda = \mu = 0$.

Consider the homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(n, \mathbb{R})$, $i = 1, \dots, 5$, where $f_i(x)$ is the matrix of the mapping $\mathrm{ad}_{\mathcal{D}\bar{\mathfrak{g}}_i} x$ in the basis $\{e_3, e_4, u_1, u_2\}$.

Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$, $i = 1, \dots, 5$, are not conjugate to each other for all values of parameters, we conclude that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1, \dots, 5$, are not equivalent.

**Proposition 4.10.** *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 4.10 is equivalent to one and only one of the following pairs:*

*1.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $e_1$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_2$ | $-e_1$ | $0$ | $0$ | $0$ | $0$ | $u_2$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $e_3$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $0$ | $0$ | $-e_3$ | $0$ | $0$ | $0$ | $u_3$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $0$ | $0$ | $-u_1$ | $-u_3$ | $0$ | $0$ | $0$ |

*2.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $e_1$ | $0$ | $0$ | $0$ | $e_2 + u_1$ | $0$ |
| $e_2$ | $-e_1$ | $0$ | $0$ | $0$ | $0$ | $u_2$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $e_3$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $0$ | $0$ | $-e_3$ | $0$ | $0$ | $0$ | $u_3$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-e_2 - u_1$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $0$ | $0$ | $-u_1$ | $-u_3$ | $0$ | $0$ | $0$ |

*3.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $e_1$ | $0$ | $0$ | $0$ | $e_2 + u_1$ | $0$ |
| $e_2$ | $-e_1$ | $0$ | $0$ | $0$ | $0$ | $u_2$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $e_3$ | $0$ | $0$ | $e_4 + \alpha u_1$ |
| $e_4$ | $0$ | $0$ | $-e_3$ | $0$ | $0$ | $0$ | $u_3$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-e_2 - u_1$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $0$ | $0$ | $-e_4 - \alpha u_1$ | $-u_3$ | $0$ | $0$ | $0$ |

, $\alpha \neq 0$

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors $e_2$ and $e_4$.

**Lemma.** *Any virtual structure $q$ on generalized module 4.10 is equivalent to one of the following:*

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_1(e_2) = 0, \quad C_1(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & s \end{pmatrix}, \quad C_1(e_4) = 0.$$

*Proof.* Let $q$ be a virtual structure on generalized module 4.10. Without loss of generality it can be assumed that $q$ is primary. Since

$$\mathfrak{g}^{(-1,0)}(\mathfrak{h}) = \mathbb{R}e_1, \quad \mathfrak{g}^{(0,0)}(\mathfrak{h}) = \mathbb{R}e_2 \oplus \mathbb{R}e_4, \quad \mathfrak{g}^{(0,-1)}(\mathfrak{h}) = \mathbb{R}e_3,$$

$$U^{(0,0)}(\mathfrak{h}) = \mathbb{R}u_1, \quad U^{(1,0)} = \mathbb{R}u_2, \quad U^{(0,1)}(\mathfrak{h}) = \mathbb{R}u_3,$$

we have

$$C(e_1) = \begin{pmatrix} c_{11}^1 & 0 & 0 \\ 0 & c_{22}^1 & 0 \\ 0 & 0 & 0 \\ 0 & c_{42}^1 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ c_{21}^2 & 0 & 0 \\ 0 & 0 & 0 \\ c_{41}^2 & 0 & 0 \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{23}^3 \\ c_{31}^3 & 0 & 0 \\ 0 & 0 & c_{43}^3 \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ c_{21}^4 & 0 & 0 \\ 0 & 0 & 0 \\ c_{41}^4 & 0 & 0 \end{pmatrix}.$$

Checking condition (6), Chapter II, we obtain $c_{21}^2 = c_{41}^2 = c_{21}^4 = c_{41}^4 = 0$ and $c_{23}^3 = -c_{11}^1$, $c_{31}^3 = -c_{42}^1$. Put

$$H = \begin{pmatrix} 0 & 0 & 0 \\ -c_{11}^1 & 0 & 0 \\ 0 & 0 & 0 \\ c_{42}^1 & 0 & 0 \end{pmatrix}$$

and $C_1(x) = C(x) + A(x)H - HB(x)$. Then

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_{22}^1 + c_{11}^1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_1(e_2) = 0,$$

$$C_1(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{43}^3 - c_{42}^1 \end{pmatrix}, \quad C_1(e_4) = 0.$$

By corollary 2, Chapter II, the virtual structures $C$ and $C_1$ on generalized module 4.10 are equivalent. This proves the Lemma.

Thus it can be assumed that the virtual structure $q$ has the form determined in the Lemma. Then

$$\begin{array}{llll}
[e_1, e_2] = e_1, & & & \\
[e_1, e_3] = 0, & [e_2, e_3] = 0, & & \\
[e_1, e_4] = 0, & [e_2, e_4] = 0, & [e_3, e_4] = e_3, & \\
[e_1, u_1] = 0, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = 0, \\
[e_1, u_2] = pe_2 + u_1, & [e_2, u_2] = u_2, & [e_3, u_2] = 0, & [e_4, u_2] = 0, \\
[e_1, u_3] = 0, & [e_2, u_3] = 0, & [e_3, u_3] = se_4 + u_1, & [e_4, u_3] = u_3.
\end{array}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 10, Chapter II). Thus

$$\bar{\mathfrak{g}}^{(-1,0)}(\mathfrak{h}) = \mathbb{R}e_1, \quad \bar{\mathfrak{g}}^{(0,-1)}(\mathfrak{h}) = \mathbb{R}e_3,$$

$$\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) = \mathbb{R}e_2 \oplus \mathbb{R}e_4 \oplus \mathbb{R}u_1,$$

$$\bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) = \mathbb{R}u_2, \quad \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}) = \mathbb{R}u_3,$$

$$\begin{array}{lcl}
[u_1, u_2] \in \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}), & & [u_1, u_2] = \alpha_2 u_2, \\
[u_1, u_3] \in \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}), & \text{and} & [u_1, u_3] = \beta_3 u_3, \\
[u_2, u_3] \in \bar{\mathfrak{g}}^{(1,1)}(\mathfrak{h}), & & [u_2, u_3] = 0.
\end{array}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | $e_1$ | 0 | 0 | 0 | $pe_2 + u_1$ | 0 |
| $e_2$ | $-e_1$ | 0 | 0 | 0 | 0 | $u_2$ | 0 |
| $e_3$ | 0 | 0 | 0 | $e_3$ | 0 | 0 | $se_4 + u_1$ |
| $e_4$ | 0 | 0 | $-e_3$ | 0 | 0 | 0 | $u_3$ |
| $u_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | $-pe_2 - u_1$ | $-u_2$ | 0 | 0 | 0 | 0 | 0 |
| $u_3$ | 0 | 0 | $-se_4 - u_1$ | $-u_3$ | 0 | 0 | 0 |

Consider the following cases:

1°. $p = s = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2°. $p \neq 0$, $s = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i, \ i = 1, \dots, 4,$$
$$\pi(u_i) = pu_i, \ i = 1, 2, 3.$$

3°. $p = 0,\ s \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_1) &= e_3, \\
\pi(e_2) &= e_4, \\
\pi(e_3) &= e_1, \\
\pi(e_4) &= e_2, \\
\pi(u_1) &= su_1, \\
\pi(u_2) &= su_3, \\
\pi(u_3) &= su_2.
\end{aligned}$$

4°. $p \neq 0,\ s \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
\pi(e_i) &= e_i,\ i = 1, \ldots, 4, \\
\pi(u_1) &= pu_1, \\
\pi(u_2) &= pu_2, \\
\pi(u_3) &= su_3.
\end{aligned}$$

Now it remains to show that the pairs determined in the Proposition are not equivalent to each other.

Since $\dim \mathcal{D}^2\bar{\mathfrak{g}}_1 = 1$, $\dim \mathcal{D}^2\bar{\mathfrak{g}}_2 = 4$, and $\dim \mathcal{D}^2\bar{\mathfrak{g}}_3 = 6$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$, and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent to each other.

This completes the proof of the Proposition.

**Proposition 4.11.** *Any pair $(\bar{\mathfrak{g}},\ \mathfrak{g})$ of type 4.11 is equivalent to one and only one of the following pairs:*

*1.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $(1-\lambda)e_4$ | $u_1$ | $0$ | $\lambda u_3$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $-\mu e_4$ | $0$ | $u_2$ | $\mu u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $(\lambda-1)e_4$ | $\mu e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-\lambda u_3$ | $-\mu u_3$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ |

*2.* $\lambda = 0,\ \mu = -1$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $e_4$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $e_4$ | $0$ | $u_2$ | $-u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $e_4$ | $e_3$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $-e_4$ | $0$ | $e_2$ |
| $u_3$ | $0$ | $u_3$ | $0$ | $-u_1$ | $-e_3$ | $-e_2$ | $0$ |

*3.* $\lambda = 0$, $\mu = 0$

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $e_4$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $0$ | $0$ | $u_2$ | $0$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_3$ | $0$ | $0$ | $0$ | $-u_1$ | $-u_1$ | $-u_2$ | $0$ |

*4.* $\lambda = \frac{1}{2}$, $\mu = -\frac{1}{2}$

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $\frac{1}{2}e_4$ | $u_1$ | $0$ | $\frac{1}{2}u_3$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $\frac{1}{2}e_4$ | $0$ | $u_2$ | $-\frac{1}{2}u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-\frac{1}{2}e_4$ | $-\frac{1}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $e_4$ |
| $u_3$ | $-\frac{1}{2}u_3$ | $\frac{1}{2}u_3$ | $0$ | $-u_1$ | $0$ | $-e_4$ | $0$ |

*5.* $\lambda = 0$, $\mu = \frac{1}{2}$

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $e_4$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $-\frac{1}{2}e_4$ | $0$ | $u_2$ | $\frac{1}{2}u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $0$ | $u_1$ | $e_4$ |
| $e_4$ | $-e_4$ | $\frac{1}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $0$ | $-\frac{1}{2}u_3$ | $-e_4$ | $-u_1$ | $0$ | $0$ | $0$ |

*6.* $\lambda = \frac{1}{2}$, $\mu = 0$

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $\frac{1}{2}e_4$ | $u_1$ | $0$ | $\frac{1}{2}u_3$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $0$ | $0$ | $u_2$ | $e_4$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-\frac{1}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-\frac{1}{2}u_3$ | $-e_4$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ |

*7.* $\lambda = \frac{1}{2}$, $\mu = 0$

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $\frac{1}{2}e_4$ | $u_1$ | $0$ | $e_4 + \frac{1}{2}u_3$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $0$ | $0$ | $u_2$ | $\alpha e_4$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-\frac{1}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-e_4 - \frac{1}{2}u_3$ | $-\alpha e_4$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ |

8. $\lambda = 1,\ \mu = -1$

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $e_4$ | $e_3+u_3$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $e_4$ | $0$ | $u_2$ | $-u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $0$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-e_4$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-e_3-u_3$ | $u_3$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ |

9. $\lambda = 1,\ \mu = -1$

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $\alpha e_4$ | $\alpha e_3+u_3$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $e_4$ | $0$ | $e_4+u_2$ | $e_3-u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $0$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-\alpha e_4$ | $-e_4-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-\alpha e_3-u_3$ | $-e_3+u_3$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ |

*Proof.* Let $\mathcal{E} = \{e_1,\ e_2,\ e_3,\ e_4\}$ be basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix},\quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\mu \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda-1 & \mu & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by vectors $e_1, e_2$.

**Lemma.** *Any virtual structure $q$ on generalized module 4.11 is equivalent to one of the following:*

*a)* $(\lambda,\ \mu) \notin \{(0, \frac{1}{2}),\ (\frac{1}{2}, 0),\ (1, -1)\}$

$$C_1(e_i) = 0, \quad i = 1, \dots, 4;$$

*b)* $\lambda = 0,\ \mu = \frac{1}{2}$

$$C_2(e_i) = 0,\ i = 1, 2, 4, \quad C_2(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix};$$

*c)* $\lambda = \frac{1}{2},\ \mu = 0$

$$C_3(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C_3(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q \end{pmatrix}, \quad C(e_3) = C(e_4) = 0;$$

*d)* $\lambda = 1,\ \mu = -1$

$$C_4(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r \\ 0 & p & 0 \end{pmatrix}, \quad C_4(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & s \\ 0 & q & 0 \end{pmatrix}, \quad C(e_3) = C(e_4) = 0.$$

*Proof.* Let $q$ be a virtual structure on the generalized module 2.19. Without loss of generality it can be assumed that $q$ is primary. Since

$$\begin{aligned} &\mathfrak{g}^{(0,0)}(\mathfrak{h}) \supseteq \mathbb{R}e_1 \oplus \mathbb{R}e_2, && U^{(1,0)}(\mathfrak{h}) \supseteq \mathbb{R}u_1, \\ &\mathfrak{g}^{(1,-1)}(\mathfrak{h}) = \mathbb{R}e_3, && U^{(0,1)}(\mathfrak{h}) = \mathbb{R}u_2, \\ &\mathfrak{g}^{(1-\lambda,\,-\mu)}(\mathfrak{h}) \supseteq \mathbb{R}e_4, && U^{(\lambda,\,\mu)}(\mathfrak{h}) \supseteq \mathbb{R}u_3, \end{aligned}$$

we have

$$C(e_1) = \begin{pmatrix} 0 & 0 & c^1_{13} \\ 0 & 0 & c^1_{23} \\ 0 & 0 & c^1_{33} \\ c^1_{41} & c^1_{42} & c^1_{43} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c^2_{13} \\ 0 & 0 & c^2_{23} \\ 0 & 0 & c^2_{33} \\ c^2_{41} & c^2_{42} & c^2_{43} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^3_{33} \\ 0 & c^3_{42} & c^3_{43} \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} c^4_{11} & 0 & 0 \\ c^4_{21} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^4_{43} \end{pmatrix}.$$

Since for any virtual structure $q$ condition (6), Chapter II, must be satisfied, after direct calculation we obtain:

$$C(e_1) = \begin{pmatrix} 0 & 0 & c^1_{13} \\ 0 & 0 & c^1_{23} \\ 0 & 0 & c^1_{33} \\ c^1_{41} & c^1_{42} & c^1_{43} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c^2_{13} \\ 0 & 0 & c^2_{23} \\ 0 & 0 & c^2_{33} \\ c^2_{41} & c^2_{42} & c^2_{43} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^3_{33} \\ 0 & c^3_{42} & c^3_{43} \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^4_{43} \end{pmatrix},$$

where the set of coefficients $c_{ij}^k$ satisfies the following system:

$$\begin{cases} c_{33}^1(1+\mu) = c_{33}^2(\lambda-1), \\ c_{42}^1(1+\mu) = c_{42}^2(\lambda-1), \\ \lambda c_{43}^3 = 0, \\ (1-2\mu)c_{43}^3 = 0, \\ (2\lambda-1)c_{43}^2 = 2\mu c_{43}^1, \\ \lambda c_{13}^2 = \mu c_{13}^1, \\ \lambda c_{23}^2 = \mu c_{23}^1, \\ c_{13}^1 = c_{23}^1 + \lambda c_{33}^3, \\ c_{13}^2 = c_{23}^2 + \mu c_{33}^3, \\ c_{41}^1 = \lambda c_{42}^3, \\ c_{41}^2 = \mu c_{42}^3, \\ \lambda c_{43}^4 = (1-\lambda)c_{13}^1 - \mu c_{23}^1 + c_{41}^1, \\ \mu c_{43}^4 = (1-\lambda)c_{13}^2 - \mu c_{23}^2 + c_{41}^2. \end{cases}$$

Consider the following cases:

1. $\lambda = 0, \ \mu = 0$. Put

$$H = \begin{pmatrix} 0 & 0 & c_{43}^4 - c_{42}^3 \\ 0 & 0 & -c_{33}^3 + c_{43}^4 - c_{42}^3 \\ 0 & 0 & 0 \\ c_{42}^3 & 0 & 0 \end{pmatrix},$$

2. $\lambda = 0, \ -1 \leqslant \mu < 0, \ 0 < \mu < 1$. Put

$$H = \begin{pmatrix} 0 & 0 & \frac{1}{\mu}c_{13}^2 \\ 0 & 0 & \frac{1}{\mu}c_{23}^2 \\ 0 & 0 & 0 \\ c_{42}^3 & 0 & 0 \end{pmatrix},$$

3. $\lambda \neq 0, \ -1 \leqslant \mu < 1$. Put

$$H = \begin{pmatrix} 0 & 0 & \frac{1}{\lambda}c_{13}^1 \\ 0 & 0 & \frac{1}{\lambda}c_{23}^1 \\ 0 & 0 & 0 \\ c_{42}^3 & 0 & 0 \end{pmatrix}$$

and put $C'(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C'(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{33}^1 \\ 0 & c_{42}^1 & c_{43}^1 \end{pmatrix}, \quad C'(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{33}^2 \\ 0 & c_{42}^2 & c_{43}^2 \end{pmatrix},$$

$$C'(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^3_{43} \end{pmatrix}, \quad C'(e_4) = 0,$$

where the set of coefficients $c^k_{ij}$ satisfies the following system:

$$\begin{cases} c^1_{33}(1+\mu) = c^2_{33}(\lambda - 1), \\ c^1_{42}(1+\mu) = c^2_{42}(\lambda - 1), \\ \lambda c^3_{43} = 0, \\ (1-2\mu)c^3_{43} = 0, \\ (2\lambda - 1)c^2_{43} = 2\mu c^1_{43}. \end{cases}$$

By corollary 2, Chapter II, the virtual structures $C$ and $C'$ are equivalent.

Consider the following cases:

*a*) $(\lambda, \mu) \notin \{(0, \frac{1}{2}), (\frac{1}{2}, 0), (1, -1)\}$. Put

$$H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{33} \\ 0 & h_{42} & h_{43} \end{pmatrix},$$

where the set of coefficients $h_{ij}$ is a solution of the following system:

$$\begin{cases} h_{33} = \frac{1}{\lambda - 1} c^1_{33} \text{ or } h_{33} = \frac{1}{1+\mu} c^2_{33} \\ h_{42} = \frac{1}{\lambda - 1} c^1_{42} \text{ or } h_{42} = \frac{1}{1+\mu} c^2_{42}, \\ h_{43} = \frac{1}{2\lambda - 1} c^1_{43} \text{ or } h_{43} = \frac{1}{2\mu} c^2_{43} \end{cases}$$

and put $C_1(x) = C'(x) + A(x)H_1 - H_1 B(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_i) = 0, \quad i = 1, \ldots, 4.$$

*b*) $\lambda = 0$, $\mu = \frac{1}{2}$. Put

$$H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -c^1_{33} \\ 0 & -c^1_{42} & -c^1_{43} \end{pmatrix}$$

and $C_2(x) = C'(x) + A(x)H_2 - H_2 B(x)$ for $x \in \mathfrak{g}$. Then

$$C_2(e_i) = 0, \ i = 1, 2, 4, \quad C_2(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^3_{43} \end{pmatrix}.$$

*c)* $\lambda = \frac{1}{2}$, $\mu = 0$. Put

$$H_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{33}^2 \\ 0 & c_{42}^2 & 0 \end{pmatrix}$$

and $C_3(x) = C'(x) + A(x)H_3 - H_3B(x)$ for $x \in \mathfrak{g}$. Then

$$C_3(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{43}^1 \end{pmatrix}, \quad C_3(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{43}^2 \end{pmatrix}, \quad C_3(e_3) = C_3(e_4) = 0.$$

*d)* $\lambda = 1$, $\mu = -1$. Put

$$H_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{43}^1 \end{pmatrix}$$

and $C_4(x) = C'(x) + A(x)H_4 - H_4B(x)$ for $x \in \mathfrak{g}$. Then

$$C_4(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{33}^1 \\ 0 & c_{42}^1 & 0 \end{pmatrix}, \quad C_4(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{33}^2 \\ 0 & c_{42}^2 & 0 \end{pmatrix}, \quad C_4(e_3) = C_4(e_4) = 0.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 4.11. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Consider the following cases:

1°. $-1 \leqslant \mu < 1$. Then

$$\begin{array}{llll}
[e_1, e_2] = 0, & & & \\
[e_1, e_3] = e_3, & [e_2, e_3] = -e_3, & & \\
[e_1, e_4] = (1-\lambda)e_4, & [e_2, e_4] = -\mu e_4, & [e_3, e_4] = 0, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = 0, \\
[e_1, u_2] = 0, & [e_2, u_2] = u_2, & [e_3, u_2] = u_1, & [e_4, u_2] = 0, \\
[e_1, u_3] = \lambda u_3, & [e_2, u_3] = \mu u_3, & [e_3, u_3] = 0, & [e_4, u_3] = u_1.
\end{array}$$

Since the virtual structure $q$ is primary, we have

$$\begin{array}{ll}
\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) \supseteq \mathbb{R}e_1 \oplus \mathbb{R}e_2, & \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) \supseteq \mathbb{R}u_1, \\
\bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) = \mathbb{R}e_3, & \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}) = \mathbb{R}u_2, \\
\bar{\mathfrak{g}}^{(1-\lambda,-\mu)}(\mathfrak{h}) \supseteq \mathbb{R}e_4, & \bar{\mathfrak{g}}^{(\lambda,\mu)}(\mathfrak{h}) \supseteq \mathbb{R}u_3.
\end{array}$$

Therefore

$$[u_1, u_2] = a_4 e_4,$$
$$[u_1, u_3] = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + \beta_1 u_1,$$
$$[u_2, u_3] = c_1 e_1 + c_2 e_2 + c_4 e_4 + \gamma_1 u_1 + \gamma_2 u_2.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $(1-\lambda)e_4$ | $u_1$ | $0$ | $\lambda u_3$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $-\mu e_4$ | $0$ | $u_2$ | $\mu u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $(\lambda-1)e_4$ | $\mu e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $c_2 e_4$ | $A$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $-c_2 e_4$ | $0$ | $B$ |
| $u_3$ | $-\lambda u_3$ | $-\mu u_3$ | $0$ | $-u_1$ | $-A$ | $-B$ | $0$ , |

$$A = c_2 e_3 + \gamma_2 u_1,$$
$$B = c_2 e_2 + c_4 e_4 + \gamma_1 u_1 + \gamma_2 u_2,$$

where the coefficients $c_2$, $c_4$, $\gamma_1$ and $\gamma_2$ satisfy the following system:

$$\begin{cases} \lambda c_2 = (1+\mu)c_2 = 0, \\ \lambda \gamma_2 = \mu \gamma_2 = 0, \\ (1-2\lambda)c_4 = (1+2\mu)c_4 = 0, \\ (1-\lambda)\gamma_1 = (1+\mu)\gamma_1 = 0. \end{cases}$$

Consider the following cases:

1.1°. $c_2 = c_4 = \gamma_1 = \gamma_2 = 0$.
Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

1.2°. $c_2 \neq 0,\ \lambda = 0,\ \mu = -1,\ c_4 = \gamma_1 = \gamma_2 = 0$.
The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i,\ i = 1, 2, 4,$$
$$\pi(e_3) = \frac{1}{c_2} e_3,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = c_2 u_2,$$
$$\pi(u_3) = u_3.$$

Since $\dim \mathcal{D}^2 \bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2 \bar{\mathfrak{g}}_2$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

1.3°. $\gamma_2 \neq 0,\ \lambda = \mu = 0,\ c_2 = c_4 = \gamma_1 = 0$.
Then the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$ such that

$$\pi(e_i) = e_i,\ i = 1, 2, 3,$$
$$\pi(e_4) = \frac{1}{\gamma_2} e_4,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2,$$
$$\pi(u_3) = \gamma_2 u_3,$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$.

It is easily proved that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent.

1.4°. $c_4 \neq 0,\ \lambda = \frac{1}{2},\ \mu = -\frac{1}{2},\ c_2 = \gamma_1 = \gamma_2 = 0.$

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_i) = e_i,\ i = 1, 2, 4,\\ &\pi(e_3) = \frac{1}{c_4} e_3,\\ &\pi(u_1) = u_1,\\ &\pi(u_2) = c_4 u_2,\\ &\pi(u_3) = u_3. \end{aligned}$$

Since $\dim \mathcal{D}^2 \bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2 \bar{\mathfrak{g}}_4$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ are not equivalent.

1.5°.$\gamma_1 \neq 0,\ \lambda = 1,\ \mu = -1,\ c_2 = c_4 = \gamma_2 = 0.$

Then the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$ such that

$$\begin{aligned} &\pi(e_i) = e_i,\ i = 1, \dots, 4,\\ &\pi(u_1) = u_1,\\ &\pi(u_2) = \gamma_1 e_4 + u_2,\\ &\pi(u_3) = u_3, \end{aligned}$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ with $\lambda = 1, \mu = -1$.

2°. $\lambda = 0,\ \mu = \frac{1}{2}$. Then

$$\begin{array}{llll} [e_1, e_2] = 0, & & & \\ [e_1, e_3] = e_3, & [e_2, e_3] = -e_3, & & \\ [e_1, e_4] = e_4, & [e_2, e_4] = -\dfrac{1}{2} e_4, & [e_3, e_4] = 0, & \\ [e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = 0, \\ [e_1, u_2] = 0, & [e_2, u_2] = u_2, & [e_3, u_2] = u_1, & [e_4, u_2] = 0, \\ [e_1, u_3] = 0, & [e_2, u_3] = \dfrac{1}{2} u_3, & [e_3, u_3] = p e_4, & [e_4, u_3] = u_1. \end{array}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1,-1/2)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(0,1/2)},$$

where

$$\begin{array}{ll} \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2, & \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) = \mathbb{R}u_1, \\ \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) = \mathbb{R}e_3, & \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}) = \mathbb{R}u_2, \\ \bar{\mathfrak{g}}^{(1,-1/2)}(\mathfrak{h}) = \mathbb{R}e_4, & \bar{\mathfrak{g}}^{(0,1/2)}(\mathfrak{h}) = \mathbb{R}u_3. \end{array}$$

Therefore

$$[u_1, u_2] = 0,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = 0.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $e_4$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $-\frac{1}{2}e_4$ | $0$ | $u_2$ | $\frac{1}{2}u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $0$ | $u_1$ | $pe_4$ |
| $e_4$ | $-e_4$ | $\frac{1}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $0$ | $-\frac{1}{2}u_3$ | $-pe_4$ | $-u_1$ | $0$ | $0$ | $0$ |

2.1°. $p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

2.2°. $p \neq 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i, \; i = 1, \ldots, 4,$$
$$\pi(u_j) = pu_j, \; j = 1, 2, 3.$$

Since $\dim \mathcal{D}^2 \bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2 \bar{\mathfrak{g}}_5$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ are not equivalent.

3°. $\lambda = \frac{1}{2}$, $\mu = 0$. Then

$$\begin{array}{llll} [e_1, e_2] = 0, & & & \\ [e_1, e_3] = e_3, & [e_2, e_3] = -e_3, & & \\ [e_1, e_4] = \dfrac{1}{2}e_4, & [e_2, e_4] = 0, & [e_3, e_4] = 0, & \\ [e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = 0, \\ [e_1, u_2] = 0, & [e_2, u_2] = u_2, & [e_3, u_2] = u_1, & [e_4, u_2] = 0, \\ [e_1, u_3] = pe_4 + \dfrac{1}{2}u_3, & [e_2, u_3] = qe_4, & [e_3, u_3] = 0, & [e_4, u_3] = u_1. \end{array}$$

Since the virtual structure $q$ is primary, we have

$$[u_1, u_2] = 0,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = 0.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $\frac{1}{2}e_4$ | $u_1$ | $0$ | $pe_4 + \frac{1}{2}u_3$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $0$ | $0$ | $u_2$ | $qe_4$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-\frac{1}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-pe_4 - \frac{1}{2}u_3$ | $-qe_4$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ |

Now we determine the group of all transformations for mappings $q$. We have

$$C(e_1)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&p\end{pmatrix},\quad C(e_2)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&q\end{pmatrix},\quad C(e_3)=C(e_4)=0.$$

Put

$$C'(e_1)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&p'\end{pmatrix},\quad C'(e_2)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&q'\end{pmatrix},\quad C'(e_3)=C'(e_4)=0.$$

The virtual structures $C$ and $C'$ are equivalent if and only if there exist matrices $P\in\mathcal{A}(\mathfrak{g})$ and $H\in\mathrm{Mat}_{4\times3}(\mathbb{R})$ such that

$$C'(x)=FC(\varphi^{-1}(x))P^{-1}+A(x)H-HB(x)\quad\text{for all}\quad x\in\mathfrak{g},$$

where $\varphi(x)=PxP^{-1}$ and $F$ is the matrix of the mapping $\varphi$. Direct calculation shows that the virtual structures $C$ and $C'$ are equivalent if and only if there exist numbers $a,b$ such that $ab\neq0$ and the following conditions are satisfied:

$$p'=\frac{a}{b^2}p,\qquad q'=\frac{a}{b^2}q.$$

Using these conditions we see that any virtual structure on the generalized module 4.11 ($\lambda=\frac{1}{2}$, $\mu=0$) is equivalent to one and only one of the following:

*a*)

$$C^1(e_i)=0,\quad i=1,\dots,4;$$

*b*)

$$C^2(e_i)=0,\ i=1,3,4,\quad C^2(e_2)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&1\end{pmatrix};$$

*c*)

$$C^3(e_1)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&1\end{pmatrix},\quad C^3(e_2)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&\alpha\end{pmatrix},\quad C^3(e_3)=C^3(e_4)=0.$$

Note that the virtual structure $C^1$ was already considered in case 1°.

For the virtual structures $C^2$, $C^3$ we obtain the following nonequivalent pairs: $(\bar{\mathfrak{g}}_6,\mathfrak{g}_6)$, $(\bar{\mathfrak{g}}_7,\mathfrak{g}_7)$.

4°. $\lambda = 1,\ \mu = -1$. Then

$$\begin{array}{llll}
[e_1, e_2] = 0, & & & \\
[e_1, e_3] = e_3, & [e_2, e_3] = -e_3, & & \\
[e_1, e_4] = 0, & [e_2, e_4] = e_4, & [e_3, e_4] = 0, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = 0, \\
[e_1, u_2] = pe_4, & [e_2, u_2] = qe_4 + u_2, & [e_3, u_2] = u_1, & [e_4, u_2] = 0, \\
[e_1, u_3] = re_3 + u_3, & [e_2, u_3] = se_3 - u_3, & [e_3, u_3] = 0, & [e_4, u_3] = u_1.
\end{array}$$

Since the virtual structure $q$ is primary, we have

$$[u_1, u_2] = 0,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = \gamma_1 u_1.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $e_3$ | 0 | $u_1$ | $pe_4$ | $pe_3+u_3$ |
| $e_2$ | 0 | 0 | $-e_3$ | $e_4$ | 0 | $qe_4+u_2$ | $qe_3-u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | 0 | 0 | 0 | $u_1$ | 0 |
| $e_4$ | 0 | $-e_4$ | 0 | 0 | 0 | 0 | $u_1$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | $-pe_4$ | $-qe_4-u_2$ | $-u_1$ | 0 | 0 | 0 | $\gamma_1 u_1$ |
| $u_3$ | $-pe_3-u_3$ | $-qe_3+u_3$ | 0 | $-u_1$ | 0 | $-\gamma_1 u_1$ | 0 |

The mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$ such that

$$\pi(e_i) = e_i,\ i = 1, \ldots, 4,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = \gamma_1 e_4 + u_2,$$
$$\pi(u_3) = u_3,$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}', \mathfrak{g}')$, where the latter has the form:

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $e_3$ | 0 | $u_1$ | $pe_4$ | $pe_3+u_3$ |
| $e_2$ | 0 | 0 | $-e_3$ | $e_4$ | 0 | $qe_4+u_2$ | $qe_3-u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | 0 | 0 | 0 | $u_1$ | 0 |
| $e_4$ | 0 | $-e_4$ | 0 | 0 | 0 | 0 | $u_1$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | $-pe_4$ | $-qe_4-u_2$ | $-u_1$ | 0 | 0 | 0 | 0 |
| $u_3$ | $-pe_3-u_3$ | $-qe_3+u_3$ | 0 | $-u_1$ | 0 | 0 | 0 |

Now we determine the group of all transformations for mappings $q$. We have

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & p & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q \\ 0 & q & 0 \end{pmatrix}, \quad C(e_3) = C(e_4) = 0.$$

Put

$$C'(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p' \\ 0 & p' & 0 \end{pmatrix}, \quad C'(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q' \\ 0 & q' & 0 \end{pmatrix}, \quad C'(e_3) = C'(e_4) = 0.$$

The virtual structures $C$ and $C'$ are equivalent if and only if there exist matrices $P \in \mathcal{A}(\mathfrak{g})$ and $H \in \mathrm{Mat}_{4\times 3}(\mathbb{R})$ such that

$$C'(x) = FC(\varphi^{-1}(x))P^{-1} + A(x)H - HB(x) \quad \text{for all} \quad x \in \mathfrak{g},$$

where $\varphi(x) = PxP^{-1}$ and $F$ is the matrix of the mapping $\varphi$. Direct calculation shows that the virtual structures $C$ and $C'$ are equivalent if and only if there exist numbers $a, b, c$ such that $abc \neq 0$ and the following conditions are satisfied:

$$p' = \frac{a}{bc}p, \qquad q' = \frac{a}{bc}q.$$

Using these conditions we see that any virtual structure on the generalized module 4.11 ($\lambda = 1,\ \mu = -1$) is equivalent to one and only one of the following ones:

*a*)

$$C^1(e_i) = 0, \quad i = 1, \ldots, 4;$$

*b*)

$$C^2(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad C^2(e_i) = 0\ i = 2, 3, 4;$$

*c*)

$$C^3(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & \alpha & 0 \end{pmatrix}, \quad C^3(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad C^3(e_3) = C^3(e_4) = 0.$$

Note that the virtual structure $C^1$ was already considered in case 1°.

For the virtual structures $C^2$, $C^3$ we obtain the following nonequivalent pairs: $(\bar{\mathfrak{g}}_8, \mathfrak{g}_8)$, $(\bar{\mathfrak{g}}_9, \mathfrak{g}_9)$.

**Proposition 4.12.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.12 is equivalent to one and only one of the following pairs:*

*1.*

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $e_3$ | $(1-\lambda)e_4$ | $u_1$ | 0 | $\lambda u_3$ |
| $e_2$ | 0 | 0 | $-e_3$ | $-e_4$ | 0 | $u_2$ | $u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | 0 | 0 | 0 | $u_1$ | 0 |
| $e_4$ | $(\lambda-1)e_4$ | $e_4$ | 0 | 0 | 0 | 0 | $u_1$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | 0 | $-u_2$ | $-u_1$ | 0 | 0 | 0 | 0 |
| $u_3$ | $-\lambda u_3$ | $-u_3$ | 0 | $-u_1$ | 0 | 0 | 0 |

*2.* $\lambda = 1$

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $e_3$ | 0 | $u_1$ | 0 | $u_3$ |
| $e_2$ | 0 | 0 | $-e_3$ | $-e_4$ | 0 | $u_2$ | $u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | 0 | 0 | 0 | $u_1$ | 0 |
| $e_4$ | 0 | $e_4$ | 0 | 0 | $e_3$ | $e_2$ | $u_1$ |
| $u_1$ | $-u_1$ | 0 | 0 | $-e_3$ | 0 | $u_3$ | 0 |
| $u_2$ | 0 | $-u_2$ | $-u_1$ | $-e_2$ | $-u_3$ | 0 | 0 |
| $u_3$ | $-u_3$ | $-u_3$ | 0 | $-u_1$ | 0 | 0 | 0 |

*Proof.*
Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda - 1 & 1 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors $e_1$ and $e_2$.

**Lemma.** *Any virtual structure $q$ on generalized module 4.6 is equivalent to one of the following:*

*a)* $\lambda = 1$

$$C(e_i) = 0, \ i = 1, 2, 3, \quad C(e_4) = \begin{pmatrix} 0 & p & 0 \\ 0 & q & 0 \\ q - p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

*b)* $\lambda \neq 1$

$$C(e_i) = 0, \quad i = 1, \ldots, 4.$$

*Proof.* Let $q$ be a virtual structure on generalized module 4.12. Without loss of generality it can be assumed that $q$ is primary. Note that

$$\mathfrak{g} = \mathfrak{g}^{(0,0)}(\mathfrak{h}) \oplus \mathfrak{g}^{(1-\lambda,-1)}(\mathfrak{h}) \oplus \mathfrak{g}^{(1,-1)}(\mathfrak{h}),$$

$$U = U^{(1,0)}(\mathfrak{h}) \oplus U^{(\lambda,1)}(\mathfrak{h}) \oplus U^{(0,1)}(\mathfrak{h}),$$

where

$$\begin{array}{lll} \mathfrak{g}^{(0,0)}(\mathfrak{h}) \supset \mathbb{R}e_1 \oplus \mathbb{R}e_2, & \mathfrak{g}^{(1,-1)}(\mathfrak{h}) \supset \mathbb{R}e_3, & \mathfrak{g}^{(1-\lambda,-1)}(\mathfrak{h}) \supset \mathbb{R}e_4, \\ U^{(1,0)}(\mathfrak{h}) \supset \mathbb{R}u_1, & U^{(0,1)}(\mathfrak{h}) \supset \mathbb{R}u_2, & U^{(\lambda,1)}(\mathfrak{h}) \supset \mathbb{R}u_3. \end{array}$$

Consider the following cases:

*a)* $\lambda = 1$. We have

$$C(e_i) = 0,\ i = 1,2,3, \quad C(e_4) = \begin{pmatrix} 0 & c^4_{12} & 0 \\ 0 & c^4_{22} & 0 \\ c^4_{31} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since for any virtual structure $q$ condition (6), Chapter II, must be satisfied, after direct calculation we obtain

$$C(e_i) = 0,\ i = 1,2,3, \quad C(e_4) = \begin{pmatrix} 0 & p & 0 \\ 0 & q & 0 \\ q-p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

*b)* $\lambda \neq 1$. We have

$$C(e_i) = 0, \quad i = 1, \ldots, 4.$$

This completes the proof of the Lemma.

Consider the following cases:

1°. $\lambda = 1$.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 4.12. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Then

$$\begin{array}{llll} [e_1, e_2] = 0, & & & \\ [e_1, e_3] = e_3, & [e_2, e_3] = -e_3, & & \\ [e_1, e_4] = 0, & [e_2, e_4] = -e_4, & [e_3, e_4] = 0, & \\ [e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = (q-p)e_3, \\ [e_1, u_2] = 0, & [e_2, u_2] = u_2, & [e_3, u_2] = u_1, & [e_4, u_2] = pe_1 + qe_2, \\ [e_1, u_3] = u_3, & [e_2, u_3] = u_3, & [e_3, u_3] = 0, & [e_4, u_3] = u_1. \end{array}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(0,-1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1,1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}),$$

where

$$\begin{array}{lll} \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) \supset \mathbb{R}e_1 \oplus \mathbb{R}e_2, & \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) \supseteq \mathbb{R}e_3, & \bar{\mathfrak{g}}^{(0,-1)}(\mathfrak{h}) \supseteq \mathbb{R}e_4, \\ \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) \supseteq \mathbb{R}u_1, & \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}) \supseteq \mathbb{R}u_2, & \bar{\mathfrak{g}}^{(1,1)}(\mathfrak{h}) \supseteq \mathbb{R}u_3. \end{array}$$

Therefore

$$[u_1, u_2] = \alpha_3 u_3,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = 0.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ | $u_3$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $-e_4$ | $0$ | $u_2$ | $u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $0$ | $e_4$ | $0$ | $0$ | $pe_3$ | $pe_2$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $-pe_3$ | $0$ | $pu_3$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $-pe_2$ | $-pu_3$ | $0$ | $0$ |
| $u_3$ | $-u_3$ | $-u_3$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ |

1.1°. $p = 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

1.2°. $p \neq 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \mathfrak{g}$, where

$$\pi(e_i) = e_i, \ i = 1, \dots, 4,$$
$$\pi(u_i) = pu_i, \ i = 1, 2, 3,$$

and since $\dim(\bar{\mathfrak{g}}_2, \mathfrak{g}_2) \neq \dim(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, we conclude that the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ are not equivalent.

2°. $\lambda \neq 1$.

Then it can be assumed that the virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Then

$$\begin{array}{llll}
[e_1, e_2] = 0, & & & \\
[e_1, e_3] = e_3, & [e_2, e_3] = -e_3, & & \\
[e_1, e_4] = (1-\lambda)e_4, & [e_2, e_4] = -e_4, & [e_3, e_4] = 0, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = 0, \\
[e_1, u_2] = 0, & [e_2, u_2] = u_2, & [e_3, u_2] = u_1, & [e_4, u_2] = 0, \\
[e_1, u_3] = \lambda u_3, & [e_2, u_3] = u_3, & [e_3, u_3] = 0, & [e_4, u_3] = u_1.
\end{array}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1-\lambda,-1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(\lambda,1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}),$$

where

$$\begin{array}{lll}
\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) \supseteq \mathbb{R}e_1 \oplus \mathbb{R}e_2, & \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) \supseteq \mathbb{R}e_3, & \bar{\mathfrak{g}}^{(1-\lambda,-1)}(\mathfrak{h}) \supseteq \mathbb{R}e_4, \\
\bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) \supseteq \mathbb{R}u_1, & \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}) \supseteq \mathbb{R}u_2, & \bar{\mathfrak{g}}^{(\lambda,1)}(\mathfrak{h}) \supseteq \mathbb{R}u_3.
\end{array}$$

Therefore

$$[u_1, u_2] = 0,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = 0.$$

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

This completes the proof of the Proposition.

**Proposition 4.13.** *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 4.13 is equivalent to one and only one of the following pairs:*

*1.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $e_2$ | $e_3$ | $0$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $-e_2$ | $0$ | $0$ | $\lambda e_2 + e_3$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $-e_3$ | $0$ | $0$ | $-e_2 + \lambda e_3$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $0$ | $-\lambda e_2 - e_3$ | $e_2 - \lambda e_3$ | $0$ | $0$ | $\lambda u_2 - u_3$ | $u_2 + \lambda u_3$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_1$ | $0$ | $u_3 - \lambda u_2$ | $0$ | $0$ | $0$ |
| $u_3$ | $0$ | $0$ | $-u_1$ | $-u_2 - \lambda u_3$ | $0$ | $0$ | $0$ |

*2.* $\lambda = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $e_2$ | $e_3$ | $0$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $-e_2$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $-e_3$ | $0$ | $0$ | $-e_2$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $0$ | $-e_3$ | $e_2$ | $0$ | $0$ | $-u_3$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $e_2$ | $e_3$ |
| $u_2$ | $0$ | $-u_1$ | $0$ | $u_3$ | $-e_2$ | $0$ | $e_4$ |
| $u_3$ | $0$ | $0$ | $-u_1$ | $-u_2$ | $-e_3$ | $-e_4$ | $0$ |

*3.* $\lambda = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $e_2$ | $e_3$ | $0$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $-e_2$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $-e_3$ | $0$ | $0$ | $-e_2$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $0$ | $-e_3$ | $e_2$ | $0$ | $0$ | $-u_3$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $-e_2$ | $-e_3$ |
| $u_2$ | $0$ | $-u_1$ | $0$ | $u_3$ | $e_2$ | $0$ | $-e_4$ |
| $u_3$ | $0$ | $0$ | $-u_1$ | $-u_2$ | $e_3$ | $e_4$ | $0$ |

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & -1 & \lambda \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & -1 & -\lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors $e_1$ and $e_2$.

Consider the complex generalized module $(\mathfrak{g}^{\mathbb{C}}, U^{\mathbb{C}})$. Put

$$\tilde{e}_i = e_i \otimes 1, \quad 1 \leqslant i \leqslant 4$$

and

$$\tilde{u}_j = u_j \otimes 1, \quad 1 \leqslant j \leqslant 3.$$

Then $\tilde{\mathcal{E}} = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ is a basis of $\mathfrak{g}^{\mathbb{C}}$. The vector space $U^{\mathbb{C}}$ can be identified with $\mathbb{C}^3$, and $\{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3\}$ is the standard basis of $U^{\mathbb{C}}$.

**Lemma.** *Any virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.13 is trivial.*

*Proof.* Suppose $(\bar{\mathfrak{g}}, \mathfrak{g})$ is a virtual pair defined by a virtual structure $q$. By Proposition 15, Chapter II, without loss of generality it can be assumed that $q^{\mathbb{C}}$ is primary virtual structure on the generalized module $(\mathfrak{g}^{\mathbb{C}}, U^{\mathbb{C}})$ (with respect to $\mathfrak{h}^{\mathbb{C}}$). Since

$$\begin{aligned}
&\mathfrak{g}^{(0,0)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}e_1 \oplus \mathbb{C}e_4,\\
&\mathfrak{g}^{(1,-\lambda+i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{e}_2 + i\tilde{e}_3),\\
&\mathfrak{g}^{(1,-\lambda-i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{e}_2 - i\tilde{e}_3),\\
&(U^{\mathbb{C}})^{(1,0)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}u_1,\\
&(U^{\mathbb{C}})^{(0,-\lambda+i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{u}_2 + i\tilde{u}_3),\\
&(U^{\mathbb{C}})^{(0,-\lambda-i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{u}_2 - i\tilde{u}_3),
\end{aligned}$$

we have

$$\begin{aligned}
&q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_1) = 0, && q^{\mathbb{C}}(\tilde{e}_2 + i\tilde{e}_3)(\tilde{u}_1) = 0,\\
&q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_2 + i\tilde{u}_3) = 0, && q^{\mathbb{C}}(\tilde{e}_2 + i\tilde{e}_3)(\tilde{u}_2 + i\tilde{u}_3) = 0,\\
&q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_2 - i\tilde{u}_3) = 0, && q^{\mathbb{C}}(\tilde{e}_2 + i\tilde{e}_3)(\tilde{u}_2 - i\tilde{u}_3) = 0,\\
&q^{\mathbb{C}}(\tilde{e}_2 - i\tilde{e}_3)(\tilde{u}_1) = 0, && q^{\mathbb{C}}(\tilde{e}_4)(\tilde{u}_1) = 0,\\
&q^{\mathbb{C}}(\tilde{e}_2 - i\tilde{e}_3)(\tilde{u}_2 + i\tilde{u}_3) = 0, && q^{\mathbb{C}}(\tilde{e}_4)(\tilde{u}_2 + i\tilde{u}_3) = 0,\\
&q^{\mathbb{C}}(\tilde{e}_2 - i\tilde{e}_3)(\tilde{u}_2 - i\tilde{u}_3) = 0, && q^{\mathbb{C}}(\tilde{e}_4)(\tilde{u}_2 - i\tilde{u}_3) = 0.
\end{aligned}$$

Therefore,

$$q^{\mathbb{C}}(\tilde{e}_i)(u_j) = 0, \quad 1 \leqslant i \leqslant 4, \quad 1 \leqslant j \leqslant 3.$$

Since the matrices of the mappings $q(e_i)$ and $q^{\mathbb{C}}(\tilde{e}_i)$, $1 \leqslant i \leqslant 4$, coincide, we obtain

$$C(e_i) = 0, \quad 1 \leqslant i \leqslant 4.$$

This completes the proof of the Lemma.

Thus, it can be assumed that the virtual structure $q$ determining the virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is the zero mapping of $\mathfrak{g}$ into $\mathcal{L}(U, \mathfrak{g})$. Then,

$$\begin{array}{llll}
[e_1, e_2] = 0, & & & \\
[e_1, e_3] = e_3, & [e_2, e_3] = 0, & & \\
[e_1, e_4] = 0, & [e_2, e_4] = \lambda e_2 + e_3, & [e_3, e_4] = -e_2 + \lambda e_3, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = 0, \\
[e_1, u_2] = 0, & [e_2, u_2] = u_1, & [e_3, u_2] = 0, & [e_4, u_2] = \lambda u_2 - u_3, \\
[e_1, u_3] = 0, & [e_2, u_3] = 0, & [e_3, u_3] = u_1, & [e_4, u_3] = u_2 + \lambda u_3.
\end{array}$$

Since $q^{\mathbb{C}}$ is a primary virtual structure on the generalized module $(\mathfrak{g}^{\mathbb{C}}, U^{\mathbb{C}})$, we have

$$(\bar{\mathfrak{g}}^{\mathbb{C}})^{\alpha}(\mathfrak{h}^{\mathbb{C}}) = (\mathfrak{g}^{\mathbb{C}})^{\alpha}(\mathfrak{h}^{\mathbb{C}}) \times (U^{\mathbb{C}})^{\alpha}(\mathfrak{h}^{\mathbb{C}}) \text{ for all } \alpha \in (\mathfrak{h}^{\mathbb{C}})^*$$

(Proposition 10, Chapter II). Thus,

$$\begin{array}{ll}
\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}e_1 \oplus \mathbb{C}e_4, & \bar{\mathfrak{g}}^{(1,-\lambda+i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{e}_2 + i\tilde{e}_3), \\
\bar{\mathfrak{g}}^{(1,-\lambda-i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{e}_2 - i\tilde{e}_3), & \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}u_1, \\
\bar{\mathfrak{g}}^{(0,-\lambda+i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{u}_2 + i\tilde{u}_3), & \bar{\mathfrak{g}}^{(0,-\lambda-i)}(\mathfrak{h}^{\mathbb{C}}) = \mathbb{C}(\tilde{u}_2 - i\tilde{u}_3),
\end{array}$$

and

$$\begin{array}{ll}
[\tilde{u}_1, \tilde{u}_2 + i\tilde{u}_3] & \in \bar{\mathfrak{g}}^{(1,\lambda+i)}(\mathfrak{h}^{\mathbb{C}}), \\
[\tilde{u}_1, \tilde{u}_2 - i\tilde{u}_3] & \in \bar{\mathfrak{g}}^{(1,\lambda-i)}(\mathfrak{h}^{\mathbb{C}}), \\
[\tilde{u}_2 + i\tilde{u}_3, \tilde{u}_2 - i\tilde{u}_3] & \in \bar{\mathfrak{g}}^{(0,2\lambda)}(\mathfrak{h}^{\mathbb{C}}).
\end{array}$$

Consider the following cases:

1°. $\lambda \neq 0$. Then

$$\begin{aligned}
&[\tilde{u}_1, \tilde{u}_2 + i\tilde{u}_3] = 0, \\
&[\tilde{u}_1, \tilde{u}_2 - i\tilde{u}_3] = 0, \\
&[\tilde{u}_2 + i\tilde{u}_3, \tilde{u}_2 - i\tilde{u}_3] = -2i[\tilde{u}_2, \tilde{u}_3] = 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&[u_1, u_2] = 0, \\
&[u_1, u_3] = 0, \\
&[u_2, u_3] = 0,
\end{aligned}$$

and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

2°. $\lambda = 0$. Then

$$\begin{array}{ll}
[\tilde{u}_1, \tilde{u}_2 + i\tilde{u}_3] = [\tilde{u}_1, \tilde{u}_2] + i[\tilde{u}_1, \tilde{u}_3] & \in \mathbb{C}(\tilde{e}_2 + i\tilde{e}_3), \\
[\tilde{u}_1, \tilde{u}_2 - i\tilde{u}_3] = [\tilde{u}_1, \tilde{u}_2] - i[\tilde{u}_1, \tilde{u}_3] & \in \mathbb{C}(\tilde{e}_2 - i\tilde{e}_3), \\
[\tilde{u}_2 + i\tilde{u}_3, \tilde{u}_2 - i\tilde{u}_3] = -2i[\tilde{u}_2, \tilde{u}_3] & \in \mathbb{C}e_1 + \mathbb{C}e_4.
\end{array}$$

Hence,

$$[u_1, u_2] = a_2 e_2 + a_3 e_3,$$
$$[u_1, u_3] = b_2 e_2 + b_3 e_3,$$
$$[u_2, u_3] = c_1 e_1 + c_4 e_4.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $e_2$ | $e_3$ | $0$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $-e_2$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $-e_3$ | $0$ | $0$ | $-e_2 e_3$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $0$ | $-e_3$ | $e_2$ | $0$ | $0$ | $-u_3$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $a_2 e_2$ | $a_2 e_3$ |
| $u_2$ | $0$ | $-u_1$ | $0$ | $u_3$ | $-a_2 e_2$ | $0$ | $a_2 e_4$ |
| $u_3$ | $0$ | $0$ | $-u_1$ | $-u_2$ | $-a_2 e_3$ | $-a_2 e_4$ | $0$ |

2.1°. $a_2 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2.2°. $a_2 > 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$ such that

$$\pi(e_i) = e_i, \ i = 1, \dots, 4,$$
$$\pi(u_j) = \frac{1}{\sqrt{a_2}} u_j, \ j = 1, 2, 3.$$

2.3°. $a_2 < 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$ such that

$$\pi(e_i) = e_i, \ i = 1, \dots, 4,$$
$$\pi(u_j) = \frac{1}{\sqrt{-a_2}} u_j, \ j = 1, 2, 3.$$

It remains to show that the pairs determined in the Proposition are not equivalent, whenever $\lambda = 0$. Since $\dim \mathcal{D}\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_2$ and $\dim \mathcal{D}\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_3$, we see that none of the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ is equivalent to the pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$. By $\mathfrak{a}_2$ and $\mathfrak{a}_3$ denote Levi subalgebras of the Lie algebras $\bar{\mathfrak{g}}_2$ and $\bar{\mathfrak{g}}_3$ respectively. Note that $\mathfrak{a}_2 \cong \mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{a}_3 \cong \mathfrak{su}(2)$. Therefore, the pairs $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent.

This completes the proof of the Proposition.

**Proposition 4.14.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.13 is equivalent to one and only one of the following pairs:*

*1.*

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $\lambda e_3 - e_4$ | $\lambda e_4 + e_3$ | $\lambda u_1$ | $-u_3$ | $u_2$ |
| $e_2$ | $0$ | $0$ | $(\mu - 1)e_3$ | $(\mu - 1)e_4$ | $\mu u_1$ | $u_2$ | $u_3$ |
| $e_3$ | $e_4 - \lambda e_3$ | $(1 - \mu)e_3$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-e_3 - \lambda e_4$ | $(1 - \mu)e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-\lambda u_1$ | $-\mu u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $u_3$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-u_2$ | $-u_3$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ |

*2.* $\lambda = 0,\ \mu = 0$

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $-e_4$ | $e_3$ | $0$ | $-u_3$ | $u_2$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $-e_4$ | $0$ | $u_2$ | $u_3$ |
| $e_3$ | $e_4$ | $e_3$ | $0$ | $0$ | $0$ | $u_1 + e_1$ | $e_2$ |
| $e_4$ | $-e_3$ | $e_4$ | $0$ | $0$ | $0$ | $-e_2$ | $u_1 + e_1$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $u_3$ | $-u_2$ | $-u_1 - e_1$ | $e_2$ | $0$ | $0$ | $0$ |
| $u_3$ | $-u_2$ | $-u_3$ | $-e_2$ | $-u_1 - e_1$ | $0$ | $0$ | $0$ |

*3.* $\lambda = 0,\ \mu = 0$

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $-e_4$ | $e_3$ | $0$ | $-u_3$ | $u_2$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $-e_4$ | $0$ | $u_2$ | $u_3$ |
| $e_3$ | $e_4$ | $e_3$ | $0$ | $0$ | $0$ | $u_1+\alpha e_1+e_2$ | $-e_1+\alpha e_2$ |
| $e_4$ | $-e_3$ | $e_4$ | $0$ | $0$ | $0$ | $e_1-\alpha e_2$ | $u_1+\alpha e_1+e_2$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $u_3$ | $-u_2$ | $-u_1-\alpha e_1-e_2$ | $\alpha e_2-e_1$ | $0$ | $0$ | $0$ |
| $u_3$ | $-u_2$ | $-u_3$ | $-\alpha e_2+e_1$ | $-u_1-\alpha e_1-e_2$ | $0$ | $0$ | $0$ |

$, \alpha \geqslant 0$

*4.* $\lambda = 0,\ \mu = 2.$

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $-e_4$ | $e_3$ | $0$ | $-u_3 + e_3$ | $u_2 + e_4$ |
| $e_2$ | $0$ | $0$ | $e_3$ | $e_4$ | $2u_1$ | $u_2$ | $u_3$ |
| $e_3$ | $e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $0$ | $-2u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $u_3 - e_3$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-u_2 - e_4$ | $-u_3$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ |

*5.* $\lambda = 0,\ \mu = 2.$

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $-e_4$ | $e_3$ | $0$ | $-u_3 + \alpha e_3$ | $u_2 + \alpha e_4$ |
| $e_2$ | $0$ | $0$ | $e_3$ | $e_4$ | $2u_1$ | $u_2 + e_3$ | $u_3 + e_4$ |
| $e_3$ | $e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $0$ | $-2u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $u_3 - \alpha e_3$ | $-u_2 - e_3$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-u_2 - \alpha e_4$ | $-u_3 - e_4$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ |

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},\quad e_2 = \begin{pmatrix} \mu & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & -1 & \lambda \end{pmatrix},\qquad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mu - 1 & 0 \\ 0 & 0 & 0 & \mu - 1 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\lambda & 1 - \mu & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -\lambda & 1 - \mu & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors $e_1$ and $e_2$. Consider the complex generalized module $(\mathfrak{g}^{\mathbb{C}}, U^{\mathbb{C}})$. Put

$$\tilde{e}_i = e_i \otimes 1,\ i = 1, \dots, 4, \qquad \text{and} \qquad \tilde{u}_j = u_j \otimes 1,\ j = 1, 2, 3.$$

Then $\tilde{\mathcal{E}} = \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$ is a basis of $\mathfrak{g}^{\mathbb{C}}$. The vector space $U^{\mathbb{C}}$ can be identified with $\mathbb{C}^3$, and $\{\tilde{u}_1, \tilde{u}_2, \tilde{u}_3\}$ is the standard basis of $U^{\mathbb{C}}$.

**Lemma.** *Any virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.14 is equivalent to one of the following :*

*a)* $(\lambda, \mu) \notin \{(0,0), (0,2)\}$

$$C_1(e_i) = 0, \quad i = 1, \dots, 4;$$

*b)* $\lambda = 0,\ \mu = 2$

$$C_2(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & q \end{pmatrix}, \quad C_2(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & r & s \\ 0 & -s & r \end{pmatrix}, \quad C_2(e_3) = C_2(e_4) = 0;$$

*c)* $\lambda = 0,\ \mu = 0$

$$C_3(e_1) = C_3(e_2) = 0, \quad C_3(e_3) = \begin{pmatrix} 0 & p & -q \\ 0 & q & p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_3(e_4) = \begin{pmatrix} 0 & q & p \\ 0 & -p & q \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

*Proof.* Suppose $(\bar{\mathfrak{g}}, \mathfrak{g})$ is a virtual pair defined by a virtual structure $q$. By Proposition 15, Chapter II, without loss of generality it can be assumed that $q^{\mathbb{C}}$ is a primary virtual structure on the generalized module $(\mathfrak{g}^{\mathbb{C}}, U^{\mathbb{C}})$ (with respect to $\mathfrak{h}^{\mathbb{C}}$). Note that

$$\begin{aligned}
&\mathfrak{g}^{(0,0)}(\mathfrak{h}^{\mathbb{C}}) \supset \mathbb{C}\tilde{e}_1 \oplus \mathbb{C}\tilde{e}_2, && (U^{\mathbb{C}})^{(}\lambda,\mu)(\mathfrak{h}^{\mathbb{C}}) \supset \mathbb{C}\tilde{u}_1, \\
&\mathfrak{g}^{(i,\mu-1)}(\mathfrak{h}^{\mathbb{C}}) \supset \mathbb{C}(\tilde{e}_3 + i\tilde{e}_4), && (U^{\mathbb{C}})^{(i,1)}(\mathfrak{h}^{\mathbb{C}}) \supset \mathbb{C}(\tilde{u}_2 + i\tilde{u}_3), \\
&\mathfrak{g}^{(-i,\mu-1)}(\mathfrak{h}^{\mathbb{C}}) \supset \mathbb{C}(\tilde{e}_3 - i\tilde{e}_4), && (U^{\mathbb{C}})^{(-i,1)}(\mathfrak{h}^{\mathbb{C}}) \supset \mathbb{C}(\tilde{u}_2 - i\tilde{u}_3).
\end{aligned}$$

Suppose $\lambda = 0$ and $\mu = 2$. Then

$$\begin{aligned}
q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_1) &= 0,\\
q^{\mathbb{C}}(\tilde{e}_3 + i\tilde{e}_4)(\tilde{u}_1) &= 0,\\
q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_2 + i\tilde{u}_3) &\in \mathbb{C}(\tilde{e}_3 + i\tilde{e}_4),\\
q^{\mathbb{C}}(\tilde{e}_3 + i\tilde{e}_4)(\tilde{u}_2 + i\tilde{u}_3) &= 0,\\
q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_2 - i\tilde{u}_3) &\in \mathbb{C}(\tilde{e}_3 - i\tilde{e}_4),\\
q^{\mathbb{C}}(\tilde{e}_3 + i\tilde{e}_4)(\tilde{u}_2 - i\tilde{u}_3) &= 0,\\
q^{\mathbb{C}}(\tilde{e}_3 - i\tilde{e}_4)(\tilde{u}_1) &= 0,\\
q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_1) &= 0,\\
q^{\mathbb{C}}(\tilde{e}_3 - i\tilde{e}_4)(\tilde{u}_2 + i\tilde{u}_3) &= 0,\\
q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_2 + i\tilde{u}_3) &\in \mathbb{C}(\tilde{e}_3 + i\tilde{e}_4),\\
q^{\mathbb{C}}(\tilde{e}_3 - i\tilde{e}_4)(\tilde{u}_2 - i\tilde{u}_3) &= 0,\\
q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_2 - i\tilde{u}_3) &\in \mathbb{C}(\tilde{e}_3 - i\tilde{e}_4),
\end{aligned}$$

and

$$\begin{aligned}
&q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_1) = q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_2) = 0,\\
&q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_2) \in \mathbb{C}\tilde{e}_3 + \mathbb{C}\tilde{e}_4,\\
&q^{\mathbb{C}}(\tilde{e}_1)(\tilde{u}_3) \in \mathbb{C}\tilde{e}_3 + \mathbb{C}\tilde{e}_4,\\
&q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_2) \in \mathbb{C}\tilde{e}_3 + \mathbb{C}\tilde{e}_4,\\
&q^{\mathbb{C}}(\tilde{e}_2)(\tilde{u}_3) \in \mathbb{C}\tilde{e}_3 + \mathbb{C}\tilde{e}_4,\\
&q^{\mathbb{C}}(\tilde{e}_i)(\tilde{u}_j) = 0, \quad i = 3,4,\ j = 1,2,3.
\end{aligned}$$

Since the matrices of the mappings $q(e_i)$ and $q^{\mathbb{C}}(\tilde{e}_i)$, $i = 1, \ldots, 4$, coincide, we get

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c^1_{32} & c^1_{33} \\ 0 & c^1_{42} & c^1_{43} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c^2_{32} & c^2_{33} \\ 0 & c^2_{42} & c^2_{43} \end{pmatrix}, \quad C(e_3) = C(e_4) = 0.$$

Checking condition (6), Chapter II, for $x, y \in \mathcal{E}$, we obtain

$$\begin{cases} c^1_{32} = c^1_{42} = 0, \\ c^2_{32} = c^2_{43}, \\ c^2_{33} = -c^2_{42}. \end{cases}$$

Finally put $C_2 = C$.

In a similar way we obtain the other results of the Lemma.

Thus, it can be assumed that any virtual structure on $(\mathfrak{g}, U)$ has the form determined in the Lemma. Since $q^{\mathbb{C}}$ is a primary virtual structure on the generalized module $(\mathfrak{g}^{\mathbb{C}}, U^{\mathbb{C}})$, we have

$$(\bar{\mathfrak{g}}^{\mathbb{C}})^{\alpha}(\mathfrak{h}^{\mathbb{C}}) = (\mathfrak{g}^{\mathbb{C}})^{\alpha}(\mathfrak{h}^{\mathbb{C}}) \times (U^{\mathbb{C}})^{\alpha}(\mathfrak{h}^{\mathbb{C}}) \text{ for all } \alpha \in (\mathfrak{h}^{\mathbb{C}})^*$$

(Proposition 10, Chapter II). Therefore,

$$\begin{aligned}
&(\bar{\mathfrak{g}}^{(0,0)})(\mathfrak{h}^{\mathbb{C}}) \supset \mathbb{C}\tilde{e}_1 \oplus \mathbb{C}\tilde{e}_2, && (\bar{\mathfrak{g}}^{\mathbb{C}})^{(\lambda,\mu)}(\mathfrak{h}^{\mathbb{C}}) \supset \mathbb{C}\tilde{u}_1,\\
&(\bar{\mathfrak{g}}^{(i,\mu-1)})(\mathfrak{h}^{\mathbb{C}}) \supset \mathbb{C}(\tilde{e}_3+i\tilde{e}_4), && (\bar{\mathfrak{g}}^{\mathbb{C}})^{(i,1)}(\mathfrak{h}^{\mathbb{C}}) \supset \mathbb{C}(\tilde{u}_2+i\tilde{u}_3),\\
&(\bar{\mathfrak{g}}^{(-i,\mu-1)})(\mathfrak{h}^{\mathbb{C}}) \supset \mathbb{C}(\tilde{e}_3-i\tilde{e}_4), && (\bar{\mathfrak{g}}^{\mathbb{C}})^{(-i,1)}(\mathfrak{h}^{\mathbb{C}}) \supset \mathbb{C}(\tilde{u}_2-i\tilde{u}_3).
\end{aligned}$$

Consider the following cases:

1°. $(\lambda,\mu) \notin \{(0,0), (0,2)\}$. Then

$$\begin{aligned}
&[e_1,e_2]=0,\\
&[e_1,e_3]=\lambda e_3-e_4, && [e_2,e_3]=(\mu-1)e_3,\\
&[e_1,e_4]=e_3+\lambda e_4, && [e_2,e_4]=(\mu-1)e_4, && [e_3,e_4]=0,\\
&[e_1,u_1]=\lambda u_1, && [e_2,u_1]=\mu u_1, && [e_3,u_1]=0, && [e_4,u_1]=0,\\
&[e_1,u_2]=-u_3, && [e_2,u_2]=u_2, && [e_3,u_2]=u_1, && [e_4,u_2]=0,\\
&[e_1,u_3]=u_2, && [e_2,u_3]=u_3, && [e_3,u_3]=0, && [e_4,u_3]=u_1.
\end{aligned}$$

We have

$$\begin{aligned}
&[\tilde{u}_1,\tilde{u}_2+i\tilde{u}_3]=[\tilde{u}_1,\tilde{u}_2]+i[\tilde{u}_1,\tilde{u}_3]=0,\\
&[\tilde{u}_1,\tilde{u}_2-i\tilde{u}_3]=[\tilde{u}_1,\tilde{u}_2]-i[\tilde{u}_1,\tilde{u}_3]=0,\\
&[\tilde{u}_2+i\tilde{u}_3,\tilde{u}_2-i\tilde{u}_3]=-2i[\tilde{u}_2,\tilde{u}_3]=0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&[u_1,u_2]=0,\\
&[u_1,u_3]=0,\\
&[u_2,u_3]=0,
\end{aligned}$$

and the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is trivial.

2°. $\lambda=\mu=0$. Then

$$\begin{aligned}
&[e_1,e_2]=0,\\
&[e_1,e_3]=-e_4, && [e_2,e_3]=-e_3,\\
&[e_1,e_4]=e_3, && [e_2,e_4]=-e_4, && [e_3,e_4]=0,\\
&[e_1,u_1]=0, && [e_2,u_1]=0, && [e_3,u_1]=0, && [e_4,u_1]=0,\\
&[e_1,u_2]=-u_3, && [e_2,u_2]=u_2, && [e_3,u_2]=u_1+pe_1+qe_2, && [e_4,u_2]=qe_1-pe_2,\\
&[e_1,u_3]=u_2, && [e_2,u_3]=u_3, && [e_3,u_3]=-qe_1+pe_2, && [e_4,u_3]=u_1+pe_1+qe_2.
\end{aligned}$$

We have

$$\begin{aligned}
&[\tilde{u}_1,\tilde{u}_2+i\tilde{u}_3]\in\mathbb{C}(\tilde{u}_2+i\tilde{u}_3),\\
&[\tilde{u}_1,\tilde{u}_2-i\tilde{u}_3]\in\mathbb{C}(\tilde{u}_2-i\tilde{u}_3),\\
&[\tilde{u}_2+i\tilde{u}_3,\tilde{u}_2-i\tilde{u}_3]=0.
\end{aligned}$$

Hence,

$$\begin{aligned}
&[u_1,u_2]=\alpha_2e_2+\alpha_3e_3,\\
&[u_1,u_3]=\beta_2u_2+\beta_3u_3,\\
&[u_2,u_3]=0.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $-e_4$ | $e_3$ | $0$ | $-u_3$ | $u_2$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $-e_4$ | $0$ | $u_2$ | $u_3$ |
| $e_3$ | $e_4$ | $e_3$ | $0$ | $0$ | $0$ | $A$ | $B$ |
| $e_4$ | $-e_3$ | $e_4$ | $0$ | $0$ | $0$ | $-B$ | $A$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $u_3$ | $-u_2$ | $-A$ | $B$ | $0$ | $0$ | $0$ |
| $u_3$ | $-u_2$ | $-u_3$ | $-B$ | $-A$ | $0$ | $0$ | $0$ , |

where

$$A = u_1 + pe_1 + qe_2,$$
$$B = -qe_1 + pe_2.$$

2.1°. $p = q = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

2.2°. $p \neq 0$, $q = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$ such that

$$\pi(e_i) = e_i, \ i = 1, \dots, 4,$$
$$\pi(u_j) = \frac{1}{p} u_j, \ j = 1, 2, 3.$$

2.3°. $q \neq 0$, $p/q \leqslant 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ $(\alpha = p/q)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$ such that

$$\pi(e_i) = e_i, \ i = 1, \dots, 4,$$
$$\pi(u_j) = \frac{1}{p} u_j, \ j = 1, 2, 3.$$

2.4°. $q \neq 0$, $p/q > 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ $(\alpha = -p/q)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$ such that

$$\pi(e_1) = -e_1,$$
$$\pi(e_2) = e_2, \quad \pi(u_1) = \frac{1}{p} u_1,$$
$$\pi(e_3) = e_4, \quad \pi(u_2) = \frac{1}{p} u_3,$$
$$\pi(e_4) = e_3, \quad \pi(u_3) = \frac{1}{p} u_2.$$

Since the virtual pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$, and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent, we see that the pairs 4.14.1, 4.14.2, and 4.14.3 are not equivalent.

3°. $\lambda = 0$, $\mu = 2$. Then

$$\begin{array}{llll}
[e_1, e_2] = 0, & & & \\
[e_1, e_3] = -e_4, & [e_2, e_3] = -e_3, & & \\
[e_1, e_4] = e_3, & [e_2, e_4] = -e_4, & [e_3, e_4] = 0, & \\
[e_1, u_1] = 0, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = 0, \\
[e_1, u_2] = -u_3, & [e_2, u_2] = u_2 + re_3 - se_4, & [e_3, u_2] = u_1, & [e_4, u_2] = 0, \\
[e_1, u_3] = u_2 + pe_3 + qe_4, & [e_2, u_3] = u_3 + se_3 + re_4, & [e_3, u_3] = 0, & [e_4, u_3] = u_1.
\end{array}$$

We have

$$[\tilde{u}_1, \tilde{u}_2 + i\tilde{u}_3] = 0,$$
$$[\tilde{u}_1, \tilde{u}_2 - i\tilde{u}_3] = 0,$$
$$[\tilde{u}_2 + i\tilde{u}_3, \tilde{u}_2 - i\tilde{u}_3] \in \mathbb{C}\tilde{u}_1.$$

Hence,

$$[u_1, u_2] = 0,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = \gamma_1 u_1.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $-e_4$ | $e_3$ | $0$ | $-u_3$ | $u_2+qe_4$ |
| $e_2$ | $0$ | $0$ | $e_3$ | $e_4$ | $2u_1$ | $u_2+re_3$ | $u_3+re_4$ |
| $e_3$ | $e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $0$ | $-2u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $u_3$ | $-u_2-re_3$ | $-u_1$ | $0$ | $0$ | $0$ | $\gamma_1 u_1$ |
| $u_3$ | $-u_2-qe_4$ | $-u_3-re_4$ | $0$ | $-u_1$ | $0$ | $-\gamma_1 u_1$ | $0$ |

Consider the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $-e_4$ | $e_3$ | $0$ | $-u_3$ | $u_2+qe_4$ |
| $e_2$ | $0$ | $0$ | $e_3$ | $e_4$ | $2u_1$ | $u_2+re_3$ | $u_3+re_4$ |
| $e_3$ | $e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $0$ | $-2u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $u_3$ | $-u_2-re_3$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-u_2-qe_4$ | $-u_3-re_4$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ |

The pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ by means of the mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$, such that

$$\pi(e_i) = e_i, \ i = 1, \dots, 4,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2 - \frac{\gamma_1}{2} e_4,$$
$$\pi(u_3) = u_3 + \frac{\gamma_1}{2} e_3.$$

3.1°. $r = q = 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is trivial.

3.2°. $q \neq 0$, $r = 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}'$ such that

$$\pi(e_i) = e_i, \ i = 1, \dots, 4,$$
$$\pi(u_1) = \frac{2}{q} u_1,$$
$$\pi(u_2) = \frac{2}{q} u_2 + \frac{1}{2} e_4,$$
$$\pi(u_3) = \frac{2}{q} u_3 + \frac{1}{2} e_3.$$

3.3°. $r \neq 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ $(\alpha = p/(2r))$ by means of the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}'$ such that

$$\pi(e_i) = e_i, \ i = 1, \dots, 4,$$
$$\pi(u_1) = \frac{1}{r} u_1,$$
$$\pi(u_2) = \frac{1}{r} u_2 + \frac{q}{4r} e_4,$$
$$\pi(u_3) = \frac{1}{r} u_3 + \frac{q}{4r} e_3.$$

Since the virtual pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$, $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$, and $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ are not equivalent, we see that the pairs 4.14.1, 4.14.4, and 4.14.5, are not equivalent.

This completes the proof of the Proposition.

**Proposition 4.15.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.15 is trivial.*

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3 - \lambda e_4$ | $e_4$ | $u_1$ | $0$ | $\lambda u_2$ |
| $e_2$ | $0$ | $0$ | $-e_3 - e_4$ | $-e_4$ | $0$ | $u_2$ | $u_2 + u_3$ |
| $e_3$ | $\lambda e_4 - e_3$ | $e_3 + e_4$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-e_4$ | $e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-\lambda u_2$ | $-u_2 - u_3$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ |

*Proof.* Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda + 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U + \varphi$, where $\varphi$ is a nilpotent endomorphism. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

**Proposition 4.16.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.16 is trivial.*

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3 - e_4$ | $e_4$ | $u_1$ | $0$ | $u_2$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $-e_4$ | $0$ | $u_2$ | $u_3$ |
| $e_3$ | $e_4 - e_3$ | $e_3$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-e_4$ | $e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-u_2$ | $-u_3$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ |

*Proof.* Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U + \varphi$, where $\varphi$ is a nilpotent endomorphism. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

And this proves the Proposition.

**Proposition 4.17.** *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 4.17 is trivial.*

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3+\lambda e_4$ | $e_4$ | $u_1$ | $\lambda u_1+u_2$ | $0$ |
| $e_2$ | $0$ | $0$ | $e_4-e_3$ | $-e_4$ | $0$ | $u_1$ | $u_3$ |
| $e_3$ | $-e_3-\lambda e_4$ | $e_3-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $-e_4$ | $e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-\lambda u_1-u_2$ | $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $0$ | $-u_3$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ |

*Proof.* Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & \lambda+1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U + \varphi$, where $\varphi$ is a nilpotent endomorphism. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

**Proposition 4.18.** *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 4.18 is trivial.*

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $e_2$ | $e_2+e_3$ | $0$ | $u_1$ | $u_1+u_2$ | $0$ |
| $e_2$ | $-e_2$ | $0$ | $0$ | $e_2$ | $0$ | $0$ | $u_1$ |
| $e_3$ | $-e_2-e_3$ | $0$ | $0$ | $e_3$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $0$ | $-e_2$ | $-e_3$ | $0$ | $0$ | $0$ | $u_3$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-u_1-u_2$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $0$ | $-u_1$ | $-u_2$ | $-u_3$ | $0$ | $0$ | $0$ |

*Proof.* Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U + \varphi$, where $\varphi$ is a nilpotent endomorphism. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

**Proposition 4.19.** *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 4.19 is equivalent to one and only one of the following pairs:*

*1.*

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $-e_3$ | $-e_4$ | $0$ | $0$ | $u_3$ |
| $e_2$ | $0$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-u_3$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ |

*2.*

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $-e_3$ | $-e_4$ | $0$ | $0$ | $u_3$ |
| $e_2$ | $0$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $e_3$ | $-e_4$ | $0$ | $0$ | $-e_4$ | $-2e_3$ | $u_2$ |
| $e_4$ | $e_4$ | $0$ | $0$ | $0$ | $0$ | $-e_4$ | $u_1$ |
| $u_1$ | $0$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $u_2$ | $0$ | $-u_1$ | $2e_3$ | $e_4$ | $-u_1$ | $0$ | $-2u_3$ |
| $u_3$ | $-u_3$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $2u_3$ | $0$ |

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector $e_1$.

**Lemma.** *Any virtual structure $q$ on generalized module 4.19 is equivalent to one of the following:*

$$C(e_1) = 0, \qquad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$C(e_3) = \begin{pmatrix} 0 & 0 & -2p \\ 0 & 0 & q \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix}.$$

*Proof.* Let $q$ be a virtual structure on generalized module 4.19. Without loss of generality it can be assumed that $q$ is primary. Since

$$\mathfrak{g}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2, \quad \mathfrak{g}^{(-1)}(\mathfrak{h}) = \mathbb{R}e_3 \oplus \mathbb{R}e_4,$$
$$U^{(0)}(\mathfrak{h}) = \mathbb{R}u_1 \oplus \mathbb{R}u_2, \quad U^{(1)}(\mathfrak{h}) = \mathbb{R}u_3,$$

we have

$$C(e_1) = \begin{pmatrix} c^1_{11} & c^1_{12} & 0 \\ c^1_{21} & c^1_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} c^2_{11} & c^2_{12} & 0 \\ c^2_{21} & c^2_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$C(e_3) = \begin{pmatrix} 0 & 0 & c^3_{13} \\ 0 & 0 & c^3_{23} \\ c^3_{31} & c^3_{32} & 0 \\ c^3_{41} & c^3_{42} & 0 \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & c^4_{13} \\ 0 & 0 & c^4_{23} \\ c^4_{31} & c^4_{32} & 0 \\ c^4_{41} & c^4_{42} & 0 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} -c_{31}^3 & -c_{32}^3 & 0 \\ c_{41}^3 & c_{42}^3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $C_1(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = \begin{pmatrix} c_{11}^1 & c_{12}^1 & 0 \\ c_{21}^1 & c_{22}^1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C_1(e_2) = \begin{pmatrix} c_{11}^2 & c_{12}^2 + c_{31}^3 & 0 \\ c_{21}^2 & c_{22}^2 - c_{41}^3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_1(e_3) = \begin{pmatrix} 0 & 0 & c_{13}^3 + c_{32}^3 \\ 0 & 0 & c_{23}^3 - c_{42}^3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & c_{13}^4 + c_{31}^3 \\ 0 & 0 & c_{23}^4 - c_{41}^3 \\ c_{31}^4 & c_{32}^4 & 0 \\ c_{41}^4 - c_{31}^3 & c_{42}^4 - c_{32}^3 & 0 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures $C$ and $C_1$ are equivalent. Checking condition (6) for the mapping $C_1$, we obtain:

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C_1(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_1(e_3) = \begin{pmatrix} 0 & 0 & -2p \\ 0 & 0 & q \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_1(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix}.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 4.19. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Then

$$\begin{array}{llll}
[e_1, e_2] = 0, & & & \\
[e_1, e_3] = -e_3, & [e_2, e_3] = e_3, & & \\
[e_1, e_4] = -e_4, & [e_2, e_4] = 0, & [e_3, e_4] = 0, & \\
[e_1, u_1] = 0, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = 0, \\
[e_1, u_2] = 0, & [e_2, u_2] = u_1 + pe_2, & [e_3, u_2] = 0, & [e_4, u_2] = pe_4, \\
[e_1, u_3] = u_3, & [e_2, u_3] = 0, & [e_3, u_3] = u_2 - 2pe_1 + qe_2, & [e_4, u_3] = u_1 + pe_2.
\end{array}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}}^\alpha(\mathfrak{h}) = \mathfrak{g}^\alpha(\mathfrak{h}) \times U^\alpha(\mathfrak{h}^*) \text{ for all } \alpha \in \mathbb{R}$$

(Proposition 10, Chapter II). Thus

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_2,$$

$$\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{R}e_3 \oplus \mathbb{R}e_4, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_3.$$

Therefore

$$\begin{aligned} [u_1, u_2] &\in \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}), \\ [u_1, u_3] &\in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), \\ [u_2, u_3] &\in \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}), \end{aligned}$$

and

$$\begin{aligned} [u_1, u_2] &= a_1 + a_2 + \alpha_1 u_1 + \alpha_2 u_2, \\ [u_1, u_3] &= \beta_3 u_3, \\ [u_2, u_3] &= \gamma_3 u_3. \end{aligned}$$

Using the Jacobi identity we obtain:

$$\begin{cases} a_1 = a_2 = \alpha_1 = \alpha_2 = 0, \\ \beta_3 = 0, \\ q = 0, \\ \gamma_3 = 0. \end{cases}$$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $-e_3$ | $-e_4$ | $0$ | $0$ | $u_3$ |
| $e_2$ | $0$ | $0$ | $e_4$ | $0$ | $0$ | $u_1 + pe_2$ | $0$ |
| $e_3$ | $e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2 - 2pe_1$ |
| $e_4$ | $e_4$ | $0$ | $0$ | $0$ | $0$ | $pe_4$ | $u_1 + pe_2$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_1 - pe_2$ | $0$ | $-pe_4$ | $0$ | $0$ | $0$ |
| $u_3$ | $-u_3$ | $0$ | $-u_2 + 2pe_1$ | $-u_1 - pe_2$ | $0$ | $0$ | $0$ |

1°. $p = 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2°. $p \neq 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_i) &= e_i, \ i = 1, \dots, 4, \\ \pi(u_1) &= \frac{1}{p} u_1 + e_2, \\ \pi(u_2) &= \frac{1}{p} u_2 - 2e_1, \\ \pi(u_3) &= \frac{1}{p} u_3. \end{aligned}$$

Since $\bar{\mathfrak{g}}_1$ is a solvable Lie algebra and $\bar{\mathfrak{g}}_2$ is unsolvable, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

This completes the proof of the Proposition.

**Proposition 4.20.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.20 is equivalent to one and only one of the following pairs:*

1.

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $-e_2$ | $-\lambda e_3$ | $(1-\lambda)e_4$ | $0$ | $u_2$ | $\lambda u_3$ |
| $e_2$ | $e_2$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $\lambda e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $(\lambda-1)e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-\lambda u_3$ | $0$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ |

2. $\lambda = -\frac{1}{2}$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $-e_2$ | $\frac{1}{2}e_3$ | $\frac{3}{2}e_4$ | $0$ | $u_2$ | $-\frac{1}{2}u_3$ |
| $e_2$ | $e_2$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $-\frac{1}{2}e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $-\frac{3}{2}e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $e_3$ |
| $u_3$ | $\frac{1}{2}u_3$ | $0$ | $-u_1$ | $-u_2$ | $0$ | $-e_3$ | $0$ |

3. $\lambda = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $-e_2$ | $0$ | $e_4$ | $0$ | $u_2$ | $0$ |
| $e_2$ | $e_2$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $-e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $e_3$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $e_4$ |
| $u_3$ | $0$ | $0$ | $-u_1$ | $-u_2$ | $-e_3$ | $-e_4$ | $0$ |

4. $\lambda = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $-e_2$ | $0$ | $e_4$ | $0$ | $u_2$ | $0$ |
| $e_2$ | $e_2$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $-e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-e_3$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $-e_4$ |
| $u_3$ | $0$ | $0$ | $-u_1$ | $-u_2$ | $e_3$ | $e_4$ | $0$ |

5. $\lambda = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $-e_2$ | $0$ | $e_4$ | $0$ | $u_2$ | $0$ |
| $e_2$ | $e_2$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $-e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\alpha e_3 + u_1$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $\alpha e_4 + u_2$ |
| $u_3$ | $0$ | $0$ | $-u_1$ | $-u_2$ | $-\alpha e_3 - u_1$ | $-\alpha e_4 - u_2$ | $0$ |

6. $\lambda = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $-e_2$ | $0$ | $e_4$ | $0$ | $u_2$ | $e_3$ |
| $e_2$ | $e_2$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $-e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-e_3$ | $0$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ |

*7.* $\lambda = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $-e_2$ | $0$ | $e_4$ | $0$ | $u_2$ | $e_3$ |
| $e_2$ | $e_2$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $-e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $e_3$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $e_4$ |
| $u_3$ | $-e_3$ | $0$ | $-u_1$ | $-u_2$ | $-e_3$ | $-e_4$ | $0$ |

*8.* $\lambda = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $-e_2$ | $0$ | $e_4$ | $0$ | $u_2$ | $e_3$ |
| $e_2$ | $e_2$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $-e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-e_3$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $-e_4$ |
| $u_3$ | $-e_3$ | $0$ | $-u_1$ | $-u_2$ | $e_3$ | $e_4$ | $0$ |

*9.* $\lambda = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $-e_2$ | $0$ | $e_4$ | $0$ | $u_2$ | $e_3$ |
| $e_2$ | $e_2$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $-e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\alpha e_3 + u_1$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $\alpha e_4 + u_2$ |
| $u_3$ | $-e_3$ | $0$ | $-u_1$ | $-u_2$ | $-\alpha e_3 - u_1$ | $-\alpha e_4 - u_2$ | $0$ |

*10.* $\lambda = 1$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $-e_2$ | $-e_3$ | $0$ | $0$ | $u_2$ | $u_3$ |
| $e_2$ | $e_2$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $e_4$ |
| $e_3$ | $e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $0$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-u_3$ | $-e_4$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ |

*11.* $\lambda = 1$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $-e_2$ | $-e_3$ | $0$ | $0$ | $u_2$ | $u_3$ |
| $e_2$ | $e_2$ | $0$ | $0$ | $e_3$ | $e_3$ | $u_1$ | $e_1$ |
| $e_3$ | $e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $0$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $0$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-u_3$ | $-e_1$ | $-u_1$ | $-u_2$ | $-u_2$ | $0$ | $0$ |

*12.* $\lambda = 1$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $-e_2$ | $-e_3$ | $0$ | $0$ | $u_2$ | $u_3$ |
| $e_2$ | $e_2$ | $0$ | $0$ | $e_3$ | $e_3$ | $u_1$ | $e_1 + e_4$ |
| $e_3$ | $e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $0$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $0$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-u_3$ | $-e_1 - e_4$ | $-u_1$ | $-u_2$ | $-u_2$ | $0$ | $0$ |

*13.* $\lambda = 3$

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $-e_2$ | $-3e_3$ | $-2e_4$ | $0$ | $u_2$ | $3u_3$ |
| $e_2$ | $e_2$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $3e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $2e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $e_2$ | $u_2$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $-e_2$ | $0$ | $0$ | $0$ |
| $u_3$ | $-3u_3$ | $0$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ |

*Proof.*
Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \lambda-1 & 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector $e_1$.

**Lemma.** *Any virtual structure $q$ on generalized module 4.20 is equivalent to one of the following:*

*a)* $\lambda \notin \{-1, 0, 1, 3\}$

$$C(e_i) = 0, \quad i = 1, \dots, 4;$$

*b)* $\lambda = -1$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & -p & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_i) = 0, \; i = 2, 3, 4;$$

*c)* $\lambda = 0$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_i) = 0, \; i = 2, 3, 4;$$

*d)* $\lambda = 1$

$$C(e_2) = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & 0 \\ p & 0 & 0 \\ 0 & 0 & q \end{pmatrix}, \quad C(e_i) = 0,\ i = 1,3,4;$$

*e)* $\lambda = 3$

$$C(e_i) = 0,\ i = 1,2,3, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

*Proof.* Put

$$C(e_i) = \left(c^i_{jk}\right)_{\substack{1\leqslant j\leqslant 4 \\ 1\leqslant k\leqslant 3}}, \quad i = 1,\dots,4.$$

Let $q$ be a virtual structure on generalized module 4.20. Without loss of generality it can be assumed that $q$ is primary. Since

$$\begin{aligned} &\mathfrak{g}^{(0)}(\mathfrak{h}) \supseteq \mathbb{R}e_1, & & \\ &\mathfrak{g}^{(-1)}(\mathfrak{h}) \supseteq \mathbb{R}e_2, & &U^{(0)}(\mathfrak{h}) \supseteq \mathbb{R}u_1, \\ &\mathfrak{g}^{(-\lambda)}(\mathfrak{h}) \supseteq \mathbb{R}e_3, & &U^{(1)}(\mathfrak{h}) \supseteq \mathbb{R}u_2, \\ &\mathfrak{g}^{(1-\lambda)}(\mathfrak{h}) \supseteq \mathbb{R}e_4, & &U^{(\lambda)}(\mathfrak{h}) \supseteq \mathbb{R}u_3, \end{aligned}$$

we have:

$$C(e_1) = \begin{pmatrix} c^1_{11} & 0 & c^1_{13} \\ 0 & 0 & c^1_{23} \\ c^1_{31} & c^1_{32} & c^1_{33} \\ c^1_{41} & c^1_{42} & c^1_{43} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & c^2_{12} & c^2_{13} \\ c^2_{21} & 0 & c^2_{23} \\ c^2_{31} & c^2_{32} & c^2_{33} \\ c^2_{41} & c^2_{42} & c^2_{43} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} c^3_{11} & c^3_{12} & c^3_{13} \\ c^3_{21} & c^3_{22} & 0 \\ c^3_{31} & 0 & c^3_{33} \\ 0 & c^3_{42} & c^3_{43} \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} c^4_{11} & c^4_{12} & 0 \\ c^4_{21} & c^4_{22} & 0 \\ 0 & 0 & c^4_{33} \\ c^4_{41} & 0 & c^4_{43} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} c^2_{12} & 0 & -c^2_{23} \\ 0 & 0 & 0 \\ c^3_{33} & c^4_{33} & 0 \\ c^3_{43} & c^4_{43} + (1-\lambda)c^2_{23} & -c^2_{33} \end{pmatrix}$$

and $C'(x) = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C'(e_1) = \begin{pmatrix} c^1_{11} & 0 & c^1_{13} \\ 0 & 0 & c^1_{23} \\ c^1_{31} & c^1_{32} & c^1_{33} \\ c^1_{41} & c^1_{42} & c^1_{43} \end{pmatrix}, \quad C'(e_2) = \begin{pmatrix} 0 & 0 & c^2_{13} \\ c^2_{21} & 0 & 0 \\ c^2_{31} & c^2_{32} & 0 \\ c^2_{41} & c^2_{42} & c^2_{43} \end{pmatrix},$$

$$C'(e_3) = \begin{pmatrix} c^3_{11} & c^3_{12} & c^3_{13} \\ c^3_{21} & c^3_{22} & 0 \\ c^3_{31} & 0 & 0 \\ 0 & c^3_{42} & 0 \end{pmatrix}, \quad C'(e_4) = \begin{pmatrix} c^4_{11} & c^4_{12} & 0 \\ c^4_{21} & c^4_{22} & 0 \\ 0 & 0 & 0 \\ c^4_{41} & 0 & 0 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures $C$ and $C'$ are equivalent.

Since for any virtual structure $q$ condition (6), Chapter II, must be satisfied, after direct calculation we obtain:

$$C'(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{23}^1 \\ 0 & -c_{23}^1 & c_{33}^1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C'(e_2) = \begin{pmatrix} 0 & 0 & c_{13}^2 \\ 0 & 0 & 0 \\ \lambda c_{13}^2 & 0 & 0 \\ 0 & 0 & c_{43}^2 \end{pmatrix},$$

$$C'(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C'(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_{22}^4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the set of coefficients $c_{ij}^k$ satisfies the following system:

$$\begin{cases} (\lambda - 1)c_{43}^2 = 0, \\ (\lambda - 3)c_{22}^4 = 0, \\ (\lambda - 1)c_{13}^2 = 0. \end{cases}$$

Consider the following cases:

*a)* $\lambda \notin \{-1, 0, 1, 3\}$. From the system it follows that $c_{12}^2 = c_{43}^2 = c_{22}^4 = 0$. Put

$$H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{1+\lambda}c_{23}^1 \\ 0 & 0 & \frac{1}{2\lambda}c_{33}^1 \\ 0 & 0 & 0 \end{pmatrix},$$

and $C_1(x) = C'(x) + A(x)H_1 - H_1B(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_i) = 0, \quad i = 1, \ldots, 4.$$

*b)* $\lambda = -1$. From the system it follows that $c_{12}^2 = c_{43}^2 = c_{22}^4 = 0$. Put

$$H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}c_{33}^1 \\ 0 & 0 & 0 \end{pmatrix},$$

and $C_2(x) = C'(x) + A(x)H_2 - H_2B(x)$ for $x \in \mathfrak{g}$. Then

$$C_2(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{23}^1 \\ 0 & -c_{23}^1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2(e_i) = 0, \ i = 2, 3, 4.$$

*c)* $\lambda = 0$. From the system it follows that $c_{12}^2 = c_{43}^2 = c_{22}^4 = 0$. Put

$$H_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{23}^1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $C_3(x) = C'(x) + A(x)H_3 - H_3B(x)$ for $x \in \mathfrak{g}$. Then

$$C_3(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{33}^1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_3(e_i) = 0,\ i = 2,3,4.$$

*d)* $\lambda = 1$. From the system it follows that $c_{22}^4 = 0$. Put

$$H_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}c_{23}^1 \\ 0 & 0 & \frac{1}{2}c_{33}^1 \\ 0 & 0 & 0 \end{pmatrix},$$

and $C_4(x) = C'(x) + A(x)H_4 - H_4B(x)$ for $x \in \mathfrak{g}$. Then

$$C_4(e_2) = \begin{pmatrix} 0 & 0 & c_{13}^2 \\ 0 & 0 & 0 \\ c_{13}^2 & 0 & 0 \\ 0 & 0 & c_{43}^2 \end{pmatrix}, \quad C_4(e_i) = 0,\ i = 1,3,4.$$

*e)* $\lambda = 3$. From the system it follows that $c_{12}^2 = c_{43}^2 = 0$. Put

$$H_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}c_{23}^1 \\ 0 & 0 & \frac{1}{6}c_{33}^1 \\ 0 & 0 & 0 \end{pmatrix},$$

and $C_5(x) = C'(x) + A(x)H_5 - H_5B(x)$ for $x \in \mathfrak{g}$. Then

$$C_5(e_i) = 0,\ i = 1,2,3, \quad C_2(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_{22}^4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 4.20. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Consider the following cases:

1°. The virtual structure $q$ is trivial. Then

$$\begin{array}{llll}
[e_1, e_2] = -e_2, & & & \\
[e_1, e_3] = -\lambda e_3, & [e_2, e_3] = 0, & & \\
[e_1, e_4] = (1-\lambda)e_4, & [e_2, e_4] = e_3, & [e_3, e_4] = 0, & \\
[e_1, u_1] = 0, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = 0, \\
[e_1, u_2] = u_2, & [e_2, u_2] = u_1, & [e_3, u_2] = 0, & [e_4, u_2] = 0, \\
[e_1, u_3] = \lambda u_3, & [e_2, u_3] = 0, & [e_3, u_3] = u_1, & [e_4, u_3] = u_2.
\end{array}$$

Since the virtual structure $q$ is primary, we have

$$\begin{aligned} &\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \supseteq \mathbb{R}e_1 \oplus \mathbb{R}u_1, && \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) \supseteq \mathbb{R}e_2, \\ &\bar{\mathfrak{g}}^{(-\lambda)}(\mathfrak{h}) \supseteq \mathbb{R}e_3, && \bar{\mathfrak{g}}^{(1-\lambda)}(\mathfrak{h}) \supseteq \mathbb{R}e_4, \\ &\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) \supseteq \mathbb{R}u_2, && \bar{\mathfrak{g}}^{(\lambda)}(\mathfrak{h}) \supseteq \mathbb{R}u_3. \end{aligned}$$

Therefore

$$\begin{aligned} [u_1, u_2] &= a_3 e_3 + a_4 e_4 + \alpha_2 u_2 + \alpha_3 u_3, \\ [u_1, u_3] &= b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3, \\ [u_2, u_3] &= c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + \gamma_1 u_1 + \gamma_2 u_2. \end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $-e_2$ | $-\lambda e_3$ | $(1-\lambda)e_4$ | $0$ | $u_2$ | $\lambda u_3$ |
| $e_2$ | $e_2$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $\lambda e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $(\lambda-1)e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $A$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $B$ |
| $u_3$ | $-\lambda u_3$ | $0$ | $-u_1$ | $-u_2$ | $-A$ | $-B$ | $0$ , |

$$\begin{aligned} A &= b_3 e_3 + \beta_1 u_1, \\ B &= c_3 e_3 + b_3 e_4 + \gamma_1 u_1 + \beta_1 u_2, \end{aligned}$$

where

$$\begin{cases} (1+2\lambda)c_3 = 0, \\ \lambda b_3 = 0, \\ \lambda \beta_1 = 0, \\ (1+\lambda)\gamma_1 = 0. \end{cases}$$

Consider the following cases:

1.1°. $\lambda \notin \{-1, -\frac{1}{2}, 0\}$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

1.2°. $\lambda = -1$. Then $c_3 = b_3 = \beta_1 = 0$ and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_i) &= e_i, \ i = 1, \dots, 4, \\ \pi(u_1) &= \frac{1}{2} u_1, \\ \pi(u_2) &= \frac{\gamma_1}{2} e_3 + \frac{1}{2} u_2, \\ \pi(u_3) &= -\frac{\gamma_1}{2} e_2 + \frac{1}{2} u_3. \end{aligned}$$

1.3°. $\lambda = -\frac{1}{2}$. Then $b_3 = \beta_1 = \gamma_1 = 0$.

1.3.1°. $c_3 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

1.3.2°. $c_3 \neq 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = \frac{1}{c_3} e_2, \quad \pi(u_1) = u_1, \\ &\pi(e_3) = e_3, \qquad \pi(u_2) = c_3 u_2, \\ &\pi(e_4) = c_3 e_4, \quad \pi(u_3) = u_3. \end{aligned}$$

It can be easily proved that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

1.4°. $\lambda = 0$. Then $c_3 = \gamma_1 = 0$.

1.4.1°. $b_3 = \beta_1 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

1.4.2°. $\beta_1 = 0$.

1.4.2.1°. $b_3 > 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = b_3 e_2, \qquad \pi(u_1) = b_3 u_1, \\ &\pi(e_3) = \sqrt{b_3} e_3, \quad \pi(u_2) = u_2, \\ &\pi(e_4) = \frac{1}{\sqrt{b_3}} e_4, \quad \pi(u_3) = \sqrt{b_3} u_3. \end{aligned}$$

1.4.2.2°. $b_3 < 0$. Then the mapping $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}$ such that

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = -b_3 e_2, \qquad \pi(u_1) = -b_3 u_1, \\ &\pi(e_3) = \sqrt{-b_3} e_3, \quad \pi(u_2) = u_2, \\ &\pi(e_4) = \frac{1}{\sqrt{-b_3}} e_4, \quad \pi(u_3) = \sqrt{-b_3} u_3, \end{aligned}$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$.

1.4.3°. $\beta_1 \neq 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_5, \mathfrak{g}_5)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_5 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = \beta_1^2 e_2, \quad \pi(u_1) = \beta_1^2 u_1, \\ &\pi(e_3) = \beta_1 e_3, \quad \pi(u_2) = u_2, \\ &\pi(e_4) = \frac{1}{\beta_1} e_4, \quad \pi(u_3) = \beta_1 u_3. \end{aligned}$$

Let $\mathfrak{a}_i$ be the ideal in $\bar{\mathfrak{g}}_i$, $i = 3, 4, 5$, spanned by $\{e_3, e_4, u_1, u_2\}$. Note, that $\mathfrak{a}_i$ is a unique 4-dimensional commutative ideal in $\mathcal{D}\bar{\mathfrak{g}}_i$. Consider the homomorphisms

$f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(4, \mathbb{R})$, $i = 3, 4, 5$, where $f_i(x)$ is the matrix of the mapping $\mathrm{ad}_{\mathfrak{a}_i} x$ in the basis $\{e_3, e_4, u_1, u_2\}$ of $\mathfrak{a}_i$, for $x \in \bar{\mathfrak{g}}_i$.

Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$, $i = 3, 4, 5$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 3, 4, 5$ are not equivalent.

2°. $\lambda = -1$. Then

$$\begin{array}{llll}
[e_1, e_2] = -e_2, & & & \\
[e_1, e_3] = e_3, & [e_2, e_3] = 0, & & \\
[e_1, e_4] = 2e_4, & [e_2, e_4] = e_3, & [e_3, e_4] = 0, & \\
[e_1, u_1] = 0, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = 0, \\
[e_1, u_2] = -pe_3 + u_2, & [e_2, u_2] = u_1, & [e_3, u_2] = 0, & [e_4, u_2] = 0, \\
[e_1, u_3] = pe_2 - u_3, & [e_2, u_3] = 0, & [e_3, u_3] = u_1, & [e_4, u_3] = u_2.
\end{array}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}),$$

where

$$\begin{gathered}
\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}u_1, \quad \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{R}e_2 \oplus \mathbb{R}u_3, \\
\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}e_3 \oplus \mathbb{R}u_2, \quad \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}) = \mathbb{R}e_4.
\end{gathered}$$

Therefore

$$\begin{aligned}
[u_1, u_2] &= a_3 e_3 + \alpha_2 u_2, \\
[u_1, u_3] &= b_2 e_2 + \beta_3 u_3, \\
[u_2, u_3] &= c_1 e_1 + \gamma_1 u_1.
\end{aligned}$$

Checking the Jacobi identity on vectors $(e_1, u_2, u_3)$.

$$[e_1, [u_2, u_3]] + [u_3, [e_1, u_2]] + [u_2, [u_3, e_1]] = 0$$

we obtain $p = 0$ and $C(e_i) = 0$, $i = 1, \dots, 4$. This virtual structure was considered in case 1°.

3°. $\lambda = 0$. Then

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_i) = 0, \ i = 2, 3, 4.$$

Put

$$C'(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p' \\ 0 & 0 & 0 \end{pmatrix}, \quad C'(e_i) = 0, \ i = 2, 3, 4.$$

The virtual structures $C$ and $C'$ are equivalent if and only if there exist matrices $P \in \mathcal{A}(\mathfrak{g})$ and $H \in Mat_{4\times 3}(\mathbb{R})$ such that

$$C'(x) = FC(\varphi^{-1}(x))P^{-1} - A(x)H + HB(x) \quad \text{for all} \quad x \in \mathfrak{g},$$

where $\varphi(x) = PxP^{-1}$ and $F$ is the matrix of the mapping $\varphi$. After some calculation we see that the virtual structures $C$ and $C'$ are equivalent if and only if there exist reals $a, b$ such that $ab \neq 0$ and the following condition is satisfied:

$$p' = p\frac{a}{b^2}.$$

Using this condition we see that any virtual structure on generalized module 4.20 ($\lambda = 0$) is equivalent to one and only one of the following:

*a*) $C^1(e_i) = 0, \quad i = 1, \ldots, 4.$

*b*) $C^2(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C^2(e_i) = 0, \ i = 2, 3, 4.$

The virtual structure $C^1$ was considered in case 1°.

For the virtual structure $C^2$ we obtain:

$$\begin{array}{llll}
[e_1, e_2] = -e_2, & & & \\
[e_1, e_3] = 0, & [e_2, e_3] = 0, & & \\
[e_1, e_4] = e_4, & [e_2, e_4] = e_3, & [e_3, e_4] = 0, & \\
[e_1, u_1] = 0, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = 0, \\
[e_1, u_2] = u_2, & [e_2, u_2] = u_1, & [e_3, u_2] = 0, & [e_4, u_2] = 0, \\
[e_1, u_3] = e_3, & [e_2, u_3] = 0, & [e_3, u_3] = u_1, & [e_4, u_3] = u_2.
\end{array}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}),$$

where

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_3 \oplus \mathbb{R}u_1 \oplus \mathbb{R}u_3,$$
$$\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{R}e_2, \qquad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}e_4 \oplus \mathbb{R}u_2.$$

Therefore

$$\begin{aligned}
[u_1, u_2] &= a_4 e_4 + \alpha_2 u_2, \\
[u_1, u_3] &= b_1 e_1 + b_3 e_3 + \beta_1 u_1 + \beta_3 u_3, \\
[u_2, u_3] &= c_4 e_4 + \gamma_2 u_2.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | $-e_2$ | 0 | $e_4$ | 0 | $u_2$ | $e_3$ |
| $e_2$ | $e_2$ | 0 | 0 | $e_3$ | 0 | $u_1$ | 0 |
| $e_3$ | 0 | 0 | 0 | 0 | 0 | 0 | $u_1$ |
| $e_4$ | $-e_4$ | $-e_3$ | 0 | 0 | 0 | 0 | $u_2$ |
| $u_1$ | 0 | 0 | 0 | 0 | 0 | 0 | $b_3e_3 + \beta_1u_1$ |
| $u_2$ | $-u_2$ | $-u_1$ | 0 | 0 | 0 | 0 | $b_3e_4 + \beta_1u_2$ |
| $u_3$ | $-e_3$ | 0 | $-u_1$ | $-u_2$ | $-b_3e_3 - \beta_1u_1$ | $-b_3e_4 - \beta_1u_2$ | 0 |

,

3.1°. $b_3 = \beta_1 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$.

3.2°. $\beta_1 = 0$.

3.2.1°. $b_3 > 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_7, \mathfrak{g}_7)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_7 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = b_3 e_2, && \pi(u_1) = b_3 u_1, \\ &\pi(e_3) = \sqrt{b_3}\, e_3, && \pi(u_2) = u_2, \\ &\pi(e_4) = \frac{1}{\sqrt{b_3}} e_4, && \pi(u_3) = \sqrt{b_3}\, u_3. \end{aligned}$$

3.2.2°. $b_3 < 0$. Then the mapping $\pi : \bar{\mathfrak{g}}_8 \to \bar{\mathfrak{g}}$ such that

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = -b_3 e_2, && \pi(u_1) = -b_3 u_1, \\ &\pi(e_3) = \sqrt{-b_3}\, e_3, && \pi(u_2) = u_2, \\ &\pi(e_4) = \frac{1}{\sqrt{-b_3}} e_4, && \pi(u_3) = \sqrt{-b_3}\, u_3, \end{aligned}$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}_8, \mathfrak{g}_8)$.

3.3°. $\beta_1 \neq 0$.

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_9, \mathfrak{g}_9)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_9 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = \beta_1^2 e_2, && \pi(u_1) = \beta_1^2 u_1, \\ &\pi(e_3) = \beta_1 e_3, && \pi(u_2) = u_2, \\ &\pi(e_4) = \frac{1}{\beta_1} e_4, && \pi(u_3) = \beta_1 u_3. \end{aligned}$$

As in case 1.4o, we can prove that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 6, 7, 8, 9$, are not equivalent.

4.° $\lambda = 1$. Then

$$\begin{aligned} &[e_1, e_2] = -e_2, \\ &[e_1, e_3] = -e_3, && [e_2, e_3] = 0, \\ &[e_1, e_4] = 0, && [e_2, e_4] = e_3, && [e_3, e_4] = 0, \\ &[e_1, u_1] = 0, && [e_2, u_1] = p e_3, && [e_3, u_1] = 0, && [e_4, u_1] = 0, \\ &[e_1, u_2] = u_2, && [e_2, u_2] = u_1, && [e_3, u_2] = 0, && [e_4, u_2] = 0, \\ &[e_1, u_3] = u_3, && [e_2, u_3] = p e_1 + q e_4, && [e_3, u_3] = u_1, && [e_4, u_3] = u_2. \end{aligned}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}),$$

where

$$\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_4 \oplus \mathbb{R}u_1,$$

$$\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{R}e_2 \oplus \mathbb{R}e_3, \quad \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_2 \oplus \mathbb{R}u_3.$$

Therefore

$$[u_1, u_2] = \alpha_2 u_2 + \alpha_3 u_3,$$
$$[u_1, u_3] = \beta_2 u_2 + \beta_3 u_3,$$
$$[u_2, u_3] = 0.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $-e_2$ | $-e_3$ | $0$ | $0$ | $u_2$ | $u_3$ |
| $e_2$ | $e_2$ | $0$ | $0$ | $e_3$ | $pe_3$ | $u_1$ | $pe_1 + qe_4$ |
| $e_3$ | $e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $0$ | $-e_3$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $0$ | $-pe_3$ | $0$ | $0$ | $0$ | $0$ | $pu_2$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-u_3$ | $-pe_1 - qe_4$ | $-u_1$ | $-u_2$ | $-pu_2$ | $0$ | $0$ |

Now we determite the group of all transformations for mappings $q$. We have

$$C(e_2) = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & 0 \\ p & 0 & 0 \\ 0 & 0 & q \end{pmatrix}, \quad C(e_i) = 0, \; i = 1, 3, 4.$$

Put

$$C'(e_2) = \begin{pmatrix} 0 & 0 & p' \\ 0 & 0 & 0 \\ p' & 0 & 0 \\ 0 & 0 & q' \end{pmatrix}, \quad C'(e_i) = 0, \; i = 1, 3, 4.$$

The virtual structures $C$ and $C'$ are equivalent if and only if there exist matrices $P \in \mathcal{A}(\mathfrak{g})$ and $H \in Mat_{4\times 3}(\mathbb{R})$ such that

$$C'(x) = FC(\varphi^{-1}(x))P^{-1} - A(x)H + HB(x) \text{ for all } x \in \mathfrak{g},$$

where $\varphi(x) = PxP^{-1}$ and $F$ is the matrix of the mapping $\varphi$. After direct calculation we see that the virtual structures $C$ and $C'$ are equivalent if and only if there exist reals $a, b, c$ such that $abc \neq 0$ and the following conditions are satisfied:

$$p' = p\frac{b}{ac}, \qquad q' = q\frac{b^2}{ac^2}.$$

Using these conditions we see that any virtual structure on generalized module 4.20 $(\lambda = 1)$ is equivalent to one and only one of the following:

*a*) $C^1(e_i) = 0, \quad i = 1, \dots, 4;$

*b*) $C^2(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C^2(e_i) = 0, \; i = 1, 3, 4;$

c) $C^3(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\quad C^3(e_i) = 0,\ i = 1,3,4;$

d) $C^4(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\quad C^4(e_i) = 0,\ i = 1,3,4.$

Note that the virtual structure $C^1$ was considered in 1°. For the virtual structures $C^2$, $C^3$, $C^4$ we obtain the following nonequivalent pairs: $(\bar{\mathfrak{g}}_{10}, \mathfrak{g}_{10})$, $(\bar{\mathfrak{g}}_{11}, \mathfrak{g}_{11})$, $(\bar{\mathfrak{g}}_{12}, \mathfrak{g}_{12})$.

5°. $\lambda = 3$. Then

$$\begin{aligned}
&[e_1,e_2] = -e_2, \\
&[e_1,e_3] = -3e_3, \quad [e_2,e_3] = 0, \\
&[e_1,e_4] = -2e_4, \quad [e_2,e_4] = e_3, \quad [e_3,e_4] = 0, \\
&[e_1,u_1] = 0, \quad [e_2,u_1] = 0, \quad [e_3,u_1] = 0, \quad [e_4,u_1] = 0, \\
&[e_1,u_2] = u_2, \quad [e_2,u_2] = u_1, \quad [e_3,u_2] = 0, \quad [e_4,u_2] = pe_2, \\
&[e_1,u_3] = 3u_3, \quad [e_2,u_3] = 0, \quad [e_3,u_3] = u_1, \quad [e_4,u_3] = u_2.
\end{aligned}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(-3)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1)} \oplus \bar{\mathfrak{g}}^{(3)},$$

where

$$\begin{aligned}
&\bar{\mathfrak{g}}^{(-3)}(\mathfrak{h}) = \mathbb{R}e_3, \quad \bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}) = \mathbb{R}e_4, \\
&\bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{R}e_2, \quad \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}u_1, \\
&\bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}u_2, \quad \bar{\mathfrak{g}}^{(3)}(\mathfrak{h}) = \mathbb{R}u_3.
\end{aligned}$$

Therefore

$$\begin{aligned}
&[u_1,u_2] = 0, \\
&[u_1,u_3] = 0, \\
&[u_2,u_3] = 0.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $-e_2$ | $-3e_3$ | $-2e_4$ | $0$ | $u_2$ | $3u_3$ |
| $e_2$ | $e_2$ | $0$ | $0$ | $e_3$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $3e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_4$ | $2e_4$ | $-e_3$ | $0$ | $0$ | $0$ | $pe_2$ | $u_2$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $-pe_2$ | $0$ | $0$ | $0$ |
| $u_3$ | $-3u_3$ | $0$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ |

5.1°. $p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

5.2°. $p \neq 0$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{13}, \mathfrak{g}_{13})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{13} \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i, \ i = 1, \dots, 4,$$
$$\pi(u_j) = pu_j, \ j = 1, 2, 3.$$

Since $\dim \mathcal{D}^2 \bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2 \bar{\mathfrak{g}}_{13}$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_{13}, \mathfrak{g}_{13})$ are not equivalent.

The proof of the Proposition is complete.

**Proposition 4.21.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.21 is equivalent to one and only one of the following pairs:*

*1.*

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $(1-\lambda)e_2$ | $(\lambda-\mu)e_3$ | $(1-\mu)e_4$ | $u_1$ | $\lambda u_2$ | $\mu u_3$ |
| $e_2$ | $(\lambda-1)e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $(\mu-\lambda)e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $(\mu-1)e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-\lambda u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-\mu u_3$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ |

2. $\lambda + 2\mu = 1$

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $(1-\lambda)e_2$ | $\frac{3\lambda-1}{2}e_3$ | $\frac{1+\lambda}{2}e_4$ | $u_1$ | $\lambda u_2$ | $\frac{1-\lambda}{2}u_3$ |
| $e_2$ | $(\lambda-1)e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $\frac{1-3\lambda}{2}e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $-\frac{1+\lambda}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-\lambda u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $e_4$ |
| $u_3$ | $\frac{\lambda-1}{2}u_3$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $-e_4$ | $0$ |

3. $\lambda = 1$, $\mu = 0$

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $0$ |
| $e_2$ | $0$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $-e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $e_4$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $e_3+e_4$ |
| $u_3$ | $0$ | $0$ | $-u_2$ | $-u_1$ | $-e_4$ | $-e_3-e_4$ | $0$ |

4. $\lambda = 1$, $\mu = 0$

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $0$ |
| $e_2$ | $0$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $-e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-e_4$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $e_4-e_3$ |
| $u_3$ | $0$ | $0$ | $-u_2$ | $-u_1$ | $e_4$ | $e_3-e_4$ | $0$ |

5. $\mu = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $(1-\lambda)e_2$ | $\lambda e_3$ | $e_4$ | $u_1$ | $\lambda u_2$ | $0$ |
| $e_2$ | $(\lambda-1)e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $-\lambda e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $e_4$ |
| $u_2$ | $-\lambda u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $e_3$ |
| $u_3$ | $0$ | $0$ | $-u_2$ | $-u_1$ | $-e_4$ | $-e_3$ | $0$ |

6. $\mu = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $(1-\lambda)e_2$ | $\lambda e_3$ | $e_4$ | $u_1$ | $\lambda u_2$ | $0$ |
| $e_2$ | $(\lambda-1)e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $-\lambda e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-e_4$ |
| $u_2$ | $-\lambda u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $-e_3$ |
| $u_3$ | $0$ | $0$ | $-u_2$ | $-u_1$ | $e_4$ | $e_3$ | $0$ |

7. $\lambda \neq 1$, $\mu = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $(1-\lambda)e_2$ | $\lambda e_3$ | $e_4$ | $u_1$ | $\lambda u_2$ | $0$ |
| $e_2$ | $(\lambda-1)e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $-\lambda e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\alpha e_4 + u_1$ |
| $u_2$ | $-\lambda u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $\alpha e_3 + u_2$ |
| $u_3$ | $0$ | $0$ | $-u_2$ | $-u_1$ | $-\alpha e_4 - u_1$ | $-\alpha e_3 - u_2$ | $0$ |

8. $\lambda = 0$, $\mu = -1$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $e_2$ | $e_3$ | $2e_4$ | $u_1$ | $0$ | $-u_3$ |
| $e_2$ | $-e_2$ | $0$ | $e_4$ | $0$ | $0$ | $e_2 + u_1$ | $0$ |
| $e_3$ | $-e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $e_2 - 2e_3$ | $u_2$ |
| $e_4$ | $-2e_4$ | $0$ | $0$ | $0$ | $0$ | $-e_4$ | $e_2 + u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-e_2 - u_1$ | $2e_3 - e_2$ | $e_4$ | $0$ | $0$ | $-2u_3$ |
| $u_3$ | $u_3$ | $0$ | $-u_2$ | $-e_2 - u_1$ | $0$ | $2u_3$ | $0$ |

9. $\mu = \frac{1}{2}$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $(1-\lambda)e_2$ | $\frac{1}{2}(2\lambda-1)e_3$ | $\frac{1}{2}e_4$ | $u_1$ | $\lambda u_2$ | $e_4 + \frac{1}{2}u_3$ |
| $e_2$ | $(\lambda-1)e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $\frac{1}{2}(1-2\lambda)e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $-\frac{1}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-\lambda u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-e_4 - \frac{1}{2}u_3$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ |

*10.* $\lambda = 0$, $\mu = \frac{1}{2}$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $e_2$ | $-\frac{1}{2}e_3$ | $\frac{1}{2}e_4$ | $u_1$ | $0$ | $e_4 + \frac{1}{2}u_3$ |
| $e_2$ | $-e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $\frac{1}{2}e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $-\frac{1}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $e_4$ |
| $u_3$ | $-e_4 - \frac{1}{2}u_3$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $-e_4$ | $0$ |

*11.* $\lambda = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $e_2$ | $-\mu e_3$ | $(1-\mu)e_4$ | $u_1$ | $0$ | $\mu u_3$ |
| $e_2$ | $-e_2$ | $0$ | $e_4$ | $0$ | $0$ | $e_2 + u_1$ | $0$ |
| $e_3$ | $\mu e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $-2e_3$ | $u_2$ |
| $e_4$ | $(\mu-1)e_4$ | $0$ | $0$ | $0$ | $0$ | $-e_4$ | $e_2 + u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-e_2 - u_1$ | $2e_3$ | $e_4$ | $0$ | $0$ | $-2u_3$ |
| $u_3$ | $-\mu u_3$ | $0$ | $-u_2$ | $-e_2 - u_1$ | $0$ | $2u_3$ | $0$ |

*12.* $\mu = 3\lambda - 1$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $(1-\lambda)e_2$ | $(1-2\lambda)e_3$ | $(2-3\lambda)e_4$ | $u_1$ | $\lambda u_2$ | $(3\lambda-1)u_3$ |
| $e_2$ | $(\lambda-1)e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $(2\lambda-1)e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $e_2$ | $u_2$ |
| $e_4$ | $(3\lambda-2)e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-\lambda u_2$ | $-u_1$ | $-e_2$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $(1-3\lambda)u_3$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ |

*13.* $\lambda = \frac{1}{3}$, $\mu = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $\frac{2}{3}e_2$ | $\frac{1}{3}e_3$ | $e_4$ | $u_1$ | $\frac{1}{3}u_2$ | $0$ |
| $e_2$ | $-\frac{2}{3}e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $-\frac{1}{3}e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $e_2$ | $u_2$ |
| $e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $e_4$ |
| $u_2$ | $-\frac{1}{3}u_2$ | $-u_1$ | $-e_2$ | $0$ | $0$ | $0$ | $e_3$ |
| $u_3$ | $0$ | $0$ | $-u_2$ | $-u_1$ | $-e_4$ | $-e_3$ | $0$ |

*14.* $\lambda = \frac{1}{3}$, $\mu = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $\frac{2}{3}e_2$ | $\frac{1}{3}e_3$ | $e_4$ | $u_1$ | $\frac{1}{3}u_2$ | $0$ |
| $e_2$ | $-\frac{2}{3}e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $-\frac{1}{3}e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $e_2$ | $u_2$ |
| $e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-e_4$ |
| $u_2$ | $-\frac{1}{3}u_2$ | $-u_1$ | $-e_2$ | $0$ | $0$ | $0$ | $-e_3$ |
| $u_3$ | $0$ | $0$ | $-u_2$ | $-u_1$ | $e_4$ | $e_3$ | $0$ |

*15.* $\lambda = \frac{3}{7}$, $\mu = \frac{2}{7}$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $\frac{4}{7}e_2$ | $\frac{1}{7}e_3$ | $\frac{5}{7}e_4$ | $u_1$ | $\frac{3}{7}u_2$ | $\frac{2}{7}u_3$ |
| $e_2$ | $-\frac{4}{7}e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $-\frac{1}{7}e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $e_2$ | $u_2$ |
| $e_4$ | $-\frac{5}{7}e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-\frac{3}{7}u_2$ | $-u_1$ | $-e_2$ | $0$ | $0$ | $0$ | $e_4$ |
| $u_3$ | $-\frac{2}{7}u_3$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $-e_4$ | $0$ |

*16.* $\lambda = \frac{1}{2}$, $\mu = \frac{1}{2}$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $\frac{1}{2}e_2$ | $0$ | $\frac{1}{2}e_4$ | $u_1$ | $\frac{1}{2}u_2$ | $e_4 + \frac{1}{2}u_3$ |
| $e_2$ | $-\frac{1}{2}e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $0$ | $-e_4$ | $0$ | $0$ | $0$ | $e_2$ | $u_2$ |
| $e_4$ | $-\frac{1}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-\frac{1}{2}u_2$ | $-u_1$ | $-e_2$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-e_4 - \frac{1}{2}u_3$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ |

*17.* $\lambda = \frac{1}{2}$, $\mu = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $\frac{1}{2}e_2$ | $\frac{1}{2}e_3$ | $e_4$ | $u_1$ | $\frac{1}{2}u_2$ | $0$ |
| $e_2$ | $-\frac{1}{2}e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $e_3$ |
| $e_3$ | $-\frac{1}{2}e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $e_4$ |
| $u_2$ | $-\frac{1}{2}u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $e_3$ |
| $u_3$ | $0$ | $-e_3$ | $-u_2$ | $-u_1$ | $-e_4$ | $-e_3$ | $0$ |

*18.* $\lambda = \frac{1}{2}$, $\mu = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $\frac{1}{2}e_2$ | $\frac{1}{2}e_3$ | $e_4$ | $u_1$ | $\frac{1}{2}u_2$ | $0$ |
| $e_2$ | $-\frac{1}{2}e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $e_3$ |
| $e_3$ | $-\frac{1}{2}e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-e_4$ |
| $u_2$ | $-\frac{1}{2}u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $-e_3$ |
| $u_3$ | $0$ | $-e_3$ | $-u_2$ | $-u_1$ | $e_4$ | $e_3$ | $0$ |

*19.* $\lambda = \frac{1}{2}$, $\mu = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $\frac{1}{2}e_2$ | $\frac{1}{2}e_3$ | $e_4$ | $u_1$ | $\frac{1}{2}u_2$ | $0$ |
| $e_2$ | $-\frac{1}{2}e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $e_3$ |
| $e_3$ | $-\frac{1}{2}e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\alpha e_4 + u_1$ |
| $u_2$ | $-\frac{1}{2}u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $\alpha e_3 + u_2$ |
| $u_3$ | $0$ | $-e_3$ | $-u_2$ | $-u_1$ | $-\alpha e_4 - u_1$ | $-\alpha e_3 - u_2$ | $0$ |

*20.* $\lambda - \mu = \frac{1}{2}$

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | $(1-\lambda)e_2$ | $\frac{1}{2}e_3$ | $\frac{3-2\lambda}{2}e_4$ | $u_1$ | $\lambda u_2$ | $\frac{2\lambda-1}{2}u_3$ |
| $e_2$ | $(\lambda-1)e_2$ | 0 | $e_4$ | 0 | 0 | $u_1$ | $e_3$ |
| $e_3$ | $-\frac{1}{2}e_3$ | $-e_4$ | 0 | 0 | 0 | 0 | $u_2$ |
| $e_4$ | $\frac{2\lambda-3}{2}e_4$ | 0 | 0 | 0 | 0 | 0 | $u_1$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | $-\lambda u_2$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 |
| $u_3$ | $\frac{1-2\lambda}{2}u_3$ | $-e_3$ | $-u_2$ | $-u_1$ | 0 | 0 | 0 |

*21.* $\lambda = \frac{2}{3}$, $\mu = \frac{1}{6}$

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | $\frac{1}{3}e_2$ | $\frac{1}{2}e_3$ | $\frac{5}{6}e_4$ | $u_1$ | $\frac{2}{3}u_2$ | $\frac{1}{6}u_3$ |
| $e_2$ | $-\frac{1}{3}e_2$ | 0 | $e_4$ | 0 | 0 | $u_1$ | $e_3$ |
| $e_3$ | $-\frac{1}{2}e_3$ | $-e_4$ | 0 | 0 | 0 | 0 | $u_2$ |
| $e_4$ | $-\frac{5}{6}e_4$ | 0 | 0 | 0 | 0 | 0 | $u_1$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | $-\frac{2}{3}u_2$ | $-u_1$ | 0 | 0 | 0 | 0 | $e_4$ |
| $u_3$ | $-\frac{1}{6}u_3$ | $-e_3$ | $-u_2$ | $-u_1$ | 0 | $-e_4$ | 0 |

*22.* $\lambda = 1$, $\mu = \frac{1}{2}$

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $\frac{1}{2}e_3$ | $\frac{1}{2}e_4$ | $u_1$ | $u_2$ | $e_4 + \frac{1}{2}u_3$ |
| $e_2$ | 0 | 0 | $e_4$ | 0 | 0 | $u_1$ | $e_3$ |
| $e_3$ | $-\frac{1}{2}e_3$ | $-e_4$ | 0 | 0 | 0 | 0 | $u_2$ |
| $e_4$ | $-\frac{1}{2}e_4$ | 0 | 0 | 0 | 0 | 0 | $u_1$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | $-u_2$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 |
| $u_3$ | $-e_4 - \frac{1}{2}u_3$ | $-e_3$ | $-u_2$ | $-u_1$ | 0 | 0 | 0 |

*23.* $\lambda = 1$, $\mu = \frac{1}{2}$

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $\frac{1}{2}e_3$ | $\frac{1}{2}e_4$ | $u_1$ | $u_2$ | $-e_4 + \frac{1}{2}u_3$ |
| $e_2$ | 0 | 0 | $e_4$ | 0 | 0 | $u_1$ | $e_3$ |
| $e_3$ | $-\frac{1}{2}e_3$ | $-e_4$ | 0 | 0 | 0 | 0 | $u_2$ |
| $e_4$ | $-\frac{1}{2}e_4$ | 0 | 0 | 0 | 0 | 0 | $u_1$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | $-u_2$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 |
| $u_3$ | $e_4 - \frac{1}{2}u_3$ | $-e_3$ | $-u_2$ | $-u_1$ | 0 | 0 | 0 |

*24.* $\lambda = 1$, $\mu = 0$

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $e_3$ | $e_4$ | $u_1$ | $u_2$ | 0 |
| $e_2$ | 0 | 0 | $e_4$ | 0 | 0 | $u_1$ | $e_2$ |
| $e_3$ | $-e_3$ | $-e_4$ | 0 | 0 | 0 | 0 | $u_2$ |
| $e_4$ | $-e_4$ | 0 | 0 | 0 | 0 | 0 | $e_4 + u_1$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | $\alpha e_4$ |
| $u_2$ | $-u_2$ | $-u_1$ | 0 | 0 | 0 | 0 | $\alpha e_3 - u_2$ |
| $u_3$ | 0 | $-e_2$ | $-u_2$ | $-e_4 - u_1$ | $-\alpha e_4$ | $-\alpha e_3 + u_2$ | 0 |

*25.* $\lambda = 1$, $\mu = 0$

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $0$ |
| $e_2$ | $0$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $e_2$ |
| $e_3$ | $-e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $e_4 + u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\alpha e_4$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $\alpha e_3 + e_4 - u_2$ |
| $u_3$ | $0$ | $-e_2$ | $-u_2$ | $-e_4 - u_1$ | $-\alpha e_4$ | $-\alpha e_3 - e_4 + u_2$ | $0$ |

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & \lambda-\mu & 0 \\ 0 & 0 & 0 & 1-\mu \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu-\lambda & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu-1 & 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

**Lemma.** *Any virtual structure $q$ on generalized module 4.21 is equivalent to one of the following:*

*a)* $\lambda = 1, \mu = 0$

$$C(e_1) = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & q \\ 0 & -p & 0 \\ -p & -q & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & r \\ 0 & 0 & s \\ 0 & -2r & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C(e_3) = 0, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -r \\ 0 & 0 & s \end{pmatrix};$$

*b)* $\lambda = \frac{3}{5}, \mu = \frac{6}{5}$

$$C(e_1) = C(e_2) = 0, \quad C(e_3) = \begin{pmatrix} 0 & -5p & 0 \\ p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

*c*) $3\lambda - 1 - \mu = 0$

$$C(e_1) = C(e_2) = C(e_4) = 0, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

*d*) $\lambda = \frac{1}{2}$

$$C(e_1) = C(e_2) = C(e_4) = 0, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix};$$

*e*) $1 - \lambda + \mu = 0$

$$C(e_1) = C(e_3) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & 0 \\ 0 & (\mu-2)p & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\mu-1)p \\ 0 & 0 & 0 \end{pmatrix};$$

*f*) $\lambda = 0$

$$C(e_1) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2p & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \\ 0 & -p & 0 \end{pmatrix};$$

*g*) $1 - 2\lambda + 2\mu = 0$

$$C(e_1) = C(e_3) = C(e_4) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix};$$

*h*) $1 - \lambda - \mu = 0$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \\ 0 & -p & 0 \end{pmatrix}, \quad C(e_2) = C(e_3) = C(e_4) = 0;$$

*i*) $1 - 2\mu = 0$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C(e_2) = C(e_3) = C(e_4) = 0;$$

*j*) $\lambda = 0, \mu = -1$

$$C(e_1) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & r \\ 0 & p & 0 \\ 0 & -3r & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \\ 0 & -2p & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & -2r \\ 0 & -p & 0 \end{pmatrix};$$

*k*) $\lambda = 0, \mu = -\frac{1}{2}$

$$C(e_1) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2p & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \\ 0 & -p & 0 \end{pmatrix};$$

*l*) $\lambda = \frac{1}{2}, \mu = -\frac{1}{2}$

$$C(e_1) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & q \\ 0 & 0 & 0 \\ 0 & -\frac{5}{2}q & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{2}q \\ 0 & 0 & 0 \end{pmatrix};$$

*m*) $\lambda = \frac{1}{4}, \mu = -\frac{1}{4}$

$$C(e_1) = C(e_4) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

*n*) $\lambda = \frac{1}{2}, \mu = 0$

$$C(e_1) = C(e_4) = 0, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix};$$

$o)\ \lambda = \frac{3}{4}, \mu = \frac{1}{4}$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \\ 0 & -p & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_3) = C(e_4) = 0;$$

$p)\ \lambda = 0, \mu = \frac{1}{2}$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2q & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q \\ 0 & 0 & 0 \\ 0 & -q & 0 \end{pmatrix};$$

$q)\ \lambda = \mu = \frac{1}{2}$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q \\ 0 & 0 & 0 \\ 0 & -q & p \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 0 \\ 0 & r & 0 \end{pmatrix}, \quad C(e_2) = C(e_4) = 0;$$

$r)\ \lambda = 1, \mu = \frac{1}{2}$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_3) = C(e_4) = 0;$$

$s)\ \lambda = \frac{3}{2}, \mu = \frac{1}{2}$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & q \\ 0 & 0 & 0 \\ 0 & -\frac{3}{2}q & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C(e_3) = 0, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}q \\ 0 & 0 & 0 \end{pmatrix};$$

$t)\ \lambda = 0, \mu = 1$

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \\ 0 & -p & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2q & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q \\ 0 & 0 & 0 \\ 0 & -q & 0 \end{pmatrix};$$

*u)* $\lambda, \mu \in \mathbb{R}$

$$C(e_i) = 0, \quad i = 1, \ldots, 4.$$

*Proof.* Put

$$C(e_i) = \left(c^i_{jk}\right)_{\substack{1 \leqslant j \leqslant 4, \\ 1 \leqslant k \leqslant 3}}, \quad i = 1, \ldots, 4.$$

Let

$$H = \begin{pmatrix} c^1_{11} & c^3_{13} & 0 \\ c^3_{41} & c^3_{23} & 0 \\ -c^2_{41} & c^3_{33} & -c^2_{43} \\ c^3_{33} + c^2_{42} & c^3_{43} & 0 \end{pmatrix},$$

and $C' = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C'(e_1) = \begin{pmatrix} 0 & c^1_{12} & c^1_{13} \\ c^1_{21} & c^1_{22} & c^1_{23} \\ c^1_{31} & c^1_{32} & c^1_{33} \\ c^1_{41} & c^1_{42} & c^1_{43} \end{pmatrix}, \quad C'(e_2) = \begin{pmatrix} c^2_{11} & c^2_{12} & c^2_{13} \\ c^2_{21} & c^2_{22} & c^2_{23} \\ c^2_{31} & c^2_{32} & c^2_{33} \\ 0 & 0 & 0 \end{pmatrix},$$

$$C'(e_3) = \begin{pmatrix} c^3_{11} & c^3_{12} & 0 \\ c^3_{21} & c^3_{22} & 0 \\ c^3_{31} & c^3_{32} & 0 \\ 0 & c^3_{42} & 0 \end{pmatrix}, \quad C'(e_4) = \begin{pmatrix} c^4_{11} & c^4_{12} & c^4_{13} \\ c^4_{21} & c^4_{22} & c^4_{23} \\ c^4_{31} & c^4_{32} & c^4_{33} \\ c^4_{41} & c^4_{42} & c^4_{43} \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures $C$ and $C'$ are equivalent.

Since for any virtual structure $q$ condition (6), Chapter II, is satisfied, after direct calculation we obtain:

system (1)

(1) $c^4_{33} = (\mu - 1)c^2_{13}$,
(2) $c^3_{21} = c^4_{22}$,
(3) $c^4_{22} = (1 - \mu)c^3_{12}$,
(4) $c^4_{23} = -c^4_{42}$,
(5) $(\lambda - 1)c^3_{12} - c^3_{21} = c^4_{22}$,
(6) $c^4_{23} = c^2_{22}$,
(7) $(\lambda - \mu)c^2_{13} + c^2_{32} = c^4_{33}$,
(8) $c^4_{42} = c^3_{32} + c^2_{22}$,
(9) $c^2_{23} = c^4_{43}$,
(10) $(2\lambda - \mu)c^4_{22} = 0$,
(11) $\lambda c^4_{23} = 0$,
(12) $(1 - \lambda + \mu)c^4_{33} = 0$,
(13) $\lambda c^4_{42} = 0$,
(14) $c^1_{41} = (\mu - 1)c^1_{13} + \mu c^4_{43}$,
(15) $(2\lambda - \mu)c^3_{12} = 0$,
(16) $(2\lambda - \mu)c^3_{21} = 0$,

(17) $(3\lambda - 1 - \mu)c^3_{22} = 0,$
(18) $\lambda c^3_{32} = 0,$
(19) $c^1_{32} = (\mu - \lambda)c^1_{13},$
(20) $(1 - 2\lambda)c^3_{42} = 0,$
(21) $c^1_{23} = -c^1_{42},$
(22) $(1 - \lambda + \mu)c^2_{13} = 0,$
(23) $\lambda c^2_{22} = 0,$
(24) $(\lambda - 1)c^1_{13} = -\mu c^2_{23},$
(25) $(1 - \lambda + \mu)c^2_{32} = 0,$
(26) $(1 - 2\lambda + 2\mu)c^2_{33} = 0,$
(27) $c^1_{41} = c^1_{32},$
(28) $c^1_{12} = c^1_{22} = c^1_{21} = c^1_{31} = c^1_{33} = 0,$
(29) $c^2_{11} = c^2_{31} = c^2_{21} = c^2_{12} = 0,$
(30) $c^3_{11} = c^3_{31} = 0,$
(31) $c^4_{11} = c^4_{21} = c^4_{31} = c^4_{13} = c^4_{32} = c^4_{41} = c^4_{12} = 0.$

Consider the following cases:
1°. $\lambda = 1, \mu = 0$. Put

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -c^1_{43} \end{pmatrix},$$

and $C_1(x) = C'(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & c^1_{13} \\ 0 & 0 & c^1_{23} \\ 0 & c^1_{32} & 0 \\ c^1_{41} & c^1_{42} & c^1_{43} \end{pmatrix}, \quad C_1(e_2) = \begin{pmatrix} 0 & 0 & c^2_{13} \\ 0 & c^2_{22} & c^2_{23} \\ 0 & c^2_{32} & c^2_{33} \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_1(e_3) = \begin{pmatrix} 0 & c^3_{12} & 0 \\ c^3_{21} & c^3_{22} & 0 \\ 0 & c^3_{32} & 0 \\ 0 & c^3_{42} & 0 \end{pmatrix}, \quad C_1(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c^4_{22} & c^4_{23} \\ 0 & 0 & c^4_{33} \\ 0 & c^4_{42} & c^4_{43} \end{pmatrix}$$

and

$$\begin{aligned}
&c^1_{43} = 0, && c^2_{22} = c^2_{33} = 0,\\
&c^3_{12} = c^3_{21} = 0, && c^3_{22} = c^3_{32} = c^3_{42} = 0,\\
&c^4_{22} = 0, && c^4_{23} = c^4_{42} = 0,\\
&c^4_{33} = -c^2_{13}, && c^3_{21} = c^4_{22},\\
&c^4_{22} = c^3_{12}, && c^4_{23} = -c^4_{42},\\
&c^4_{22} = -c^3_{21}, && c^4_{23} = c^2_{22},\\
&c^4_{33} = c^2_{32} + c^2_{13}, && c^4_{42} = c^3_{32} + c^2_{22},\\
&c^2_{23} = c^4_{43}, && c^1_{41} = -c^1_{13},\\
&c^1_{32} = -c^1_{13}, && c^1_{23} = -c^1_{42},\\
&c^1_{41} = c^1_{32},
\end{aligned}$$

Direct calculation shows that

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & c^1_{13} \\ 0 & 0 & c^1_{23} \\ 0 & -c^1_{13} & 0 \\ -c^1_{13} & -c^1_{23} & 0 \end{pmatrix}, \quad C_1(e_2) = \begin{pmatrix} 0 & 0 & c^2_{13} \\ 0 & 0 & c^2_{23} \\ 0 & -2c^2_{13} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_1(e_3)=0,\qquad C_1(e_4)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&-c_{13}^2\\0&0&c_{23}^2\end{pmatrix}.$$

The virtual structures $C_1$ and $C'$ are equivalent.

2°. $(\lambda-1)^2+\mu^2\neq 0$.

2.1°. Suppose $\mu=0$. Then from equations (19), (24) and (27) of system (1) it follows that $c_{13}^1=c_{41}^1=c_{32}^1=0$. Put

$$H=\begin{pmatrix}0&0&\frac{1}{1-\lambda}c_{23}^2\\0&0&0\\0&\frac{\lambda}{\lambda-1}c_{23}^2&0\\\frac{\lambda}{\lambda-1}c_{23}^2&0&0\end{pmatrix}$$

and $C_2(x)=C'(x)+A(x)H-HB(x)$ for $x\in\mathfrak{g}$. Then

$$C_2(e_i)=\left(c_{jk}^i\right)_{\substack{1\leqslant j\leqslant 4,\\1\leqslant k\leqslant 3}},\qquad i=1,\dots,4,$$

where $c_{jk}^i$ satisfy system (1) and $c_{23}^2=c_{43}^4=0$.

2.2°. Suppose $\mu\neq 0$. Put

$$H=\begin{pmatrix}0&0&\frac{1}{\mu}c_{13}^1\\0&0&0\\0&\frac{\mu-\lambda}{\mu}c_{13}^1&0\\\frac{\mu-\lambda}{\mu}c_{13}^1&0&0\end{pmatrix}$$

and $C_3(x)=C'(x)+A(x)H-HB(x)$ for $x\in\mathfrak{g}$. Then

$$C_3(e_i)=\left(c_{jk}^i\right)_{\substack{1\leqslant j\leqslant 4,\\1\leqslant k\leqslant 3}},\qquad i=1,\dots,4,$$

where $c_{jk}^i$ satisfy system (1) and $c_{13}^1=0$. From equations (9), (19), (24) and (27) of system (1) it follows that $c_{23}^2=c_{43}^4=c_{41}^1=c_{32}^1=0$.

Hence, if $(\lambda-1)^2+\mu^2\neq 0$, then $c_{23}^2=c_{43}^4=c_{13}^1=c_{41}^1=c_{32}^1=0$. It follows that the matrices $C_3(e_i)$, $i=1,\dots,4$, have the form:

$$C_3(e_1)=\begin{pmatrix}0&0&0\\0&0&c_{23}^1\\0&0&0\\0&c_{42}^1&c_{43}^1\end{pmatrix},\qquad C_3(e_2)=\begin{pmatrix}0&0&c_{13}^2\\0&c_{22}^2&0\\0&c_{32}^2&c_{33}^2\\0&0&0\end{pmatrix},$$

$$C_3(e_3)=\begin{pmatrix}0&c_{12}^3&0\\c_{21}^3&c_{22}^3&0\\0&c_{32}^3&0\\0&c_{42}^3&0\end{pmatrix},\qquad C_3(e_4)=\begin{pmatrix}0&0&0\\0&c_{22}^4&c_{23}^4\\0&0&c_{33}^4\\0&c_{42}^4&0\end{pmatrix},$$

where the coefficients $c_{jk}^i$ satisfy the following system of linear equations system (2)

(1) $c_{42}^4 = -c_{22}^2$,
(2) $c_{23}^4 = c_{22}^2$,
(3) $c_{32}^3 = -2c_{22}^2$,
(4) $c_{23}^1 = -c_{42}^1$,
(5) $c_{21}^3 = (1-\mu)c_{12}^3$,
(6) $c_{22}^4 = (1-\mu)c_{12}^3$,
(7) $c_{33}^4 = (\mu-1)c_{13}^2$,
(8) $c_{32}^2 = (\mu-2)c_{13}^2$,
(9) $(\lambda+2\mu-3)c_{12}^3 = 0$,
(10) $(2\lambda-\mu)c_{12}^3 = 0$,
(11) $(3\lambda-1-\mu)c_{22}^3 = 0$,
(12) $(1-2\lambda)c_{42}^3 = 0$,
(13) $(1-\lambda+\mu)c_{13}^2 = 0$,
(14) $\lambda c_{22}^2 = 0$,
(15) $(1-\lambda+\mu)c_{32}^2 = 0$,
(16) $(1-2\lambda+2\mu)c_{33}^2 = 0$.

Consider the following cases:
3°. $\lambda = \frac{3}{5}, \mu = \frac{6}{5}$. Then

$$C_4(e_1) = C_4(e_2) = 0, \quad C_4(e_3) = \begin{pmatrix} 0 & c_{12}^3 & 0 \\ -\frac{1}{5}c_{12}^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ C_4(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{5}c_{12}^3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The virtual structures $C_4$ and $C'$ are equivalent.

4°. $(\lambda, \mu) \neq (\frac{3}{5}, \frac{6}{5})$. From equations (9) and (10) of system (2) it follows that $c_{12}^3 = 0$.

Hence, system (2) is equivalent to the following system

(1) $c_{21}^3 = c_{12}^3 = c_{22}^4 = 0$,
(2) $c_{42}^4 = -c_{22}^2$,
(3) $c_{23}^4 = c_{22}^2$,
(4) $c_{32}^3 = -2c_{22}^2$,
(5) $c_{23}^1 = -c_{42}^1$,
(6) $c_{33}^4 = (\mu-1)c_{13}^2$,
(7) $c_{32}^2 = (\mu-2)c_{13}^2$,
(8) $(3\lambda-1-\mu)c_{22}^3 = 0$,
(9) $(1-2\lambda)c_{42}^3 = 0$,
(10) $(1-\lambda+\mu)c_{13}^2 = 0$,
(11) $\lambda c_{22}^2 = 0$,
(12) $(1-2\lambda+2\mu)c_{33}^2 = 0$.

The other results of the Lemma can be obtained using this system of linear equations.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 4.21. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the

Lemma. Put

$$[u_1,u_2] = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3,$$
$$[u_1,u_3] = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3,$$
$$[u_2,u_3] = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + \gamma_1u_1 + \gamma_2u_2 + \gamma_3u_3.$$

1°. $\lambda = 1, \mu = 0$. Then

$$\begin{array}{llll}
[e_1,e_2] = 0, & & & \\
[e_1,e_3] = e_3, & [e_2,e_3] = e_4, & & \\
[e_1,e_4] = e_4, & [e_2,e_4] = 0, & [e_3,e_4] = 0, & \\
[e_1,u_1] = -pe_4+u_1, & [e_2,u_1] = 0, & [e_3,u_1] = 0, & [e_4,u_1] = 0, \\
[e_1,u_2] = -pe_3-qe_4+u_2, & [e_2,u_2] = -2re_3+u_1, & [e_3,u_2] = 0 & [e_4,u_2] = 0, \\
[e_1,u_3] = pe_1+qe_2, & [e_2,u_3] = re_1+se_2, & [e_3,u_3] = 0, & [e_4,u_3] = -re_3+se_4+u_1.
\end{array}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $e_3$ | $e_4$ | $u_1$ | $u_2$ | 0 |
| $e_2$ | 0 | 0 | $e_4$ | 0 | 0 | $u_1$ | $se_2$ |
| $e_3$ | $-e_3$ | $-e_4$ | 0 | 0 | 0 | 0 | $u_2$ |
| $e_4$ | $-e_4$ | 0 | 0 | 0 | 0 | 0 | $se_4+u_1$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | $A$ |
| $u_2$ | $-u_2$ | $-u_1$ | 0 | 0 | 0 | 0 | $B$ |
| $u_3$ | 0 | $-se_2$ | $-u_2$ | $-se_4-u_1$ | $-A$ | $-B$ | 0 |

, (3)

where

$$A = b_4e_4 + \beta_1u_1,$$
$$B = b_4e_3 + c_4e_4 + \gamma_1u_1 + (\beta_1 - s)u_2,$$
$$s \neq 0.$$

The mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$ such that

$$\pi(e_i) = e_i, \; i = 1, \ldots, 4,$$
$$\pi(u_j) = \frac{1}{s}u_j, \; j = 1, 2, 3,$$

establishes the equivalence of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}', \mathfrak{g}')$, where the latter has form (3) for $s = 1$.

The mapping $\pi : \bar{\mathfrak{g}}'' \to \bar{\mathfrak{g}}'$ such that

$$\pi(e_i) = e_i, \; i = 1, 2, 4,$$
$$\pi(e_3) = e_3 - \frac{1}{2}\gamma_1 e_4,$$
$$\pi(u_1) = u_1,$$
$$\pi(u_2) = u_2 - \frac{1}{2}\gamma_1 e_4 - \frac{1}{2}\gamma_1 u_1,$$
$$\pi(u_3) = u_3,$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}'', \mathfrak{g}'')$ and $(\bar{\mathfrak{g}}', \mathfrak{g}')$, where the latter has form (3) for $s = 1, \gamma_1 = 0$.

The mapping $\pi : \bar{\mathfrak{g}}''' \to \bar{\mathfrak{g}}''$ such that

$$\begin{aligned}
&\pi(e_i) = e_i, \; i = 1, \dots, 4,\\
&\pi(u_1) = u_1 - \frac{1}{2}\beta_1 e_4,\\
&\pi(u_2) = u_2 - \frac{1}{2}\beta_1 e_3,\\
&\pi(u_3) = u_3 + \frac{1}{2}\beta_1 e_1,
\end{aligned}$$

establishes the equivalence of pairs $(\bar{\mathfrak{g}}'', \mathfrak{g}'')$ and $(\bar{\mathfrak{g}}''', \mathfrak{g}''')$, where the latter has form (3) for $s = 1, \gamma_1 = \beta_1 = 0$.

1.1°. $c_4 = 0$. Then the pair $(\bar{\mathfrak{g}}''', \mathfrak{g}''')$ is equivalent to the pair $(\bar{\mathfrak{g}}_{24}, \mathfrak{g}_{24})$.

1.2°. $c_4 \neq 0$. Then the pair $(\bar{\mathfrak{g}}''', \mathfrak{g}''')$ is equivalent to the pair $(\bar{\mathfrak{g}}_{25}, \mathfrak{g}_{25})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{25} \to \bar{\mathfrak{g}}'''$, where

$$\begin{aligned}
&\pi(e_i) = e_i, \; i = 1, 2, 4,\\
&\pi(e_3) = \frac{1}{c_4} e_3,\\
&\pi(u_1) = u_1,\\
&\pi(u_2) = \frac{1}{c_4} u_2,\\
&\pi(u_3) = u_3.
\end{aligned}$$

Consider the pairs $(\bar{\mathfrak{g}}_{24}, \mathfrak{g}_{24})$ and $(\bar{\mathfrak{g}}'_{24}, \mathfrak{g}'_{24})$ with parameters $\alpha$ and $\alpha'$ respectively.

Let us show that these pairs are not equivalent, whenever $\alpha \neq \alpha'$. Suppose $\bar{\mathfrak{a}} = \bar{\mathfrak{g}}_{24}/\mathcal{D}^2\bar{\mathfrak{g}}_{24}$ and $\varphi$ is a natural projection of $\bar{\mathfrak{g}}_{24}$ to $\bar{\mathfrak{a}}$. Then $\mathfrak{a} = \varphi(\mathfrak{g}_{24})$ for the pair $(\bar{\mathfrak{g}}_{24}, \mathfrak{g}_{24})$. Similarly define $\bar{\mathfrak{a}}'$ and $\mathfrak{a}'$ for the pair $(\bar{\mathfrak{g}}'_{24}, \mathfrak{g}'_{24})$. Let us show that the pairs $(\bar{\mathfrak{a}}, \mathfrak{a})$ and $(\bar{\mathfrak{a}}', \mathfrak{a}')$ are not equivalent, whenever $\alpha \neq \alpha'$.

Suppose these pairs are equivalent by means of a mapping $\pi : \bar{\mathfrak{a}} \to \bar{\mathfrak{a}}'$. Since

$$\pi(D\bar{\mathfrak{a}}) \subset D\bar{\mathfrak{a}}', \quad \pi(\mathfrak{a}) = \mathfrak{a}', \quad \pi(\mathcal{D}\mathfrak{a}) \subset \mathcal{D}\mathfrak{a}' \quad \text{and} \quad \pi(\mathcal{Z}(\mathfrak{a})) \subset \mathcal{Z}(\mathfrak{a}'),$$

the matrix of $\pi$ has the form:

$$\begin{pmatrix}
k_1 & 0 & 0 & 0 & a_1\\
k_2 & b & 0 & d_2 & a_2\\
k_3 & 0 & c_3 & d_3 & a_3\\
0 & 0 & 0 & \delta_2 & \alpha_2\\
0 & 0 & 0 & 0 & \alpha_3
\end{pmatrix}.$$

Let us check the following equality

$$\pi([x, y]) = [\pi(x), \pi(y)]$$

for basis vectors of $\bar{\mathfrak{a}}$.

$$[\pi(e_2), \pi(u_3)] = [be_2, \alpha_3 u_3] = \alpha_3 be_2 = \pi([e_2, u_3]) = be_2 \Leftrightarrow \alpha_3 = 1.$$

$$[\pi(e_3), \pi(u_3)] = -c_3 a_1 e_3 + c_3 u_2 = \pi([e_3, u_3]) = d_2 e_2 + d_3 e_3 + \delta_2 u_2 \Leftrightarrow$$

$$\Leftrightarrow d_2 = 0, d_3 = -c_3 a_1, \delta_2 = c_3.$$

$$[\pi(u_2), \pi(u_3)] = d_2 e_2 - d_3 a_1 e_3 + d_3 u_2 - \delta_2 a_1 u_2 + \alpha \delta_2 e_3 - \delta_2 u_2 =$$

$$= \pi([u_2, u_3]) = -de_2 + \alpha' c_3 e_3 - d_3 e_3 - \delta_2 u_2 \Leftrightarrow$$

$$\Leftrightarrow -\delta_2 a_1 = \delta_2 a_1, \alpha' - a_1 = \alpha - a_1^2 \Leftrightarrow a_1 = 0, \alpha' = \alpha.$$

It follows that the pairs $(\bar{\mathfrak{a}}, \mathfrak{a})$ and $(\bar{\mathfrak{a}}', \mathfrak{a}')$ are not equivalent if $\alpha \neq \alpha'$. Hence, the pairs $(\bar{\mathfrak{g}}_{24}, \mathfrak{g}_{24})$ and $(\bar{\mathfrak{g}}'_{24}, \mathfrak{g}'_{24})$ are not equivalent whenever $\alpha \neq \alpha'$.

Similarly it is possible to show that the pairs $(\bar{\mathfrak{g}}_{25}, \mathfrak{g}_{25})$ and $(\bar{\mathfrak{g}}'_{25}, \mathfrak{g}'_{25})$ with parameters $\alpha$ and $\alpha'$ respectively are not equivalent, whenever $\alpha \neq \alpha'$, and that the pairs $(\bar{\mathfrak{g}}_{25}, \mathfrak{g}_{25})$ and $(\bar{\mathfrak{g}}_{24}, \mathfrak{g}_{24})$ are not equivalent too.

2°. $\lambda = \frac{3}{5}, \mu = \frac{6}{5}$. Using the Jacobi identity we see that there does not exist any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ corresponding to this virtual structure.

In a similar way for the virtual structures $e), h), d), k), l), m), n), o), p), s)$ and $t)$ of the Lemma there also does not exist any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$.

3°. $3\lambda - 1 - \mu = 0$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $(1-\lambda)e_2$ | $(1-2\lambda)e_3$ | $(2-3\lambda)e_4$ | $u_1$ | $\lambda u_2$ | $(3\lambda-1)u_3$ |
| $e_2$ | $(\lambda-1)e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $(2\lambda-1)e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $pe_2$ | $u_2$ |
| $e_4$ | $(3\lambda-2)e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $b_4 e_4$ |
| $u_2$ | $-\lambda u_2$ | $-u_1$ | $-pe_2$ | $0$ | $0$ | $0$ | $B$ |
| $u_3$ | $(1-3\lambda)u_3$ | $0$ | $-u_2$ | $-u_1$ | $-b_4 e_4$ | $-B$ | $0$ |

,

$$B = b_4 e_3 + c_4 e_4 + \gamma_1 u_1,$$

where $b_4(3\lambda - 1) = 0$, $c_4(7\lambda - 3) = 0$, $\gamma_1(2\lambda - 1) = 0$, $p \neq 0$.

The mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$ such that

$$\pi(e_i) = e_i, \; i = 1, \ldots, 4,$$

$$\pi(u_j) = \frac{1}{p} u_j, \; j = 1, 2, 3,$$

establishes the equivalence of pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}', \mathfrak{g}')$, where the latter has form of $(\bar{\mathfrak{g}}, \mathfrak{g})$ for $p = 1$.

Now it remains to show that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}', \mathfrak{g}')$ are not equivalent.

*i)* $\lambda \neq \frac{1}{2}$. Since

$$\dim \mathcal{D}^2 \bar{\mathfrak{g}}' \neq \dim \mathcal{D}^2 \bar{\mathfrak{g}}_1,$$

we see that the pairs $(\bar{\mathfrak{g}}', \mathfrak{g}')$ and $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ are not equivalent.

*ii)* $\lambda = \frac{1}{2}$. Suppose

$$\bar{\mathfrak{g}}_1^* = \bar{\mathfrak{g}}_1/\mathcal{D}^2\bar{\mathfrak{g}}_1$$

and $\varphi$ is a natural projection of $\bar{\mathfrak{g}}_1$ to $\bar{\mathfrak{g}}_1^*$. Then $\mathfrak{g}_1^* = \varphi(\mathfrak{g}_1)$.

Suppose

$$\bar{\mathfrak{g}}_2^* = \bar{\mathfrak{g}}'/\mathcal{D}^2\bar{\mathfrak{g}}'$$

and $\varphi$ is a natural projection of $\bar{\mathfrak{g}}'$ to $\bar{\mathfrak{g}}_2^*$. Then $\mathfrak{g}_2^* = \varphi(\mathfrak{g}')$ .

Consider the homomorphisms

$$f_i : \mathfrak{g}_i^* \to \mathfrak{gl}(4, \mathbb{R}), \quad i = 1, 2,$$

where $f_i(x)$ is the matrix of the mapping $\mathrm{ad}_{\mathcal{D}\bar{\mathfrak{g}}_i^*} x$ in the basis

$$\{e_2 + \mathbb{R}u_1, e_4 + \mathbb{R}u_1, u_2 + \mathbb{R}u_1, u_3 + \mathbb{R}u_1\},$$

$x \in \mathfrak{g}_i^*$. Since the subalgebras $f_1(\mathfrak{g}_1^*)$ and $f_2(\mathfrak{g}_2^*)$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}', \mathfrak{g}')$ and $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ are not equivalent.

3.1°. $\lambda = \frac{1}{3}$, $c_4 = \gamma_1 = 0$, $b_4 \neq 0$.

3.1.1°. $b_4 > 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_{13}, \mathfrak{g}_{13})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{13} \to \bar{\mathfrak{g}}'$, where

$$\begin{aligned}
&\pi(e_1) = e_1, \\
&\pi(e_2) = \frac{1}{\sqrt{b_4}}e_2, \quad \pi(u_1) = \frac{1}{b_4}u_1, \\
&\pi(e_3) = e_3, \qquad\quad \pi(u_2) = \frac{1}{\sqrt{b_4}}u_2, \\
&\pi(e_4) = \frac{1}{\sqrt{b_4}}e_4, \quad \pi(u_3) = \frac{1}{\sqrt{b_4}}u_3.
\end{aligned}$$

3.1.2°. $b_4 < 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_{14}, \mathfrak{g}_{14})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{14} \to \bar{\mathfrak{g}}'$, where

$$\begin{aligned}
&\pi(e_1) = e_1, \\
&\pi(e_2) = \frac{1}{\sqrt{-b_4}}e_2, \quad \pi(u_1) = -\frac{1}{b_4}u_1, \\
&\pi(e_3) = e_3, \qquad\quad \pi(u_2) = \frac{1}{\sqrt{-b_4}}u_2, \\
&\pi(e_4) = \frac{1}{\sqrt{-b_4}}e_4, \quad \pi(u_3) = \frac{1}{\sqrt{-b_4}}u_3.
\end{aligned}$$

3.2°. $\lambda = \frac{3}{7}$, $\gamma_1 = b_4 = 0$, $c_4 \neq 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_{15}, \mathfrak{g}_{15})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{15} \to \bar{\mathfrak{g}}'$, where

$$\begin{aligned}
&\pi(e_1) = e_1, \\
&\pi(e_2) = \frac{1}{c_4}e_2, \quad \pi(u_1) = \frac{1}{c_4^2}u_1, \\
&\pi(e_3) = e_3, \qquad \pi(u_2) = \frac{1}{c_4}u_2, \\
&\pi(e_4) = \frac{1}{c_4}e_4, \quad \pi(u_3) = \frac{1}{c_4}u_3.
\end{aligned}$$

3.3°. $\lambda = \frac{1}{2}$, $b_4 = c_4 = 0$, $\gamma_1 \neq 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_{12}, \mathfrak{g}_{12})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{12} \to \bar{\mathfrak{g}}'$, where

$$\begin{aligned}
&\pi(e_i) = e_i, \ i = 1, \dots, 4, \\
&\pi(u_1) = u_1, \\
&\pi(u_2) = u_2 - \frac{1}{2}\gamma_1 e_4, \\
&\pi(u_3) = u_3 + \frac{1}{2}\gamma_1 e_2.
\end{aligned}$$

3.4°. $c_4 = b_4 = \gamma_1 = 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_{12}, \mathfrak{g}_{12})$.

4°. $\lambda = 0$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $e_2$ | $-\mu e_3$ | $(1-\mu)e_4$ | $u_1$ | $0$ | $\mu u_3$ |
| $e_2$ | $-e_2$ | $0$ | $e_4$ | $0$ | $0$ | $pe_2+u_1$ | $0$ |
| $e_3$ | $\mu e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $-2pe_3$ | $u_2$ |
| $e_4$ | $(\mu-1)e_4$ | $0$ | $0$ | $0$ | $0$ | $-pe_4$ | $pe_2+u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-pe_2-u_1$ | $2pe_3$ | $pe_4$ | $0$ | $0$ | $C$ |
| $u_3$ | $-\mu u_3$ | $0$ | $-u_2$ | $-pe_2-u_1$ | $0$ | $-C$ | $0$ |

,

$$C = c_4 e_4 + \gamma_1 u_1 - 2pu_3,$$

where $c_4(2\mu - 1) = 0$, $\gamma_1(\mu - 1) = 0$.

4.1°. $\mu = \frac{1}{2}, c_4 \neq 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_{11}, \mathfrak{g}_{11})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{11} \to \bar{\mathfrak{g}}'$, where

$$\begin{aligned}
&\pi(e_1) = e_1, \\
&\pi(e_2) = \frac{c_4}{p^2} e_2, \quad \pi(u_1) = \frac{c_4}{p^3} u_1, \\
&\pi(e_3) = e_3, \qquad \pi(u_2) = \frac{1}{p} u_2, \\
&\pi(e_4) = \frac{c_4}{p^2} e_4, \quad \pi(u_3) = \frac{1}{p} u_3 - \frac{c_4}{3p^2} e_4.
\end{aligned}$$

4.2°. $\mu = 1, \gamma_1 \neq 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_{11}, \mathfrak{g}_{11})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{11} \to \bar{\mathfrak{g}}'$, where

$$\begin{aligned}
&\pi(e_1) = e_1, \\
&\pi(e_2) = \frac{\gamma_1}{p} e_2, \quad \pi(u_1) = \frac{\gamma_1}{p^2} u_1, \\
&\pi(e_3) = e_3, \qquad \pi(u_2) = \frac{1}{p} u_2 - \frac{\gamma_1}{2p} e_4, \\
&\pi(e_4) = \frac{\gamma_1}{p} e_4, \quad \pi(u_3) = \frac{1}{p} u_3 + \frac{\gamma_1}{2p} e_2.
\end{aligned}$$

4.3°. $\gamma_1 = c_4 = 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_{11}, \mathfrak{g}_{11})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{11} \to \bar{\mathfrak{g}}'$, where

$$\pi(e_i) = e_i,\ i = 1, \dots, 4,$$
$$\pi(u_j) = \frac{1}{p} u_j,\ j = 1, 2, 3.$$

5°. $1 - 2\lambda + 2\mu = 0$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $(\frac{1}{2}-\mu)e_2$ | $\frac{1}{2}e_3$ | $(1-\mu)e_4$ | $u_1$ | $(\mu+\frac{1}{2})u_2$ | $\mu u_3$ |
| $e_2$ | $(\mu-\frac{1}{2})e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $pe_3$ |
| $e_3$ | $-\frac{1}{2}e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $(\mu-1)e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $b_4e_4+\beta_1u_1$ |
| $u_2$ | $-(\mu+\frac{1}{2})u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $C$ |
| $u_3$ | $-\mu u_3$ | $-pe_3$ | $-u_2$ | $-u_1$ | $-b_4e_4-\beta_1u_1$ | $-C$ | $0$ , |

$$C = b_4e_3 + \gamma_1u_1 + \beta_1u_2 + c_4e_4,$$

where $\beta_1\mu = 0$, $b_4\mu = 0$, $c_4(1 - 6\mu) = 0$, $\gamma_1(1 - 4\mu) = 0$, $p \neq 0$.

5.1°. $\mu = 0, \beta_1^2 + b_4^2 \neq 0, c_4 = \gamma_1 = 0$.

5.1.1°. $\beta_1 = 0, b_4 > 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{17}, \mathfrak{g}_{17})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{17} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = e_2, && \pi(u_1) = \frac{p}{b_4}u_1, \\ &\pi(e_3) = \frac{|p|}{\sqrt{b_4}}e_3, && \pi(u_2) = \frac{p}{b_4}u_2, \\ &\pi(e_4) = \frac{|p|}{\sqrt{b_4}}e_4, && \pi(u_3) = \frac{|p|}{p\sqrt{b_4}}u_3. \end{aligned}$$

5.1.2°. $\beta_1 = 0, b_4 < 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{18}, \mathfrak{g}_{18})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{18} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = e_2, && \pi(u_1) = -\frac{p}{b_4}u_1, \\ &\pi(e_3) = \frac{|p|}{\sqrt{-b_4}}e_3, && \pi(u_2) = -\frac{p}{b_4}u_2, \\ &\pi(e_4) = \frac{|p|}{\sqrt{-b_4}}e_4, && \pi(u_3) = \frac{|p|}{p\sqrt{-b_4}}u_3. \end{aligned}$$

5.1.3°. $\beta_1 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{19}, \mathfrak{g}_{19})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{19} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = e_2, && \pi(u_1) = \frac{p}{\beta_1^2} u_1, \\ &\pi(e_3) = \frac{p}{\beta_1} e_3, && \pi(u_2) = \frac{p}{\beta_1^2} u_2, \\ &\pi(e_4) = \frac{p}{\beta_1} e_4, && \pi(u_3) = \frac{1}{\beta_1} u_3. \end{aligned}$$

5.2°. $\mu = \frac{1}{6}, c_4 \neq 0, \beta_1 = b_4 = \gamma_1 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{21}, \mathfrak{g}_{21})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{21} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = e_2, && \pi(u_1) = pc_4^2 u_1, \\ &\pi(e_3) = pc_4 e_3, && \pi(u_2) = pc_4 u_2, \\ &\pi(e_4) = pc_4^2 e_4, && \pi(u_3) = u_3. \end{aligned}$$

5.3°. $\mu = \frac{1}{4}, c_4 = \beta_1 = b_4 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{20}, \mathfrak{g}_{20})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{20} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_i) = e_i, \ i = 1, \dots, 4, \\ &\pi(u_1) = \frac{1}{p} u_1, \\ &\pi(u_2) = \frac{1}{p}(u_2 - \gamma_1 e_4), \\ &\pi(u_3) = \frac{1}{p}(u_3 + \gamma_1 e_2). \end{aligned}$$

5.4°. $c_4 = \beta_1 = b_4 = \gamma_1 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{20}, \mathfrak{g}_{20})$.

6°. $1 - 2\mu = 0$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $(1-\lambda)e_2$ | $(\lambda-\frac{1}{2})e_3$ | $\frac{1}{2}e_4$ | $u_1$ | $\lambda u_2$ | $pe_4+\frac{1}{2}u_3$ |
| $e_2$ | $(\lambda-1)e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $(\frac{1}{2}-\lambda)e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $-\frac{1}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-\lambda u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $c_4e_4+\gamma_1u_1$ |
| $u_3$ | $-pe_4-\frac{1}{2}u_3$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $-c_4e_4-\gamma_1u_1$ | $0$ , |

where $c_4\lambda = 0$, $\gamma_1(\lambda - \frac{1}{2}) = 0$, $p \neq 0$.

6.1°. $\lambda = 0, c_4 \neq 0, \gamma_1 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{11}, \mathfrak{g}_{11})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{11} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
&\pi(e_1) = e_1, \\
&\pi(e_2) = c_4 e_2, \quad \pi(u_1) = p u_1, \\
&\pi(e_3) = \frac{p}{c_4} e_3, \quad \pi(u_2) = \frac{p}{c_4} u_2, \\
&\pi(e_4) = p e_4, \quad \pi(u_3) = u_3.
\end{aligned}$$

6.2°. $\lambda = \frac{1}{2}, c_4 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{10}, \mathfrak{g}_{10})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{10} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
&\pi(e_i) = e_i, \ i = 1, \dots, 4, \\
&\pi(u_1) = \frac{1}{p} u_1, \\
&\pi(u_2) = \frac{1}{p}(u_2 - \gamma_1 e_4), \\
&\pi(u_3) = \frac{1}{p}(u_3 + \gamma_1 e_2).
\end{aligned}$$

6.3°. $c_4 = \gamma_1 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{10}, \mathfrak{g}_{10})$.

7°. $\lambda = 0, \mu = -1$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $e_2$ | $e_3$ | $2e_4$ | $u_1$ | $0$ | $-u_3$ |
| $e_2$ | $-e_2$ | $0$ | $e_4$ | $0$ | $0$ | $pe_2+u_1$ | $0$ |
| $e_3$ | $-e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $qe_2-2pe_3$ | $u_2$ |
| $e_4$ | $-2e_4$ | $0$ | $0$ | $0$ | $0$ | $-pe_4$ | $pe_2+u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-pe_2-u_1$ | $2pe_3-qe_2$ | $pe_4$ | $0$ | $0$ | $-2pu_3$ |
| $u_3$ | $u_3$ | $0$ | $-u_2$ | $-pe_2-u_1$ | $0$ | $2pu_3$ | $0$ , |

where $pq \neq 0$.

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_8, \mathfrak{g}_8)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_8 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
&\pi(e_1) = e_1, \\
&\pi(e_2) = \frac{q}{p} e_2, \quad \pi(u_1) = \frac{q}{p^2} u_1, \\
&\pi(e_3) = e_3, \quad \pi(u_2) = \frac{1}{p} u_2, \\
&\pi(e_4) = \frac{q}{p} e_4, \quad \pi(u_3) = \frac{1}{p} u_3.
\end{aligned}$$

8°. $\lambda = \frac{1}{2}, \mu = \frac{1}{2}$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $\frac{1}{2}e_2$ | $0$ | $\frac{1}{2}e_4$ | $u_1$ | $\frac{1}{2}u_2$ | $pe_4+\frac{1}{2}u_3$ |
| $e_2$ | $-\frac{1}{2}e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $0$ | $-e_4$ | $0$ | $0$ | $0$ | $se_2$ | $u_2$ |
| $e_4$ | $-\frac{1}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-\frac{1}{2}u_2$ | $-u_1$ | $-se_2$ | $0$ | $0$ | $0$ | $\gamma_1 u_1$ |
| $u_3$ | $-pe_4-\frac{1}{2}u_3$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $-\gamma_1 u_1$ | $0$ , |

where $ps \neq 0$.

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{16}, \mathfrak{g}_{16})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{16} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
&\pi(e_1) = e_1, \\
&\pi(e_2) = \sqrt[3]{p^2 s}\, e_2, \quad \pi(u_1) = pu_1, \\
&\pi(e_3) = \sqrt[3]{\frac{p}{s}}\, e_3, \quad \pi(u_2) = \sqrt[3]{\frac{p}{s}} \left( u_2 - \frac{1}{2}\gamma_1 e_4 \right), \\
&\pi(e_4) = pe_4, \qquad \pi(u_3) = u_3 + \frac{1}{2}\gamma_1 e_2.
\end{aligned}$$

9°. $\lambda = 1, \mu = \frac{1}{2}$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $\frac{1}{2}e_3$ | $\frac{1}{2}e_4$ | $u_1$ | $u_2$ | $pe_4+\frac{1}{2}u_3$ |
| $e_2$ | $0$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $qe_3$ |
| $e_3$ | $-\frac{1}{2}e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $-\frac{1}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-pe_4-\frac{1}{2}u_3$ | $-qe_3$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ , |

where $pq \neq 0$.

9.1°. $pq > 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{22}, \mathfrak{g}_{22})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{22} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
&\pi(e_1) = e_1, \\
&\pi(e_2) = \sqrt{\frac{p}{q}}\, e_2, \quad \pi(u_1) = pu_1, \\
&\pi(e_3) = q\sqrt{\frac{p}{q}}\, e_3, \quad \pi(u_2) = q\sqrt{\frac{p}{q}}\, u_2, \\
&\pi(e_4) = pe_4, \qquad \pi(u_3) = u_3.
\end{aligned}$$

9.2°. $pq < 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_{23}, \mathfrak{g}_{23})$ by means of the mapping $\pi : \bar{\mathfrak{g}}_{23} \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_1) = e_1, \\ &\pi(e_2) = \sqrt{-\frac{p}{q}}e_2, && \pi(u_1) = -pu_1, \\ &\pi(e_3) = q\sqrt{-\frac{p}{q}}e_3, && \pi(u_2) = q\sqrt{-\frac{p}{q}}u_2, \\ &\pi(e_4) = -pe_4, && \pi(u_3) = u_3. \end{aligned}$$

10°. $\lambda, \mu \in \mathbb{R}$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $(1-\lambda)e_2$ | $(\lambda-\mu)e_3$ | $(1-\mu)e_4$ | $u_1$ | $\lambda u_2$ | $\mu u_3$ |
| $e_2$ | $(\lambda-1)e_2$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $(\mu-\lambda)e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $(\mu-1)e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $a_4e_4$ | $A$ |
| $u_2$ | $-\lambda u_2$ | $-u_1$ | $0$ | $0$ | $-a_4e_4$ | $0$ | $B$ |
| $u_3$ | $-\mu u_3$ | $0$ | $-u_2$ | $-u_1$ | $-A$ | $-B$ | $0$ , |

where

$$\begin{cases} A &= b_2e_2 + b_4e_4 + \beta_1u_1, \\ B &= c_1e_1 + c_3e_3 + c_4e_4 + \gamma_1u_1 + \gamma_2u_2, \end{cases}$$

and

$$\begin{aligned} &a_4(\lambda+\mu) = b_2(\lambda+\mu) = c_1(\lambda+\mu) = 0, \\ &b_2 + c_1(1-\lambda) = 0, \\ &c_3 = b_4, \\ &\mu b_4 = \mu\beta_1 = \mu c_3 = \mu\gamma_2 = 0, \\ &\gamma_2 = \beta_1, \\ &b_2 = -a_4, \\ &c_1(\lambda-\mu) = 0, \\ &a_4 = c_1(1-\mu), \\ &(1-\lambda-2\mu)c_4 = 0, \\ &b_2 = c_1 + a_4, \\ &(1-\mu-\lambda)\gamma_1 = 0, \\ &a_4(\gamma_2+\beta_1) = 0. \end{aligned}$$

10.1°. $1 - \lambda - \mu = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_i) = e_i, \ i = 1, \dots, 4, \\ &\pi(u_1) = u_1, \\ &\pi(u_2) = u_2 - \frac{1}{2}\gamma_1e_4, \\ &\pi(u_3) = u_3 + \frac{1}{2}\gamma_1e_2. \end{aligned}$$

10.2°. $1-\lambda-2\mu=0$.

10.2.1°. $c_4=0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_1,\mathfrak{g}_1)$.

10.2.2°. $c_4\neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2,\mathfrak{g}_2)$ by means of the mapping $\pi:\bar{\mathfrak{g}}_2\to\bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_1)=e_1,\\ &\pi(e_2)=c_4e_2, && \pi(u_1)=u_1,\\ &\pi(e_3)=\frac{1}{c_4}e_3, && \pi(u_2)=\frac{1}{c_4}u_2,\\ &\pi(e_4)=e_4, && \pi(u_3)=u_3. \end{aligned}$$

10.3°. $\mu=0, b_4^2+\beta_1^2\neq 0$. Suppose $\alpha=b_4,\beta=\beta_1$.

10.3.1°. $\lambda\neq 1,\beta\neq 0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_7,\mathfrak{g}_7)$ by means of the mapping $\pi:\bar{\mathfrak{g}}_7\to\bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_i)=e_i,\ i=1,\dots,4,\\ &\pi(u_j)=\frac{1}{\beta}u_j,\ j=1,2,3. \end{aligned}$$

10.3.2°. $\lambda=1,\beta\neq 0$. Then the mapping $\pi:\bar{\mathfrak{g}}'\to\bar{\mathfrak{g}}$ such that

$$\begin{aligned} &\pi(e_i)=e_i,\ i=1,\dots,4,\\ &\pi(u_1)=u_1-\frac{1}{2}\beta e_4,\\ &\pi(u_2)=u_2-\frac{1}{2}\beta e_2,\\ &\pi(u_3)=u_3+\frac{1}{2}\beta e_1, \end{aligned}$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}},\mathfrak{g})$ and $(\bar{\mathfrak{g}}',\mathfrak{g}')$, where the latter has the form

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $e_3$ | $e_4$ | $u_1$ | $u_2$ | 0 |
| $e_2$ | 0 | 0 | $e_4$ | 0 | 0 | $u_1$ | 0 |
| $e_3$ | $-e_3$ | $-e_4$ | 0 | 0 | 0 | 0 | $u_2$ |
| $e_4$ | $-e_4$ | 0 | 0 | 0 | 0 | 0 | $u_1$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | $\alpha' e_4$ |
| $u_2$ | $-u_2$ | $-u_1$ | 0 | 0 | 0 | 0 | $\alpha' e_3$ |
| $u_3$ | 0 | 0 | $-u_2$ | $-u_1$ | $-\alpha' e_4$ | $-\alpha' e_3$ | 0 , |

$\alpha'=\alpha+\frac{1}{4}\beta^2$.

10.3.3°. $\beta=0,\alpha>0$. Then the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_5,\mathfrak{g}_5)$ by means of the mapping $\pi:\bar{\mathfrak{g}}_5\to\bar{\mathfrak{g}}$, where

$$\begin{aligned} &\pi(e_i)=e_i,\ i=1,\dots,4,\\ &\pi(u_j)=\frac{1}{\alpha}u_j,\ j=1,2,3. \end{aligned}$$

10.3.4°. $\beta = 0, \alpha < 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_6, \mathfrak{g}_6)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_6 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i, \ i = 1, \ldots, 4,$$
$$\pi(u_j) = \frac{1}{\alpha} u_j, \ j = 1, 2, 3.$$

10.4°. $\mu = 0, \lambda = 1, c_4^2 + \gamma_1^2 \neq 0$. Suppose $\sigma = c_4, \rho = \gamma_1$. Then the mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$ such that

$$\pi(e_i) = e_i, \ i = 1, \ldots, 4,$$
$$\pi(u_1) = u_1 - \frac{1}{2}\beta e_4,$$
$$\pi(u_2) = u_2 - \frac{1}{2}\rho e_4 - \frac{1}{2}\beta e_3,$$
$$\pi(u_3) = u_3 + \frac{1}{2}\rho e_2 + \frac{1}{2}\beta e_1,$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $(\bar{\mathfrak{g}}', \mathfrak{g}')$, where the latter has the form

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $0$ |
| $e_2$ | $0$ | $0$ | $e_4$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $-e_3$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\alpha' e_4$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $\alpha' e_3 + \sigma e_4$ |
| $u_3$ | $0$ | $0$ | $-u_2$ | $-u_1$ | $-\alpha' e_4$ | $-\alpha' e_3 - \sigma e_4$ | $0$ |

,

$\alpha' = \alpha + \frac{1}{4}\beta^2, \sigma \neq 0$.

10.4.1°. $\alpha' = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1,$$
$$\pi(e_2) = \sigma e_2, \quad \pi(u_1) = u_1,$$
$$\pi(e_3) = \frac{1}{\sigma} e_3, \quad \pi(u_2) = \frac{1}{\sigma} u_2,$$
$$\pi(e_4) = e_4, \quad \pi(u_3) = u_3.$$

10.4.2°. $\alpha' > 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$\pi(e_1) = e_1,$$
$$\pi(e_2) = \frac{\sigma}{\alpha'} e_2, \quad \pi(u_1) = \frac{1}{\alpha'} u_1,$$
$$\pi(e_3) = \frac{\sqrt{\alpha'}}{\sigma} e_3, \quad \pi(u_2) = \frac{1}{\sigma} u_2,$$
$$\pi(e_4) = \frac{1}{\sqrt{\alpha'}} e_4, \quad \pi(u_3) = \frac{1}{\sqrt{\alpha'}} u_3.$$

10.4.3°. $\alpha' < 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_4 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned}
&\pi(e_1) = e_1, \\
&\pi(e_2) = -\frac{\sigma}{\alpha'} e_2, \qquad \pi(u_1) = -\frac{1}{\alpha'} u_1, \\
&\pi(e_3) = \frac{\sqrt{-\alpha'}}{\sigma} e_3, \quad \pi(u_2) = \frac{1}{\sigma} u_2, \\
&\pi(e_4) = \frac{1}{\sqrt{-\alpha'}} e_4, \quad \pi(u_3) = \frac{1}{\sqrt{-\alpha'}} u_3.
\end{aligned}$$

10.5°. $A = B = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

Now it remains to show that the pairs determined in the Proposition are not equivalent to each other.

Suppose $\mathfrak{a}_i = [\bar{\mathfrak{g}}_i, \mathcal{D}^2 \bar{\mathfrak{g}}_i]$, where $i = 12, 13, 14$. Consider the homomorphisms

$$f_i : \mathfrak{a}_i \to \mathfrak{gl}(2, \mathbb{R}), \; i = 12, 13, 14,$$

where $f_i(x)$ is the matrix of the mapping $\operatorname{ad}_{\mathcal{D}\mathfrak{a}_i} x$ in the basis

$$\{e_3 + \mathcal{D}^2(\bar{\mathfrak{g}}_i), u_2 + \mathcal{D}^2(\bar{\mathfrak{g}}_i)\},$$

$x \in \mathfrak{a}_i$. Since the subalgebras $f_i(\mathfrak{a}_i)$ are not conjugated to each other, we conclude that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$ are not equivalent to each other.

Since $\dim \mathcal{D}^2(\bar{\mathfrak{g}}_1) \neq \dim \mathcal{D}^2(\bar{\mathfrak{g}}_{20})$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_{20}, \mathfrak{g}_{20})$ are not equivalent.

Similarly we prove that all the other pairs are not equivalent to each other.

The proof of the Proposition is up.

**Proposition 4.22.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 4.22 is trivial.*

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | 0 | 0 | $u_1$ | $u_2$ | $u_3$ |
| $e_2$ | 0 | 0 | $e_3$ | $2e_4$ | $u_1$ | 0 | $-u_3$ |
| $e_3$ | 0 | $-e_3$ | 0 | 0 | 0 | $u_1$ | $u_2$ |
| $e_4$ | 0 | $-2e_4$ | 0 | 0 | 0 | 0 | $u_1$ |
| $u_1$ | $-u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | $-u_2$ | 0 | $-u_1$ | 0 | 0 | 0 | 0 |
| $u_3$ | $-u_3$ | $u_3$ | $-u_2$ | $-u_1$ | 0 | 0 | 0 |

*Proof.* Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \operatorname{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

## 5. Five-dimensional case

**Proposition 5.1.** *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 5.1 is trivial.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $0$ | $e_4$ | $-e_5$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $0$ | $-e_4$ | $e_5$ | $0$ | $u_2$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_3$ |
| $e_4$ | $-e_4$ | $e_4$ | $0$ | $0$ | $e_1 - e_2$ | $0$ | $u_1$ | $0$ |
| $e_5$ | $e_5$ | $-e_5$ | $0$ | $e_2 - e_1$ | $0$ | $u_2$ | $0$ | $0$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $0$ | $0$ | $-u_3$ | $0$ | $0$ | $0$ | $0$ | $0$ |

*Proof.* Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

**Proposition 5.2.** *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 5.2 is equivalent to one and only one of the following pairs:*

*1.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $2e_2$ | $-2e_3$ | $e_4$ | $-e_5$ | $u_1$ | $-u_2$ | $0$ |
| $e_2$ | $-2e_2$ | $0$ | $e_1$ | $0$ | $e_4$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $2e_3$ | $-e_1$ | $0$ | $e_5$ | $0$ | $u_2$ | $0$ | $0$ |
| $e_4$ | $-e_4$ | $0$ | $-e_5$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_5$ | $e_5$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $0$ | $0$ | $0$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ |

*2.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $2e_2$ | $-2e_3$ | $e_4$ | $-e_5$ | $u_1$ | $-u_2$ | $0$ |
| $e_2$ | $-2e_2$ | $0$ | $e_1$ | $0$ | $e_4$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $2e_3$ | $-e_1$ | $0$ | $e_5$ | $0$ | $u_2$ | $0$ | $0$ |
| $e_4$ | $-e_4$ | $0$ | $-e_5$ | $0$ | $0$ | $0$ | $0$ | $e_4 + u_1$ |
| $e_5$ | $e_5$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $e_5 + u_2$ |
| $u_1$ | $-u_1$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $\alpha u_1$ |
| $u_2$ | $u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\alpha u_2$ |
| $u_3$ | $0$ | $0$ | $0$ | $-e_4 - u_1$ | $-e_5 - u_2$ | $-\alpha u_1$ | $-\alpha u_2$ | $0$ |

, $|\alpha| \leqslant 1$

*3.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $2e_2$ | $-2e_3$ | $e_4$ | $-e_5$ | $u_1$ | $-u_2$ | $0$ |
| $e_2$ | $-2e_2$ | $0$ | $e_1$ | $0$ | $e_4$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $2e_3$ | $-e_1$ | $0$ | $e_5$ | $0$ | $u_2$ | $0$ | $0$ |
| $e_4$ | $-e_4$ | $0$ | $-e_5$ | $0$ | $0$ | $0$ | $0$ | $u_1 + \alpha e_4$ |
| $e_5$ | $e_5$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_2 + \alpha e_5$ |
| $u_1$ | $-u_1$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $\alpha u_1 - e_4$ |
| $u_2$ | $u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\alpha u_2 - e_5$ |
| $u_3$ | $0$ | $0$ | $0$ | $-u_1 - \alpha e_4$ | $-u_2 - \alpha e_5$ | $-\alpha u_1 + e_4$ | $-\alpha u_2 + e_5$ | $0$ |

, $\alpha \geqslant 0$

*Proof.*
Let $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$A(e_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector $e_1$.

**Lemma.** *Any virtual structure $q$ on generalized module 5.2 is trivial.*

*Proof.* Let $q$ be a virtual structure on generalized module 5.2. Without loss of generality it can be assumed that $q$ is primary. Since

$$\bar{\mathfrak{g}} = \bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) \oplus \bar{\mathfrak{g}}^{(2)}(\mathfrak{h}),$$

where

$$\begin{aligned} &\bar{\mathfrak{g}}^{(-2)}(\mathfrak{h}) = \mathbb{R}e_3, && \bar{\mathfrak{g}}^{(-1)}(\mathfrak{h}) = \mathbb{R}e_5 \oplus \mathbb{R}u_2, \\ &\bar{\mathfrak{g}}^{(0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}u_3, && \bar{\mathfrak{g}}^{(1)}(\mathfrak{h}) = \mathbb{R}e_4 \oplus \mathbb{R}u_1, \\ &\bar{\mathfrak{g}}^{(2)}(\mathfrak{h}) = \mathbb{R}e_2, \end{aligned}$$

we have

$$C(e_i) = 0, i = 1,2,3, \quad C(e_4) = \begin{pmatrix} 0 & 0 & c_{12}^4 \\ c_{21}^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{43}^4 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_5) = \begin{pmatrix} c_{51}^5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{32}^5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{53}^5 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -c_{53}^5 & 0 & 0 \\ 0 & -c_{53}^5 & 0 \end{pmatrix},$$

and $C_1(x) = C(x) - A(x)H + HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_i) = 0, \;\; i = 1,2,3, \quad C_1(e_4) = \begin{pmatrix} 0 & c_{12}^4 & 0 \\ c_{21}^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{43}^4 \\ 0 & 0 & 0 \end{pmatrix}, \; C_1(e_5) = \begin{pmatrix} c_{11}^5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{32}^5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures $C$ and $C_1$ are equivalent.

Since for any virtual structure $q$ condition (6), Chapter II, must be satisfied, after direct calculation we obtain

$$C_1(e_i) = 0, \quad i = 1, \dots, 5.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 5.2. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Then

$$\begin{array}{lllll}
[e_1, e_2] = 2e_2, & & & & \\
[e_1, e_3] = -2e_3, & [e_2, e_3] = e_1, & & & \\
[e_1, e_4] = e_4, & [e_2, e_4] = 0, & [e_3, e_4] = e_5, & & \\
[e_1, e_5] = -e_5, & [e_2, e_5] = e_4, & [e_3, e_5] = 0, & [e_4, e_5] = 0, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = u_2, & [e_4, u_1] = 0, & [e_5, u_1] = 0, \\
[e_1, u_2] = -u_2, & [e_2, u_2] = u_1, & [e_3, u_2] = 0, & [e_4, u_2] = 0, & [e_5, u_2] = 0, \\
[e_1, u_3] = 0, & [e_2, u_3] = 0, & [e_3, u_3] = 0, & [e_4, u_3] = u_1, & [e_5, u_3] = 0.
\end{array}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 10, Chapter II). Therefore

$$[u_1, u_2] = a_1 e_1 + \alpha_3 u_3,$$
$$[u_1, u_3] = b_4 e_4 + \beta_1 u_1,$$
$$[u_2, u_3] = c_5 e_5 + \gamma_2 u_2.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $2e_2$ | $-2e_3$ | $e_4$ | $-e_5$ | $u_1$ | $-u_2$ | $0$ |
| $e_2$ | $-2e_2$ | $0$ | $e_1$ | $0$ | $e_4$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $2e_3$ | $-e_1$ | $0$ | $e_5$ | $0$ | $u_2$ | $0$ | $0$ |
| $e_4$ | $-e_4$ | $0$ | $-e_5$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_5$ | $e_5$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $\alpha e_4 + \beta u_1$ |
| $u_2$ | $u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\alpha e_5 + \beta u_2$ |
| $u_3$ | $0$ | $0$ | $0$ | $-u_1$ | $-u_2$ | $-\alpha e_4 - \beta u_1$ | $-\alpha e_5 - \beta u_2$ | $0$ |

Consider the following cases:

1°. $\alpha = \beta = 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2°. $\beta^2 + 4\alpha \geqslant 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \mathfrak{g}$, where

$$\pi(e_1) = e_1, \quad \pi(u_1) = e_4 + u_1,$$
$$\pi(e_2) = e_2, \quad \pi(u_2) = e_5 + u_2,$$
$$\pi(e_3) = e_3, \quad \pi(u_3) = \lambda u_3,$$
$$\pi(e_4) = \lambda e_4,$$
$$\pi(e_5) = \lambda e_5,$$

and $\lambda = \frac{\beta \pm \sqrt{\beta^2 + 4\alpha}}{2} \neq 0$.

3°. $\beta^2 + 4\alpha < 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \mathfrak{g}$, where

$$\pi(e_1) = e_1, \quad \pi(u_1) = u_1 + \frac{\beta\lambda}{2} e_4,$$
$$\pi(e_2) = e_2, \quad \pi(u_2) = u_2 + \frac{\beta\lambda}{2} e_5,$$
$$\pi(e_3) = e_3, \quad \pi(u_3) = \lambda^{-1} u_3,$$
$$\pi(e_4) = \lambda e_4,$$
$$\pi(e_5) = \lambda e_5,$$

and $\lambda = \frac{2}{\sqrt{-\beta^2 - 4\alpha}}$.

Let $\mathfrak{r}_i$ be the radical of $\bar{\mathfrak{g}}_i$ for $i = 1, 2, 3$. Consider the homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(4, \mathbb{R})$, $i = 1, 2, 3$, where $f_i(x)$ is the matrix of the mapping $\operatorname{ad}_{\bar{\mathfrak{r}}_i} x$ in the basis $\{e_4, e_5, u_1, u_2\}$ of $\bar{\mathfrak{r}}_i$, $x \in \bar{\mathfrak{g}}_i$.

Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$, $i = 1, 2, 3$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1, 2, 3$ are not equivalent.

This completes the proof of the Proposition.

**Proposition 5.3.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 5.3 is equivalent to one and only one of the following pairs:*

*1.*

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_2$ | $0$ | $e_1$ | $0$ | $u_1$ | $0$ |
| $e_2$ | $0$ | $0$ | $0$ | $e_1$ | $-e_2$ | $0$ | $0$ | $u_1$ |
| $e_3$ | $-e_2$ | $0$ | $0$ | $e_5$ | $-2e_3$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $0$ | $-e_1$ | $-e_5$ | $0$ | $2e_4$ | $0$ | $u_3$ | $0$ |
| $e_5$ | $-e_1$ | $e_2$ | $2e_3$ | $-2e_4$ | $0$ | $0$ | $u_2$ | $-u_3$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-u_1$ | $0$ | $0$ | $-u_3$ | $-u_2$ | $0$ | $0$ | $0$ |
| $u_3$ | $0$ | $-u_1$ | $-u_2$ | $0$ | $u_3$ | $0$ | $0$ | $0$ |

*2.*

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_2$ | $0$ | $e_1$ | $-3e_1$ | $-\frac{1}{2}e_5+\frac{1}{2}u_1$ | $-e_4$ |
| $e_2$ | $0$ | $0$ | $0$ | $e_1$ | $-e_2$ | $-3e_2$ | $-e_3$ | $\frac{1}{2}e_5+\frac{1}{2}u_1$ |
| $e_3$ | $-e_2$ | $0$ | $0$ | $e_5$ | $-2e_3$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $0$ | $-e_1$ | $-e_5$ | $0$ | $2e_4$ | $0$ | $u_3$ | $0$ |
| $e_5$ | $-e_1$ | $e_2$ | $2e_3$ | $-2e_4$ | $0$ | $0$ | $u_2$ | $-u_3$ |
| $u_1$ | $3e_1$ | $3e_2$ | $0$ | $0$ | $0$ | $0$ | $-3u_2$ | $-3u_3$ |
| $u_2$ | $\frac{1}{2}e_5-\frac{1}{2}u_1$ | $e_3$ | $0$ | $-u_3$ | $-u_2$ | $3u_2$ | $0$ | $0$ |
| $u_3$ | $e_4$ | $-\frac{1}{2}e_5-\frac{1}{2}u_1$ | $-u_2$ | $0$ | $u_3$ | $3u_3$ | $0$ | $0$ |

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$A(e_5) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vector $e_5$.

**Lemma.** *Any virtual structure $q$ on generalized module 5.3 is equivalent to one of the following:*

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 2p & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 3p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -3p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_5) = 0.$$

*Proof.* Put

$$C(e_i) = \left(c^i_{jk}\right)_{\substack{1 \leqslant j \leqslant 5 \\ 1 \leqslant k \leqslant 3}}, \quad i = 1, \dots, 5.$$

Let $q$ be a virtual structure on generalized module 5.3. Without loss of generality it can be assumed that $q$ is primary. Since

$$\begin{aligned} &\mathfrak{g}^{(-1)}(\mathfrak{h}) = \mathbb{R}e_1, && U^{(0)}(\mathfrak{h}) = \mathbb{R}u_1, \\ &\mathfrak{g}^{(1)}(\mathfrak{h}) = \mathbb{R}e_2, && U^{(1)}(\mathfrak{h}) = \mathbb{R}u_2, \\ &\mathfrak{g}^{(2)}(\mathfrak{h}) = \mathbb{R}e_3, && U^{(-1)}(\mathfrak{h}) = \mathbb{R}u_3, \\ &\mathfrak{g}^{(-2)}(\mathfrak{h}) = \mathbb{R}e_4, \\ &\mathfrak{g}^{(0)}(\mathfrak{h}) = \mathbb{R}e_5, \end{aligned}$$

we have

$$C(e_1) = \begin{pmatrix} c^1_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^1_{43} \\ 0 & c^1_{52} & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ c^2_{21} & 0 & 0 \\ 0 & c^2_{32} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^2_{53} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c^3_{23} \\ c^3_{31} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & c^4_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ c^4_{41} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C(e_5) = \begin{pmatrix} 0 & 0 & c_{13}^5 \\ 0 & c_{22}^5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_{51}^5 & 0 & 0 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & 0 & -c_{12}^4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -c_{21}^2 & 0 & 0 \end{pmatrix},$$

and $C_1(x) = C(x) - A(X)H + HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = \begin{pmatrix} c_{11}^1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{43}^1 \\ 0 & c_{52}^1 & 0 \end{pmatrix}, \quad C_1(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{32}^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{53}^2 \end{pmatrix},$$

$$C_1(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{23}^3 \\ c_{31}^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_1(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_{41}^4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_1(e_5) = \begin{pmatrix} 0 & 0 & c_{13}^5 \\ 0 & c_{22}^5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_{51}^5 & 0 & 0 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures $C$ and $C_1$ are equivalent.

Since for any virtual structure $q$ condition (6), Chapter II, must be satisfied, after direct calculation we see that $C_1$ has the form indicated in the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 5.3. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Then

$$\begin{array}{lllll}
[e_1, e_2] = 0, & & & & \\
[e_1, e_3] = e_2, & [e_2, e_3] = -e_3, & & & \\
[e_1, e_4] = 0, & [e_2, e_4] = e_1, & [e_3, e_4] = e_5, & & \\
[e_1, e_5] = e_1, & [e_2, e_5] = -e_2, & [e_3, e_5] = -2e_3, & [e_4, e_5] = 2e_4, & \\
[e_1, u_1] = 0, & [e_2, u_1] = 3pe_2, & [e_3, u_1] = 3pe_3, & [e_4, u_1] = -3pe_4, & [e_5, u_1] = 0, \\
[e_1, u_2] = 2pe_5 + u_1, & [e_2, u_2] = pe_3, & [e_3, u_2] = 0, & [e_4, u_2] = 0, & [e_5, u_2] = u_2, \\
[e_1, u_3] = 3pe_4, & [e_2, u_3] = pe_5 + u_1, & [e_3, u_3] = u_2, & [e_4, u_3] = 0, & [e_5, u_3] = -u_3.
\end{array}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 10, Chapter II). Therefore

$$\begin{aligned} [u_1, u_2] &= a_2 e_2 + \alpha_2 u_2, \\ [u_1, u_3] &= b_1 e_1 + \beta_3 u_3, \\ [u_2, u_3] &= c_5 e_5 + \gamma_1 u_1. \end{aligned}$$

Using the Jacobi identity we see that

$$\begin{aligned} a_2 &= -\frac{3p\gamma_1}{2}, & \alpha_2 &= 0, \\ b_1 &= \frac{3p\gamma_1}{2}, & \beta_3 &= 3p, \\ c_5 &= \frac{3p\gamma_1}{2}. \end{aligned}$$

Consider the following cases:

1°. $p = 0$.

Then the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$ such that

$$\begin{aligned} \pi(e_i) &= e_i, \ i = 1, \dots, 5, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= u_2 - \gamma_1 e_2, \\ \pi(u_3) &= u_3 + \gamma_1 e_1, \end{aligned}$$

establishes the equivalence of the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}, \mathfrak{g})$.

2°. $p \neq 0$.

Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_1) &= \frac{1}{3} e_1, & \pi(u_1) &= -\frac{p}{2} u_1 - \frac{3p}{2} e_5, \\ \pi(e_2) &= \frac{1}{3} e_2, & \pi(u_2) &= 3pu_2 + \frac{1}{3} e_2, \\ \pi(e_3) &= e_3, & \pi(u_3) &= 3pu_3 - \frac{p}{6} e_1. \\ \pi(e_4) &= e_4, \\ \pi(e_5) &= e_5, \end{aligned}$$

Since the Lie algebra $\bar{\mathfrak{g}}_2$ is simple, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

This completes the proof of the Proposition.

**Proposition 5.4.** *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 5.4 is trivial.*

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $0$ | $0$ | $e_5$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $0$ | $e_4$ | $0$ | $0$ | $u_2$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $-e_4$ | $-e_5$ | $0$ | $0$ | $u_3$ |
| $e_4$ | $0$ | $-e_4$ | $e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_5$ | $-e_5$ | $0$ | $e_5$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $0$ | $0$ | $-u_3$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ |

*Proof.* Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

**Proposition 5.5.** *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 5.5 is trivial.*

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $0$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $0$ |
| $e_2$ | $0$ | $0$ | $0$ | $-e_5$ | $e_4$ | $-u_2$ | $u_1$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $-e_4$ | $-e_5$ | $0$ | $0$ | $u_3$ |
| $e_4$ | $-e_4$ | $e_5$ | $e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_5$ | $-e_5$ | $-e_4$ | $e_5$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $u_2$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $0$ | $0$ | $-u_3$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ |

*Proof.* Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

**Proposition 5.6.** *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 5.6 is trivial.*

| $[,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $0$ | $e_4$ | $e_5$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $0$ | $-e_4$ | $0$ | $0$ | $u_2$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $0$ | $-e_5$ | $0$ | $0$ | $u_3$ |
| $e_4$ | $-e_4$ | $e_4$ | $0$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_5$ | $-e_5$ | $0$ | $e_5$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $0$ | $0$ | $-u_3$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ |

*Proof.* Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

**Proposition 5.7.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 5.7 is trivial.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | 0 | $e_4$ | $e_5$ | $u_1$ | 0 | 0 |
| $e_2$ | 0 | 0 | 0 | $-e_4$ | $-e_5$ | 0 | $u_2$ | $u_3$ |
| $e_3$ | 0 | 0 | 0 | $-e_5$ | $e_4$ | 0 | $-u_3$ | $u_2$ |
| $e_4$ | $-e_4$ | $e_4$ | $e_5$ | 0 | 0 | 0 | $u_1$ | 0 |
| $e_5$ | $-e_5$ | $e_5$ | $-e_4$ | 0 | 0 | 0 | 0 | $u_1$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | 0 | $-u_2$ | $u_3$ | $-u_1$ | 0 | 0 | 0 | 0 |
| $u_3$ | 0 | $-u_3$ | $-u_2$ | 0 | $-u_1$ | 0 | 0 | 0 |

*Proof.* Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

**Proposition 5.8.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 5.8 is trivial.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $-e_3$ | 0 | $e_5$ | 0 | $u_2$ | 0 |
| $e_2$ | 0 | 0 | 0 | $-e_4$ | $-e_5$ | 0 | 0 | $u_3$ |
| $e_3$ | $e_3$ | 0 | 0 | 0 | $e_4$ | 0 | $u_1$ | 0 |
| $e_4$ | 0 | $e_4$ | 0 | 0 | 0 | 0 | 0 | $u_1$ |
| $e_5$ | $-e_5$ | $e_5$ | $-e_4$ | 0 | 0 | 0 | 0 | $u_2$ |
| $u_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | $-u_2$ | 0 | $-u_1$ | 0 | 0 | 0 | 0 | 0 |
| $u_3$ | 0 | $-u_3$ | 0 | $-u_1$ | $-u_2$ | 0 | 0 | 0 |

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(e_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors $e_1$ and $e_2$.

**Lemma.** *Any virtual structure $q$ on generalized module 5.8 is trivial.*

*Proof.* Let $q$ be a virtual structure on generalized module 5.8. Without loss of generality it can be assumed that $q$ is primary. Since

$$\begin{aligned} &\mathfrak{g}^{(0,0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2, && \mathfrak{g}^{(1,0)}(\mathfrak{h}) = \mathbb{R}e_3 \\ &\mathfrak{g}^{(0,-1)}(\mathfrak{h}) = \mathbb{R}e_4, && \mathfrak{g}^{(1,-1)}(\mathfrak{h}) = \mathbb{R}e_5, \\ &U^{(0,0)} = \mathbb{R}u_1, && U^{(1,0)} = \mathbb{R}u_2, \\ &U^{(0,1)}(\mathfrak{h}) = \mathbb{R}u_3, \end{aligned}$$

we have

$$C(e_1) = \begin{pmatrix} c_{11}^1 & 0 & 0 \\ c_{21}^1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} c_{11}^2 & 0 & 0 \\ c_{21}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & c_{12}^3 & 0 \\ 0 & c_{22}^3 & 0 \\ c_{31}^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C(e_4) = \begin{pmatrix} 0 & 0 & c_{13}^4 \\ 0 & 0 & c_{23}^4 \\ 0 & 0 & 0 \\ c_{41}^4 & 0 & 0 \\ 0 & c_{52}^4 & 0 \end{pmatrix}, \quad C(e_5) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_{51}^5 & 0 & 0 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} -c_{12}^3 & 0 & 0 \\ -c_{22}^3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $C_1 = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = \begin{pmatrix} c^1_{11} & 0 & 0 \\ c^1_{21} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_1(e_2) = \begin{pmatrix} c^2_{11} & 0 & 0 \\ c^2_{21} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C_1(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c^3_{31} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_1(e_4) = \begin{pmatrix} 0 & 0 & c^4_{13} \\ 0 & 0 & c^4_{23} \\ 0 & 0 & 0 \\ c^4_{41} & 0 & 0 \\ 0 & c^4_{52} & 0 \end{pmatrix}, C_1(e_5) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ c^5_{51} & 0 & 0 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures $C$ and $C_1$ are equivalent.

Since for any virtual structure $q$ condition (6), Chapter II, must be satisfied, after direct calculation we obtain:

$$C_1(e_i) = 0, \quad i = 1, \dots, 5.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 5.8. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial. Then

$$\begin{array}{lllll}
[e_1, e_2] = 0, & & & & \\
[e_1, e_3] = -e_3, & [e_2, e_3] = 0, & & & \\
[e_1, e_4] = 0, & [e_2, e_4] = -e_4, & [e_3, e_4] = 0, & & \\
[e_1, e_5] = e_5, & [e_2, e_5] = -e_5, & [e_3, e_5] = e_4, & [e_4, e_5] = 0, & \\
[e_1, u_1] = 0, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = 0, & [e_5, u_1] = 0, \\
[e_1, u_2] = u_2, & [e_2, u_2] = 0, & [e_3, u_2] = u_1, & [e_4, u_2] = 0, & [e_5, u_2] = 0, \\
[e_1, u_3] = 0, & [e_2, u_3] = u_3, & [e_3, u_3] = 0, & [e_4, u_3] = u_1, & [e_5, u_3] = u_2.
\end{array}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 10, Chapter II). Thus

$$\begin{array}{ll}
\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}u_1, & \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) = \mathbb{R}e_3 \oplus \mathbb{R}u_2, \\
\bar{\mathfrak{g}}^{(0,-1)}(\mathfrak{h}) = \mathbb{R}e_4, & \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) = \mathbb{R}e_5, \\
\bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}) = \mathbb{R}u_3, &
\end{array}$$

$$\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}), \\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}), \\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(1,1)}(\mathfrak{h}),
\end{aligned}$$

and

$$[u_1, u_2] = a_3 e_3 + \alpha_2 u_2,$$
$$[u_1, u_3] = \beta_3 u_3,$$
$$[u_2, u_3] = 0.$$

Using the Jacobi identity we see that the pair is trivial.

This completes the proof of the Proposition.

**Proposition 5.9.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 5.9 is equivalent to one and only one of the following pairs:*

*1.*

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $(1-\lambda)e_3$ | $\lambda e_4$ | $e_5$ | $u_1$ | $\lambda u_2$ | 0 |
| $e_2$ | 0 | 0 | 0 | $-e_4$ | $-e_5$ | 0 | 0 | $u_3$ |
| $e_3$ | $(\lambda-1)e_3$ | 0 | 0 | $e_5$ | 0 | 0 | $u_1$ | 0 |
| $e_4$ | $-\lambda e_4$ | $e_4$ | $-e_5$ | 0 | 0 | 0 | 0 | $u_2$ |
| $e_5$ | $-e_5$ | $e_5$ | 0 | 0 | 0 | 0 | 0 | $u_1$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | $-\lambda u_2$ | 0 | $-u_1$ | 0 | 0 | 0 | 0 | 0 |
| $u_3$ | 0 | $-u_3$ | 0 | $-u_2$ | $-u_1$ | 0 | 0 | 0 |

*2.* $\lambda = 0$

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $e_3$ | 0 | $e_5$ | $u_1$ | 0 | 0 |
| $e_2$ | 0 | 0 | 0 | $-e_4$ | $-e_5$ | 0 | 0 | $u_3$ |
| $e_3$ | $-e_3$ | 0 | 0 | $e_5$ | 0 | 0 | $u_1$ | 0 |
| $e_4$ | 0 | $e_4$ | $-e_5$ | 0 | 0 | $e_5$ | $2e_4$ | $u_2$ |
| $e_5$ | $-e_5$ | $e_5$ | 0 | 0 | 0 | 0 | $e_5$ | $u_1$ |
| $u_1$ | $-u_1$ | 0 | 0 | $-e_5$ | 0 | 0 | $-u_1$ | 0 |
| $u_2$ | 0 | 0 | $-u_1$ | $-2e_4$ | $-e_5$ | $u_1$ | 0 | $2u_3$ |
| $u_3$ | 0 | $-u_3$ | 0 | $-u_2$ | $-u_1$ | 0 | $-2u_3$ | 0 |

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \lambda-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix},$$

$$A(e_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors $e_1$ and $e_2$.

**Lemma.** *Any virtual structure $C$ on generalized module 5.9 is equivalent to one of the following:*

*a)* $\lambda \neq 0, \frac{1}{2}$

$$C_1(e_i) = 0, \quad i = 1, \dots, 5;$$

*b)* $\lambda = \frac{1}{2}$

$$C_2(e_i) = 0,\ i = 1,2,3,5, \quad C_2(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

*c)* $\lambda = 0$

$$C_3(e_i) = 0,\ i = 1,2,3,$$

$$C_3(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2p & 0 \\ p & 0 & 0 \end{pmatrix}, \quad C_3(e_5) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p & 0 \end{pmatrix}.$$

*Proof.* Let $q$ be a virtual structure on the generalized module 5.9. Without loss of generality it can be assumed that $q$ is primary. Since

$$\begin{aligned} &\mathfrak{g}^{(0,0)}(\mathfrak{h}) \supset \mathbb{R}e_1 \oplus \mathbb{R}e_2, \\ &\mathfrak{g}^{(1-\lambda,0)}(\mathfrak{h}) \supset \mathbb{R}e_3, && U^{(1,0)}(\mathfrak{h}) \supset \mathbb{R}u_1, \\ &\mathfrak{g}^{(\lambda,-1)}(\mathfrak{h}) \supset \mathbb{R}e_4, && U^{(\lambda,0)}(\mathfrak{h}) \supset \mathbb{R}u_2, \\ &\mathfrak{g}^{(1,-1)}(\mathfrak{h}) \supset \mathbb{R}e_5, && U^{(0,1)}(\mathfrak{h}) \supset \mathbb{R}u_3, \end{aligned}$$

we have

1°. $\lambda = 0$

$$C(e_1) = \begin{pmatrix} 0 & c_{12}^1 & 0 \\ 0 & c_{22}^1 & 0 \\ c_{31}^1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & c_{12}^2 & 0 \\ 0 & c_{22}^2 & 0 \\ c_{31}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{32}^3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C(e_4)=\begin{pmatrix}0&0&c_{13}^4\\0&0&c_{23}^4\\0&0&0\\0&c_{42}^4&0\\c_{51}^4&0&0\end{pmatrix},\quad C(e_5)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&c_{33}^5\\0&0&0\\0&c_{52}^5&0\end{pmatrix}.$$

2°. $\lambda=\frac{1}{2}$

$$C(e_1)=\begin{pmatrix}0&0&0\\0&0&0\\0&c_{32}^1&0\\0&0&0\\0&0&0\end{pmatrix},\quad C(e_2)=\begin{pmatrix}0&0&0\\0&0&0\\0&c_{32}^2&0\\0&0&0\\0&0&0\end{pmatrix},\quad C(e_4)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&c_{33}^4\\0&0&0\\0&c_{52}^4&0\end{pmatrix},$$

$$C(e_3)=C(e_5)=0.$$

3°. $\lambda=2$

$$C(e_i)=0,\ i=1,2,4,\qquad C(e_3)=\begin{pmatrix}c_{11}^3&0&0\\c_{21}^3&0&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad C(e_5)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\c_{41}^5&0&0\\0&0&0\end{pmatrix}.$$

4°. $\lambda\notin\{0,\frac{1}{2},2\}$.

$$C(e_i)=0,\quad i=1,\dots,5.$$

Checking condition (6), Chapter II, after diect calculation we obtain

$\lambda=2$: $\quad C(e_i)=0,\ i=1,\dots,5;$

$\lambda=0$:

$$C(e_1)=C(e_2)=0,\quad C(e_3)=\begin{pmatrix}0&0&0\\0&0&0\\0&c_{13}^4+c_{33}^5&0\\0&0&0\\0&0&0\end{pmatrix},$$

$$C(e_4)=\begin{pmatrix}0&0&c_{13}^4\\0&0&c_{23}^4\\0&0&0\\0&c_{42}^4&0\\c_{33}^5+\frac{c_{42}^4+c_{23}^4}{2}&0&0\end{pmatrix},\quad C(e_5)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&c_{33}^5\\0&0&0\\0&c_{13}^4+\frac{c_{42}^4-c_{23}^4}{2}&0\end{pmatrix}.$$

$\lambda=\frac{1}{2}$:

$$C(e_i)=0,\quad i=1,2,3,5,\qquad C(e_4)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&c_{33}^4\\0&0&0\\0&c_{52}^4&0\end{pmatrix}.$$

Consider the following cases:

1°. $\lambda = 0$. Put

$$H = \begin{pmatrix} 0 & c_{13}^4 & 0 \\ 0 & c_{23}^4 & 0 \\ c_{33}^5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $C_1(x) = C(x) + A(x)H - HB(x)$. Then

$$C_1(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{42}^4 + c_{23}^4 & 0 \\ \frac{c_{42}^4+c_{23}^4}{2} & 0 & 0 \end{pmatrix}, \quad C_1(e_5) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{c_{42}^4+c_{23}^4}{2} & 0 \end{pmatrix},$$

$$C_1(e_i) = 0, \quad i = 1, 2, 3.$$

By corollary 2, Chapter II, the virtual structures defined by $C$ and $C_1$ are equivalent.

2°. $\lambda = \frac{1}{2}$. Put

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{52}^4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $C_2(x) = C(x) + A(x)H - HB(x)$. Then

$$C_2(e_i) = 0, \; i = 1, 2, 3, 5, \quad C_2(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{33}^4 - c_{52}^4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures defined by the mappings $C$ and $C_2$ are equivalent.
The proof of the Lemma is complete.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 5.9. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma.

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}}^\alpha(\mathfrak{h}) = \mathfrak{g}^\alpha(\mathfrak{h}) \times U^\alpha(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*.$$

Thus,

$$\begin{aligned} &\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2, \\ &\bar{\mathfrak{g}}^{(1-\lambda,0)}(\mathfrak{h}) = \mathbb{R}e_3, && \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) = \mathbb{R}u_1, \\ &\bar{\mathfrak{g}}^{(\lambda,-1)}(\mathfrak{h}) = \mathbb{R}e_4, && \bar{\mathfrak{g}}^{(\lambda,0)}(\mathfrak{h}) = \mathbb{R}u_2, \\ &\bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) = \mathbb{R}e_5, && \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}) = \mathbb{R}u_3, \end{aligned}$$

and

$$[u_1, u_2] \in \bar{\mathfrak{g}}^{(1+\lambda,0)}(\mathfrak{h}),$$
$$[u_1, u_3] \in \bar{\mathfrak{g}}^{(1,1)}(\mathfrak{h}),$$
$$[u_2, u_3] \in \bar{\mathfrak{g}}^{(\lambda,1)}(\mathfrak{h}).$$

Consider the following cases:

1°. $\lambda = 0$. Then

$$[u_1, u_2] = a_3 e_3 + \alpha_1 u_1,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = \gamma_3 u_3.$$

Using the Jacobi identity we see that $a_3 = 0$, $\alpha_1 = -p$, $\gamma_3 = 2p$, and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $0$ | $e_5$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $0$ | $-e_4$ | $-e_5$ | $0$ | $0$ | $u_3$ |
| $e_3$ | $-e_3$ | $0$ | $0$ | $e_5$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $0$ | $e_4$ | $-e_5$ | $0$ | $0$ | $pe_5$ | $2pe_4$ | $u_2$ |
| $e_5$ | $-e_5$ | $e_5$ | $0$ | $0$ | $0$ | $0$ | $pe_5$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $-pe_5$ | $0$ | $0$ | $-pu_1$ | $0$ |
| $u_2$ | $0$ | $0$ | $-u_1$ | $-2pe_4$ | $-pe_5$ | $u_1$ | $0$ | $2pu_3$ |
| $u_3$ | $0$ | $-u_3$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $-2pu_3$ | $0$ . |

1.1°. $p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

1.2°. $p \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\pi(e_i) = e_i, \;\; i = 1, \dots, 5,$$
$$\pi(u_j) = pu_j, \;\; j = 1, 2, 3.$$

2°. $\lambda = \frac{1}{2}$. Then

$$[u_1, u_2] = 0,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = 0.$$

Using the Jacobi identity we obtain that $p = 0$, and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

3°. $\lambda \notin \{0, \frac{1}{2}\}$.

3.1°. $\lambda = -1$. Then

$$[u_1, u_2] = a_1 e_1 + a_2 e_2,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = 0.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

3.2°. $\lambda \notin \{-1, 0, \frac{1}{2}\}$. Then

$$[u_1, u_2] = 0,$$
$$[u_1, u_3] = 0,$$
$$[u_2, u_3] = 0,$$

and the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

Since $\dim \mathcal{D}^2 \bar{\mathfrak{g}}_1 = 3$, $\dim \mathcal{D}^2 \bar{\mathfrak{g}}_2 = 6$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

**Proposition 5.10.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 5.10 is equivalent to one and only one of the following pairs:*

*1.*

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $e_3$ | $(1-\lambda)e_4$ | $-\lambda e_5$ | $u_1$ | 0 | $\lambda u_3$ |
| $e_2$ | 0 | 0 | $-e_3$ | $-\mu e_4$ | $(1-\mu)e_5$ | 0 | $u_2$ | $\mu u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | 0 | 0 | $e_4$ | 0 | $u_1$ | 0 |
| $e_4$ | $(\lambda-1)e_4$ | $\mu e_4$ | 0 | 0 | 0 | 0 | 0 | $u_1$ |
| $e_5$ | $\lambda e_5$ | $(\mu-1)e_5$ | $-e_4$ | 0 | 0 | 0 | 0 | $u_2$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | 0 | $-u_2$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 |
| $u_3$ | $-\lambda u_3$ | $-\mu u_3$ | 0 | $-u_1$ | $-u_2$ | 0 | 0 | 0 |

*2.* $\lambda = \frac{1}{2}$, $\mu = -\frac{1}{2}$

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $e_3$ | $\frac{1}{2}e_4$ | $-\frac{1}{2}e_5$ | $u_1$ | 0 | $\frac{1}{2}u_3$ |
| $e_2$ | 0 | 0 | $-e_3$ | $\frac{1}{2}e_4$ | $\frac{3}{2}e_5$ | 0 | $u_2$ | $-\frac{1}{2}u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | 0 | 0 | $e_4$ | 0 | $u_1$ | 0 |
| $e_4$ | $-\frac{1}{2}e_4$ | $-\frac{1}{2}e_4$ | 0 | 0 | 0 | 0 | 0 | $u_1$ |
| $e_5$ | $\frac{1}{2}e_5$ | $-\frac{3}{2}e_5$ | $-e_4$ | 0 | 0 | 0 | 0 | $u_2$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | 0 | $-u_2$ | $-u_1$ | 0 | 0 | 0 | 0 | $e_4$ |
| $u_3$ | $-\frac{1}{2}u_3$ | $\frac{1}{2}u_3$ | 0 | $-u_1$ | $-u_2$ | 0 | $-e_4$ | 0 |

*3.* $\lambda = -1$, $\mu = 1$

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $e_3$ | $2e_4$ | $e_5$ | $u_1$ | 0 | $-u_3$ |
| $e_2$ | 0 | 0 | $-e_3$ | $-e_4$ | 0 | 0 | $u_2$ | $u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | 0 | 0 | $e_4$ | $e_4$ | $u_1$ | $e_2$ |
| $e_4$ | $-2e_4$ | $e_4$ | 0 | 0 | 0 | 0 | 0 | $u_1$ |
| $e_5$ | $-e_5$ | 0 | $-e_4$ | 0 | 0 | 0 | 0 | $u_2$ |
| $u_1$ | $-u_1$ | 0 | $-e_4$ | 0 | 0 | 0 | 0 | $u_2$ |
| $u_2$ | 0 | $-u_2$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 |
| $u_3$ | $u_3$ | $-u_3$ | $-e_2$ | $-u_1$ | $-u_2$ | $-u_2$ | 0 | 0 |

*4.* $\lambda = -1$, $\mu = 3$

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $e_3$ | $2e_4$ | $e_5$ | $u_1$ | 0 | $-u_3$ |
| $e_2$ | 0 | 0 | $-e_3$ | $-3e_4$ | $-2e_5$ | 0 | $u_2$ | $3u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | 0 | 0 | $e_4$ | 0 | $u_1$ | 0 |
| $e_4$ | $-2e_4$ | $3e_4$ | 0 | 0 | 0 | 0 | 0 | $u_1$ |
| $e_5$ | $-e_5$ | $2e_5$ | $-e_4$ | 0 | 0 | 0 | $e_3$ | $u_2$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | 0 | $-u_2$ | $-u_1$ | 0 | $-e_3$ | 0 | 0 | 0 |
| $u_3$ | $u_3$ | $-3u_3$ | 0 | $-u_1$ | $-u_2$ | 0 | 0 | 0 |

5. $\lambda = 0,\ \mu = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $e_4$ | $0$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $0$ | $e_5$ | $0$ | $u_2$ | $0$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $e_4$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_5$ | $0$ | $-e_5$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $e_4$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $e_5$ |
| $u_3$ | $0$ | $0$ | $0$ | $-u_1$ | $-u_2$ | $-e_4$ | $-e_5$ | $0$ |

6. $\lambda = 0,\ \mu = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $e_4$ | $0$ | $u_1$ | $0$ | $0$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $0$ | $e_5$ | $0$ | $u_2$ | $0$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $e_4$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_5$ | $0$ | $-e_5$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-e_4$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $-e_5$ |
| $u_3$ | $0$ | $0$ | $0$ | $-u_1$ | $-u_2$ | $e_4$ | $e_5$ | $0$ |

7. $\lambda = \frac{1}{2},\ \mu = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $\frac{1}{2}e_4$ | $-\frac{1}{2}e_5$ | $u_1$ | $0$ | $\frac{1}{2}u_3$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $0$ | $e_5$ | $0$ | $u_2$ | $e_4$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $e_4$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-\frac{1}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_5$ | $\frac{1}{2}e_5$ | $-e_5$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-\frac{1}{2}u_3$ | $-e_4$ | $0$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ |

8. $\lambda = \frac{1}{2},\ \mu = 0$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $\frac{1}{2}e_4$ | $-\frac{1}{2}e_5$ | $u_1$ | $0$ | $\frac{1}{2}u_3 + e_4$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $0$ | $e_5$ | $0$ | $u_2$ | $\alpha e_4$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $e_4$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-\frac{1}{2}e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_5$ | $\frac{1}{2}e_5$ | $-e_5$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-\frac{1}{2}u_3 - e_4$ | $-\alpha e_4$ | $0$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ |

9. $\lambda = -\frac{1}{2},\ \mu = 1$

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $\frac{3}{2}e_4$ | $\frac{1}{2}e_5$ | $u_1$ | $0$ | $-\frac{1}{2}u_3$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $-e_4$ | $0$ | $0$ | $u_2$ | $u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $e_4$ | $0$ | $u_1$ | $e_5$ |
| $e_4$ | $-\frac{3}{2}e_4$ | $e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_5$ | $-\frac{1}{2}e_5$ | $0$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $\frac{1}{2}u_3$ | $-u_3$ | $-e_5$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ |

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 0 & -\lambda \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 & 1-\mu \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \lambda-1 & \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(e_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ \lambda & \mu-1 & 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by $e_1$ and $e_2$.

**Lemma.** *Any virtual structure $q$ on the generalized module 5.10 is equivalent to one of the following:*

*a)* $(\lambda, \mu) \notin \{(0,0), (-1,1), (0,2), (-1,3), (1,-1), (\frac{1}{2}, 0), (-\frac{1}{2}, 1)\}$

$$C_1(e_i) = 0, \quad i = 1, \dots, 5;$$

*b)* $(\lambda, \mu) = (-1, 1)$

$$C_2(e_i) = 0,\ i = 1,2,5,\ C_2(e_3) = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & q \\ 0 & 0 & 0 \\ q-2p-r & 0 & 0 \\ 0 & r-p & 0 \end{pmatrix}, \quad C_2(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r \end{pmatrix};$$

*c)* $(\lambda, \mu) = (0, 2)$

$$C_3(e_i) = 0,\ i = 1,2,3, \quad C_3(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{q-p}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_3(e_5) = \begin{pmatrix} 0 & p & 0 \\ 0 & q & 0 \\ \frac{q-p}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

*d)* $(\lambda,\mu)=(-1,3)$

$$C_4(e_i)=0,\ i=1,\dots,4,\quad C_4(e_5)=\begin{pmatrix}0&0&0\\0&0&0\\0&p&0\\0&0&0\\0&0&0\end{pmatrix};$$

*e)* $(\lambda,\mu)=(0,0)$

$$C_5(e_1)=\begin{pmatrix}0&0&p\\0&0&p\\0&0&0\\-p&0&0\\0&-p&0\end{pmatrix},\quad C_5(e_2)=\begin{pmatrix}0&0&q\\0&0&q\\0&0&0\\-q&0&0\\0&-q&0\end{pmatrix},\quad C_5(e_i)=0,\ i=3,4,5;$$

*f)* $(\lambda,\mu)=(1,-1)$

$$C_6(e_1)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&p\\0&-p&0\\0&0&0\end{pmatrix},\quad C_6(e_2)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&q\\0&-q&0\\0&0&0\end{pmatrix},\quad C_6(e_i)=0,\ i=3,4,5;$$

*g)* $(\lambda,\mu)=(\frac{1}{2},0)$

$$C_7(e_1)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&p\\0&0&0\end{pmatrix},\quad C_7(e_2)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&q\\0&0&0\end{pmatrix},\quad C_7(e_i)=0,\ i=3,4,5;$$

*h)* $(\lambda,\mu)=(-\frac{1}{2},1)$

$$C_8(e_i)=0,\ i=1,2,4,5,\quad C_8(e_3)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&p\end{pmatrix}.$$

*Proof.* Let $q$ be a virtual structure on the generalized module 5.10. Without loss of generality it can be assumed that $q$ is primary. Since

$$\begin{aligned}
&\mathfrak{g}^{(0,0)}(\mathfrak{h})\supset\mathbb{R}e_1\oplus\mathbb{R}e_2, && U^{(1,0)}(\mathfrak{h})\supset\mathbb{R}u_1,\\
&\mathfrak{g}^{(1,-1)}(\mathfrak{h})\supset\mathbb{R}e_3, && U^{(0,1)}(\mathfrak{h})\supset\mathbb{R}u_2,\\
&\mathfrak{g}^{(1-\lambda,-\mu)}(\mathfrak{h})\supset\mathbb{R}e_4, && U^{(\lambda,\mu)}(\mathfrak{h})\supset\mathbb{R}u_3,\\
&\mathfrak{g}^{(-\lambda,1-\mu)}(\mathfrak{h})\supset\mathbb{R}e_5,
\end{aligned}$$

we have:

$$C(e_1) = \begin{pmatrix} 0 & 0 & c^1_{13} \\ 0 & 0 & c^1_{23} \\ 0 & 0 & c^1_{33} \\ c^1_{41} & c^1_{42} & c^1_{43} \\ c^1_{51} & c^1_{52} & c^1_{53} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c^2_{13} \\ 0 & 0 & c^2_{23} \\ 0 & 0 & c^2_{33} \\ c^2_{41} & c^2_{42} & c^2_{43} \\ c^2_{51} & c^2_{52} & c^2_{53} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & c^3_{13} \\ 0 & 0 & c^3_{23} \\ 0 & 0 & c^3_{33} \\ c^3_{41} & c^3_{42} & c^3_{43} \\ c^3_{51} & c^3_{52} & c^3_{53} \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} c^4_{11} & c^4_{12} & 0 \\ c^4_{21} & c^4_{22} & 0 \\ c^4_{31} & c^4_{32} & 0 \\ 0 & 0 & c^4_{43} \\ 0 & 0 & c^4_{53} \end{pmatrix},$$

$$C(e_5) = \begin{pmatrix} c^5_{11} & c^5_{12} & 0 \\ c^5_{21} & c^5_{22} & 0 \\ c^5_{31} & c^5_{32} & 0 \\ 0 & 0 & c^5_{43} \\ 0 & 0 & c^5_{53} \end{pmatrix}.$$

Checking condition (6), Chapter II, we obtain:

*a*) $(\lambda, \mu) \notin \{(0,0), (-1,1), (0,2), (-1,3), (1,-1), (\frac{1}{2},0), (-\frac{1}{2},1)\}$

$$C(e_1) = \begin{pmatrix} 0 & 0 & c^1_{13} \\ 0 & 0 & c^1_{23} \\ 0 & 0 & c^1_{33} \\ c^1_{41} & c^1_{42} & c^1_{43} \\ (1+\lambda)c^4_{53} & c^1_{52} & c^1_{53} \end{pmatrix}, \quad C(e_2) = \begin{pmatrix} 0 & 0 & c^2_{13} \\ 0 & 0 & c^2_{23} \\ 0 & 0 & c^2_{33} \\ c^2_{41} & c^2_{42} & c^2_{43} \\ (1-\mu)c^4_{53} & c^2_{52} & c^2_{53} \end{pmatrix},$$

where

$$\begin{cases} \mu c^1_{13} = \lambda c^2_{13}, \\ \mu c^1_{41} = \lambda c^2_{41}, \\ \mu c^1_{52} = \lambda c^2_{52}, \\ (\mu+1)c^1_{42} = (\lambda-1)c^2_{42}, \\ \mu c^1_{23} = \lambda c^2_{23}, \\ (\mu+1)c^1_{33} = (\lambda-1)c^2_{33}, \\ 2\mu c^1_{43} = (2\lambda-1)c^2_{43}, \\ (2\mu-1)c^1_{53} = 2\lambda c^2_{53}. \end{cases}$$

Now put $a_{13} = c^1_{13}/\lambda$, if $\lambda \neq 0$, or $a_{13} = c^2_{13}/\mu$, if $\mu \neq 0$. Similarly define

$a_{23}, a_{33}, a_{41}, a_{42}, a_{43}, a_{52}, a_{53}$. We obtain

$$C(e_1) = \begin{pmatrix} 0 & 0 & \lambda a_{13} \\ 0 & 0 & \lambda a_{23} \\ 0 & 0 & (\lambda-1)a_{33} \\ \lambda a_{41} & (\lambda-1)a_{42} & (2\lambda-1)a_{43} \\ (\lambda+1)c^4_{53} & \lambda a_{52} & 2\lambda a_{53} \end{pmatrix},$$

$$C(e_2) = \begin{pmatrix} 0 & 0 & \mu a_{13} \\ 0 & 0 & \mu a_{23} \\ 0 & 0 & (1+\mu)a_{33} \\ \mu a_{41} & (1+\mu)a_{42} & 2\mu a_{43} \\ (1-\mu)c^4_{53} & \mu a_{52} & (2\mu-1)a_{53} \end{pmatrix},$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a_{23}+a_{13} \\ -c^4_{53} & a_{41}-a_{52} & -a_{53} \\ 0 & c^4_{53} & 0 \end{pmatrix},$$

$$C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (1-\lambda)a_{13}-\mu a_{23}+a_{41} \\ 0 & 0 & c^4_{53} \end{pmatrix},$$

$$C(e_5) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{33}+a_{42} \\ 0 & 0 & -\lambda a_{13}+(1-\mu)a_{23}+a_{52} \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \\ a_{41} & a_{42} & a_{43} \\ c^4_{53} & a_{52} & a_{53} \end{pmatrix},$$

and $C_1(x) = C(x) + A(x)H - HB(x)$. Then

$$C_1(e_i) = 0, \quad i = 1, \dots, 5.$$

By corollary 2, Chapter II, the virtual structures $C$ and $C_1$ are equivalent.

*b*) $(\lambda, \mu) = (-1, 1)$

$$C(e_1) = C(e_2) = C(e_5) = 0,$$

$$C(e_3) = \begin{pmatrix} 0 & 0 & c^3_{13} \\ 0 & 0 & c^3_{23} \\ 0 & 0 & 0 \\ -c^4_{53}-2c^3_{13}+c^3_{23} & 0 & 0 \\ 0 & c^4_{53}-c^3_{13} & 0 \end{pmatrix}, \quad C(e_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^4_{53} \end{pmatrix}.$$

Now we can put $C_2 = C$.

Similarly we obtain the other results of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 5.10. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Consider the following cases:

1°. $(\lambda, \mu) \notin \{(0,0), (-1,1), (0,2), (-1,3), (1,-1), (\frac{1}{2}, 0), (-\frac{1}{2}, 1)\}$. Then

$$\begin{array}{lllll}
[e_1, e_2] = 0, & & & & \\
[e_1, e_3] = e_3, & [e_2, e_3] = -e_3, & & & \\
[e_1, e_4] = (1-\lambda)e_4, & [e_2, e_4] = -\mu e_4, & [e_3, e_4] = 0, & & \\
[e_1, e_5] = -\lambda e_5, & [e_2, e_5] = (1-\mu)e_5, & [e_3, e_5] = e_4, & [e_4, e_5] = 0, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = 0, & [e_5, u_1] = 0, \\
[e_1, u_2] = 0, & [e_2, u_2] = u_2, & [e_3, u_2] = u_1, & [e_4, u_2] = 0, & [e_5, u_2] = 0, \\
[e_1, u_3] = \lambda u_3, & [e_2, u_3] = \mu u_3, & [e_3, u_3] = 0, & [e_4, u_3] = u_1, & [e_5, u_3] = u_2.
\end{array}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 10, Chapter II). Thus

$$\begin{array}{ll}
\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) \supset \mathbb{R}e_1 \oplus \mathbb{R}e_2, & \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) \supset \mathbb{R}e_3, \\
\bar{\mathfrak{g}}^{(1-\lambda,-\mu)}(\mathfrak{h}) \supset \mathbb{R}e_4, & \bar{\mathfrak{g}}^{(-\lambda,1-\mu)}(\mathfrak{h}) \supset \mathbb{R}e_5, \\
\bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) \supset \mathbb{R}u_1, & \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}) \supset \mathbb{R}u_2, \\
\bar{\mathfrak{g}}^{(\lambda,\mu)}(\mathfrak{h}) \supset \mathbb{R}u_3, &
\end{array}$$

$$\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(1,1)}(\mathfrak{h}), \\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(\lambda+1,\mu)}(\mathfrak{h}), \\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(\lambda,\mu+1)}(\mathfrak{h}),
\end{aligned}$$

and

$$\begin{aligned}
&[u_1, u_2] = a_4 e_4 + a_5 e_5 + \alpha_3 u_3, \\
&[u_1, u_3] = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_3 e_3 + b_5 e_5 + \beta_2 u_2, \\
&[u_2, u_3] = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5 + \gamma_1 u_1.
\end{aligned}$$

Let $(\lambda, \mu) \neq (\frac{1}{2}, -\frac{1}{2})$. Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$. Otherwise the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $e_3$ | $\frac{1}{2}e_4$ | $-\frac{1}{2}e_5$ | $u_1$ | 0 | $\frac{1}{2}u_3$ |
| $e_2$ | 0 | 0 | $-e_3$ | $\frac{1}{2}e_4$ | $\frac{3}{2}e_5$ | 0 | $u_2$ | $-\frac{1}{2}u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | 0 | 0 | $e_4$ | 0 | $u_1$ | 0 |
| $e_4$ | $-\frac{1}{2}e_4$ | $-\frac{1}{2}e_4$ | 0 | 0 | 0 | 0 | 0 | $u_1$ |
| $e_5$ | $\frac{1}{2}e_5$ | $-\frac{3}{2}e_5$ | $-e_4$ | 0 | 0 | 0 | 0 | $u_2$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | 0 | $-u_2$ | $-u_1$ | 0 | 0 | 0 | 0 | $c_4 e_4$ |
| $u_3$ | $-\frac{1}{2}u_3$ | $\frac{1}{2}u_3$ | 0 | $-u_1$ | $-u_2$ | 0 | $-c_4 e_4$ | 0 . |

Consider the following cases:

1.1°. $c_4 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

1.2°. $c_4 \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_i) &= e_i, \ i = 1, 2, 5, \\ \pi(e_3) &= \sqrt[3]{c_4}\, e_3, \\ \pi(e_4) &= \sqrt[3]{c_4}\, e_4, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= \frac{1}{\sqrt[3]{c_4}} u_2, \\ \pi(u_3) &= \frac{1}{\sqrt[3]{c_4}} u_3. \end{aligned}$$

2°. $(\lambda, \mu) = (-1, 1)$. Then

$$\begin{array}{lllll} [e_1, e_2] = 0, & & & & \\ [e_1, e_3] = e_3, & [e_2, e_3] = -e_3, & & & \\ [e_1, e_4] = 2e_4, & [e_2, e_4] = -e_4, & [e_3, e_4] = 0, & & \\ [e_1, e_5] = e_5, & [e_2, e_5] = 0, & [e_3, e_5] = e_4, & [e_4, e_5] = 0, & \\ [e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = (q-r-2p)e_4, & [e_4, u_1] = 0, & [e_5, u_1] = 0, \\ [e_1, u_2] = 0, & [e_2, u_2] = u_2, & [e_3, u_2] = u_1 - pe_4 - re_5, & [e_4, u_2] = 0, & [e_5, u_2] = 0, \\ [e_1, u_3] = -u_3, & [e_2, u_3] = u_3, & [e_3, u_3] = pe_1 + qe_2, & [e_4, u_3] = u_1 + re_5, & [e_5, u_3] = u_2. \end{array}$$

Since virtual structure $q$ is primary, we have

$$\begin{aligned} &\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2, && \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}) = \mathbb{R}e_3, \\ &\bar{\mathfrak{g}}^{(2,-1)}(\mathfrak{h}) = \mathbb{R}e_4, && \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) = \mathbb{R}e_5 \oplus \mathbb{R}u_1, \\ &\bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}) = \mathbb{R}u_2, && \bar{\mathfrak{g}}^{(-1,1)}(\mathfrak{h}) = \mathbb{R}u_3, \end{aligned}$$

$$\begin{aligned} [u_1, u_2] &\in \bar{\mathfrak{g}}^{(1,1)}(\mathfrak{h}), \\ [u_1, u_3] &\in \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}), \\ [u_2, u_3] &\in \bar{\mathfrak{g}}^{(-1,2)}(\mathfrak{h}), \end{aligned}$$

and

$$\begin{aligned} [u_1, u_2] &= 0, \\ [u_1, u_3] &= \beta_2 u_2, \\ [u_2, u_3] &= 0. \end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $e_3$ | $2e_4$ | $e_5$ | $u_1$ | $0$ | $-u_3$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $-e_4$ | $0$ | $0$ | $u_2$ | $u_3$ |
| $e_3$ | $-e_3$ | $e_3$ | $0$ | $0$ | $e_4$ | $(q-r)e_4$ | $u_1+re_5$ | $qe_2$ |
| $e_4$ | $-2e_4$ | $e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1+re_5$ |
| $e_5$ | $-e_5$ | $0$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $(r-q)e_4$ | $0$ | $0$ | $0$ | $0$ | $(q-r)u_2$ |
| $u_2$ | $0$ | $-u_2$ | $-u_1-re_5$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $u_3$ | $-u_3$ | $-qe_2$ | $-u_1-re_5$ | $-u_2$ | $(r-q)u_2$ | $0$ | $0$ |

Consider the following cases:

2.1°. $q = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_i) &= e_i, \; i = 1, \dots, 5, \\ \pi(u_1) &= u_1 + re_5, \\ \pi(u_2) &= u_2, \\ \pi(u_3) &= u_3. \end{aligned}$$

2.2°. $q \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_3 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_i) &= e_i, \; i = 1, 2, 3, \\ \pi(e_4) &= qe_4, \\ \pi(e_5) &= qe_5, \\ \pi(u_1) &= u_1 + re_5, \\ \pi(u_2) &= u_2, \\ \pi(u_3) &= \frac{1}{q} u_3. \end{aligned}$$

The other cases can be considered similarly.

Now it remains to show that the pairs determined in the Proposition are not equivalent to each other.

Since

$$\dim \mathcal{Z}(\mathcal{D}\bar{\mathfrak{g}}_1 / \mathcal{Z}(\mathcal{D}\bar{\mathfrak{g}}_1)) \neq \dim \mathcal{Z}(\mathcal{D}\bar{\mathfrak{g}}_2 / \mathcal{Z}(\mathcal{D}\bar{\mathfrak{g}}_2)),$$

we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

Since $\dim \mathcal{D}\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}\bar{\mathfrak{g}}_3$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ are not equivalent.

Since $\dim \mathcal{D}^2\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2\bar{\mathfrak{g}}_4$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_4, \mathfrak{g}_4)$ are not equivalent.

Consider the homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(2, \mathbb{R})$, $i = 1, 5, 6$, where $f_i(x)$ is the matrix of the mapping $\operatorname{ad}|_{\mathcal{D}^2\bar{\mathfrak{g}}_i} x$ in the basis $\{e_4, u_1\}$ of $\bar{\mathfrak{g}}_i$. Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$, $i = 1, 5, 6$, are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$, $i = 1, 5, 6$, are not equivalent.

Since $\dim \mathcal{D}^2\bar{\mathfrak{g}}_1 \neq \dim \mathcal{D}^2\bar{\mathfrak{g}}_9$, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_9, \mathfrak{g}_9)$ are not equivalent.

Consider the homomorphisms

$$f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(5, \mathbb{R}), \; i = 1, 7, 8,$$

where $f_i(x)$ is the matrix of the mapping

$$\mathcal{D}\bar{\mathfrak{g}}_i / \mathcal{Z}(\mathcal{D}\bar{\mathfrak{g}}_i) \to \mathcal{D}\bar{\mathfrak{g}}_i / \mathcal{Z}(\mathcal{D}\bar{\mathfrak{g}}_i), \quad y + \mathcal{Z}(\mathcal{D}\bar{\mathfrak{g}}_i) \mapsto [x, y] + \mathcal{Z}(\mathcal{D}\bar{\mathfrak{g}}_i)$$

in the basis $\{e_3 + \mathcal{Z}(\mathcal{D}\bar{\mathfrak{g}}_i), e_4 + \mathcal{Z}(\mathcal{D}\bar{\mathfrak{g}}_i), e_5 + \mathcal{Z}(\mathcal{D}\bar{\mathfrak{g}}_i), u_2 + \mathcal{Z}(\mathcal{D}\bar{\mathfrak{g}}_i), u_3 + \mathcal{Z}(\mathcal{D}\bar{\mathfrak{g}}_i)\}$.

Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$ are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g}_i)$ are not equivalent.

Thus the proof of the Proposition is complete.

## 6. Six-dimensional case

**Proposition 6.1.** *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 6.1 is equivalent to one and only one of the following pairs:*

*1.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $2e_2$ | $-2e_3$ | $0$ | $e_5$ | $-e_6$ | $u_1$ | $-u_2$ | $0$ |
| $e_2$ | $-2e_2$ | $0$ | $e_1$ | $0$ | $0$ | $e_5$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $2e_3$ | $-e_1$ | $0$ | $0$ | $e_6$ | $0$ | $u_2$ | $0$ | $0$ |
| $e_4$ | $0$ | $0$ | $0$ | $0$ | $e_5$ | $e_6$ | $u_1$ | $u_2$ | $0$ |
| $e_5$ | $-e_5$ | $0$ | $-e_6$ | $-e_5$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_6$ | $e_6$ | $-e_5$ | $0$ | $-e_6$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $u_2$ | $-u_1$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $0$ | $0$ | $0$ | $0$ | $-u_1$ | $-u_2$ | $0$ | $0$ | $0$ |

*2.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $2e_2$ | $-2e_3$ | $0$ | $e_5$ | $-e_6$ | $u_1$ | $-u_2$ | $0$ |
| $e_2$ | $-2e_2$ | $0$ | $e_1$ | $0$ | $0$ | $e_5$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $2e_3$ | $-e_1$ | $0$ | $0$ | $e_6$ | $0$ | $u_2$ | $0$ | $0$ |
| $e_4$ | $0$ | $0$ | $0$ | $0$ | $e_5$ | $e_6$ | $u_1$ | $u_2$ | $0$ |
| $e_5$ | $-e_5$ | $0$ | $-e_6$ | $-e_5$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_6$ | $e_6$ | $-e_5$ | $0$ | $-e_6$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $e_5$ |
| $u_2$ | $u_2$ | $-u_1$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $e_6$ |
| $u_3$ | $0$ | $0$ | $0$ | $0$ | $-u_1$ | $-u_2$ | $-e_5$ | $-e_6$ | $0$ |

*3.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $2e_2$ | $-2e_3$ | $0$ | $e_5$ | $-e_6$ | $u_1$ | $-u_2$ | $0$ |
| $e_2$ | $-2e_2$ | $0$ | $e_1$ | $0$ | $0$ | $e_5$ | $0$ | $u_1$ | $0$ |
| $e_3$ | $2e_3$ | $-e_1$ | $0$ | $0$ | $e_6$ | $0$ | $u_2$ | $0$ | $0$ |
| $e_4$ | $0$ | $0$ | $0$ | $0$ | $e_5$ | $e_6$ | $u_1$ | $u_2$ | $0$ |
| $e_5$ | $-e_5$ | $0$ | $-e_6$ | $-e_5$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_6$ | $e_6$ | $-e_5$ | $0$ | $-e_6$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-u_1$ | $0$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $-e_5$ |
| $u_2$ | $u_2$ | $-u_1$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ | $-e_6$ |
| $u_3$ | $0$ | $0$ | $0$ | $0$ | $-u_1$ | $-u_2$ | $e_5$ | $e_6$ | $0$ |

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1)=\begin{pmatrix}0&0&0&0&0&0\\0&2&0&0&0&0\\0&0&-2&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&-1\end{pmatrix},\quad A(e_2)=\begin{pmatrix}0&0&1&0&0&0\\-2&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{pmatrix},$$

$$A(e_3)=\begin{pmatrix}0&-1&0&0&0&0\\0&0&0&0&0&0\\2&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\end{pmatrix},\quad A(e_4)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},$$

$$A(e_5)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\-1&0&0&-1&0&0\\0&0&-1&0&0&0\end{pmatrix},\quad A(e_6)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&-1&0&0&0&0\\1&0&0&-1&0&0\end{pmatrix},$$

and for $x\in\mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors $e_1$ and $e_4$.

**Lemma.** *Any virtual structure $q$ on generalized module 6.1 is equivalent to one of the following:*

$$C(e_i)=0,\ i=1,2,3,5,6,\quad C(e_4)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&-p\\p&0&0\\0&p&0\end{pmatrix}.$$

*Proof.* Let $q$ be a virtual structure on generalized module 6.1. Without loss of generality it can be assumed that $q$ is primary. Since

$$\begin{aligned}
&\mathfrak{g}^{(0,0)}(\mathfrak{h})=\mathbb{R}e_1\oplus\mathbb{R}e_4,\\
&\mathfrak{g}^{(2,0)}(\mathfrak{h})=\mathbb{R}e_2,\\
&\mathfrak{g}^{(-2,0)}(\mathfrak{h})=\mathbb{R}e_3, && U^{(1,1)}(\mathfrak{h})=\mathbb{R}u_1,\\
&\mathfrak{g}^{(1,1)}(\mathfrak{h})=\mathbb{R}e_5, && U^{(-1,1)}(\mathfrak{h})=\mathbb{R}u_2,\\
&\mathfrak{g}^{(-1,1)}(\mathfrak{h})=\mathbb{R}e_6, && U^{(0,0)}(\mathfrak{h})=\mathbb{R}u_3,
\end{aligned}$$

we have

$$C(e_1)=\begin{pmatrix}0&0&c^1_{13}\\0&0&0\\0&0&0\\0&0&c^1_{43}\\c^1_{51}&0&0\\0&c^1_{62}&0\end{pmatrix},\quad C(e_2)=\begin{pmatrix}0&0&0\\0&0&c^2_{23}\\0&0&0\\0&0&0\\0&c^2_{52}&0\\0&0&0\end{pmatrix},$$

$$C(e_3)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&c^3_{33}\\0&0&0\\0&0&0\\c^3_{61}&0&0\end{pmatrix},\quad C(e_4)=\begin{pmatrix}0&0&c^4_{13}\\0&0&0\\0&0&0\\0&0&c^4_{43}\\c^4_{51}&0&0\\0&c^4_{62}&0\end{pmatrix},$$

$$C(e_5)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&c^5_{53}\\0&0&0\end{pmatrix},\quad C(e_6)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&c^6_{63}\end{pmatrix}.$$

Put

$$H=\begin{pmatrix}0&0&h_{13}\\0&0&0\\0&0&0\\0&0&h_{43}\\h_{51}&0&0\\0&h_{62}&0\end{pmatrix},$$

where the set of coefficients $h_{ij}$ is a solution of the following system:

$$\begin{cases}c^3_{33}=2h_{13}\\c^2_{52}=h_{62}-h_{51}\\c^6_{63}=h_{13}-h_{43}-h_{62}\\c^5_{53}=-h_{13}-h_{43}-h_{51}\end{cases}$$

Note that the solution exists, since the matrix of the system is non-singular.

Now put $C_1(x)=C(x)-A(X)H+HB(x)$ for $x\in\mathfrak{g}$. Then

$$C_1(e_1)=\begin{pmatrix}0&0&c^1_{13}\\0&0&0\\0&0&0\\0&0&c^1_{43}\\c^1_{51}&0&0\\0&c^1_{62}&0\end{pmatrix},\quad C_1(e_2)=\begin{pmatrix}0&0&0\\0&0&c^2_{23}\\0&0&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},$$

$$C_1(e_3)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&0\\p&0&0\\c^3_{61}&0&0\end{pmatrix},\quad C_1(e_4)=\begin{pmatrix}0&0&c^4_{13}\\0&0&0\\0&0&0\\0&0&c^4_{43}\\c^4_{51}&0&0\\0&c^4_{62}&0\end{pmatrix},$$

$$C_1(e_5) = C_1(e_6) = 0.$$

By corollary 2, Chapter II, the virtual structures $C$ and $C_1$ are equivalent.

Since for any virtual structure $q$ condition (6), Chapter II, must be satisfied, after direct calculation we see that $C_1$ has the form determined in the Lemma.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 6.1. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Then

$$\begin{array}{llllll}
[e_1,e_2]=2e_2, & & & & & \\
[e_1,e_3]=-2e_3, [e_2,e_3]=e_1, & & & & & \\
[e_1,e_4]=0, & [e_2,e_4]=0, & [e_3,e_4]=0, & & & \\
[e_1,e_5]=e_5, & [e_2,e_5]=0, & [e_3,e_5]=e_6, & [e_4,e_5]=e_5, & & \\
[e_1,e_6]=-e_6, & [e_2,e_6]=e_5, & [e_3,e_6]=0, & [e_4,e_6]=e_6, & [e_5,e_6]=0 & \\
[e_1,u_1]=u_1, & [e_2,u_1]=0, & [e_3,u_1]=u_2, & [e_4,u_1]=pe_5+u_1, & [e_5,u_1]=0, & [e_6,u_1]=0, \\
[e_1,u_2]=-u_2, & [e_2,u_2]=u_1, & [e_3,u_2]=0, & [e_4,u_2]=pe_6+u_2, & [e_5,u_2]=0, & [e_6,u_2]=0, \\
[e_1,u_3]=0, & [e_2,u_3]=0, & [e_3,u_3]=0, & [e_4,u_3]=-pe_4, & [e_5,u_3]=u_1, & [e_6,u_3]=u_2.
\end{array}$$

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}}^\alpha(\mathfrak{h}) = \mathfrak{g}^\alpha(\mathfrak{h}) \times U^\alpha(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 10, Chapter II). Therefore

$$\begin{aligned}
[u_1, u_2] &= 0, \\
[u_1, u_3] &= b_5 e_5 + \beta_1 u_1, \\
[u_2, u_3] &= c_6 e_6 + \gamma_2 u_2.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | $2e_2$ | $-2e_3$ | 0 | $e_5$ | $-e_6$ | $u_1$ | $-u_2$ | 0 |
| $e_2$ | $-2e_2$ | 0 | $e_1$ | 0 | 0 | $e_5$ | 0 | $u_1$ | 0 |
| $e_3$ | $2e_3$ | $-e_1$ | 0 | 0 | $e_6$ | 0 | $u_2$ | 0 | 0 |
| $e_4$ | 0 | 0 | 0 | 0 | $e_5$ | $e_6$ | $u_1$ | $u_2$ | 0 |
| $e_5$ | $-e_5$ | 0 | $-e_6$ | $-e_5$ | 0 | 0 | 0 | 0 | $u_1$ |
| $e_6$ | $e_6$ | $-e_5$ | 0 | $-e_6$ | 0 | 0 | 0 | 0 | $u_2$ |
| $u_1$ | $-u_1$ | 0 | $-u_2$ | $-u_1$ | 0 | 0 | 0 | 0 | $b_5e_5+\beta_1u_1$ |
| $u_2$ | $u_2$ | $-u_1$ | 0 | $-u_2$ | 0 | 0 | 0 | 0 | $b_5e_6+\beta_1u_2$ |
| $u_3$ | 0 | 0 | 0 | 0 | $-u_1$ | $-u_2$ | $-b_5e_5-\beta_1u_1$ | $-b_5e_6-\beta_1u_2$ | 0 |

Then the mapping $\pi : \bar{\mathfrak{g}}' \to \bar{\mathfrak{g}}$ such that

$$\begin{aligned}
\pi(e_i) &= e_i, \ i = 1, \ldots, 6, \\
\pi(u_1) &= u_1 + \frac{\beta_1}{2} e_5, \\
\pi(u_2) &= u_2 + \frac{\beta_1}{2} e_6, \\
\pi(u_3) &= u_3 - \frac{\beta_1}{2} e_4,
\end{aligned}$$

establishes the equivalence of pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}', \mathfrak{g}')$, where the latter has the form:

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | $2e_2$ | $-2e_3$ | 0 | $e_5$ | $-e_6$ | $u_1$ | $-u_2$ | 0 |
| $e_2$ | $-2e_2$ | 0 | $e_1$ | 0 | 0 | $e_5$ | 0 | $u_1$ | 0 |
| $e_3$ | $2e_3$ | $-e_1$ | 0 | 0 | $e_6$ | 0 | $u_2$ | 0 | 0 |
| $e_4$ | 0 | 0 | 0 | 0 | $e_5$ | $e_6$ | $u_1$ | $u_2$ | 0 |
| $e_5$ | $-e_5$ | 0 | $-e_6$ | $-e_5$ | 0 | 0 | 0 | 0 | $u_1$ |
| $e_6$ | $e_6$ | $-e_5$ | 0 | $-e_6$ | 0 | 0 | 0 | 0 | $u_2$ |
| $u_1$ | $-u_1$ | 0 | $-u_2$ | $-u_1$ | 0 | 0 | 0 | 0 | $b_5e_5$ |
| $u_2$ | $u_2$ | $-u_1$ | 0 | $-u_2$ | 0 | 0 | 0 | 0 | $b_5e_6$ |
| $u_3$ | 0 | 0 | 0 | 0 | $-u_1$ | $-u_2$ | $-b_5e_5$ | $-b_5e_6$ | 0 |

Consider the following cases:

1°. $b_5 = 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2°. $b_5 > 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}'$ ,where

$$\pi(e_i) = e_i,\ i = 1, \ldots, 6,$$
$$\pi(u_j) = \sqrt{b_5}u_j,\ j = 1, 2, 3.$$

3°. $b_5 < 0$. Then the pair $(\bar{\mathfrak{g}}', \mathfrak{g}')$ is equivalent to the pair $(\bar{\mathfrak{g}}_3, \mathfrak{g}_3)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}'$, where

$$\pi(e_i) = e_i,\ i = 1, \ldots, 6,$$
$$\pi(u_j) = \sqrt{-b_5}u_j,\ j = 1, 2, 3.$$

Now it remains to show that the pairs determined in the Proposition are not equivalent to each other.

Let $\mathfrak{a}_i$ denote the radical of $\mathcal{D}\bar{\mathfrak{g}}_i$ for $i = 1, 2, 3$. Consider the homomorphisms $f_i : \bar{\mathfrak{g}}_i \to \mathfrak{gl}(4, \mathbb{R})$, $i = 1, 2, 3$, where $f_i(x)$ is the matrix of the mapping $\mathrm{ad}_{\mathfrak{a}_i} x$ in the basis $\{e_6, e_6, u_1, u_2\}$, $x \in \bar{\mathfrak{g}}_i$.

Since the subalgebras $f_i(\bar{\mathfrak{g}}_i)$, $i = 1, 2, 3$, are not conjugate, we conclude that the pairs $(\bar{\mathfrak{g}}_i, \mathfrak{g})$, $i = 1, 2, 3$, are not equivalent to each other.

**Proposition 6.2.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 6.2 is trivial.*

| [ , ] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | 0 | $(\lambda-1)e_4$ | 0 | $(\lambda-1)e_6$ | $\lambda u_1$ | $\lambda u_2$ | $u_3$ |
| $e_2$ | 0 | 0 | $2e_3$ | $e_4$ | $-2e_5$ | $-e_6$ | $u_1$ | $-u_2$ | 0 |
| $e_3$ | 0 | $-2e_3$ | 0 | 0 | $e_2$ | $e_4$ | 0 | $u_1$ | 0 |
| $e_4$ | $(1-\lambda)e_4$ | $-e_4$ | 0 | 0 | $-e_6$ | 0 | 0 | 0 | $u_1$ |
| $e_5$ | 0 | $2e_5$ | $-e_2$ | $e_6$ | 0 | 0 | $u_2$ | 0 | 0 |
| $e_6$ | $(1-\lambda)e_6$ | $e_6$ | $-e_4$ | 0 | 0 | 0 | 0 | 0 | $u_2$ |
| $u_1$ | $-\lambda u_1$ | $-u_1$ | 0 | 0 | $-u_2$ | 0 | 0 | 0 | 0 |
| $u_2$ | $-\lambda u_2$ | $u_2$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_3$ | $-u_3$ | 0 | 0 | $-u_1$ | 0 | $-u_2$ | 0 | 0 | 0 |

*Proof.* Let $\mathcal{E}=\{e_1,e_2,e_3,e_4,e_5,e_6\}$ be a basis of $\mathfrak{g}$, where

$$e_1=\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&1\end{pmatrix},\quad e_2=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\quad e_3=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},$$
$$e_4=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\quad e_5=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\quad e_6=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}.$$

Then

$$A(e_1)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&\lambda-1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&\lambda-1\end{pmatrix},\quad A(e_2)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&2&0&0&0\\0&0&0&1&0&0\\0&0&0&0&-2&0\\0&0&0&0&0&-1\end{pmatrix},$$

$$A(e_3)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&1&0\\0&-2&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\quad A(e_4)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1-\lambda&-1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\end{pmatrix},$$

$$A(e_5)=\begin{pmatrix}0&0&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&2&0&0&0&0\\0&0&0&1&0&0\end{pmatrix},\quad A(e_6)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&0&0&0\\1-\lambda&1&0&0&0&0\end{pmatrix},$$

and for $x\in\mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

**Lemma.** *Any virtual structure $q$ on generalized module 6.2 is trivial.*

*Proof.* Let $q$ be a virtual structure on generalized module 6.2. Note that

$$\mathfrak{a}=\mathbb{R}e_2\oplus\mathbb{R}e_3\oplus\mathbb{R}e_5$$

is a semisimple subalgebra of the Lie algebra $\mathfrak{g}$. Without loss of generality it can be assumed that $q(\mathfrak{a})=\{0\}$. Therefore

$$C(e_i)=(c^i_{jk})_{\substack{1\leqslant j\leqslant 6\\1\leqslant k\leqslant 3}},\ i=1,4,6,\quad C(e_2)=C(e_3)=C(e_5)=0.$$

Checking condition (6), Chapter II, we obtain:

$$C(e_1)=\begin{pmatrix}0&0&c^1_{13}\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad C(e_4)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&(\lambda-1)c^1_{13}\\0&0&0\\0&0&0\end{pmatrix},$$

$$C(e_6) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\lambda-1)c^1_{13} \end{pmatrix},$$

and $C(e_2) = C(e_3) = C_(e_5) = 0$. Put

$$H = \begin{pmatrix} 0 & 0 & c^1_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now put $C_1(x) = C(x) + A(x)H - HB(x)$. Then

$$C_1(e_i) = 0, \quad i = 1, \ldots, 6.$$

By corollary 2, Chapter II, the virtual structures $C$ and $C_1$ are equivalent.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 6.2. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by the virtual structure determined in the Lemma. Then

$[e_1, e_2] = 0,$

$[e_1, e_3] = 0,\quad [e_2, e_3] = 2e_3,$

$[e_1, e_4] = (\lambda{-}1)e_4,\ [e_2, e_4] = e_4,\quad [e_3, e_4] = 0,$

$[e_1, e_5] = 0,\quad [e_2, e_5] = -2e_5,\ [e_3, e_5] = e_2,\ [e_4, e_5] = -e_6,$

$[e_1, e_6] = (\lambda{-}1)e_6,\ [e_2, e_6] = -e_6,\quad [e_3, e_6] = e_4,\ [e_4, e_6] = 0,\quad [e_5, e_6] = 0,$

$[e_1, u_1] = \lambda u_1,\quad [e_2, u_1] = u_1,\quad [e_3, u_1] = 0,\ [e_4, u_1] = 0,\ [e_5, u_1] = u_2, [e_6, u_1] = 0,$

$[e_1, u_2] = \lambda u_2,\quad [e_2, u_2] = -u_2,\ [e_3, u_2] = u_1, [e_4, u_2] = 0,\ [e_5, u_2] = 0,\ [e_6, u_2] = 0,$

$[e_1, u_3] = u_3,\quad [e_2, u_3] = 0,\quad [e_3, u_3] = 0,\ [e_4, u_3] = u_1, [e_5, u_3] = 0,\ [e_6, u_3] = u_2.$

Put

$$[u_1, u_2] = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3,$$
$$[u_1, u_3] = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5 + b_6e_6 + \beta_1u_1 + \beta_2u_2 + \beta_3u_3,$$
$$[u_2, u_3] = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 + c_6e_6 + \gamma_1u_1 + \gamma_2u_2 + \gamma_3u_3.$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $0$ | $(\lambda-1)e_4$ | $0$ | $(\lambda_1)e_6$ | $\lambda u_1$ | $\lambda u_2$ | $u_3$ |
| $e_2$ | $0$ | $0$ | $2e_3$ | $e_4$ | $-2e_5$ | $-e_6$ | $u_1$ | $-u_2$ | $0$ |
| $e_3$ | $0$ | $-2e_3$ | $0$ | $0$ | $e_2$ | $e_4$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $(1-\lambda)e_4$ | $-e_4$ | $0$ | $0$ | $-e_6$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_5$ | $0$ | $2e_5$ | $-e_2$ | $e_6$ | $0$ | $0$ | $u_2$ | $0$ | $0$ |
| $e_6$ | $(1-\lambda)e_6$ | $e_6$ | $-e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_2$ |
| $u_1$ | $-\lambda u_1$ | $-u_1$ | $0$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-\lambda u_2$ | $u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-u_3$ | $0$ | $0$ | $-u_1$ | $0$ | $-u_2$ | $0$ | $0$ | $0$ |

The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

**Proposition 6.3.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 6.3 is equivalent to one and only one of the following pairs:*

*1.*

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $2e_2$ | $-2e_3$ | $0$ | $-e_5$ | $e_6$ | $0$ | $u_2$ | $-u_3$ |
| $e_2$ | $-2e_2$ | $0$ | $e_1$ | $0$ | $-e_6$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_3$ | $2e_3$ | $-e_1$ | $0$ | $0$ | $0$ | $-e_5$ | $0$ | $u_3$ | $0$ |
| $e_4$ | $0$ | $0$ | $0$ | $0$ | $-e_5$ | $-e_6$ | $0$ | $u_2$ | $u_3$ |
| $e_5$ | $e_5$ | $e_6$ | $0$ | $e_5$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_6$ | $-e_6$ | $0$ | $e_5$ | $e_6$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-u_2$ | $0$ | $-u_3$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $u_3$ | $-u_2$ | $0$ | $-u_3$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ |

*2.*

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $2e_2$ | $-2e_3$ | $0$ | $-e_5$ | $e_6$ | $0$ | $u_2$ | $-u_3$ |
| $e_2$ | $-2e_2$ | $0$ | $e_1$ | $0$ | $-e_6$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_3$ | $2e_3$ | $-e_1$ | $0$ | $0$ | $0$ | $-e_5$ | $0$ | $u_3$ | $0$ |
| $e_4$ | $0$ | $0$ | $0$ | $0$ | $-e_5$ | $-e_6$ | $0$ | $u_2$ | $u_3$ |
| $e_5$ | $e_5$ | $e_6$ | $0$ | $e_5$ | $0$ | $0$ | $0$ | $e_1+3e_4+u_1$ | $2e_3$ |
| $e_6$ | $-e_6$ | $0$ | $e_5$ | $e_6$ | $0$ | $0$ | $0$ | $2e_2$ | $-e_1+3e_4+u_1$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-u_2$ | $0$ | $-u_3$ | $-u_2$ | $-e_1-3e_4-u_1$ | $-2e_2$ | $0$ | $0$ | $0$ |
| $u_3$ | $u_3$ | $-u_2$ | $0$ | $-u_3$ | $-2e_3$ | $e_1-3e_4-u_1$ | $0$ | $0$ | $0$ |

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1)=\begin{pmatrix}0&0&0&0&0&0\\0&2&0&0&0&0\\0&0&-2&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&1\end{pmatrix},\quad A(e_2)=\begin{pmatrix}0&0&1&0&0&0\\-2&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\end{pmatrix},$$

$$A(e_3)=\begin{pmatrix}0&-1&0&0&0&0\\0&0&0&0&0&0\\2&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&-1\\0&0&0&0&0&0\end{pmatrix},\quad A(e_4)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&-1\end{pmatrix},$$

$$A(e_5)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&0&0&1&0&0\\0&1&0&0&0&0\end{pmatrix},\quad A(e_6)=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&1&0&0&0\\-1&0&0&1&0&0\end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors $e_1$ and $e_4$.

**Lemma.** *Any virtual structure $q$ on generalized module 6.3 is equivalent to one of the following:*

$$C(e_i)=0,\ i=1,2,3,4,\quad C(e_5)=\begin{pmatrix}0&p&0\\0&0&0\\0&0&2p\\0&3p&0\\0&0&0\\0&0&0\end{pmatrix},\quad C(e_6)=\begin{pmatrix}0&0&-p\\0&2p&0\\0&0&0\\0&0&3p\\0&0&0\\0&0&0\end{pmatrix}.$$

*Proof.* Let $q$ be a virtual structure on generalized module 6.3. Without loss of generality it can be assumed that $q$ is primary. Since

$$\begin{aligned}
&\mathfrak{g}^{(0,0)}(\mathfrak{h})=\mathbb{R}e_1\oplus\mathbb{R}e_4,\\
&\mathfrak{g}^{(2,0)}(\mathfrak{h})=\mathbb{R}e_2,\\
&\mathfrak{g}^{(-2,0)}(\mathfrak{h})=\mathbb{R}e_3, && U^{(0,0)}(\mathfrak{h})=\mathbb{R}u_1,\\
&\mathfrak{g}^{(-1,-1)}(\mathfrak{h})=\mathbb{R}e_5, && U^{(1,1)}(\mathfrak{h})=\mathbb{R}u_2,\\
&\mathfrak{g}^{(1,-1)}(\mathfrak{h})=\mathbb{R}e_6, && U^{(-1,1)}(\mathfrak{h})=\mathbb{R}u_3,
\end{aligned}$$

we have

$$C(e_1) = C(e_2) = C(e_3) = 0, \quad C(e_4) = \begin{pmatrix} c_{11}^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_{41}^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C(e_5) = \begin{pmatrix} 0 & c_{12}^5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{33}^5 \\ 0 & c_{42}^5 & 0 \\ c_{51}^5 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_6) = \begin{pmatrix} 0 & 0 & c_{13}^6 \\ 0 & c_{22}^6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{43}^6 \\ 0 & 0 & 0 \\ c_{61}^6 & 0 & 0 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_{51}^5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $C_1(x) = C(x) - A(X)H + HB(x)$ for $x \in \mathfrak{g}$. Then

$$C_1(e_1) = C_1(e_2) = C_1(e_3) = 0, \quad C_1(e_4) = \begin{pmatrix} c_{11}^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_{41}^4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_1(e_5) = \begin{pmatrix} 0 & c_{12}^5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{33}^5 \\ 0 & c_{42}^5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C_1(e_6) = \begin{pmatrix} 0 & 0 & c_{13}^6 \\ 0 & c_{22}^6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{43}^6 \\ 0 & 0 & 0 \\ c_{61}^6 & 0 & 0 \end{pmatrix}.$$

By corollary 2, Chapter II, the virtual structures $C$ and $C_1$ are equivalent.

Since for any virtual structure $q$ condition (6), Chapter II, must be satisfied, after direct calculation we see that $C_1$ has the form determined in the Lemma.

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 6.3. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the

Lemma. Then

$$
\begin{aligned}
&[e_1,e_2]=2e_2,\\
&[e_1,e_3]=-2e_3, [e_2,e_3]=e_1,\\
&[e_1,e_4]=0, \quad [e_2,e_4]=0, \quad [e_3,e_4]=0,\\
&[e_1,e_5]=-e_5, [e_2,e_5]=-e_6, [e_3,e_5]=0, \quad [e_4,e_5]=-e_5,\\
&[e_1,e_6]=e_6, \quad [e_2,e_6]=0, \quad [e_3,e_6]=-e_5, [e_4,e_6]=-e_6, [e_5,e_6]=0,\\
&[e_1,u_1]=0, \quad [e_2,u_1]=0, \quad [e_3,u_1]=0, \quad [e_4,u_1]=0, \quad [e_5,u_1]=0, \quad [e_6,u_1]=0,\\
&[e_1,u_2]=u_2, \quad [e_2,u_2]=0, \quad [e_3,u_2]=u_3, \quad [e_4,u_2]=u_2, \quad [e_5,u_2]=pe_1+A, [e_6,u_2]=2pe_2,\\
&[e_1,u_3]=-u_3, [e_2,u_3]=u_2, \quad [e_3,u_3]=0, \quad [e_4,u_3]=u_3, \quad [e_5,u_3]=2pe_3, \quad [e_6,u_3]=-pe_1+A,
\end{aligned}
$$

where $A = 3pe_4 + u_1$.

Since the virtual structure $q$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 10, Chapter II). Therefore

$$
\begin{aligned}
[u_1, u_2] &= \alpha_2 u_2,\\
[u_1, u_3] &= \beta_3 u_3,\\
[u_2, u_3] &= 0.
\end{aligned}
$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $2e_2$ | $-2e_3$ | $0$ | $-e_5$ | $e_6$ | $0$ | $u_2$ | $-u_3$ |
| $e_2$ | $-2e_2$ | $0$ | $e_1$ | $0$ | $-e_6$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_3$ | $2e_3$ | $-e_1$ | $0$ | $0$ | $0$ | $-e_5$ | $0$ | $u_3$ | $0$ |
| $e_4$ | $0$ | $0$ | $0$ | $0$ | $-e_5$ | $-e_6$ | $0$ | $u_2$ | $u_3$ |
| $e_5$ | $e_5$ | $e_6$ | $0$ | $e_5$ | $0$ | $0$ | $0$ | $pe_1 + A$ | $2pe_3$ |
| $e_6$ | $-e_6$ | $0$ | $e_5$ | $e_6$ | $0$ | $0$ | $0$ | $2pe_2$ | $-pe_1 + A$ |
| $u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-u_2$ | $0$ | $-u_3$ | $-u_2$ | $-pe_1 - A$ | $-2pe_2$ | $0$ | $0$ | $0$ |
| $u_3$ | $u_3$ | $-u_2$ | $0$ | $-u_3$ | $-2pe_3$ | $pe_1 - A$ | $0$ | $0$ | $0$ |

where $A = 3pe_4 + u_1$.

Consider the following cases:

1°. $p = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2°. $p \neq 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_2 \to \bar{\mathfrak{g}}$, where

$$
\begin{aligned}
\pi(e_i) &= e_i, \ i = 1, \ldots, 6,\\
\pi(u_j) &= \frac{1}{p} u_j, \ j = 1, 2, 3.
\end{aligned}
$$

Since the Lie algebra $\bar{\mathfrak{g}}_1$ is reductive, and $\bar{\mathfrak{g}}_2$ is nonreductive, we see that the pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are not equivalent.

**Proposition 6.4.** *Any pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ *of type 6.4 is trivial.*

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $0$ | $0$ | $(1-\lambda)e_5$ | $(1-\lambda)e_6$ | $u_1$ | $\lambda u_2$ | $\lambda u_3$ |
| $e_2$ | $0$ | $0$ | $2e_3$ | $-2e_4$ | $-e_5$ | $e_6$ | $0$ | $u_2$ | $-u_3$ |
| $e_3$ | $0$ | $-2e_3$ | $0$ | $e_2$ | $-e_6$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $0$ | $2e_4$ | $-e_2$ | $0$ | $0$ | $-e_5$ | $0$ | $u_3$ | $0$ |
| $e_5$ | $(\lambda-1)e_5$ | $e_5$ | $e_6$ | $0$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_6$ | $(\lambda-1)e_6$ | $-e_6$ | $0$ | $e_5$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-\lambda u_2$ | $-u_2$ | $0$ | $-u_3$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ |
| $u_3$ | $-\lambda u_3$ | $u_3$ | $-u_2$ | $0$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ |

*Proof.* Let $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\lambda \end{pmatrix}, \quad A(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad A(e_4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(e_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda-1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad A(e_6) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \lambda-1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

**Lemma.** *Any virtual structure $q$ on generalized module 6.4 is equivalent to one of the following:*

*a)* $\lambda = \frac{1}{2}$.

$$C(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & -p & 0 \end{pmatrix}, \quad C(e_i) = 0, \quad i = 2, \dots, 6;$$

*b)* $\lambda \neq \frac{1}{2}$.

$$C(e_i) = 0, \quad i = 1, \dots, 6.$$

*Proof.* Put

$$C(e_i) = \left(c^i_{jk}\right)_{\substack{1 \leqslant j \leqslant 6 \\ 1 \leqslant k \leqslant 3}}, \quad i = 1, \dots, 6.$$

Note that $\mathfrak{a} = \mathbb{R}e_2 \oplus \mathbb{R}e_3 \oplus \mathbb{R}e_4$ is a semisimple subalgebra of the Lie algebra $\mathfrak{g}$. By Proposition 12, Chapter II, without loss of generality it can be assumed that $C(\mathfrak{a}) = \{0\}$. Therefore

$$C(e_2) = C(e_3) = C(e_4) = 0, \quad C(e_i) = \left(c^i_{jk}\right)_{\substack{1 \leqslant j \leqslant 6 \\ 1 \leqslant k \leqslant 3}}, \quad i = 1, 5, 6.$$

Since for any virtual structure $q$ condition (6), Chapter II, must be satisfied, after direct calculation we obtain:

$$C(e_2) = C(e_3) = C(e_4) = 0, \qquad C(e_1) = \begin{pmatrix} c^1_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^1_{53} \\ 0 & -c^1_{53} & 0 \end{pmatrix},$$

$$C(e_5) = \begin{pmatrix} 0 & c^1_{11} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ (1-\lambda)c^1_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C(e_6) = \begin{pmatrix} 0 & 0 & c^1_{11} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ (1-\lambda)c^1_{11} & 0 & 0 \end{pmatrix}.$$

Put

$$H = \begin{pmatrix} c^1_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h \\ 0 & -h & 0 \end{pmatrix},$$

and $C_1 = C(x) + A(x)H - HB(x)$ for $x \in \mathfrak{g}$. Then, by corollary 2, Chapter II, the virtual structures $C$ and $C_1$ are equivalent.

Consider the following cases:

1°. $\lambda \neq \frac{1}{2}$. Suppose $h = \frac{1}{2\lambda-1} c^1_{53}$. Then $C_1(e_i) = 0,\ i = 1, \dots, 6$.

2°. $\lambda = \frac{1}{2}$. Then $C_1(e_i) = 0,\ i = 2, \dots, 6$,

$$C_1(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & -p & 0 \end{pmatrix}, \text{ where } p = c^1_{53}.$$

This completes the proof of the Lemma.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 6.4. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is defined by one of the virtual structures determined in the Lemma. Then

$$\begin{array}{llllll}
[e_1, e_2] = 0, & & & & & \\
[e_1, e_3] = 0, & [e_2, e_3] = 2e_3, & & & & \\
[e_1, e_4] = 0, & [e_2, e_4] = -2e_4, & [e_3, e_4] = e_2, & & & \\
[e_1, e_5] = (1-\lambda)e_5, & [e_2, e_5] = -e_5, & [e_3, e_5] = -e_6, & [e_4, e_5] = 0, & & \\
[e_1, e_6] = (1-\lambda)e_6, & [e_2, e_6] = e_6, & [e_3, e_6] = 0, & [e_4, e_6] = -e_5, & [e_5, e_6] = 0, & \\
[e_1, u_1] = u_1, & [e_2, u_1] = 0, & [e_3, u_1] = 0, & [e_4, u_1] = 0, & [e_5, u_1] = 0, & [e_6, u_1] = 0, \\
[e_1, u_2] = \lambda u_2 - p e_6, & [e_2, u_2] = u_2, & [e_3, u_2] = 0, & [e_4, u_2] = u_3, & [e_5, u_2] = u_1, & [e_6, u_2] = 0, \\
[e_1, u_3] = \lambda u_3 + p e_5, & [e_2, u_3] = -u_3, & [e_3, u_3] = u_2, & [e_4, u_3] = 0, & [e_5, u_3] = 0, & [e_6, u_3] = u_1.
\end{array}$$

Put

$$\begin{aligned}
[u_1, u_2] &= a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3, \\
[u_1, u_2] &= b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5 + b_6 e_6 + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3, \\
[u_1, u_2] &= c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5 + c_6 e_6 + \gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3.
\end{aligned}$$

Using the Jacobi identity we see that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the form:

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $0$ | $0$ | $(1-\lambda)e_5$ | $(1-\lambda)e_6$ | $u_1$ | $\lambda u_2$ | $\lambda u_3$ |
| $e_2$ | $0$ | $0$ | $2e_3$ | $-2e_4$ | $-e_5$ | $e_6$ | $0$ | $u_2$ | $-u_3$ |
| $e_3$ | $0$ | $-2e_3$ | $0$ | $e_2$ | $-e_6$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_4$ | $0$ | $2e_4$ | $-e_2$ | $0$ | $0$ | $-e_5$ | $0$ | $u_3$ | $0$ |
| $e_5$ | $(\lambda-1)e_5$ | $e_5$ | $e_6$ | $0$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_6$ | $(\lambda-1)e_6$ | $-e_6$ | $0$ | $e_5$ | $0$ | $0$ | $0$ | $0$ | $u_1$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $u_2$ | $-\lambda u_2$ | $-u_2$ | $0$ | $-u_3$ | $-u_1$ | $0$ | $0$ | $0$ | $\gamma_1 u_1$ |
| $u_3$ | $-\lambda u_3$ | $u_3$ | $-u_2$ | $0$ | $0$ | $-u_1$ | $0$ | $-\gamma_1 u_1$ | $0$ , |

where $\gamma_1(\lambda - \frac{1}{2}) = 0$.

Consider the following cases:

1°. $\gamma_1 = 0$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the trivial pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$.

2°. $\gamma_1 \neq 0$, $\lambda = \frac{1}{2}$. Then the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is equivalent to the pair $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ by means of the mapping $\pi : \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}$, where

$$\begin{aligned} \pi(e_i) &= e_i, \ i = 1, \dots, 6, \\ \pi(u_1) &= u_1, \\ \pi(u_2) &= u_2 - \frac{\gamma_1}{2} e_6, \\ \pi(u_3) &= u_3 + \frac{\gamma_1}{2} e_5. \end{aligned}$$

This completes the proof of the Proposition.

**Proposition 6.5.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 6.5 is trivial.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | $e_2$ | 0 | $e_4$ | 0 | 0 | $u_1$ | 0 | 0 |
| $e_2$ | $-e_2$ | 0 | $e_4$ | 0 | $e_2$ | 0 | 0 | $u_1$ | 0 |
| $e_3$ | 0 | $-e_4$ | 0 | 0 | $-e_3$ | $e_3$ | 0 | 0 | $u_2$ |
| $e_4$ | $-e_4$ | 0 | 0 | 0 | 0 | $e_4$ | 0 | 0 | $u_1$ |
| $e_5$ | 0 | $-e_2$ | $e_3$ | 0 | 0 | 0 | 0 | $u_2$ | 0 |
| $e_6$ | 0 | 0 | $-e_3$ | $-e_4$ | 0 | 0 | 0 | 0 | $u_3$ |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | 0 | $-u_1$ | 0 | 0 | $-u_2$ | 0 | 0 | 0 | 0 |
| $u_3$ | 0 | 0 | $-u_2$ | $-u_1$ | 0 | $-u_3$ | 0 | 0 | 0 |

*Proof.* Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

## 7. Pairs with subalgebra of dimension higher then 6

**Proposition 7.1.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 7.1 is trivial.*

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | 0 | 0 | 0 | $e_6$ | $e_7$ | $u_1$ | $u_2$ | 0 |
| $e_2$ | 0 | 0 | 0 | $2e_4$ | $-2e_5$ | $e_6$ | $-e_7$ | $u_1$ | $-u_2$ | 0 |
| $e_3$ | 0 | 0 | 0 | 0 | 0 | $-e_6$ | $-e_7$ | 0 | 0 | $u_3$ |
| $e_4$ | 0 | $-2e_4$ | 0 | 0 | $e_2$ | 0 | $e_6$ | 0 | $u_1$ | 0 |
| $e_5$ | 0 | $2e_5$ | 0 | $-e_2$ | 0 | $e_7$ | 0 | $u_2$ | 0 | 0 |
| $e_6$ | $-e_6$ | $-e_6$ | $e_6$ | 0 | $-e_7$ | 0 | 0 | 0 | 0 | $u_1$ |
| $e_7$ | $-e_7$ | $e_7$ | $e_7$ | $-e_6$ | 0 | 0 | 0 | 0 | 0 | $u_2$ |
| $u_1$ | $-u_1$ | $-u_1$ | 0 | 0 | $-u_2$ | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | $-u_2$ | $u_2$ | 0 | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_3$ | 0 | 0 | $-u_3$ | 0 | 0 | $-u_1$ | $-u_2$ | 0 | 0 | 0 |

*Proof.* Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

**Proposition 7.2.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 7.2 is trivial.*

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | $e_2$ | $e_3$ | 0 | 0 | 0 | 0 | $u_1$ | 0 | 0 |
| $e_2$ | $-e_2$ | 0 | 0 | 0 | $e_2$ | $e_3$ | 0 | 0 | $u_1$ | 0 |
| $e_3$ | $-e_3$ | 0 | 0 | $e_3$ | $-e_3$ | 0 | $e_2$ | 0 | 0 | $u_1$ |
| $e_4$ | 0 | 0 | $-e_3$ | 0 | 0 | $-e_6$ | $e_7$ | 0 | 0 | $u_3$ |
| $e_5$ | 0 | $-e_2$ | $e_3$ | 0 | 0 | $2e_6$ | $-2e_7$ | 0 | $u_2$ | $-u_3$ |
| $e_6$ | 0 | $-e_3$ | 0 | $e_6$ | $-2e_6$ | 0 | $e_5$ | 0 | 0 | $u_2$ |
| $e_7$ | 0 | 0 | $-e_2$ | $-e_7$ | $2e_7$ | $-e_5$ | 0 | 0 | $u_3$ | 0 |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $u_2$ | 0 | $-u_1$ | 0 | 0 | $-u_2$ | 0 | $-u_3$ | 0 | 0 | 0 |
| $u_3$ | 0 | 0 | $-u_1$ | $-u_3$ | $u_3$ | $-u_2$ | 0 | 0 | 0 | 0 |

*Proof.* Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

**Proposition 8.1.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 8.1 is trivial.*

| [,] | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | 0 | 0 | $2e_3$ | $e_4$ | $-2e_5$ | $-e_6$ | $-e_7$ | $e_8$ | $u_1$ | $-u_2$ | 0 |
| $e_2$ | 0 | 0 | $-e_3$ | $e_4$ | $e_5$ | $2e_6$ | $-e_7$ | $-2e_8$ | 0 | $u_2$ | $-u_3$ |
| $e_3$ | $-2e_3$ | $e_3$ | 0 | 0 | $e_1$ | $e_4$ | $-e_8$ | 0 | 0 | $u_1$ | 0 |
| $e_4$ | $-e_4$ | $-e_4$ | 0 | 0 | $-e_6$ | 0 | $e_1 + e_2$ | $e_3$ | 0 | 0 | $u_1$ |
| $e_5$ | $2e_5$ | $-e_5$ | $-e_1$ | $e_6$ | 0 | 0 | 0 | $-e_7$ | $u_2$ | 0 | 0 |
| $e_6$ | $e_6$ | $-2e_6$ | $-e_4$ | 0 | 0 | 0 | $e_5$ | $e_2$ | 0 | 0 | $u_2$ |
| $e_7$ | $e_7$ | $e_7$ | $e_8$ | $-e_1 - e_2$ | 0 | $-e_5$ | 0 | 0 | $u_3$ | 0 | 0 |
| $e_8$ | $-e_8$ | $2e_8$ | 0 | $-e_3$ | $e_7$ | $-e_2$ | 0 | 0 | 0 | $u_3$ | 0 |
| $u_1$ | $-u_1$ | 0 | 0 | 0 | $-u_2$ | 0 | $-u_3$ | 0 | 0 | 0 | 0 |
| $u_2$ | $u_2$ | $-u_2$ | $-u_1$ | 0 | 0 | 0 | 0 | $-u_3$ | 0 | 0 | 0 |
| $u_3$ | 0 | $u_3$ | 0 | $-u_1$ | 0 | $-u_2$ | 0 | 0 | 0 | 0 | 0 |

*Proof.* Let $\mathcal{E} = \{e_1, \dots, e_8\}$ be a basis of $\mathfrak{g}$, where

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$e_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad e_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and for $x \in \mathfrak{g}$ the matrix $B(x)$ is identified with $x$.

By $\mathfrak{h}$ denote the nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ spanned by the vectors $e_1$ and $e_2$.

**Lemma.** *Any virtual structure $q$ on generalized module 8.1 is trivial.*

*Proof.* Note that $\mathfrak{g}$ is a semisimple Lie algebra. By statement 12, Charter II, without loss of generality it can be assumed that $C(\mathfrak{g}) = \{0\}$.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of type 8.1. Then it can be assumed that the corresponding virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

Since the mapping $q : \mathfrak{g} \to \mathcal{L}(U, \mathfrak{g})$ corresponding to the virtual pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is primary, we have

$$\bar{\mathfrak{g}}^{\alpha}(\mathfrak{h}) = \mathfrak{g}^{\alpha}(\mathfrak{h}) \times U^{\alpha}(\mathfrak{h}) \text{ for all } \alpha \in \mathfrak{h}^*$$

(Proposition 10, Charter II). Thus,

$$\begin{aligned}
&\bar{\mathfrak{g}}^{(0,0)}(\mathfrak{h}) = \mathbb{R}e_1 \oplus \mathbb{R}e_2, && \bar{\mathfrak{g}}^{(2,-1)}(\mathfrak{h}) = \mathbb{R}e_3, \\
&\bar{\mathfrak{g}}^{(1,1)}(\mathfrak{h}) = \mathbb{R}e_4, && \bar{\mathfrak{g}}^{(-2,1)}(\mathfrak{h}) = \mathbb{R}e_5, \\
&\bar{\mathfrak{g}}^{(-1,2)}(\mathfrak{h}) = \mathbb{R}e_6, && \bar{\mathfrak{g}}^{(-1,-1)}(\mathfrak{h}) = \mathbb{R}e_7, \\
&\bar{\mathfrak{g}}^{(1,-2)}(\mathfrak{h}) = \mathbb{R}e_8, && \bar{\mathfrak{g}}^{(1,0)}(\mathfrak{h}) = \mathbb{R}u_1, \\
&\bar{\mathfrak{g}}^{(-1,1)}(\mathfrak{h}) = \mathbb{R}u_2, && \bar{\mathfrak{g}}^{(0,-1)}(\mathfrak{h}) = \mathbb{R}u_3,
\end{aligned}$$

$$\begin{aligned}
&[u_1, u_2] \in \bar{\mathfrak{g}}^{(0,1)}(\mathfrak{h}), \\
&[u_1, u_3] \in \bar{\mathfrak{g}}^{(1,-1)}(\mathfrak{h}), \\
&[u_2, u_3] \in \bar{\mathfrak{g}}^{(-1,0)}(\mathfrak{h}),
\end{aligned}$$

and

$$\begin{aligned}
&[u_1, u_2] = 0, \\
&[u_1, u_3] = 0, \\
&[u_2, u_3] = 0.
\end{aligned}$$

It follows that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.

**Proposition 9.1.** *Any pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of type 9.1 is trivial.*

| $[\,,]$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $u_1$ | $u_2$ | $u_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $2e_3$ | $e_4$ | $-2e_5$ | $-e_6$ | $-e_7$ | $e_8$ | $0$ | $u_1$ | $-u_2$ | $0$ |
| $e_2$ | $0$ | $0$ | $-e_3$ | $e_4$ | $e_5$ | $2e_6$ | $-e_7$ | $-2e_8$ | $0$ | $0$ | $u_2$ | $-u_3$ |
| $e_3$ | $-2e_3$ | $e_3$ | $0$ | $0$ | $e_1$ | $e_4$ | $-e_8$ | $0$ | $0$ | $0$ | $u_1$ | $0$ |
| $e_4$ | $-e_4$ | $-e_4$ | $0$ | $0$ | $-e_6$ | $0$ | $e_1 + e_2$ | $e_3$ | $0$ | $0$ | $0$ | $u_1$ |
| $e_5$ | $2e_5$ | $-e_5$ | $-e_1$ | $e_6$ | $0$ | $0$ | $0$ | $-e_7$ | $0$ | $u_2$ | $0$ | $0$ |
| $e_6$ | $e_6$ | $-2e_6$ | $-e_4$ | $0$ | $0$ | $0$ | $e_5$ | $e_2$ | $0$ | $0$ | $0$ | $u_2$ |
| $e_7$ | $e_7$ | $e_7$ | $e_8$ | $-e_1 - e_2$ | $0$ | $-e_5$ | $0$ | $0$ | $0$ | $u_3$ | $0$ | $0$ |
| $e_8$ | $-e_8$ | $2e_8$ | $0$ | $-e_3$ | $e_7$ | $-e_2$ | $0$ | $0$ | $0$ | $0$ | $u_3$ | $0$ |
| $e_9$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $u_1$ | $u_2$ | $u_3$ |
| $u_1$ | $-u_1$ | $0$ | $0$ | $0$ | $-u_2$ | $0$ | $-u_3$ | $0$ | $-u_1$ | $0$ | $0$ | $0$ |
| $u_2$ | $u_2$ | $-u_2$ | $-u_1$ | $0$ | $0$ | $0$ | $0$ | $-u_3$ | $-u_2$ | $0$ | $0$ | $0$ |
| $u_3$ | $0$ | $u_3$ | $0$ | $-u_1$ | $0$ | $-u_2$ | $0$ | $0$ | $-u_3$ | $0$ | $0$ | $0$ |

*Proof.* Consider $x \in \mathfrak{g}$ such that

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $x_U = \mathrm{id}_U$. Then, by Proposition 13, Chapter II, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is trivial.