With this paper we start a series of papers on quantizations of differential operators and differential structures.

In this first paper we introduce our main objects: differential operators in braid tensor (or quasitensor) categories and quantizations. We propose a quantization scheme which unifies classical "quantizations" and anomalies.

I would like to point out that it is necessary to introduce braid category for the following two reasons: First, a rich and satisfied enough theory of differential operators and equations is only possible with some generalized "commutativity" conditions. And second, a quantization is from our opinion a "controled deviation" of a commutativity law. For example, one can consider the famous Heisenberg relation of quantum mechanics, $[\hat{p}, \hat{q}] = -i\hbar$, for operators $x = \exp(i\hat{q}), y = \exp(i\hat{p})$ and obtain the relations $xy = \lambda yx$, with $\lambda = \exp(i\hbar)$, (this gives the so-called quantum plane), which control the degree of non-commutativity of our quantum algebra.

1 Braided tensor categories

In this section we collect and recall some notions and facts about tensor categories, internal homomorphisms and braidings [1,2,5,9].

1.1. Let $C$ be an Abelian category. We denote by $\text{Mor}(X,Y)$ the set of all morphisms in $C$ from $X$ into $Y$, $X,Y \in \text{Ob}(C)$. We call $C$ a monoidal category if we introduce

1) a covariant bifunctor

$\otimes : C \times C \to C,$

$\otimes : X \times Y \mapsto X \otimes Y,$

which is called the tensor product,
2) a natural isomorphism

\[ \alpha(X, Y, Z) : X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z, \]

which is called the \textit{associativity constraint}.

3) a unit object \( k \), with natural isomorphisms

\[ \eta^*(X) : X \otimes k \rightarrow X, \eta^f(X) : k \otimes X \rightarrow X, \]

such that the following MacLane coherence conditions hold:

(i) the pentagon axiom

\[
\begin{array}{c}
\xymatrix{X \otimes (Y \otimes (Z \otimes T)) \ar[rr]^{\alpha} & & ((X \otimes Y) \otimes Z) \otimes T \ar[ll]^{\alpha} \\
\downarrow 1 \otimes \alpha & & \downarrow \alpha \otimes 1 \\
X \otimes ((Y \otimes Z) \otimes T) \ar[r]^{\alpha} & (X \otimes (Y \otimes Z)) \otimes T
}\end{array}
\]

(ii) the unity axiom

\[
\begin{array}{c}
\xymatrix{X \otimes k \otimes Y \ar[r]^{\alpha} & X \otimes (k \otimes Y) \\
\eta^* \otimes 1 \downarrow & & \downarrow 1 \otimes \eta^f \\
X \otimes Y
}\end{array}
\]

The MacLane coherence conditions mean that all diagrams built up from associativity and unity constraints by tensoring, substitution, and composition are commutative. Hence, all objects obtained by computing tensor products with different bracketing and cancelling units are naturally isomorphic.
1.2. A braiding [5] in a monoidal category is a natural isomorphism

\[ \sigma(X, Y) : X \otimes Y \rightarrow Y \otimes X, \]

also called a commutativity constraint, such that the following hexagonal diagrams commute

\[
\begin{array}{ccccccc}
X \otimes (Y \otimes Z) & X \otimes (Z \otimes Y) & (X \otimes Y) \otimes Z & (X \otimes Z) \otimes Y & (X \otimes Y) \otimes Z & (Y \otimes Z) \otimes X & Y \otimes (Z \otimes X) \\
1 \otimes \sigma \swarrow & \alpha & \downarrow & \downarrow & \alpha^{-1} & \downarrow & \alpha^{-1} \\
\sigma \otimes 1 \searrow & \sigma & (Z \otimes X) \otimes Y & (X \otimes Z) \otimes Y & X \otimes (Y \otimes Z) & Y \otimes (X \otimes Z) \\
\end{array}
\]

H(1) H(2)

A braiding \( \sigma \) is called a symmetry if

\[ \sigma(X, Y) \circ \sigma(Y, X) = 1. \]

Note that conditions H(1) and H(2) are equivalent if the braiding \( \sigma \) is a symmetry.

Definition

1. A monoidal category equipped with a symmetry is called a tensor category.

2. A monoidal category equipped with a braiding is called a braided tensor (or quasitensor) category.
1.3. Remark. The hexagon conditions yield the commutativity of the following Yang-Baxter diagram.

\[
\begin{array}{c}
\text{\textbf{Remark.}} \text{ The hexagon conditions yield the commutativity of the following Yang-Baxter diagram.} \\
X \otimes (Y \otimes Z) \\
\alpha \nearrow \quad 1 \otimes \sigma \\
(X \otimes Y) \otimes Z \quad X \otimes (Z \otimes Y) \\
\nearrow \sigma \otimes 1 \quad \nearrow \alpha \\
(Y \otimes X) \otimes Z \quad (X \otimes Z) \otimes Y \\
\nearrow \alpha^{-1} \quad \nearrow \sigma \otimes 1 \\
Y \otimes (X \otimes Z) \quad (Z \otimes X) \otimes Y \\
\nearrow 1 \otimes \sigma \quad \nearrow \alpha^{-1} \\
Y \otimes (Z \otimes X) \quad Z \otimes (X \otimes Y) \\
\nearrow \alpha \quad \nearrow 1 \otimes \sigma \\
(Y \otimes Z) \otimes X \quad Z \otimes (Y \otimes X) \\
\nearrow \sigma \otimes 1 \quad \nearrow \alpha \\
(Z \otimes Y) \otimes X.
\end{array}
\]

For the trivial associativity constraint \( \alpha = 1 \), the diagram reduces to the Yang-Baxter hexagon:

\[
\begin{array}{c}
\text{\textbf{For the trivial associativity constraint}} \text{ \( \alpha = 1 \), the diagram reduces to the Yang-Baxter hexagon:} \\
X \otimes Y \otimes Z \\
\sigma \otimes 1 \nearrow \quad \nearrow 1 \otimes \sigma \\
Y \otimes X \otimes Z \quad X \otimes Z \otimes Y \\
1 \otimes \sigma \downarrow \quad \downarrow \sigma \otimes 1 \\
Y \otimes Z \otimes X \quad Z \otimes X \otimes Y \\
\sigma \otimes 1 \nearrow \quad \nearrow 1 \otimes \sigma \\
Z \otimes Y \otimes X.
\end{array}
\]

And the last diagram gives the Yang-Baxter equation for the case \( X = Y = Z \):

\[
(\sigma \otimes 1)(1 \otimes \sigma)(\sigma \otimes 1) = (1 \otimes \sigma)(\sigma \otimes 1)(1 \otimes \sigma) : X^{\otimes 3} \to X^{\otimes 3}.
\]
1.4. Internal homomorphisms

Any pair \( X, Y \in Ob(C) \) defines a functor \( M_{X,Y} : C^{op} \to Set \), where

\[
M_{X,Y} : Z \to \text{Mor}(Z \otimes X, Y).
\]

Assume that \( M_{X,Y} \) is a representable functor. Then we have

1) the representing object \( \text{Hom}(X, Y) \) called an internal Hom, and

2) the morphism

\[
ev_{X,Y} : \text{Hom}(X, Y) \otimes X \to Y
\]

is called an evaluation map,

such that any morphism \( f : Z \otimes X \to Y \) can be represented by the composition:

\[
Z \otimes X \xrightarrow{\hat{f} \otimes 1} \text{Hom}(X, Y) \otimes X \xrightarrow{ev_{X,Y}} Y,
\]

for a unique morphism

\[
\hat{f} : Z \to \text{Hom}(X, Y).
\]

Hence we get an isomorphism

\[
\text{Mor}(Z, \text{Hom}(X, Y)) \xrightarrow{\text{j}} \text{Mor}(Z \otimes X, Y),
\]

\[
j \mapsto f = ev_{X,Y} \circ (\hat{f} \otimes 1).
\]

We will assume that the representation by interval \( \text{Hom}(X, Y) \) exists for all objects.

Let \( g : Y \to Y' \) be a morphism than the composition

\[
f : \text{Hom}(X, Y) \otimes X \xrightarrow{ev_{X,Y}} Y \xrightarrow{g} Y'
\]

defines the morphism \( \hat{f} \) which we denote by

\[
g_\ast : \text{Hom}(X, Y) \to \text{Hom}(X, Y').
\]
If \( g : X' \to X \), consider the composition

\[
f : \text{Hom}(X, Y) \otimes X' \xrightarrow{\otimes g} \text{Hom}(X, Y) \otimes X \xrightarrow{\text{ev}_X} Y
\]

we get the correspondence morphism

\[
g^* : \text{Hom}(X, Y) \to \text{Hom}(X', Y).
\]

Moreover, the composition

\[
\begin{align*}
(Hom(Y, Z) \otimes Hom(X, Y)) \otimes X & \xrightarrow{\alpha^{-1}} Hom(Y, Z) \otimes (Hom(X, Y) \otimes X) \\
1 \otimes \text{ev}_{X,Y} & \downarrow \\
Hom(Y, Z) \otimes Y & \xrightarrow{\text{ev}_Y} Z
\end{align*}
\]

defines a composition map

\[
\text{Hom}(Y, Z) \otimes \text{Hom}(X, Y) \to \text{Hom}(X, Z).
\]

Consider the following chain of isomorphisms

\[
\begin{align*}
\text{Mor}(T, \text{Hom}(Z, \text{Hom}(X, Y))) & \simeq \text{Mor}(T \otimes Z, \text{Hom}(X, Y)) \\
\downarrow f & \\
\text{Mor}((T \otimes Z) \otimes X, Y) & \simeq \text{Mor}(T \otimes (Z \otimes X), Y)
\end{align*}
\]

we obtain the isomorphism

\[
\text{Hom}(Z, \text{Hom}(X, Y)) \simeq \text{Hom}(Z \otimes X, Y).
\]

A unity object \( k \) yields the isomorphisms

\[
\eta^r : X \otimes k \cong X
\]

and an isomorphism

\[
\ell_X : X \cong \text{Hom}(k, X)
\]

corresponding to \( \eta \) by the isomorphism:

\[
\text{Mor}(X, \text{Hom}(k, X)) \simeq \text{Mor}(X \otimes k, X)
\]
By using isomorphisms $\ell$ one can consider the evaluation map

$$ev_{X,Y} : \text{Hom}(X,Y) \otimes X \to Y$$

as the composition

$$\text{Hom}(X,Y) \otimes X \xrightarrow{ev_{X,Y}} Y$$

$$\downarrow \ell_x \quad \downarrow \ell_y$$

$$\text{Hom}(X,Y) \otimes \text{Hom}(k,X) \to \text{Hom}(k,Y).$$

By considering the following composition

$$\text{Hom}(X_1,Y_1) \otimes \text{Hom}(X_2,Y_2) \otimes (X_1 \otimes X_2)$$

$$\downarrow \alpha^{-1}$$

$$\text{Hom}(X_1,Y_1) \otimes (\text{Hom}(X_2,Y_2) \otimes (X_1 \otimes X_2))$$

$$\downarrow 1 \otimes \alpha$$

$$\text{Hom}(X_1,Y_1) \otimes ((\text{Hom}(X_2,Y_2) \otimes X_1) \otimes X_2)$$

$$\downarrow 1 \otimes (\sigma \otimes 1)$$

$$\text{Hom}(X_1,Y_1) \otimes ((X_1 \otimes \text{Hom}(X_2,Y_2)) \otimes X_2)$$

$$\downarrow 1 \otimes \alpha^{-1}$$

$$(\text{Hom}(X_1,Y_1) \otimes (\text{Hom}(X_2,Y_2) \otimes X_2)$$

$$\downarrow \text{ev}_{x_1,y_1 \otimes \text{ev}_{x_2,y_2}}$$

$$Y_1 \otimes Y_2$$

we obtain a morphism

$$\text{Hom}(X_1,Y_1) \otimes \text{Hom}(X_2,Y_2) \to \text{Hom}(X_1 \otimes X_2, Y_1 \otimes Y_2).$$

### 1.5. Examples

1.5.1. Let $k$ be a commutative ring with unity. The category $\text{Mod}_k$ of finitely generated modules over $k$ is a tensor category with the usual tensor product and the obvious constraints. The internal $\text{Hom}(X,Y)$ in the category coincides with the module of $k$-homomorphisms $\text{Hom}(X,Y) = \text{Hom}_k(X,Y)$ and the evaluation map

$$ev_{X,Y} : f \otimes x \mapsto f(x), \quad f \in \text{Hom}_k(X,Y), \ x \in X,$$

is a usual one.
1.5.2. Let $A$ be a bialgebra over $k$ with a multiplication $\mu : A \otimes A \to A$, comultiplication or diagonal $s : A \to A \otimes A$, a unit $\eta : k \to A$, and a counit $\varepsilon : A \to k$. We will denote the category of left $A$-modules with $\text{Mor}(X, Y) = \text{Hom}_A(X, Y)$ as $\mathcal{A}\text{Mod}$. A tensor product in $\mathcal{A}\text{Mod}$ is a usual product over $k$, with obvious constraints and the following $A$-action

$$a(x \otimes y) = \sum a(1) x \otimes a(2) y,$$

where $x \in X, y \in Y, a \in A$, and

$$\Delta(a) = \sum a(1) \otimes a(2)$$

in the Sweedler notation.

1.5.3. Let $A$ be a $k$-algebra with a counit $\varepsilon : A \to k$ and comultiplication $\Delta : A \to A \otimes A$. Assume that $\varepsilon$ and $\Delta$ are $k$-algebra morphisms. Then we can define the tensor product as above, but with a new associativity constraint.

Note that any associativity constraint $\alpha(X, Y, Z)$ is completely determined by the element

$$\alpha = \alpha(A, A, A)(1 \otimes (1 \otimes 1)) \in (A \otimes A) \otimes A.$$

Indeed, we can identify arbitrary elements $x \in X, y \in Y, z \in Z$ with morphisms

$$\varphi_x : A \to X, \quad \varphi_y : A \to Y, \quad \varphi_z : A \to Z,$$

where, for example,

$$\varphi_x(a) = ax.$$

Because $\alpha$ is a natural isomorphism we have the following commutative diagram

$$\begin{array}{ccc}
X \otimes (Y \otimes Z) & \xrightarrow{\alpha(X,Y,Z)} & (X \otimes Y) \otimes Z \\
\uparrow \varphi_x \otimes (\varphi_y \otimes \varphi_z) & & \uparrow (\varphi_x \otimes \varphi_y) \otimes \varphi_z \\
A \otimes (A \otimes A) & \xrightarrow{\alpha(A,A,A)} & (A \otimes A) \otimes A.
\end{array}$$

Hence, $\alpha(X, Y, Z)(x \otimes (y \otimes z))$ is completely determined by $\alpha = \alpha(A, A, A)(1 \otimes (1 \otimes 1))$. 
Moreover, if $\alpha = \sum a' \otimes a'' \otimes a'''$, then

$$\alpha(X,Y,Z)(x \otimes (y \otimes z)) = \sum (a'x \otimes a''y) \otimes a'''z.$$  

From the condition that $\alpha$ is an $A$-homomorphism we obtain the following condition on $\alpha$ and $\Delta$:

$$\alpha \cdot ((1 \otimes \Delta)\Delta(a)) = ((\Delta \otimes 1)\Delta(a))\alpha,$$

for all $a \in A$.

The pentagon axiom yields the next condition on $\alpha$:

$$(1 \otimes 1 \otimes \Delta) \cdot (\Delta \otimes 1 \otimes 1)(\alpha) = (1 \otimes \alpha)(1 \otimes \Delta \otimes 1)(\alpha)(\alpha \otimes 1),$$

and the unity axiom gives

$$(1 \otimes \varepsilon \otimes 1)(\alpha) = 1.$$  

**Definition.** [2,4] A $k$-algebra $A$ with augmentation $\varepsilon : A \to k$ and diagonal $\Delta : A \to A \otimes A$ is called a quasibialgebra if

1. $\Delta$ and $\varepsilon$ are $k$-algebra morphisms,

2. there exists an element $\alpha \in A^{\otimes 3}$ such that conditions (1)–(3) hold.

**Theorem.** The category $A\text{Mod}$ of left $A$-modules over quasibialgebra $(A, \alpha)$ is a monoidal with associativity constraint defined by $\alpha$.

1.5.4. Consider a commutativity constraint in the category $A\text{Mod}$ of left $A$-modules over a bialgebra $A$. We identify elements $x \in X, y \in Y$ with morphisms $\varphi_x : A \to X, \varphi_y : A \to Y$ as above. By using naturality of $\sigma$ we get the following commutative diagram

$$
\begin{array}{ccc}
X \otimes Y & \xrightarrow{\sigma(X,Y)} & Y \otimes X \\
\uparrow \varphi_x \otimes \varphi_y & & \uparrow \varphi_y \otimes \varphi_x \\
A \otimes A & \xrightarrow{\sigma(A,A)} & A \otimes A.
\end{array}
$$
Hence,
\[ \sigma(X,Y)(x \otimes y) = \sigma \cdot (y \otimes x) \]
where \( \sigma = \sigma(A,A)(1 \otimes 1) \in A^{\otimes 2} \).

The condition that \( \sigma(X,Y) \) is an \( A \)-algebra homomorphism means
\[ \sigma \cdot \tau(\Delta(a)) = \Delta(a) \cdot \sigma, \quad (1) \]
for all \( a \in A \). The hexagon conditions yield
\[ (1 \otimes \Delta)(\sigma) = (\sigma \otimes 1)\sigma_{13}, \quad (2) \]
\[ (\Delta \otimes 1)(\sigma) = (1 \otimes \sigma)\sigma_{13}, \quad (3) \]
where \( \sigma_{13} = \sum \sigma' \otimes 1 \otimes \sigma'' \), if \( \sigma = \sum \sigma' \otimes \sigma'' \).

A bialgebra \( A \) is called a quasitriangular \([4]\) if \( A \) equipped with an universal element \( \sigma \in A^{\otimes 2} \), such that the conditions \((1)-(3)\) hold.

**Theorem.** Let \( (A,\sigma) \) be a quasitriangular bialgebra. Then the category \( _A\text{Mod} \) is a braided tensor category with braiding defined by the element \( \sigma \in A^{\otimes 2} \). \( _A\text{Mod} \) is a tensor category iff \( \sigma \cdot \tau(\sigma) = 1 \), i.e. when \( (A,\sigma) \) is a triangular bialgebra.

For the case where \( A = H \) is a quasitriangular Hopf algebra \([4]\) internal \( \text{Hom}(X,Y) \) coincides with usual \( \text{Hom}_k(X,Y) \) equipped with the following \( H \)-action
\[ h(f)(x) = \sum h_{(1)} f(S(h_{(2)}) x), \]
where \( S : H \to H \) is an antipode and \( x \in X, h \in H, f \in \text{Hom}_k(X,Y) \).

1.5.5. Let \( G \) be a finite group and \( A = k[G] \) be a group algebra. The \( _A\text{Mod} \) is a category \( \text{Mod}_G \) of left \( G \)-modules over \( k \).

A braiding \( \sigma \in k[G] \otimes k[G] \) (we take \( \alpha = 1 \)) is an element
\[ \sigma = \sum_{x,y \in G} \sigma(x,y)x \otimes y, \]
such that the above conditions \((1)-(3)\) hold.
(1) We have
\[ \Delta(g)\sigma = \sum_{x,y} \sigma(x,y)gx \otimes gy, \]
\[ \sigma\Delta(g) = \sum_{x,y} \sigma(x,y)xg \otimes yg, \]
Hence,
\[ \sigma(gxg^{-1},gyg^{-1}) = \sigma(x,y), \] (1)
for all \( x, y, g \in G \).

(2) Considering the 1-st hexagon axiom we get
\[ (1 \otimes \Delta)(\sigma) = \sum \sigma(x,y)x \otimes y \otimes y, \]
\[ (\sigma \otimes 1)\sigma_{13} = \sum \sigma(xf^{-1},y)\sigma(f,g)x \otimes y \otimes g. \]
Hence,
\[ \sum_{f \in G} \sigma(xf^{-1},y)\sigma(f,g) = 0 \] (2)
if \( y \neq g \), and
\[ \sum_{f \in G} \sigma(xf^{-1},y)\sigma(f,y) = \sigma(x,y). \]

(3) From the 2-nd hexagon axiom we have
\[ \sum_{g \in G} \sigma(x,yg^{-1})\sigma(f,g) = 0, \] (3)
when \( f \neq x \), and
\[ \sum_{g \in G} \sigma(x,yg^{-1})\sigma(x,g) = \sigma(x,y). \]

1.5.6. Let \( G \) be a finite group and \( A = k(G) = \text{Map}(G,k) \) be a bialgebra of \( k \)-valued functions on \( G \) with

i) multiplication
\[ (f_1 \cdot f_2)(x) = f_1(x)f_2(x), \]

ii) comultiplication
\[ \Delta f(x,y) = f(xy), \]
iii) antipode
\[ S(f)(x) = f(x^{-1}), \]

iv) unit \( \eta(1)(x) = 1 \), and counit \( \varepsilon(f) = f(1) \).

The algebra \( A \) generated “\( \delta \)-functions” \( \delta_x \in A, x \in G, \) such that \( \delta_x(y) = 0 \), if \( x \neq y \), and \( \delta_x(x) = 1 \). In terms of the \( \delta \)-functions we have

(1) \( \delta_x \cdot \delta_y = 0 \), when \( x \neq y \), and \( \delta_x \cdot \delta_x = \delta_x \),

(2) \( \Delta(\delta_x) = \sum_{g \in G} \delta_g \otimes \delta_{g^{-1}x} \),

(3) \( \varepsilon(\delta_x) = 0 \), if \( x \neq 1 \), and \( \varepsilon(\delta_1) = 1 \).

The category \( AMod \) coincides with the category of \( G \)-graded \( k \)-modules
\[ X = \sum_{g \in G} X_g \]
with the following \( A \)-action:
\[ \delta_g \left( \sum_{h \in G} x_h \right) = x_g. \]

Morphisms in the category are \( G \)-graded \( k \)-homomorphisms. Tensor products \( X \otimes Y \) are the usual \( G \)-graded products with the grading
\[ (X \otimes Y)_g = \sum_{h \in G} X_h \otimes Y_{h^{-1}g}. \]

Internal Hom\( (X, Y) \) also coincides with a module of \( k \)-homomorphisms, and has additional \( A \)-structure:
\[ \text{Hom}(X, Y) = \sum_{g \in G} \text{Hom}_g(X, Y). \]

where
\[ \text{Hom}_g(X, Y) = \{ f \in \text{Hom}_k(X, Y) | f(X_h) \subset Y_{gh}, \forall h \in G \}. \]

A braiding \( \sigma \) is defined by an element
\[ \sigma = \sum_{x, y \in G} \sigma(x, y) \delta_x \otimes \delta_y. \]
The condition 1.5.4. (1) holds if and only if $G$ is an Abelian group.

The first hexagon condition gives

\[(\Delta \otimes 1)(\sigma) = \sum \sigma(x, y)\delta_x \otimes \delta_{x^{-1}x} \otimes \delta_y,\]
\[(1 \otimes \sigma)\sigma_{13} = \sum \sigma(x^{-1}x, y)\sigma(z, y)\delta_z \otimes \delta_{z^{-1}x} \otimes \delta_y.\]

Hence,

\[\sigma(ab, c) = \sigma(a, c)\sigma(b, c),\]  \hspace{1cm} (2)

for all $a, b, c \in G$.

From the second hexagon axiom we get

\[\sigma(c, ab) = \sigma(c, a)\sigma(c, b),\]  \hspace{1cm} (3)

for all $a, b, c \in G$.

Therefore, any braiding $\sigma$ for the category is given by the group bihomomorphism

\[\sigma : G \times G \to \mathcal{U}(k),\]

where $\mathcal{U}(k)$ is the unit group of the ring $k$.

The braiding $\sigma$ is a symmetry if and only if the following multiplicative skew-symmetry property

\[\sigma(a, b)\sigma(b, a) = 1,\]  \hspace{1cm} (4)

holds.

2  A symmetrization and differential approximations.

2.1. Let $C$ be a braided tensor category, and $A$ be an algebra in the category with multiplication $\mu_A : A \otimes A \to A$, and unit $\eta : k \to A$.

We say that $A$ is a $\sigma$-commutative algebra if the following diagram

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{\sigma(A, A)} & A \otimes A \\
\mu_A \setminus & & \setminus \mu_A \\
\downarrow & & \downarrow \\
A & & A
\end{array}
\]

commutes.
2.2. Example

Take the category of $G$-graded modules (1.5.6). An algebra $A$ in the category is a $G$-graded algebra:

$$A = \sum_{g \in G} A_g, \quad A_g A_h \subset A_{gh}.$$  

The $\sigma$-commutativity condition of $A$ implies

$$a_g a_h = \sigma(g, h) a_h a_g, \quad (1)$$

for all $a_g \in A_g, a_h \in A_h$.

Hence, if we wish to get non-trivial $\sigma$-commutative algebras in the category, we must require to $G$ be an Abelian group. Then (1) defines $\sigma$-commutative algebras or algebras with controlled non-commutative multiplication.

There is another possibility. One can introduce an action of some new group $S$ (or Hopf algebra) to compare elements $a_g a_h$ and $a_h a_g$. It is equivalent to an action of a smash-product $k[S] * k(G)$, and yields a "rich" calculus (see [7]).

2.3. We fix a $\sigma$-commutative algebra $A$ and consider $A - A$ bimodules over $A$. Let $X$ be such a bimodule with left and right multiplications

$$\mu^l : A \otimes X \to X, \quad \text{and} \quad \mu^r : X \otimes A \to X$$

respectively.

Say that $X$ is a $\sigma$-symmetric bimodule of the following diagrams

$$\mu^l \downarrow \mu^r, \quad \mu^r \downarrow \mu^l$$

are commutative.

For arbitrary $A - A$ bimodule $X$ we define "$\sigma$-symmetric part" $X_\sigma \subset X$ in the following way

$$X_\sigma = \left\{ x \in X \mid (\mu^r - \mu^l \circ \sigma(A, X))(x \otimes a) = 0, \quad (\mu^l - \mu^r \circ \sigma(A, X))(a \otimes x) = 0, \quad \forall a \in A \right\}.$$
Lemma. For any $A - A$ bimodule $X$ over a $\sigma$-commutative algebra $A$, the set $X_\sigma \subset X$ is a $\sigma$-symmetric bimodule.

Proof. To simplify our diagrams we suppose that the associativity constraint $\alpha$ is trivial.

Let $x \in X_\sigma, b \in A$. We show how one can check, for example, that

$$(\mu^* - \mu^1 \circ \sigma(X, A))(bx \otimes a) = 0,$$

for any $a \in A$.

We have

$$\mu^*(bx \otimes a) = \mu^*(\mu^1 \otimes 1)(b \otimes x \otimes a) = \mu^1(1 \otimes \mu^*)(b \otimes x \otimes a) =$$

$$= \mu^1(1 \otimes \mu^1 \circ \sigma(X, A)(b \otimes x \otimes a)) = \mu^1(1 \otimes \mu^1)(1 \otimes \sigma(X, A))(b \otimes x \otimes a) =$$

$$= \mu^1(\mu_A \otimes 1)(1 \otimes \sigma(X, A))(b \otimes x \otimes a),$$

and

$$(\mu^1 \circ \sigma(X, A))(bx \otimes a) = \mu^1 \circ \sigma(X, A) \circ (\mu^1 \otimes 1)(b \otimes x \otimes a).$$

To see that

$$\mu^1 \circ \sigma(X, A) \circ (\mu^1 \otimes 1) = \mu^1 \circ (\mu_A \otimes 1)(1 \otimes \sigma(X, A)), \quad (1)$$

we use the hexagon axiom and consider the following commutative diagram

$$\begin{array}{ccc}
A \otimes A \otimes X & \xrightarrow{1 \otimes \sigma} & A \otimes A \otimes X \\
\| & & \| \\
A \otimes X \otimes A & \xrightarrow{\sigma} & A \otimes A \otimes X \\
\| & & \| \\
X \otimes A & \xrightarrow{\sigma} & A \otimes X \\
\| & & \| \\
& \xrightarrow{\mu^1} & X
\end{array}$$

The exterior contour of the diagram yields formula (1).

2.4. Now we consider the factor bimodule $X/X_\sigma$, and define a bimodule $X_\sigma^{(1)} \subset X$ as the inverse image of the $\sigma$-symmetric bimodule $(X/X_\sigma)_{\sigma} \subset X/X_\sigma$ with respect to the natural projection $X \to X/X_\sigma$. If we apply the procedure to the bimodule $X/X_\sigma^{(1)}$ we get a bimodule $X_\sigma^{(2)}$ and so on.
Continuing in this way we get a filtration of the bimodule $X$ by bimodules $X^{(i)}$:

$$0 = X^{(-1)} \subset X^{(0)} \subset X^{(1)} \subset \cdots \subset X^{(*)} = \bigcup_{i \geq 0} X^{(i)} \subset X,$$

where by definition $X^{(i+1)}$ is an inverse image of $(X/X^{(i)})_{\sigma}$.

We will call a bimodule $X^{(*)}$ a differential approximation of the $A - A$ bimodule $X$.

2.5. Theorem. A graded object

$$\text{Gr}(X) = \sum_{i \geq 0} \text{Gr}_i(X),$$

where

$$\text{Gr}_i(X) = X^{(i)}/X^{(i-1)},$$

related to any $A - A$ bimodule $X$ over a $\sigma$-commutative algebra $A$ is a $\sigma$-symmetric $A - A$ bimodule.

2.6. Examples

2.6.1. Let $\text{HMod}$ be the category of left modules over the quasitriangular Hopf algebra $(H, \sigma)$ with the braiding defined by $\sigma \in H^{\otimes 2}$ (see 1.5.4), $\sigma = \Sigma \sigma' \otimes \sigma''$. Let $A$ be an algebra in $\text{HMod}$. To define the $\sigma$-commutativity we get

$$a \# b = (\mu \circ \sigma)(a \otimes b) = \sum (\sigma'a)(\sigma''b).$$

Then $A$ is a $\sigma$-commutative algebra iff

$$a \cdot b = b \# a.$$

If $X$ is an $A - A$ bimodule we get

$$a \# x = \sum (\sigma'a) \cdot (\sigma''x),$$

$$x \# a = \sum (\sigma'x) \cdot (\sigma''a).$$
Then $X$ is a $\sigma$-symmetric bimodule iff the following conditions hold

\begin{align*}
x \cdot a &= a x \sigma, \\
a \cdot x &= x \sigma a.
\end{align*}

(1)

For any $A - A$ bimodule $X$ we define two types of morphisms

\[ \delta^l_a(x) = ax - x \sigma a, \]

and

\[ \delta^r_a(x) = xa - a \sigma x. \]

Then by definition we have

\[ X_\sigma = \{ x \in X \mid \delta^l_a(x) = \delta^r_a(x) = 0, \forall a \in A \}. \]

and

\[ X^{(i+1)}_\sigma = \{ x \in X \mid \delta^l_a(x) \in X^{(i)}_\sigma, \delta^r_a(x) \in X^{(i)}_\sigma, \forall a \in A \}. \]

2.6.2. In the category of $G$-graded modules with commutative group $G$, the $A - A$ bimodule $X$ is a $G$-graded bimodule

\[ X = \sum_{g \in G} X_g, \quad A_g X_h \subset X_{gh}, \quad X_h A_g \subset X_{hg}. \]

$X$ is $\sigma$-symmetric if and only if

\[ a_g \cdot x_h = \sigma(g, h)x_h a_g \]

for all $x_h \in X_h$, $a_g \in A_g$, $g, h \in G$. The $\sigma$-commutative part $X_\sigma = \sum_{g \in G} (X_\sigma)_g$ is

\[ (X_\sigma)_g = \{ x_g \in X_g \mid a_h x_g = \sigma(h, g)x_g a_h, x_g a_h = \sigma(g, h)a_h x_g \}. \]

If we denote by $\delta^l_{ah}$ and $\delta^r_{ah}$ the following maps

\[ \delta^l_{ah}(x_g) = a_h x_y - \sigma(h, g)x_g a_h, \]

\[ \delta^r_{ah}(x_h) = x_g a_h - \sigma(g, h)a_h x_g, \]

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then

\[(X_\sigma)_g = \{ x_g \in X_g | \delta_a^l(x_g) = \delta_a^r(x_g) = 0, \forall a \in A, h \in G \}, \]

and

\[X_\sigma^{(i)} = \sum_{g \in G} (X_\sigma^{(i)})_g, \]

where

\[(X_\sigma^{(i)})_g = \{ x_g \in X_g | \delta_a^l(x_g) \in X_\sigma^{(i-1)}, \delta_a^r(x_g) \in X_\sigma^{(i-1)}, \forall a \in A, h \in G \}. \]

3 Differential operators

In this section we apply the above procedure to bimodules of internal homomorphisms and obtain modules of differential operators in braided tensor category.

3.1. Let \( A \) be a \( \sigma \)-commutative algebra and let \( X, Y \) be left \( A \)-modules. The following composition of morphisms

\[(A \otimes \text{Hom}(X, Y)) \otimes X \xrightarrow{\alpha^{-1}} A \otimes (\text{Hom}(X, Y) \otimes X) \]
\[\downarrow \otimes \text{ev}_{x,y} \]
\[A \otimes Y \xrightarrow{\mu} Y. \]

defines a left \( A \)-module structure on the internal \( \text{Hom}(X, Y) \):

\[l_A : A \otimes \text{Hom}(X, Y) \rightarrow \text{Hom}(X, Y). \]

Similarly, we get a right \( A \)-module structure

\[r_A : \text{Hom}(X, Y) \otimes A \rightarrow \text{Hom}(X, Y) \]

from the following composition

\[(\text{Hom}(X, Y) \otimes A) \otimes X \xrightarrow{\alpha^{-1}} \text{Hom}(X, Y) \otimes (A \otimes X) \]
\[\downarrow \otimes \mu \]
\[\text{Hom}(X, Y) \otimes X \xrightarrow{\text{ev}_{x,y}} Y. \]
Lemma. Morphisms $l_A$ and $r_A$ define an $A - A$ bimodule structure on an internal $\text{Hom}(X, Y)$.

3.2. Example

Let $C = \text{Mod}_k$, $\alpha = 1$, then morphisms

$l_A$ and $r_A$ are

$l_A(a \otimes f)(x) = af(x),

r_A(f \otimes a)(x) = f(ax),$

for all $x \in X, a \in A, f \in \text{Hom}_k(X, Y)$.

3.3. Definition. The differential approximations $(\text{Hom}(X, Y))^{(n)}_\sigma$ of the $A_A$ bimodule $\text{Hom}(X, Y)$ will be called modules of $\sigma$-differential operators of order $\leq n$ and denoted by $\text{Diff}_n^{\sigma}(X, Y)$.

Recalling the definition of differential approximations, we can define modules of differential operators in direct way.

Denote by

$\delta_A^l : A \otimes \text{Hom}(X, Y) \to \text{Hom}(X, Y),$

and

$\delta_A^r : \text{Hom}(X, Y) \otimes A \to \text{Hom}(X, Y),$

the following morphisms

$\delta_A^l = l_A - r_A \circ \sigma(A, \text{Hom}(X, Y)),

\delta_A^r = r_A - l_A \circ \sigma(\text{Hom}(X, Y), A).$

Then set

$\text{Diff}^{\sigma}_n(X, Y) = \text{Hom}^{\sigma}_A(X, Y),$

where $\text{Hom}^{\sigma}_A(X, Y)(= \text{Hom}(X, Y)_\sigma)$ is a module of $\sigma$-homomorphisms, i.e. a maximal submodule with the following conditions

$\delta_A^l(A \otimes \text{Hom}^{\sigma}_A(X, Y)) = 0,

\delta_A^r(\text{Hom}^{\sigma}_A(X, Y) \otimes A) = 0.$

To define modules $\text{Diff}^{\sigma}_n(X, Y), n = 1, 2, \ldots$, we use induction on $n$ and define $\text{Diff}^{\sigma}_{n+1}(X, Y) \subset \text{Hom}(X, Y)$ as a maximal submodule such that

$\delta_A^l(A \otimes \text{Diff}^{\sigma}_{n+1}(X, Y)) \subset \text{Diff}^{\sigma}_n(X, Y)$  (1)
\[ \delta^\sigma_n (\text{Diff}^\sigma_n (X, Y) \otimes A) \subset \text{Diff}^\sigma_n (X, Y) \]  
\[ (2) \]

Note that we need only one condition (1) if the braiding \( \sigma \) is a symmetry.

We denote the graded object \( \text{Gr}(\text{Hom}(X, Y)) \) by

\[ \text{Smbl}^\sigma (X, Y) = \sum_{i \geq 0} \text{Smbl}^\sigma_i (X, Y), \]

\[ \text{Smbl}^\sigma_i (X, Y) = \text{Diff}^\sigma_i (X, Y)/\text{Diff}^\sigma_{i-1} (X, Y), \]

and call \( \text{Smbl}^\sigma (X, Y) \) a symbol module.

3.4. Theorem. (1) The internal composition

\[ \text{Hom}(Y, Z) \otimes \text{Hom}(X, Y) \rightarrow \text{Hom}(X, Z) \]

generates a composition of differential operators

\[ \text{Diff}^\sigma_i (Y, Z) \otimes \text{Diff}^\sigma_j (X, Y) \rightarrow \text{Diff}^\sigma_{i+j} (X, Z). \]

(2) The module of \( \sigma \)-differential operators \( \text{Diff}^\sigma (A, A) \) is an algebra in the category \( C \).

(3) The symbol modules \( \text{Smbl}^\sigma (X, Y) \) are \( \sigma \)-symmetric \( A - A \) bimodules, and the symbol algebra \( \text{Smbl}^\sigma (A, A) \) is a \( \sigma \)-commutative algebra.

3.5. \( \sigma \)-Poisson structure

Using the \( \sigma \)-commutative algebra structure in \( \text{Smbl}^\sigma (A, A) \) we can usually define some additional structure. For this purpose we take two \( \sigma \)-differential operators, \( f \in \text{Diff}^\sigma_i (A, A) \) and \( g \in \text{Diff}^\sigma_j (A, A) \) with symbols \( \bar{f} \in \text{Smbl}^\sigma_i (A, A), \)

\( \bar{g} \in \text{Smbl}^\sigma_j (A, A) \) respectively. Then the \( \sigma \)-commutator

\[ [f, g]^\sigma = \mu^D (f \otimes g - \sigma(\text{Hom}(A, A), \text{Hom}(A, A)(g \otimes f)), \]

where

\[ \mu^D : \text{Diff}^\sigma_i (A, A) \otimes \text{Diff}^\sigma_j (A, A) \rightarrow \text{Diff}^\sigma_{i+j} (A, A) \]

is the multiplication morphism, has order \( \leq i + j - 1 \).

Hence, one can define the symbol

\[ [\bar{f}, \bar{g}]^\sigma \in \text{Smbl}^\sigma_{i+j-1} (A, A) \]
which really depends only on $\bar{f}$ and $\bar{g}$. Therefore we obtain a Poisson bracket

$$[\cdot, \cdot]^\sigma : \text{Smbl}_{i}^\sigma(A, A) \otimes \text{Smbl}_{j}^\sigma(A, A) \to \text{Smbl}_{i+j-1}^\sigma(A, A).$$

Like the usual Poisson bracket the Poisson bracket $[\cdot, \cdot]^\sigma$ is a 1-st order $\sigma$-differential operator over the $\sigma$-commutative algebra $\text{Smbl}_{i}^\sigma(A, A)$ converting $\text{Smbl}_{i}^\sigma(A, A)$ into a $\sigma$-Lie algebra.

### 3.6. Examples

#### 3.6.1.

Let $(H, \sigma)$ be a quasitriangular Hopf algebra and $H\text{Mod}$ be the category of left $H$-modules with trivial associativity constraint, and with braiding defined by $\sigma$ (1.5.4). Using example 2.6.1 we introduce two types of morphisms in $\text{Hom}(X, Y)$:

$$\delta_a^l(f)(x) = af(x) - \sum \sigma'(1) f(S(\sigma'(2))\sigma''(a)x),$$

$$\delta_a^r(f)(x) = f(ax) - \sum \sigma'(a)\sigma''(1) f(S(\sigma''(2)x)),$$

for all $a \in A$, $x \in X$, $f \in \text{Hom}(X, Y)$.

We have

$$\text{Hom}^\sigma(X, Y) = \text{Diff}^\sigma_0(X, Y) = \{ f \in \text{Hom}(X, Y) | \delta_a^l(f) = \delta_a^r(f) = 0, \forall a \in A \},$$

and

$$\text{Diff}^\sigma_{i+1}(X, Y) = \left\{ f \in \text{Hom}(X, Y) \left| \begin{array}{c}
\delta_a^l(f) \in \text{Diff}^\sigma_{i-1}(X, Y), \\
\delta_a^r(f) \in \text{Diff}^\sigma_{i-1}(X, Y), \forall a \in A.
\end{array} \right. \right\}$$

#### 3.6.2.

In the category of $G$-graded modules (2.6.2) we have

$$\delta_a^l(f)(x) = a_s f(x) - \sigma(s, t) f(a_s x),$$

$$\delta_a^r(f)(x) = f(a \cdot x) - \sigma(t, s) a_t f(x),$$

where $f \in \text{Hom}(X, Y)_t$, and $a_s \in A_s$, $s, t \in G$.

### 4 Quantizations

Now we realize our image of quantizations as "controlled variations" of commutativity by the following definition.
4.1. Definition. A quantization $Q$ of a monoidal category $C$ is a natural isomorphism of the tensor product functor

$$Q(X,Y) : X \otimes Y \to X \otimes Y$$

such that the following diagrams

\begin{align*}
(1) \quad X \otimes (Y \otimes Z) \xrightarrow{\Delta \otimes Q(Y,Z)} X \otimes (Y \otimes Z) \xrightarrow{Q(X,Y \otimes Z)} X \otimes (Y \otimes Z) \\
\quad \downarrow \alpha(X,Y,Z) \quad \downarrow \alpha(X,Y,Z) \\
(X \otimes Y) \otimes Z \xrightarrow{Q(X,Y) \otimes 1} (X \otimes Y) \otimes Z \xrightarrow{Q(X \otimes Y,Z)} (X \otimes Y) \otimes Z
\end{align*}

and

\begin{align*}
(2) \quad X \otimes k \xrightarrow{Q(X,k)} X \otimes k \\
\quad \eta^\flat \downarrow \quad \eta^\flat \downarrow \\
\quad X \otimes k \xrightarrow{Q(k,X)} k \otimes X \\
\quad \eta^\flat \downarrow \quad \eta^\flat \downarrow \\
\quad X \otimes k \xrightarrow{Q(k,X)} k \otimes X
\end{align*}

all commute.

4.2. Consider a braiding $\sigma(X,Y) : X \otimes Y \to Y \otimes X$ of the category.

Let us define a quantization $\sigma^q$ of the braiding $\sigma$ from the following commutative diagram

\begin{align*}
X \otimes Y \quad \xrightarrow{\sigma(X,Y)} \\ \downarrow Q(X,Y) \quad \downarrow Q(Y,X) \\
X \otimes Y \quad \xrightarrow{\sigma(X,Y)} \\
\end{align*}

Theorem. Let $Q$ be a quantization of a braided tensor category with a braiding $\sigma(X,Y)$, then

$$\sigma^q(X,Y) = Q^{-1}(Y,X)\sigma(X,Y)Q(X,Y)$$

is a braiding of the category.

Proof. We get the commutativity of the 1-st hexagon diagram from the viewing of the following commutative "double-hexagon" diagram.

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4.3. By using the following commutative diagram

\[
\begin{array}{ccc}
\text{Mor}(Z, \text{Hom}(X,Y)) & \xrightarrow{\phi} & \text{Mor}(Z \otimes X,Y) \\
\uparrow Q_h & & \uparrow (Q^{-1})^* \\
\text{Mor}(Z, \text{Hom}(X,Y)) & \xrightarrow{\phi} & \text{Mor}(Z \otimes X,Y)
\end{array}
\]

where \((Q^{-1})^*(f) = f \circ Q^{-1}\), we can define natural isomorphism

\[Q_h(X,Y): \text{Hom}(X,Y) \rightarrow \text{Hom}(X,Y)\,.
\]

In terms of evaluation maps \(Q_h(X,Y)\) may be defined by means of the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(X,Y) \otimes X & \xrightarrow{\text{ev}_{X,Y}} & Y \\
\uparrow Q^{-1}(\text{Hom}(X,Y),X) & & \uparrow \text{ev}_{X,Y} \\
\text{Hom}(X,Y) \otimes X & \xrightarrow{Q_h(X,Y) \otimes 1} & \text{Hom}(X,Y) \otimes X
\end{array}
\]

4.4. For any algebra \((A,\mu)\) with multiplication \(\mu : A \otimes A \rightarrow A\) in the category we define a quantization \((A_q,\mu_q)\) as the object \(A_q = A\) with new multiplication

\[\mu_q = \mu \circ Q(A,A)\,.
\]

Moreover, for any left \(A\)-module \((X,\mu_X)\), where \(\mu_X : A \otimes X \rightarrow X\) is a multiplication in \(X\), we define a quantization \((X_q,\mu_X)\), where \(X_q = X, \mu_X^q = \mu_0Q(A,X)\). The procedure we apply to right \(A\)-modules and \(A-A\) bimodules.

**Theorem.** Let \(A\) be an algebra in a braid tensor category with a quantization \(Q\) and let \(X\) be a left (right, bimodule) \(A\)-module in the category. Then

1) \((A_q,\mu_q^A)\) is an algebra, and \((X_q,\mu_X^q)\) is an \(A_q\)-module.

2) if \(A\) is a \(\sigma\)-commutative algebra, then \(A_q\) is a \(\sigma^q\)-commutative algebra and if \(X\) is a \(\sigma\)-symmetric \(A-A\)-bimodule then \(X_q\) is a \(\sigma^q\)-symmetric \(A_q-A_q\)-bimodule.
Proof. We consider the last part of the theorem only and the following commutative diagram

\[
\begin{array}{ccc}
X \otimes A & \xrightarrow{\sigma^*} & A \otimes X \\
\mu_q^r \downarrow & & \downarrow \mu_q^l \\
Q \downarrow & & \downarrow Q \\
X \otimes A & \xrightarrow{\sigma} & X \otimes A.
\end{array}
\]

4.5. Now we apply the results of 4.4. to \( A \rightleftharpoons A \) bimodules \( \text{Hom}(X, Y) \), where \( X, Y \) are left \( A \)-modules over \( \sigma \)-commutative algebra \( A \). After quantization we get \( A_q \rightleftharpoons A_q \) bimodule \( \text{Hom}(X_q, Y_q) \). Morphism

\[ Q^{-1}_h(X, Y) : \text{Hom}(X, Y) \to \text{Hom}(X, Y) \]

is a morphism of \( A \rightleftharpoons A \)-bimodule \( \text{Hom}(X, Y) \) on \( A_q \rightleftharpoons A_q \) bimodule \( \text{Hom}(X_q, Y_q) \).

Moreover \( Q^{-1}_h \) preserves differential approximations of the bimodules and defines isomorphisms of modules of differential operators

\[ Q^{-1}_h(X, Y) : \text{Diff}^\sigma_*(X, Y) \to \text{Diff}^\sigma_*(X_q, Y_q). \] (1)

Theorem. 1) Any quantization \( Q \) defines natural isomorphisms \( Q^{-1}_h \) (1),

2) \( Q^{-1}_h \) preserves filtration (2.4) and defines a “quasiclassical” approximation

\[ \text{Smbl}(Q^{-1}_h) : \text{Smbl}^\sigma_*(X, Y) \to \text{Smbl}^\sigma_*(X, Y), \]

3) In the case \( X = Y = A \) \( \text{Smbl}(Q^{-1}_h) \) is a morphism of Poisson algebras.

4.6. Definition. A quantization \( Q \) is called a quantum symmetry if \( \sigma^q = \sigma \).

4.7. Theorem. Quantum symmetries yield isomorphisms of modules of \( \sigma \)-differential operators.
4.8. Examples

4.8.1. Consider $H_{\text{Mod}}$ over quasitriangular Hopf algebra $(H, \sigma)$. Any quantization $Q$ is defined by an invertible element $q \in H^{\otimes 2}$ (see 1.5.3) such that

i) $Q\Delta(h) = \Delta(h)q$, $\forall h \in H$,

ii) $(\Delta \otimes 1)(q)(q \otimes 1) = (1 \otimes \Delta)(q)(1 \otimes q),$

iii) $\varepsilon(q) = 1$.

Then $\sigma^q = q\sigma(q^{-1})$ is a new braiding.

We should note that for the case of universal enveloping algebra $H = \mathcal{U}(\mathfrak{g})$, and $\sigma = \tau$, $\sigma^q$ coincides with the element $R$ in the Drinfeld definition of quasitriangular algebras [2, 4] and $q$ coincides with the function $F$ [3].

4.8.2. For the case $\mathfrak{g} = \mathbb{R}^n$ elements $q \in H^{\otimes 2}$ give the Moayl-Vey quantization [12].

4.8.3. Let $H = k(G)$, where $G$ is a finite group (see 2.6.2).

Then

$$q = \sum_{x,y \in G} q(x,y)\delta_x \otimes \delta_y$$

and

$$q : G \times G \to \mathcal{U}(k)$$

is a multiplicator.

References


