

Deformation Theory of Sandwiched Singularities

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Abstract

In this article we describe the deformation theory of sandwiched singularities in terms of δ -constant deformations of plane curves, and a divisor of points on it. This leads to an immediate understanding of the smoothings of sandwiched singularities in terms of *pictures*: certain configurations of points and curves with only d -fold points in the plane. The topology of a smoothing can be described completely in terms of the associated picture.

Introduction

A sandwiched singularity is, by definition, a normal surface singularity which admits a birational map to $(\mathbb{C}^2, 0)$. They therefore belong to the simplest class of rational surface singularities. A surprisingly large number of geometrically relevant singularities are sandwiched, for example cyclic quotient singularities, or more generally rational singularities with reduced fundamental cycle. Sandwiched singularities were studied by various authors like Zariski [35], Lipman [22], Hironaka [13] and Spivakovsky [28], who also seems to have invented the name. In this article we study *deformations* of sandwiched singularities. Our main result is a geometric interpretation of deformations of sandwiched singularities, the *picture method*, which we will describe now.

Let \tilde{Z} be obtained from $Z = (\mathbb{C}^2, 0)$ by a finite sequence of blow-ups. Any sandwiched singularity can be obtained from some \tilde{Z} by *blowing down* the non (-1) curves. Any \tilde{Z} as above can be obtained as the total space of a (not necessarily minimal) embedded resolution of a plane curve singularity C . We therefore can assign to every sandwiched singularity a so-called *decorated curve* (C, l) . Here l is a function, assigning to each branch C_i of C a number $l(i)$ which expresses how non-minimal the embedded resolution of C is. For a precise definition, see (1.3). Conversely, any decorated curve (C, l) gives rise to a sandwiched singularity $X(C, l)$. A representation of a singularity X as an $X(C, l)$ we call a *sandwiched representation*. A sandwiched representation is not given naturally, and in fact, it usually happens that there are many different ways to get a sandwiched representation for a given singularity.

We can interpret the function l as defining a subscheme of length $l(i)$ on the normalisation of each branch C_i of C . We define the notion of one-parameter deformation

of decorated curve in (4.2) as a δ -constant deformation of C , and a deformation of the subscheme l which satisfies a simple condition. The main result of this article (4.4) could be stated as:

Any one-parameter deformation of a decorated curve (C, l) gives rise to a one-parameter deformation of the corresponding sandwiched singularity $X(C, l)$, and all one-parameter deformations of $X(C, l)$ can be obtained this way.

If the general fibre of the deformation of the subscheme is *reduced* (from which it follows that the general fiber of the deformation of the curve only has d -fold points), then the corresponding deformation of the sandwiched singularity is a *smoothing*. Therefore, by looking at special configurations of curves in the plane, we can construct many interesting smoothings components. The Milnor fibre can be understood completely from the associated picture. We describe H_1 , H_2 , the intersection form and in some cases π_1 of the Milnor fibre.

In order to prove (4.4) we use the so-called *projection method*, which we review in the appendix. Consider a projection Y in \mathbb{C}^{n+1} of a normal CM singularity X of dimension n , in such a way that X can be obtained as normalisation of Y . If one considers so-called R.C. deformations of Y , then the total space can be simultaneously normalised. Moreover, any deformation of X is obtained from an R.C. deformation of Y . In the second section we therefore consider a very special projection of the surface $X(C, l)$ into \mathbb{C}^3 . The equations of these projected sandwiched singularities are ridiculously simple: they are just of the form:

$$zf(x, y) = g(x, y)$$

Here $f(x, y) = 0$ is a defining equation for C , and the vanishing order of the function $g(x, y)$ on the normalisation of branch C_i of C is related to the number $l(i)$. The main point concerning deformations of sandwiched singularities is, as proved in section 3, that they can all be obtained from normalising R.C. deformations of Y of the form (S is some parameter space):

$$zf_S(x, y) - g_S(x, y) = 0$$

Here $f_S(x, y)$ defines a δ -constant deformation of C . So we have a ridiculously simple equation for a projection of any *deformation* of a sandwiched singularity as well! This immediately leads to the picture method, see section 4.

The structure of the paper is as follows. In the first section we review some notions related to sandwiched singularities. In the second section we consider the very special projection of the surface $X(C, l)$ into \mathbb{C}^3 . In the third and fourth section we use our theory of R.C. deformations to establish the picture method. In section 5 we give a more detailed account of the topological aspects of the situation. In section 6 we give examples and applications. Finally, there is an appendix reviewing the most important aspects of R.C. deformations.

What is missing in this paper is a discussion of *Kollár's conjectures*. According to these conjectures, smoothings of rational surface singularities should correspond to so-called *P-resolutions*. The existence of P-resolutions depends on the finite

generation of the relative canonical ring of a smoothing. Using our picture method, we can prove that this finite generation is equivalent to the finite generation of the *symbolic algebra* of the curve in $(\mathbb{C}^3, 0)$, swept out by the points of the picture deformation, counted with certain multiplicities. Hopefully this can be used to shed light on Kollár's conjectures for sandwiched singularities. These matters will be discussed in a future paper.

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§1 Sandwiched Singularities

In this paragraph we review the basic construction and properties of a special class of surface singularities called *sandwiched singularities*. We refer to [28] for all unproven statements about sandwiched singularities

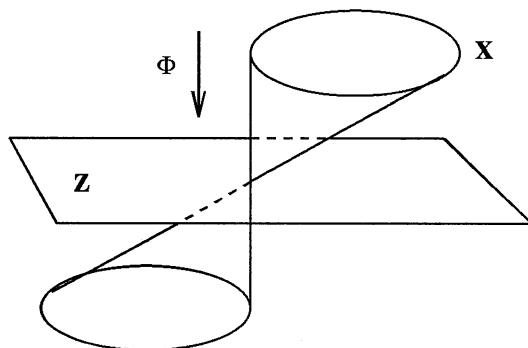
Consider a normal surface singularity $X = (X, p)$ and a resolution

$$\pi : M \longrightarrow X.$$

If X admits a birational map ϕ to $Z := (\mathbb{C}^2, 0)$, then we get a diagram

$$M \xrightarrow{\pi} X \xrightarrow{\phi} Z.$$

So X is “sandwiched” between two smooth spaces via birational maps, and this is the reason for calling such singularities sandwiched singularities. The simplest example of a sandwiched singularity is the A_1 -singularity $X = \{(x, y, z) \in \mathbb{C}^3 : xz - y^2 = 0\}$. The projection onto the x, y -plane Z gives a birational isomorphism $\phi : X \longrightarrow Z$.



A_1 as sandwiched singularity

It is easy to see that such a sandwiched singularity must be *rational* (use Leray's spectral sequence), but not all rational surface singularities are sandwiched. For example, it follows from the construction that a general hyperplane section of such a singularity is a curve singularity that has a *smooth branch*. From this it follows that the D_4 -singularity is *not sandwiched*. Being a sandwiched singularity is a property of the dual resolution graph Γ , so it makes sense to talk about sandwiched *graphs*. The class of sandwiched graphs is closed under taking sub-graphs and decreasing self intersections. On the other hand, a non-sandwiched sub-graph makes a graph non-sandwiched. As a consequence, of the rational double points only the A_k 's are sandwiched, because the others have a D_4 -sub-graph. In general it is rather cumbersome to recognize sandwiched graphs; in fact we do not know any other algorithm than just trying. In any case, the class of sandwiched graphs is surprisingly big. For example it includes the cyclic quotient singularities and more general, the rational surface singularities with a reduced fundamental cycle (sometimes called *minimal singularities*). So we have the following hierarchy of rational surface singularities.

$$\{A_k\} \subset \{\text{cyclic quotients}\} \subset \{\text{minimal}\} \subset \{\text{sandwiched}\} \subset \{\text{rational}\}$$

Each of the inclusions is proper.

Decorated Curves

Let $\rho : \tilde{Z} \longrightarrow Z = (\mathbb{C}^2, 0)$ be a sequence of point blow-ups with exceptional set $F := \rho^{-1}(0)$.

Definition (1.1):

The sandwiched singularity X determined by $\rho : \tilde{Z} \longrightarrow Z$ is obtained by contracting the set E of all non (-1) -curves of \tilde{Z} . (We assume for now that this configuration is connected).

So if we choose some neighbourhood M of E , we get the minimal resolution

$$\pi : (M, E) \longrightarrow (X, 0)$$

Let T be the set of (-1) -curves in \tilde{Z} . For $i \in T$ choose a *curvetta* \tilde{C}_i transverse to the (-1) -curve E_i . We put $\tilde{C} = \cup_{i \in T} \tilde{C}_i$ and $C = \rho(\tilde{C}) = \cup_{i \in T} C_i$ where $C_i = \rho(\tilde{C}_i)$. It is well known that $\rho : \tilde{Z} \longrightarrow Z$ can be seen as a good (but not necessarily minimal) embedded resolution of C . So we have a diagram

$$\begin{array}{ccc} \tilde{C} & \hookrightarrow & \tilde{Z} \\ \rho \downarrow & & \rho \downarrow \\ C & \hookrightarrow & Z \end{array}$$

As any embedded resolution of C is obtained from the minimal resolution by a number of further blowing ups at points on the branches of the strict transforms, we can label modifications $\tilde{Z} \longrightarrow Z$ by what we call a *decorated curve*.

Definition (1.2):

For a plane curve germ $C = \cup_{i \in T} C_i$ we define numbers:

1. $m(i)$ = sum of multiplicities of branch i in the multiplicity sequence of the minimal resolution of C .

2. $M(i)$ = sum of multiplicities of branch i in the multiplicity sequence of the minimal good resolution of C .

For example, for the ordinary cusp we have $m = 2$, $M = 4$.

Definition (1.3): A decorated curve is a pair (C, l) consisting of:

1. A curve singularity $C = \cup_{i \in T} C_i \subset (\mathbb{C}^2, 0)$.
2. A function $l : T \rightarrow \mathbb{Z}$ assigning to each branch of C a number.
3. With the condition that $l(i) \geq m(i)$.

The decoration l defines a unique sub-schemes of length $l(i)$ in \tilde{C}_i . So we could as well define a decorated curve as a curve, together with a sub-scheme of the normalisation that maps to the singular point. This point of view will be useful in §4.

Definition (1.4): Let (C, l) be a decorated curve.

1. The modification $\tilde{Z}(C, l) \rightarrow Z$ determined by (C, l) is obtained from the minimal embedded resolution of C by $l(i) - m(i)$ consecutive blow-ups at the i -th branch of C .
2. The analytic space $X(C, l)$ is obtained from $\tilde{Z}(C, l) - \tilde{C}$ by blowing down the maximal compact set, that is, the union of all exceptional divisors not intersecting the strict transform $\tilde{C} \subset \tilde{Z}(C, l)$.

The analytic space $X(C, l)$ can be smooth, or have several singularities. If however the decoration satisfies the stronger condition

$$l(i) \geq M(i) + 1,$$

then the space $\tilde{Z}(C, l)$ lies over the minimal *good* resolution and the maximal compact set is *connected*, hence $X(C, l)$ has a unique singular point, which by abuse of notation we call the sandwiched singularity $X(C, l)$. It is clear that every sandwiched singularity is of the form $X(C, l)$ for certain C and $l(i) \geq M(i) + 1$. However, a singularity X can have very well have many different representations as $X(C, l)$ with various (C, l) . We now will give some examples to clarify these definitions.

Examples (1.5):

- 1) $A_k = X(\text{Line}, k + 1)$.

Indeed, after blowing up $(k + 1)$ times, we create a chain of k (-2) -curves (and one (-1) -curve).

- 2) Let C be an ordinary m -fold point, that is, a union of m smooth branches with distinct tangents.

If $l(i) = 2$ for each branch, then $X(C, l)$ is isomorphic to the cone over the rational normal curve of degree $m + 1$. If $l(i) = 1$ or 2 (but at least one of them is 1), then $\tilde{Z}(C, l) - \tilde{C}$ does not contain any exceptional curves, so $X(C, l)$ is smooth.

- 3) Let C be the ordinary cusp $y^2 - x^3 = 0$. Then $X(C, 2)$ and $X(C, 3)$ are smooth, $X(C, 4)$ contains two singular points, $X(C, 5)$ is a cyclic quotient and $X(C, 6)$ has non-reduced fundamental cycle.

4) If X is a rational surface singularity with reduced fundamental cycle, then it has a sandwiched representation with a curve C all whose branches are *smooth*. In fact, the strict transform of the generic hyperplane section consists of $\text{mult}(X)$ curvetta's. If we pick out one of these, and replace the others by (-1) -curves, we get a space \tilde{Z} that contracts to \mathbb{C}^2 . If we blow down curvetta's \tilde{C}_i transverse to the (-1) -curves, we get our sandwiched representation as $X(C, l)$ with smooth branches, and where $l(i)$ is the length between the picked curvetta of the hyperplane section and \tilde{C}_i . From this it is already clear that X has many different sandwiched representations, by picking other branches of the hyperplane section. The first blown-up curve in the sandwiched representation is the exceptional curve that intersects the chosen branch of the general hypersurface section. But note also that $X(\mathbf{A}_2, 5)$ of example 3) is isomorphic to $X(\mathbf{A}_1, 2, 4)$! So a singularity with reduced fundamental cycle can very well have representations with non-smooth branches.

The ideals $I(C, l)$

Another way to describe a sandwiched singularity is as the singularity occurring in the blow-up of Z in a complete ideal. We will denote by $I(C, l)$ the ideal needed to get $X(C, l)$. This ideal can be described in several ways.

Proposition (1.6):

1. Let $(\cdot)_c$ denote the compact part of the divisor of the pull-back of a function to $\tilde{Z}(C, l)$. Then:

$$I(C, l) = \{g \in \mathbb{C}\{x, y\} \mid (g)_c \geq (f)_c\}$$

Here $f = 0$ is a defining equation for C .

2. Let $(t_1^{c_1}, \dots, t_r^{c_r}) = I \subset \mathcal{O}_C \subset \mathcal{O}_{\tilde{C}} = \prod_{i=1}^r \mathbb{C}\{t_i\}$ be the conductor ideal of $n : \tilde{C} \rightarrow C$.

$$I(C, l) = \{g \in \mathbb{C}\{x, y\} \mid (C_i \cdot (g = 0)) \geq c_i + l(i)\}$$

3. If we "shift" the curvetta's \tilde{C}_i on $\tilde{Z}(C, l)$ transverse to themselves, we get by blowing down a curve C' , defined by some equation $g = 0$. $I(C, l)$ is the ideal generated by these g 's.

Obviously, $I(C, l') \subseteq I(C, l)$ if $l'(i) \geq l(i)$. The largest of these ideals is $I(C, m)$, with $m(i)$ as in (1.2). This ideal is also exactly the ideal I^{ev} , introduced in the appendix, and plays an important role in this article.

Multiplicity Matrix

Consider as before an embedded resolution of the curve C :

$$\begin{array}{ccc} \tilde{C} & \hookrightarrow & \tilde{Z} \\ \rho \downarrow & & \rho \downarrow \\ C & \hookrightarrow & Z \end{array}$$

The map $\rho : \tilde{Z} \rightarrow Z$ can be factored into a finite sequence of blow-ups $\rho = \rho_N \circ \rho_{N-1} \circ \rho_{N-2} \dots \rho_1$ where $\rho_k : Z_k \rightarrow Z_{k-1}$ is the blow up in a finite number

of points of Z_{k-1} . The totality of points in which we blow up, that is, the set of *infinitely near points*, we denote by \mathcal{I} . For $p \in \mathcal{I} \cap Z_k$, we put:

$$\begin{aligned} E_p &:= \text{strict transform of } (\rho_{k+1})^{-1}(p) \text{ in } \tilde{Z} \\ E_p^* &:= \text{total transform of } E_p \text{ in } \tilde{Z} \end{aligned}$$

It is usual to identify the E_p 's on the different Z_k 's. Let $\mathbf{P} := \text{Pic}(\tilde{Z}/Z)$ be the lattice of divisors contracted by ρ . It is clear that both $E_p, p \in \mathcal{I}$ and $E_p^*, p \in \mathcal{I}$ form bases for \mathbf{P} . The E_p are \mathbb{P}^1 's, whereas the E_p^* in general are reducible, but have self-intersection -1 . The relation between the E_p and E_p^* is expressed in terms of the multiplicity and proximity matrices.

Definition/Proposition (1.7):

The multiplicity matrix is defined as:

$$(M_{p,q})_{p,q \in \mathcal{I}} := \text{multiplicity of } C_k \text{ in } E_p$$

Here C_k is the strict transform of C in Z_k , where $q \in Z_k$.

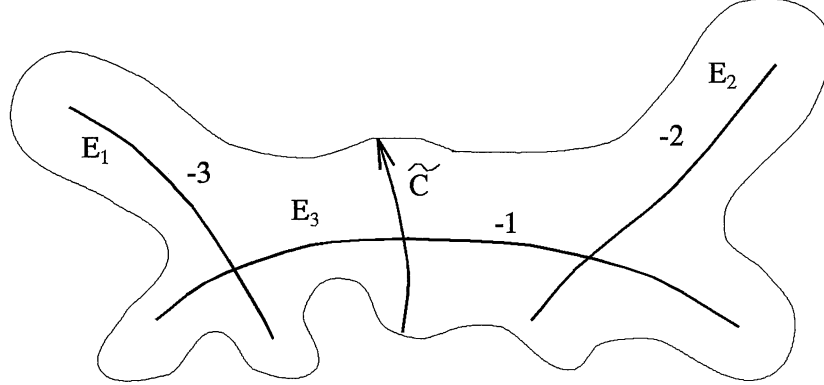
One says that $q \in \mathcal{I}$ is **proximate** to $p \in \mathcal{I}$, notation $q \rightarrow p$, if q is on E_p . One then has:

$$E_p = E_p^* - \sum_{q \rightarrow p} E_q^* =: \sum_q \Pi_{q,p} E_q^*$$

$\Pi_{p,q}$ is called the **proximity matrix**. One has $E_p^* = \sum_q M_{q,p} E_q$ so the multiplicity matrix is the inverse of the proximity matrix.

Example (1.8):

Take the minimal resolution of the ordinary cusp to a divisor with normal crossings. We have to blow up three times. Denote the arising exceptional divisors by E_1, E_2, E_3 .



The multiplicity and proximity matrices are:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}, \Pi = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

We let \mathbf{L} be the free \mathbb{Z} -module spanned by the (-1) -curves E_i , $i \in T$. There is a natural map

$$\mathbf{I} : \mathbf{P} \longrightarrow \mathbf{L}; x \mapsto \sum_{i \in T} (\tilde{C}_i \cdot x) \cdot E_i$$

Clearly, the kernel H of this map is the sub-lattice spanned by all E_p , where p is $\notin T$. So this is the lattice of the resolution graph of $X(C, l)$. If we choose as basis for \mathbf{P} the divisors E_p^* , then the intersection form becomes *diagonal*. If we choose for H the natural basis consisting of E_p , $p \notin T$, then matrix of the inclusion $H \hookrightarrow \mathbf{P}$ is described by the restricted proximity matrix, obtained by removing all columns corresponding to a (-1) -curve. The natural basis for \mathbf{L} is $E_i, i \in T$. The matrix of the map $\mathbf{P} \rightarrow \mathbf{L}$ with respect to these bases now is that part of the restricted multiplicity matrix, obtained by keeping only the rows corresponding to the (-1) -curves. This state of affairs can be formulated as:

The rows of the restricted multiplicity matrix are the coefficients of the equations for the resolution graph inside the trivial diagonal lattice P .

This restricted multiplicity matrix is essentially the same thing as the multiplicity sequence of C .

Example (1.9): For the cusp the map $\mathbf{I} : \mathbf{P} \rightarrow \mathbf{L}$ is given by looking at the third row of multiplicity matrix, that is, the multiplicity sequence of C .

$$\mathbb{Z}^3 \xrightarrow{(2,1,1)} \mathbb{Z}$$

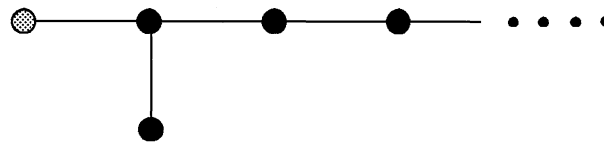
Indeed, the resolution graph can be obtained by looking at the first two columns of the matrix \mathbf{II} . One finds a (-2) and a disjoint (-3) . If we blow up further, we get

$$\mathbb{Z}^3 \xrightarrow{(2,1,1,\dots,1)} \mathbb{Z}$$

which has as kernel elements of the form

$$\begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

These vectors make up precisely the diagram



● : (-2) -curve ● : (-3) -curve

The series of the ordinary cusp

Note that the map $\mathbf{I} : \mathbf{P} \rightarrow \mathbf{L}$ defines a priori its kernel H as lattice, but from the structure of the proximity matrix we in fact can find a natural basis in it.

The Infinitely Near Points made Visible

The set \mathcal{I} of infinitely near points are points on some blow up. However, one can make these points visible by a small deformation of the curve. This was first

described in a nice paper by Scott, [27]. For this reason we will call it informally the *Scott-deformation*. The same deformation was also used by A'Campo [1] and Gusein-Zade [12]. For convenience of the reader we include a proof:

Proposition (1.10): *Let $C \subset \mathbb{C}^2$ be an isolated plane curve singularity of multiplicity m . Then there exists a 1-parameter δ -constant deformation of C , such that on a general fibre one has the following singularities:*

1. *The singularities occurring on the strict transform of C under the blow-up of \mathbb{C}^2 at the origin.*
2. *A singularity consisting of m smooth branches intersecting mutually transverse. (We will call such a singularity an ordinary m -fold point from now on.)*

Proof: After a change of coordinates, we may assume that C is given by the zero set of a Weierstrass polynomial f :

$$f(x, y) = y^m + a_1(x)y^{m-1} + \dots + a_m(x) = 0$$

Because we assumed C to have multiplicity m , the vanishing order of a_i at the origin is at least i . The total transform after the blow-up (in the interesting chart) is given by:

$$x^m(y^m + \frac{a_1(x)}{x}y^{m-1} + \dots + \frac{a_m(x)}{x^m}) = 0$$

The intersection multiplicity of the exceptional divisor with the strict transform is m . Now move the strict transform "down" by replacing x by $x - s$. We then have as singularities the singularities of the strict transform, and we have m intersection points with the exceptional divisor. Blowing down gives the ordinary m -fold point. In terms of f itself, we are looking at the deformation:

$$f_s = y^m + \frac{a_1(x-s)}{x-s}y^{m-1}x + \dots + \frac{a_m(x-s)}{(x-s)^m}x^m = 0$$

We may assume that each branch C_i of C (whose multiplicity is m_i) is given by a parametrization of the form:

$$x = t_i^{m_i}; y = t_i^{m_i} \phi_i(t_i)$$

One then checks that the deformation of this parametrisation

$$x = t_i^{m_i} + s; y = (t_i^{m_i} + s)\phi_i(t_i)$$

is a parametrisation of $f_s(x, y) = 0$. □

From a repeated application of the above proposition it now follows that:

Corollary (1.11): *There exists a one-parameter deformation of the curve C such that for generic $s \neq 0$ there are points $P_q, q \in \mathcal{I}$ in the plane, which are ordinary $M_{i,q}$ -fold points of $C_{i,s}$.*

For pictures we refer to [27], [1], [12], [26].

The corollary has a very nice interpretation: the matrix of the map \mathbf{I} , that is, the multiplicity matrix, is the **incidence matrix** of a set of points $P_q, q \in \mathcal{I}$ and the curve C_s . The δ -constancy of the family is then equivalent to the classical formula of M. Noether for the δ -invariant of the curve singularity C :

$$\delta(C) = \sum_{q \in \mathcal{I}} \frac{1}{2} \cdot m_q \cdot (m_q - 1),$$

with $m_q = \sum_{i \in T} M_{i,q}$.

We will see later that this particular deformation *corresponds* to the Artin-component deformation of $X(C, l)$. Moreover, in a similar way *every* smoothing of $X(C, l)$ corresponds to a certain δ -constant deformation of C , and certain points on it. This description of the smoothings of $X(C, l)$ is what we call the *picture method*, because the curves and points are conveniently drawn in the plane. We consider it as the most important result of this paper. A precise statement and the proof will be given in section 4.

§2 The Shape of the Surface

To get a feeling for what is going on, we need some insight into the *shape* of the surface.

The spaces $X(C)$ and $Y(C)$

Remark that sandwiched singularities come naturally in *series*, indexed essentially by the $l(i)$. So what will happen if we let the $l(i)$ go to infinity? According to [32], we can refine the series by deforming the *improvement* of the limit which we call $X(C)$. This improvement can be described as follows. Take for each branch $i \in T$ a smooth plane $(\mathbb{C}^2, 0)_i$ and an embedding $\widetilde{C}_i \hookrightarrow (\mathbb{C}^2, 0)_i$. Take

$$\widetilde{Z} \coprod_{i \in T} (\mathbb{C}^2, 0)_i$$

and identify the curves \widetilde{C}_i in \widetilde{Z} and $(\mathbb{C}^2, 0)_i$. It is now clear that this space is the improvement of the following singularity:

Definition (2.1):

Let $C = \cup_{i \in T} C_i \subset Z$. Consider the normalisation map

$$n : \widetilde{C} = \coprod_{i \in T} \widetilde{C}_i \longrightarrow \cup_{i \in T} C_i = C.$$

Choose an embedding

$$\coprod_{i \in T} \widetilde{C}_i \hookrightarrow \coprod_{i \in T} (\mathbb{C}^2, 0)_i$$

Then the space $X(C)$ is defined to be the push-out:

$$\begin{array}{ccc} \coprod_{i \in T} \widetilde{C}_i & \hookrightarrow & \widetilde{Z} \coprod_{i \in T} (\mathbb{C}^2, 0)_i \\ n \downarrow & & \downarrow \\ \cup_{i \in T} C_i & \hookrightarrow & X(C) \end{array}$$

We say: $X(C)$ is the space obtained by glueing planes along the branches $X(C)$. This $X(C)$ will naturally be a weakly normal Cohen-Macaulay space, with singular locus C . Note that under the identification map the smooth planes $(\mathbb{C}^2, 0)$ get mapped in general to something singular in $X(C)$; the corresponding components will then be non-Cohen-Macaulay.

The use of *projections* of surface singularities into three space in order to understand the deformations and equations has turned out to be very fruitful: in [17] this idea was used to obtain the structure of the base space for rational quadruple points. Here we use the same method to study sandwiched singularities. For a review of the method of projections we refer to the appendix. We start with a special projection of the limit $X(C)$, which is a surface that is very easy to define.

Definition (2.3): Let C be an isolated plane curve singularity defined by $f = 0$, where $f \in \mathbb{C}\{x, y\} \subset \mathbb{C}\{x, y, z\}$. We put

$$Y(C) := \{(x, y, z) : zf(x, y) = 0\} \subset (\mathbb{C}^3, 0)$$

So $Y(C)$ consists of a smooth plane $\{z = 0\}$ together with the product of the z -axis with the curve C . The singular locus of $Y(C)$ therefore consists of two parts: 1) the curve C in the plane $\{z = 0\}$ and 2) the z -axis through the singular point of C . It is easy to construct a finite, generically 1:1 map from $X(C)$ to $Y(C)$: we can resolve the curve C by a sequence of point blow-ups. We now can apply the same sequence of blow-ups crossed with the z -axis to $Y(C)$ to construct a modification

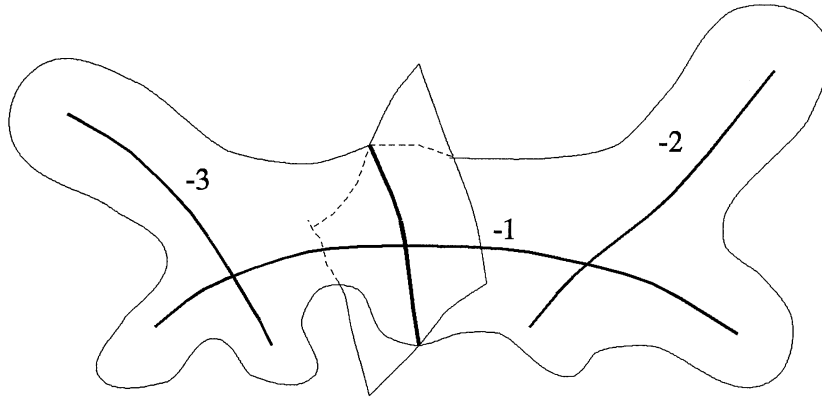
$$Z(C) \longrightarrow Y(C)$$

This $Z(C)$ is exactly the improvement of $X(C)$ constructed above, and by the universal property of glueing and blowing down, we get a factorization

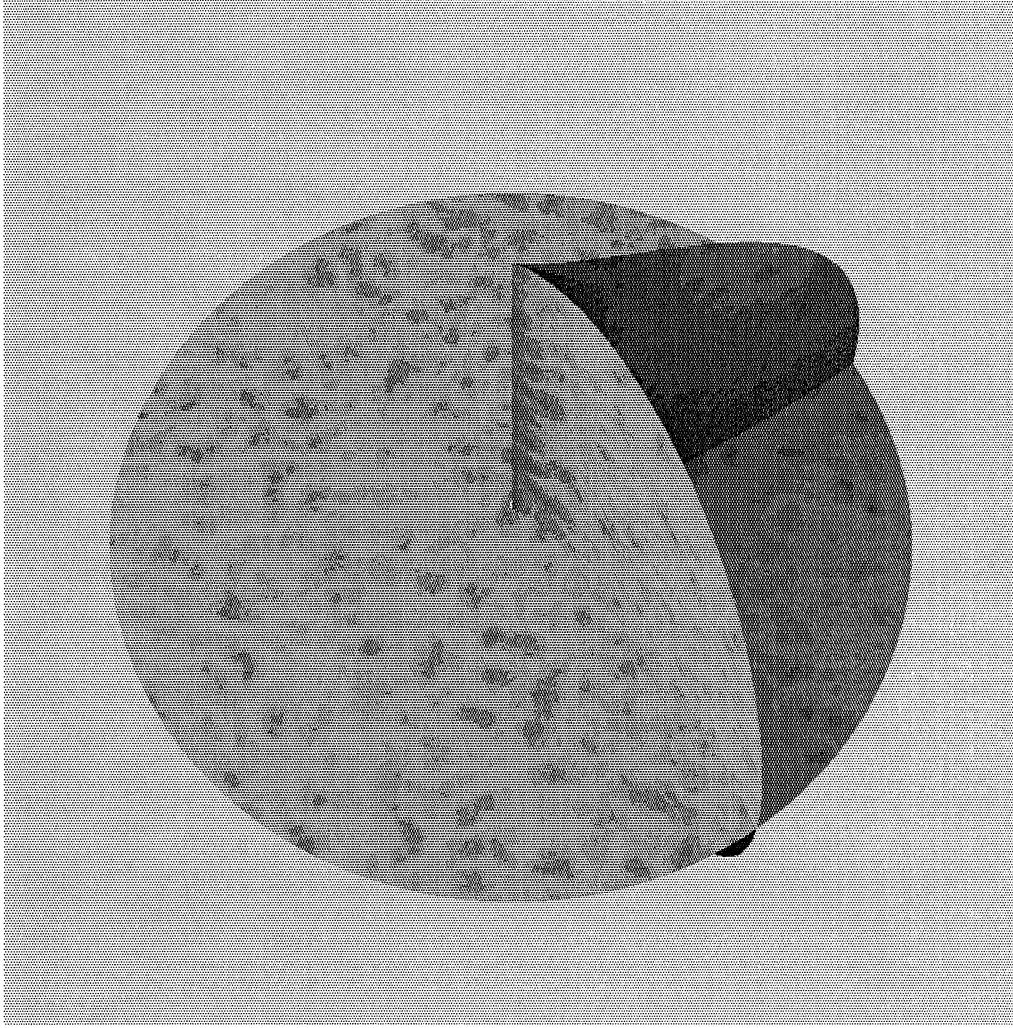
$$Z(C) \longrightarrow X(C) \longrightarrow Y(C)$$

To put it in another way, $X(C)$ is obtained from $Y(C)$ by a *partial normalisation* that removes only the singularities on the z -axis.

Example (2.4): We consider the ordinary cusp $C : x^2 - y^3 = 0$:



Improvement of the surface $X(C)$



The surface $Y(C) : z(y^2 - x^3) = 0$

The spaces $X(C, l)$ and $Y(C, l)$

As $X(C, l)$ should be a small deformation of $X(C)$, one expects to be able to define a $Y(C, l)$ as a small deformation of $Y(C)$ (which in fact is a so called R.C.-deformation, see the appendix), from which $X(C, l)$ can be obtained as normalisation. This in fact is the case, as we will show now.

Theorem/Definition (2.5): Let (C, l) be a decorated curve, and $X(C, l)$ the analytic space determined by it.

Let $(t_1^{c_1}, \dots, t_r^{c_r}) = I \subset \mathcal{O}_C \subset \mathcal{O}_{\tilde{C}} = \prod_{i=1}^r \mathbb{C}\{t_i\}$ be the conductor ideal of $n : \tilde{C} \rightarrow C$.

Then, for every function $g \in \mathbb{C}\{x, y\}$ such that its restriction g_i has exact vanishing order $c_i + l(i)$ on \tilde{C}_i , $X(C, l)$ is the normalisation of the surface

$$Y(C, l) = \{(x, y, z) | zf(x, y) - g(x, y) = 0\} \subset \mathbb{C} \times Z$$

proof: Let $\tilde{Z} = \tilde{Z}(C, l)$ be the modification of Z determined by (C, l) . On it we have functions x and y , pull-backs of the functions x and y on \mathbb{C}^2 . The function $f(x, y)$ on \tilde{Z} vanishes exactly on $F \cup \tilde{C}$.

Then we define a meromorphic function z on \tilde{Z} by

$$z = g/f.$$

Replacing g by $g + \alpha f$ we can arrange that the divisors (f) and (g) on \tilde{Z} have the same compact part. As the vanishing order of g_i on \tilde{C}_i is assumed to be *exactly* $c_i + l(i)$, it follows that the non-compact parts of (f) and (g) are disjoint, see (1.6). So z has a simple pole along \tilde{C} and is zero along the non-compact part of (g) . In particular, z is non-constant on every compact curve in \tilde{Z} intersecting \tilde{C} . So we get a *holomorphic* map

$$(x, y, z) : \tilde{Z} - \tilde{C} \longrightarrow \mathbb{C} \times Z$$

As $X(C, l)$ is obtained from $\tilde{Z} - \tilde{C}$ by contracting the maximal compact set, the above map factorizes to give a map:

$$p : X(C, l) \longrightarrow Y(C, l) \subset \mathbb{C} \times Z$$

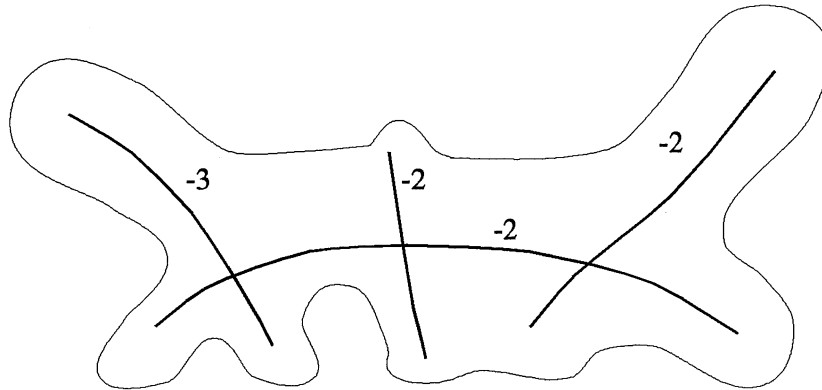
Clearly, p is birational, as the map to Z already is birational. The inverse image on \tilde{Z} of the z -axis $\subset \mathbb{C}^3$ is the set F . The function z is finite (because non-constant) on each of the exceptional curves intersecting \tilde{C} . It follows that $p : X(C, l) \rightarrow Y(C, l)$ is the normalisation map. \square

Remarks (2.6):

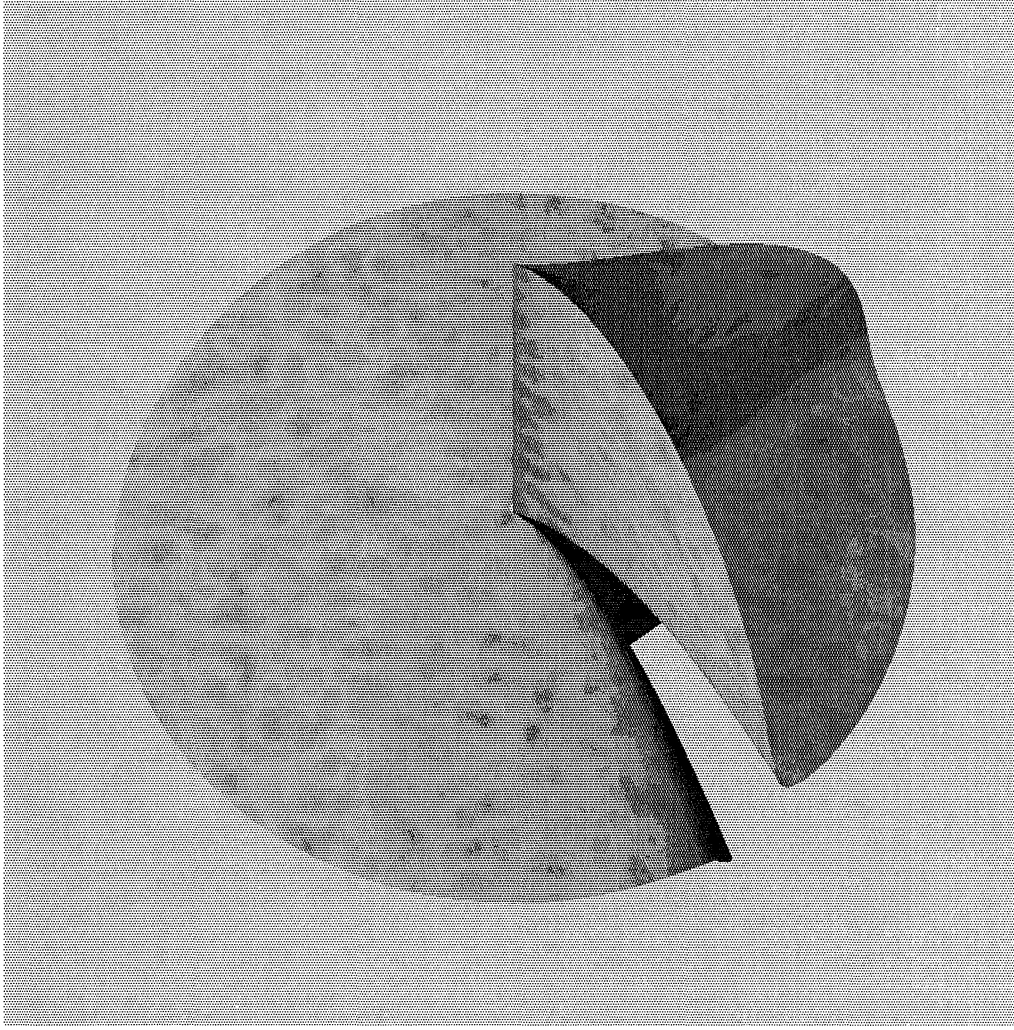
We note that strictly speaking the space $Y(C, l)$ depends on the choice of a g . However, its normalisation $X(C, l)$ only depends on g via its vanishing orders encoded in the $l(i)$ and therefore we don't mind.

Note also that the surface $Y(C, l)$ has a natural partial compactification $\bar{Y}(C, l) \subset \mathbb{P} \times Z$. The normalisation of this space could be called $\bar{X}(C, l)$, which is also precisely the space obtained by blowing up Z in the complete ideal $I(C, l)$.

Example (2.7): We consider the decorated curve $(C, 6)$, where C is the cusp $x^2 - y^3 = 0$:



Resolution of the surface $X(C, 6)$



The surface $Y(C, 6) : z(y^2 - x^3) + (0.05)x^4 = 0$

We now turn to the algebraic relation between $X(C)$ and $Y(C)$ and between $X(C, l)$ and $Y(C, l)$.

Let I be the conductor of the normalisation map $\tilde{C} \rightarrow C$. Consider the extension of I to $\mathbb{C}\{x, y, z\}$, which we denote by \tilde{I} . The ideal of functions

$$I^{ev} := \{g \in \mathbb{C}\{x, y\} \mid \text{ord}(g_i) \geq c_i + m(i)\}$$

plays an important role in the R.C.-description of the δ -constant deformations of the curve C , see appendix. We note that because $l(i) \geq m(i)$ the particular g constructed in (2.5) is $\in I^{ev}$. Because \tilde{I}^{ev} is an *ideal* we conclude that both zf and $zf - g$ are elements of \tilde{I}^{ev} . So the ideal \tilde{I} satisfies R.C. both for $Y(C)$ and $Y(C, l)$.

Theorem (2.8):

$$\mathcal{O}_{X(C)} = \text{Hom}_{Y(C)}(\tilde{I}, \tilde{I}); \quad \mathcal{O}_{X(C, l)} = \text{Hom}_{Y(C, l)}(\tilde{I}, \tilde{I})$$

proof: We give the proof of the second statement only. The singular locus of $Y(C, l)$ is exactly the z -axis. Taking $\text{Hom}_{Y(C, l)}(\tilde{I}, \tilde{I})$ commutes with localization, so

for a generic z , $zf - g$ gives the general fiber of a δ -constant deformation of C , so for generic z , $\text{Hom}_{Y(C,l)}(\tilde{I}, \tilde{I})$ describes a smooth space. We conclude that the space defined by $\text{Hom}_{Y(C,l)}(\tilde{I}, \tilde{I})$ is CM and has codimension two singular locus. Thus $\text{Hom}_{Y(C,l)}(\tilde{I}, \tilde{I})$ is the normalization of $\mathcal{O}_{Y(C,l)}$ which by (2.5) is $\mathcal{O}_{X(C,l)}$. \square

This description of $\mathcal{O}_{X(C)}$ and $\mathcal{O}_{X(C,l)}$ is very useful to get explicit equations for these spaces in ambient space. This is explained in the appendix, and illustrated by the following example.

Example (2.9): Take the decorated curve C consisting of the E_6 singularity, defined by $y^3 - x^4 = 0$, and the function l defined by the number 8 attached to its only branch. The equation for $Y(C)$ is $z(y^3 - x^4) = 0$. Equations for the limit $X(C)$ in the space with coordinates x, y, z, u, v can be obtained, as explained in the appendix, as follows. The conductor $I = (x, y)^2$ of the curve is obtained as the ideal of minors of the matrix

$$\begin{pmatrix} x & y & 0 \\ 0 & x & y \end{pmatrix}$$

A presentation matrix of $\mathcal{O}_{X(C)}$ as $\mathcal{O}_{(\mathbb{C}^3, 0)}$ -module is then :

$$\begin{pmatrix} zy & 0 & -zx^2 \\ x & y & 0 \\ 0 & x & y \end{pmatrix}$$

Thus, we get linear equations:

$$zy + ux = 0; \quad uy + vx = 0; \quad -zx^2 + vy = 0,$$

and the quadratic equations are:

$$u^2 = zv; \quad uv = -z^2x; \quad v^2 = z^2y$$

Now we have to choose a g . The E_6 singularity is parametrized by $x = t^3; y = t^4$. The conductor is given by the ideal $(t^6) = (x, y)^2$. Hence the function g has to have vanishing order $6 + 8 = 14$. So $x^2y^2 = t^{14}$ will do. We conclude that the projection $Y(C, 8)$ has equation:

$$z(y^3 - x^4) = x^2y^2$$

From this information equations of $X(C, 8)$ can be computed. For the corresponding presentation matrix of $\mathcal{O}_{X(C,8)}$ we get:

$$\begin{pmatrix} zy & xy & -zx^2 \\ x & y & 0 \\ 0 & x & y \end{pmatrix}$$

From this we get linear equations:

$$zy + ux = 0; \quad xy + uy + vx = 0; \quad -zx^2 + vy = 0,$$

and quadratic equations:

$$u^2 = zv + yz; \quad uv = -z^2x; \quad v^2 = z^2y - x^2z$$

for the sandwiched singularity $X(C, 8)$.

In general, the equations for the limit $X(C)$ are easy to describe in terms of the equations of \tilde{C} as in the appendix.

Proposition (2.10):

If

$$\sum_{i=0}^t M_{ij} u_i = 0 ; \quad u_k u_l = \sum_{i=0}^t g_{kl}^i u_i$$

are the linear resp. quadratic equations for \tilde{C} , then the linear resp. quadratic equations for $X(C)$ are:

$$zM_{1j} + \sum_{i=1}^t M_{ij} u_i = 0 ; \quad u_k u_l = z^2 g_{kl}^1 + \sum_{i=1}^t z g_{kl}^i u_i$$

(Recall that in the module basis $u_0 = 1$).

Remarks (2.11):

a) There is a one parameter R.C. deformation:

$$zf(x, y) - sg(x, y) = 0$$

For the special fibre $s = 0$ we have the space $Y(C)$, and for all $s \neq 0$ the fibre is a $Y(C, l)$. If we consider the normalization, we get a one parameter deformation of $X(C)$ such that for all $s \neq 0$ the fibre is isomorphic to $X(C, l)$. We remark that this deformation can be obtained from the minimal improvement of $X(C)$ by deforming away the A_∞ -singularity at \tilde{C}_i to an $A_{l(i)-m(i)}$ -singularity, in the way also described in [32].

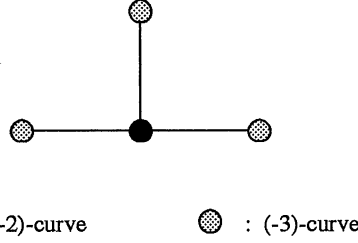
Similarly, there is a one-parameter deformation $X_S \rightarrow S$ with zero-fibre $X(C, l)$ and all other fibres isomorphic to $X(C, l')$ if $l(i) \geq l'(i)$ for all i : just look at

$$zf(x, y) - sg'(x, y) - g(x, y).$$

These deformations are useful in various situations.

b) Although in Example (2.8) the computation of equations of $X(C, l)$ was quite easy, the computation of equations for $X(C, l)$ will in general become very lengthy and boring. This will even be more true for deformations, which we will consider later. The crux of the theory of R.C. deformations is that one can circumvent these calculations. One only needs to take care of the linear equations, while the R.C. condition exactly says that one can compute the quadratic equations, without actually doing so.

c) The resulting embedding of $X(C, l)$ need not to be minimal, although it will be in most cases. For example, take the sandwiched singularity with the following dual resolution graph.



There exists no sandwiched representation for this singularity such that the resulting embedding is minimal. Note that this example is the standard counterexample to the T^1 and T^2 formulae, see [6] and [4].

§3 Deformations of Sandwiched Singularities

In section 2 we have seen how to get equations for $X(C, l)$ using a projection to a surface $Y(C, l)$. With the same ease, the theory of R.C. deformations can be used to describe the *deformations* of $X(C, l)$ in terms of $Y(C, l)$. The main result of this section is the theorem, that expresses the stability of the normal form $zf - g = 0$ of the projection under arbitrary deformations, in a strong sense. To formulate this appropriately, we need to define a new deformation functor.

To simplify notation we shall put $X = X(C, l)$ and $Y = Y(C, l)$.

The Functor $Def(\Sigma, C, g)$

Let Y be defined by an equation of the form $zf - g = 0$. As usual, $f = 0$ is an equation for the curve C , Σ the fat point defined by the conductor I of the normalisation. We denote by $\tilde{\Sigma}$ the fat line defined by I in \mathbb{C}^3 . We will define a functor $Def(\Sigma, C, g)$ of what we call *normal form deformations*.

Definition (3.1): Let S be an local analytic space. A triple (Σ_S, C_S, g_S) called a nice triple if and only if:

- 1) (Σ_S, C_S) is an R.C. deformation of (Σ, C) over S .
- 2) (Σ_S, g_S) is an R.C. deformation of (Σ, g) over S .

Two nice triples (Σ_S, C_S, g_S) and (Σ'_S, C'_S, g'_S) are called isomorphic if there is a coordinate transformation in the x, y -plane over S which maps (Σ_S, C_S) to (Σ'_S, C'_S) and g_S to g'_S modulo some multiple of f'_S .

We define the functor $Def(\Sigma, C, g)$ by putting:

$$Def(\Sigma, C, g)(S) := \{(\Sigma_S, C_S, g_S); \text{ nice triple over } S\} / \{\text{isomorphism}\}$$

It is easy to see that this is a semi-homogeneous functor, and we will see in a moment that $Def(\Sigma, C, g)(\mathbb{C}[\epsilon]/(\epsilon^2))$ is finite dimensional, so that by Schlessinger's theorem it has a hull.

Proposition (3.2): There is a natural transformation of functors

$$Def(\Sigma, C, g) \longrightarrow Def(\tilde{\Sigma}, Y(C, l))$$

$$(\Sigma_S, C_S, g_S) \mapsto (\Sigma_S, Y_S = \{zf_S - g_S = 0\})$$

proof: Given a nice triple (Σ_S, C_S, g_S) , the function f_S defining C_S is determined up to a unit u , and g_S is determined up to a multiple $a \in m_{\mathcal{O}_S\{x, y\}}$ of f_S . In the

equation $z f_S - g_S = 0$, these ambiguities can be absorbed in z by the replacement $z \mapsto uz + a$, so Y_S is well-defined. The pair (Σ_S, Y_S) satisfies R.C. if and only if the evaluation map $ev_{z f_S - g_S}$ is the zero map. We have: $ev_{z f_S - g_S} = z \cdot ev_{f_S} - ev_{g_S}$. As both (Σ_S, f_S) and (Σ, g_S) satisfy R.C., we have $ev_{f_S} = ev_{g_S} = 0$. So indeed $(\Sigma_S, z f_S - g_S)$ satisfies R.C. and we get a well-defined transformation of functors. \square

Hence, we have a chain of transformations of functors

$$Def(\Sigma, C, g) \longrightarrow Def(\tilde{\Sigma}, Y) \xrightarrow{\cong} Def(X \longrightarrow Y) \longrightarrow Def(X)$$

The main result of this section is:

Theorem (3.3): *The composed transformation of functors*

$$Def(\Sigma, C, g) \longrightarrow Def(X)$$

is formally smooth.

This formal smoothness is a strong form of surjectivity. It means in particular that every flat deformation of X over S can be projected into three-space to an R.C.-admissible family of the form $z f_S - g_S = 0$.

Infinitesimal Deformations

As before, let C be described by an equation $f = 0$, $f \in \mathbb{C}\{x, y\} =: \mathcal{O}$, and let $(\Delta) = (\Delta_1, \dots, \Delta_t) = I \subset \mathcal{O}$ be the conductor ideal, Σ the fat point it defines. The infinitesimal deformations of the functor $Def(\Sigma, C, g)$ are represented by *admissible triples*

$$\mathcal{A}(I, f, g) = \{(n, f_1, g_1) \mid (n, f_1) \in \mathcal{A}(I, f) \text{ and } (n, g_1) \in \mathcal{A}(I, g)\} \subset N_\Sigma \oplus \mathcal{O} \oplus \mathcal{O}$$

The infinitesimal coordinate transformations, i.e. vector fields $\theta \in \Theta := \Theta_{\mathbb{C}^2, 0}$ give a submodule of triples of the form

$$(\theta(\Delta), \theta(f), \theta(g))$$

Furthermore, the equation of C is determined up to a unit, and the function g up to multiples of f . As a consequence, the triples $(0, f, 0)$, $(0, 0, f)$ are zero in $T^1(\Sigma, C, g) := Def(\Sigma, C, g)(\mathbb{C}[\epsilon]/(\epsilon^2))$. As a result we have

Proposition (3.4): *There is an exact sequence of the form:*

$$0 \longrightarrow I^{ev}/(f, \Theta_C(g)) \longrightarrow T^1(\Sigma, C, g) \longrightarrow T^1(\Sigma, C)$$

Here $\Theta_C = \{\theta \in \Theta \mid \theta(f) \in (f)\}$ is the module of vector fields tangent to the curve C . In particular, $T^1(\Sigma, C, g)$ is a finite dimensional vector space as soon as g is not identically zero on any branch of C .

Our next aim is to prove that the map $T^1(\Sigma, C, g) \longrightarrow T^1(X)$ is surjective. For this it is useful first to look at the limit $X = X(C)$, for which $T^1(X)$ can be understood completely. We will use the following notation: if M is any $\mathbb{C}\{x, y\}$ -module, we write \tilde{M} for its extension to $\mathbb{C}\{x, y, z\}$. Similarly for spaces.

Proposition (3.5): Let $T_a^1 = \text{Ker}(h_f : T^1(\Sigma) \rightarrow N^*/I)$. Then there are exact sequences:

$$\begin{aligned} 0 \longrightarrow \widetilde{I^{ev}}/(f, zJ_\Sigma(f)) \longrightarrow T^1(\widetilde{\Sigma}, Y(C)) \longrightarrow \widetilde{T_a^1} \longrightarrow 0 \\ 0 \longrightarrow (I^{ev}/(f)) \oplus (zI^{ev}/(zJ_\Sigma(f), zf)) \longrightarrow T^1(X(C)) \longrightarrow T_a^1 \longrightarrow 0 \end{aligned}$$

Proof: We first determine the admissible pairs $\mathcal{A}(\widetilde{I}, zf) \subset \widetilde{N_\Sigma} \times N_{Y(C)}$ of the function zf . Elements of the form $(0, h)$ are admissible iff $h \in \widetilde{I^{ev}}$. Furthermore, if $(n, f_n) \in \mathcal{A}(I, f)$, then $(n, zf_n) \in \mathcal{A}(\widetilde{I}, zf)$. One has $(n, h) \in \mathcal{A}(\widetilde{I}, zf)$ if and only if $h_{zf}(n) = 0$, where $h_{zf} : \widetilde{N_\Sigma} \rightarrow \widetilde{N^*/I}$ is the hessian map. As $h_{zf}(n) = zh_f(n)$, and $\widetilde{N^*/I}$ is $\mathbb{C}\{z\}$ -free, one sees $\text{ker}(h_{zf}) = \widetilde{\text{ker}(h_f)}$ which means that there is nothing else. To obtain the first exact sequence, one has to note that that ∂_z maps to the pair $(0, f)$.

For the second exact sequence, we recall that, according to the appendix, one obtains $T^1(X(C))$ as a quotient of $T^1(\widetilde{\Sigma}, Y(C))$ by dividing out the image of the vector fields of the form $u_k \partial_x$, etc, in the space of admissible pairs. As \widetilde{C} is smooth, we have $T^1(\widetilde{C}) = 0$. This means that all of $\mathcal{A}(I, f)$ is obtained by applying $u_k \partial_x$, etc. From the explicit equations for $X(C)$ in terms of the equations of \widetilde{C} , (2.10) and the description of the R.C. admissible pairs obtained from vector fields in the appendix, one concludes that if (n, h) is an R.C. admissible pair for C , then (zn, z^2h) gives an R.C. admissible pair for $X(C)$, which then is a trivial infinitesimal deformation of $X(C)$. Dividing out these elements in the first sequence gives the second sequence. \square

Corollary (3.6):

The map $T^1(\Sigma, C, 0) \longrightarrow T^1(X)$ is an isomorphism. It sits in a diagram:

$$\begin{array}{ccccccc} 0 \rightarrow & I^{ev}/(f) & \rightarrow & T^1(\Sigma, C, 0) & \rightarrow & T^1(\Sigma, C) & \rightarrow 0 \\ & \downarrow & & \cong \downarrow & & \downarrow & \\ 0 \rightarrow & (I^{ev}/(f)) \oplus (zI^{ev}/(zJ_\Sigma(f), zf)) & \rightarrow & T^1(X(C)) & \rightarrow & T_a^1 & \rightarrow 0 \end{array}$$

Here we see one of the main reasons for introducing $\text{Def}(\Sigma, C, g)$: it maps naturally to $\text{Def}(\Sigma, C)$, whereas $T^1(X(C))$ only maps to T_a^1 . Using the isomorphism, $T^1(X(C))$ gets a beautiful structure: it has a finite dimensional piece $T^1(\Sigma, C)$ corresponding to the δ -constant deformations of C , and an infinite dimensional piece $I^{ev}/(f)$ corresponding to the "series" deformations, the deformations which deform $X(C)$ to sandwiched singularities $X(C, l)$. Remark that this infinite dimensional part $I^{ev}/(f)$ has support on C , the singular locus of $X(C)$.

Let us turn to sandwiched singularities, so we let $X = X(C, l)$, $Y = Y(C, l)$, etc. For these we do not have such an explicit description of T^1 . But what matters for now is:

Proposition (3.7) The map

$$T^1(\Sigma, C, g) \longrightarrow T^1(X)$$

is surjective.

In other words, the vector space $T^1(X)$ of infinitesimal deformations has basis represented by admissible pairs of the form:

$$(0, h), h \in I^{ev} \oplus zI^{ev}$$

$$(n, zf_n - g_n) \text{ with } (n, f_n) \in \mathcal{A}(I, f) \text{ and } (n, g_n) \in \mathcal{A}(I, g)$$

proof: Recall from (2.9) that the normalisation $X_S \rightarrow S$ of the one-parameter family $\{zf - sg = 0\}$ has special fibre $X(C)$ and all other fibres isomorphic to $X(C, l)$. It follows from general principles that there is a relative T^1 -sequence that reads as follows:

$$\dots \rightarrow T_{rel}^1(X_S) \xrightarrow{s} T_{rel}^1(X_S) \rightarrow T^1(X(C)) \rightarrow \dots$$

By proposition (3.4), there is a basis of $T^1(X(C))$ consisting of elements of the form $(0, h) \in I^{ev} \oplus zI^{ev}$ and elements of the form (n, zf_n) with $(n, f_n) \in \mathcal{A}(I, f)$. The question now first is which of these elements can be lifted to $T_{rel}^1(X_S)$. Obviously, the elements $(0, h)$ can be lifted in the trivial way. Elements of the form (n, zf_n) can be lifted if $h_g(n) = 0$, because then for some g_n one has $(n, g_n) \in \mathcal{A}(I, g)$, so $(n, zf_n - sg_n)$ is a lift to $\mathcal{A}(I, zf - sg)$. The condition $h_g(n) = 0$ is also necessary: take any lift $n + sm$, $m \in \tilde{N}$ of n . Then $n + sm$ can be extended to an element of $\mathcal{A}(\tilde{I}, zf - sg)$ if and only if $h_{zf-sg}(n + sm) = 0$ in the free $\mathbb{C}\{z, s\}$ -module \tilde{N}^*/I . But this is $zh_f(n) + zsh_f(m) - sh_g(n) - s^2h_g(m)$. The coefficient of s , $h_g(n)$, has to vanish. So $T_{rel}^1(X_S)$ has S -module generators of the stated type. Now we can restrict to any fibre, e.g. $s = 1$, to get the result for $T^1(X(C, l))$. \square

Proof of Theorem (3.3) for $l(i)$ big

We first prove Theorem(3.3) in the case that all the $l(i)$ are big. In fact we need that for all i :

$$l(i) \geq c_i.$$

Then one can choose $g(x, y) \in I^2$. The crucial consequence of this condition is that the hessian map

$$h_g : N \rightarrow N^*/I$$

is the *zero-map*. Recall that formal smoothness of a transformation of functors $F \rightarrow G$ means that for all small extensions $0 \rightarrow V \rightarrow T \rightarrow S \rightarrow 0$ the canonical map

$$F(T) \rightarrow F(S) \times_{G(S)} G(T)$$

is surjective. Assume that we have (Σ_S, C_S, g_S) representing an element of $Def(\Sigma, C, g)(S)$. We have corresponding elements $(\Sigma_S, Y_S) \in Def(\Sigma, Y)(S)$ and $Def(X)(S)$. Assume that we *can* lift the corresponding deformation of X to T . We will show that then we can find (Σ_T, C_T, g_T) lifting (Σ_S, C_S, g_S) and mapping to the corresponding deformation of X over T .

The functor of δ -constant deformations of the curve C is *unobstructed*, so we can lift f_S to f_T and Σ_S to Σ_T in an R.C.-admissible way. Consider an arbitrary lift of g_S to g_T . The condition of R.C. admissibility of the family $zf_T - g_T$ is expressed by the vanishing of the evaluation map

$$ev_{zf_T - g_T} : \widetilde{N}_\Sigma \rightarrow \widetilde{\mathcal{O}}_\Sigma \otimes_{\mathbb{C}} V$$

By construction of the lift (Σ_T, f_T) , $ev_{f_T} = 0$, and so the condition for admissibility on g_T becomes *independent* of z , that is,

$$ev_{zf_T - g_T} = -ev_{g_T} \in N^* \otimes_{\mathbb{C}} V \subset \widetilde{N}^* \otimes_{\mathbb{C}} V$$

The obstruction element of the family $zf_S - g_S = 0 \in \text{Def}(\tilde{\Sigma}, Y)(S)$ is given by the class of $ev_{zf_T - g_T} = -ev_{g_T}$ in the obstruction space:

$$\left(\widetilde{N^*} / (\tilde{I} + h_{zf-g}(\tilde{N})) \right) \otimes_{\mathbb{C}} V$$

Because $l(i) \geq c_i$ we know that $h_g(\tilde{N}) = 0$, hence the obstruction space is equal to:

$$\left(\widetilde{N^*} / (\tilde{I} + h_{zf}(\tilde{N})) \right) \otimes_{\mathbb{C}} V$$

As the element of $\text{Def}(X)(S)$ can be lifted to $\text{Def}(X)(T)$ by assumption, this obstruction element in fact has to vanish. This means that

$$ev_{g_T} \in I^{**} \otimes_{\mathbb{C}} V$$

Hence ev_{g_T} is of the form $n \rightarrow n(h)$ for some $h \in I \otimes V$. Change the chosen lift to $g'_T = g_T - h$ to get $ev_{g'_T} = 0$. This means that we have lifted (Σ_S, C_S, g_S) to T . As the possible lifts of $\text{Def}(X)(S)$ to $\text{Def}(X)(T)$ form a principal homogeneous space for $T^1 \otimes V$, the result follows from (3.7). \square

The argument for $l(i)$ small

As the transformation $\text{Def}(\Sigma, C, g) \rightarrow \text{Def}(X)$ is smooth for large l , one has that the complete local ring R_1 of the base space of the formal semi-universal deformation of $\text{Def}(\Sigma, C, g)$ is of the form $R[[s_1, \dots, s_N]]$, where R is the complete local ring of the base space of $X(C, l)$. As we know that one can take R to be the completion of an analytic local ring, the same is true for R_1 . Because the R.C. conditions are expressed by polynomial equations, it follows from the ordinary Artin approximation theorem that one can construct an analytic family that is formally semi-universal for $\text{Def}(\Sigma, C, g)$. (Alternatively, one could argue as in [16].) So we get a *smooth* map of analytic spaces

$$B(\Sigma, C, g) \rightarrow B(X)$$

as base spaces for $\text{Def}(\Sigma, C, g) \rightarrow \text{Def}(X)$. Recall that according to (2.9), $X(C, l')$ occurs as a small deformation of $X(C, l)$ if $l'(i) \leq l(i)$. We can apply the theorem of openness of versality to conclude that theorem (3.3) is true for small l as well. \square

Remark (3.8): Also in case that all the $l(i)$ are big, we can get a clearer description of it T^1 . In fact one has a diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & I^{ev}/(f, \Theta_C(g)) & \rightarrow & T^1(\Sigma, C, g) & \rightarrow & T^1(\Sigma, C) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \cong \downarrow \\ 0 & \rightarrow & I^{ev}/(f, \Theta_{\Sigma}(g)) & \rightarrow & T^1(X) & \rightarrow & T^1(\Sigma, C) \rightarrow 0 \end{array}$$

where Θ_{Σ} is the module of vector fields on \tilde{C} generated by $t\partial/\partial t$. In particular one has a dimension formula:

$$\dim(T^1(X)) = \sum_{i \in T} (l(i) - m(i)) + \dim(T^1(\Sigma, C))$$

It is also known that $\dim(T^1(\Sigma, C)) = \tau(C) - \delta$.

It is unclear, however, how big one has to take the $l(i)$, to have the above formula.

For lack of better place, we state and prove here the stability result:

Theorem (3.9): Consider two decorated curves (C, l) and (C, l') . Suppose that for all i one of the following cases occurs:

1. $l(i) = l'(i)$
2. $l(i) \geq c_i, l'(i) \geq c_i$

Then the base spaces of a semi-universal deformation of $X(C, l)$ and $X(C, l')$ are isomorphic up to a smooth factor.

proof: Under the assumption of the theorem one choose a g for $Y(C, l)$ and g' for $Y(C, l')$ with the property that $g - g' \in I^2$. The theorem follows from the principle of I^2 -equivalence [14] (1.16). \square

In case that for all i the second case of (3.9) occurs, the theorem is sharp, as then there always is a special smoothing existing if $l(i) = c_i$ for all i , but not existing if for at least one i we have $l(i) < c_i$, see (4.13).

We will refer to the singularities with $l(i) \geq c_i$ as being in the *stable range*, because here the general phenomenon of stability has set in: if we go higher in the series, the base space gets crossed with a smooth factor, and hence the component structure is the same.

§4 Pictures and Components

We have seen in §3 that the base space of the semi-universal deformation $B(X)$ of a sandwiched singularity $X = X(C, l)$ is, up to a smooth factor, the same as the base space $B(\Sigma, C, g)$. This will lead us to a description of smoothing of X in terms of geometry in the plane.

Decoration as Divisor on \tilde{C}

When a sandwiched singularity $X(C, l)$ is constructed as the normalisation of $zf - g = 0$, it is only the *vanishing order* of g on \tilde{C} that matters. In other words, the functor $Def(\Sigma, C, g)$ contains a little bit too much information.

It will be useful to change the perspective, and try to reformulate everything in terms of *divisors* on \tilde{C} . For example, in (1.3) we introduced the concept of a decorated curve as a curve with *numbers* attached to its branches. From now on we will think of the $l(i)$ as information encoding the unique sub-scheme $\subset \tilde{C}$, whose components $\subset \tilde{C}_i$ have length $l(i)$. Equivalently, we may think of it as a divisor on \tilde{C} . We will be sloppy and denote this subscheme or divisor by the same symbol: $l \subset \tilde{C}$, with components $l(i) \subset \tilde{C}_i$.

The divisor (g) of the function g on \tilde{C} consists of subschemes of length $ord(g, C_i) = c_i + l(i)$, or in terms of divisors:

$$(g) = c + l$$

In our construction of sandwiched singularities, we had to assume that $l(i) \geq m(i)$. This means that l has to contain a certain other scheme:

$$m \subset l$$

Here of course $m := m(C, p)$ is the unique subscheme on \tilde{C} with length $m(i)$ on branch i . These concepts now can be globalised as follows:

Definition (4.1): Let $n : \tilde{C} \rightarrow C$ be the normalisation of any plane curve. We define its multiplicity scheme $m(C) \subset \tilde{C}$ as

$$m(C) = \bigcup_{p \in C} m(C, p).$$

Here $m(C, p)$ denotes the local multiplicity scheme of length $m(i)$ on the i -th branch of \tilde{C}_i .

A pair (C, l) consisting of a curve and a subscheme $l \subset \tilde{C}$ is called a **decorated curve** if

$$m(C) \subset l.$$

We also can define what we call a *one-parameter deformation* of a decorated curve. For this, let S be the germ of a smooth curve, $\{0\}$ is the special point, and $S^* = S - \{0\}$ the set of generic points.

Definition (4.2): A one-parameter deformation of a decorated curve (C, l) over S consists of

- 1) A δ -constant deformation $C_S \rightarrow S$ of C .
- 2) A flat deformation $l_S \subset \tilde{C}_S = \tilde{C} \times S$ of the scheme l .
- 3) With the condition that

$$m_S \subset l_S$$

Here we define the relative m_S of $\tilde{C}_S \rightarrow C_S$ as

$$\overline{\bigcup_{s \in S^*} m(C_s)}$$

We want to stress here that the formation of m is in general **not** compatible with base change in the sense that $m(C_0) \neq (m_S)_0$. But in any case we have an inclusion $m(C_0) \subset (m_S)_0$.

The idea of course is that a one-parameter deformation of a decorated curve gives rise to a one-parameter deformation of the corresponding sandwiched singularity. We first construct geometrically the fibre $X(C_s, D_s)$, $s \in S^*$, which is nothing but a global version of the construction of a sandwiched singularity, see (1.4) and (2.5).

Construction (4.3): Let (C, l) be a decorated curve (in the sense of (4.1)), where C is a curve in a smooth surface Z . Locally on Z we have the situation as in (1.4), so we can construct a modification

$$\rho : \tilde{Z}(C, l) \rightarrow Z$$

by blowing up in points $p = n(q)$, $q \in \text{supp}(l)$. The analytic space $X(C, l)$ is obtained from $\tilde{Z}(C, l) \setminus \tilde{C}$ by blowing down the maximal compact set.

As the above construction involves blowing up, it is not obvious how to obtain a *flat* family of surfaces $X(C_S, l_S)$ directly from any one-parameter deformation of decorated curves (C_S, l_S) . The problem of obtaining the deformation directly via blow-up is related to the problem of finding P -resolutions. We hope to come back

to this theme on a future occasion.

The central theorem of this section is the following

Theorem (4.4): *For any one-parameter deformation (C_S, l_S) of a decorated curve (C, D) there exist a flat one-parameter deformation*

$$X_S \longrightarrow S$$

with the property that:

- 1) $X_0 = X(C, l)$.
- 2) $X_s = X(C_s, l_s)$ for all $s \in S^*$.

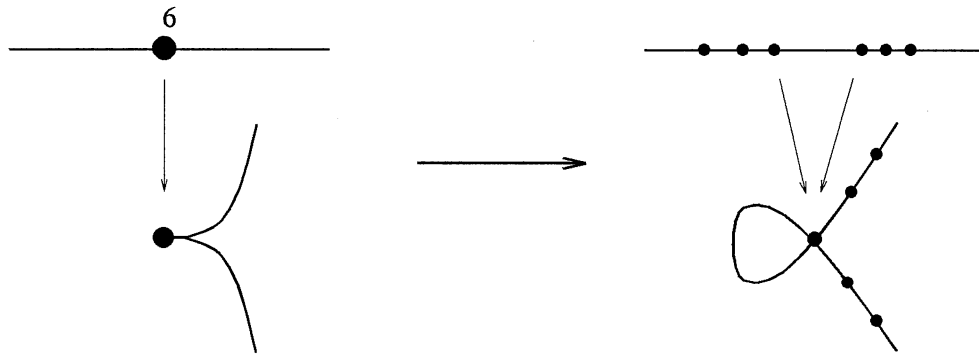
Moreover, every one-parameter deformation of $X(C, l)$ is obtained in this way.

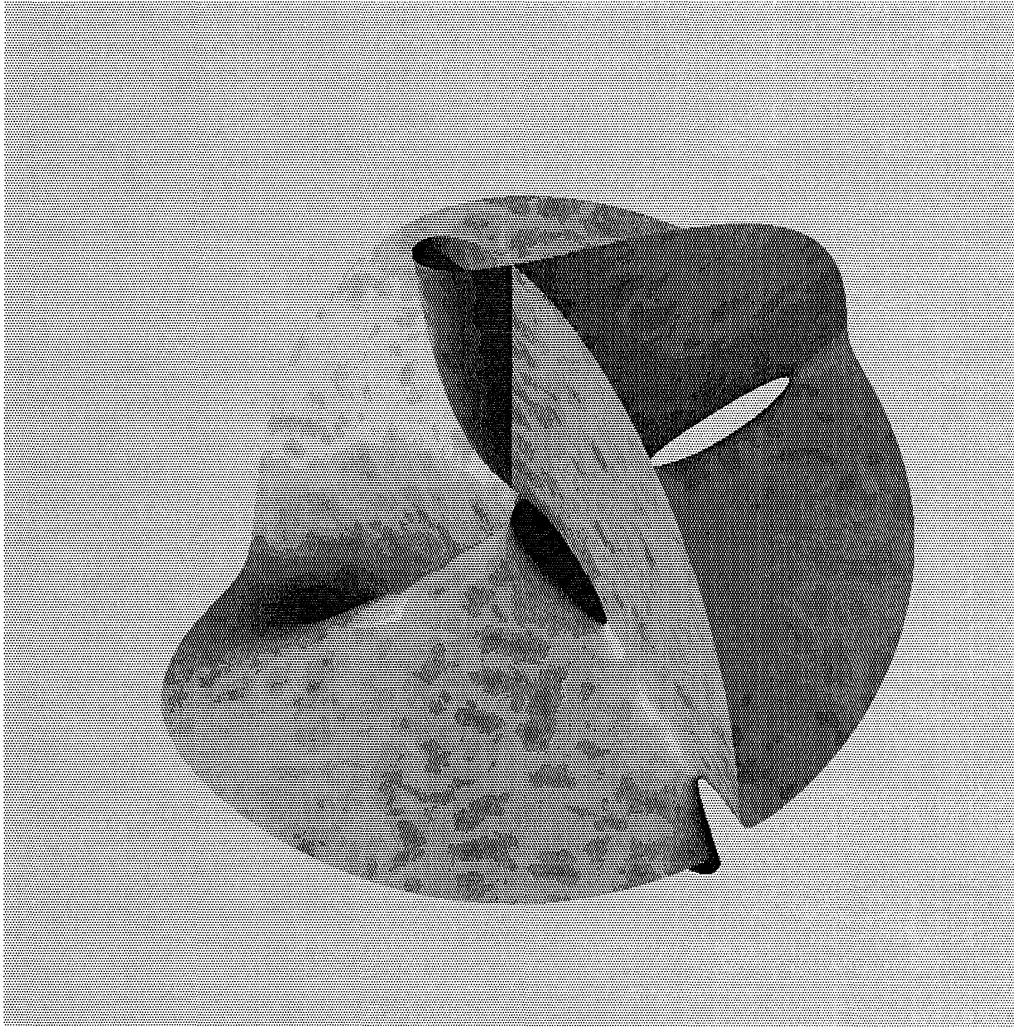
proof: For $X(C, l)$ we choose $g \in \mathbb{C}\{x, y\}$, so that $n : X(C, l) \longrightarrow Y(C, l)$, where $Y(C, l) = \{(x, y, z) : zf(x, y) - g(x, y) = 0\}$, as in §2. Write $g = ah$, with $a, h \in \mathcal{O}_{\tilde{C}}$ such that a generates the conductor $I \subset \mathcal{O}_{\tilde{C}}$, and the divisor of h is l . This is possible, because the vanishing order of g_i was $c_i + l(i)$.

Suppose we are given a one-parameter deformation (C_S, l_S) of (C, l) . The δ -constancy implies that the conductor $I_S \subset \mathcal{O}_{C_S} \subset \mathcal{O}_{\tilde{C}_S}$ is S -flat. So we can lift a, h to elements $a_S, h_S \in \mathcal{O}_{\tilde{C}_S}$. Pick any $g_S \in \mathcal{O}_S\{x, y\}$ lifting g and $h_S a_S$. Define a family of surfaces $Y_S \subset \mathbb{C}^3 \times S$ by the equation $zf_S - g_S = 0$. It follows from the description of I^{ev} from the appendix and the fact that $m(C_s) \subset l_s$ that in fact $(\Sigma_S, Y_S) \in Def(\Sigma, Y(C, l))$. So we can normalise over S to obtain a family $n_S : X_S \longrightarrow Y_S$, which is a flat deformation of $X(C, l)$. As the construction of $X(C_s, l_s)$ is local on Z , one can use theorem (2.5) to conclude that the normalisation of $Y(C_s, l_s) = \{(x, y, z) | zf_s - g_s = 0\}$ is $X(C_s, l_s)$.

Conversely, any deformation $X_S \longrightarrow S$ of $X(C, l)$ can be obtained from $Def(\Sigma, C, g)$ by (3.3), so is of the above type. \square

Examples (4.5): A first remark is that it follows from the theorem that sandwiched singularities only deform into sandwiched singularities. We consider Example (2.7) and the following one-parameter deformation of the decorated curve:

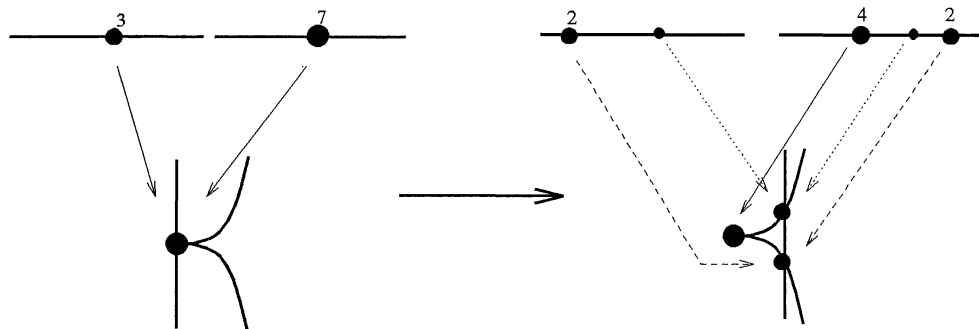




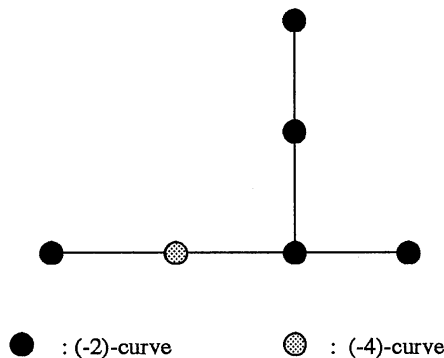
**Deformation of $Y(C, 6)$ corresponding
to smoothing of $X(C, 6)$.**

$$z(y^2 - x^2(x + (0.3))) + (0.2)(x^2)(x - (0.3))(x - (0.6)) = 0$$

As a second example, we consider the following deformation of a decorated curve:



This gives a deformation of the rational surface singularity with the following dual resolution graph:



On the general fibre we have singularities coming from $X(A_2, 4)$, $X(A_1, 2, 2)$ and $X(A_1, 1, 1)$, so we have *two* cones over the rational normal curve of degree 3 and an A_1 singularity, see (1.5).

Picture deformations and Smoothing components

Every singularity X has a collection $\mathcal{S}(X)$ of *smoothing components*, that is irreducible components of its base space $B(X)$ over which smoothing occurs. It is well known that for rational singularities *all* components of $B(X)$ are smoothing components.

Let us try to describe the smoothing components for a sandwiched singularity. Any (non-trivial) one-parameter deformation $X_S \rightarrow S$ is induced from a map $j : S \rightarrow B(X)$. If $X_S \rightarrow S$ is a smoothing, then $j(S)$ is contained in a well-defined component of $B(X)$. In principle, theorem (4.4) gives us a complete description of all one-parameter deformations, in particular of all smoothings. The general fibre C_s of the corresponding family of curves can have all sorts of singularities, see for example (1.5) 3). We will show however that a *generic* one-parameter smoothing is of a very particular type.

Definition (4.6): A one-parameter deformation (C_S, l_S) is called a **picture deformation** if for generic $s \neq 0$ the following holds:

The divisor l_s on \tilde{C}_s is reduced

This implies in particular that the singularities of C_s only consist of ordinary m -tuple points, for various m . By convention, we call an ordinary 1-tuple point a *free point*. So these are the points of the divisor l_s that map to a non-singular point of C_s .

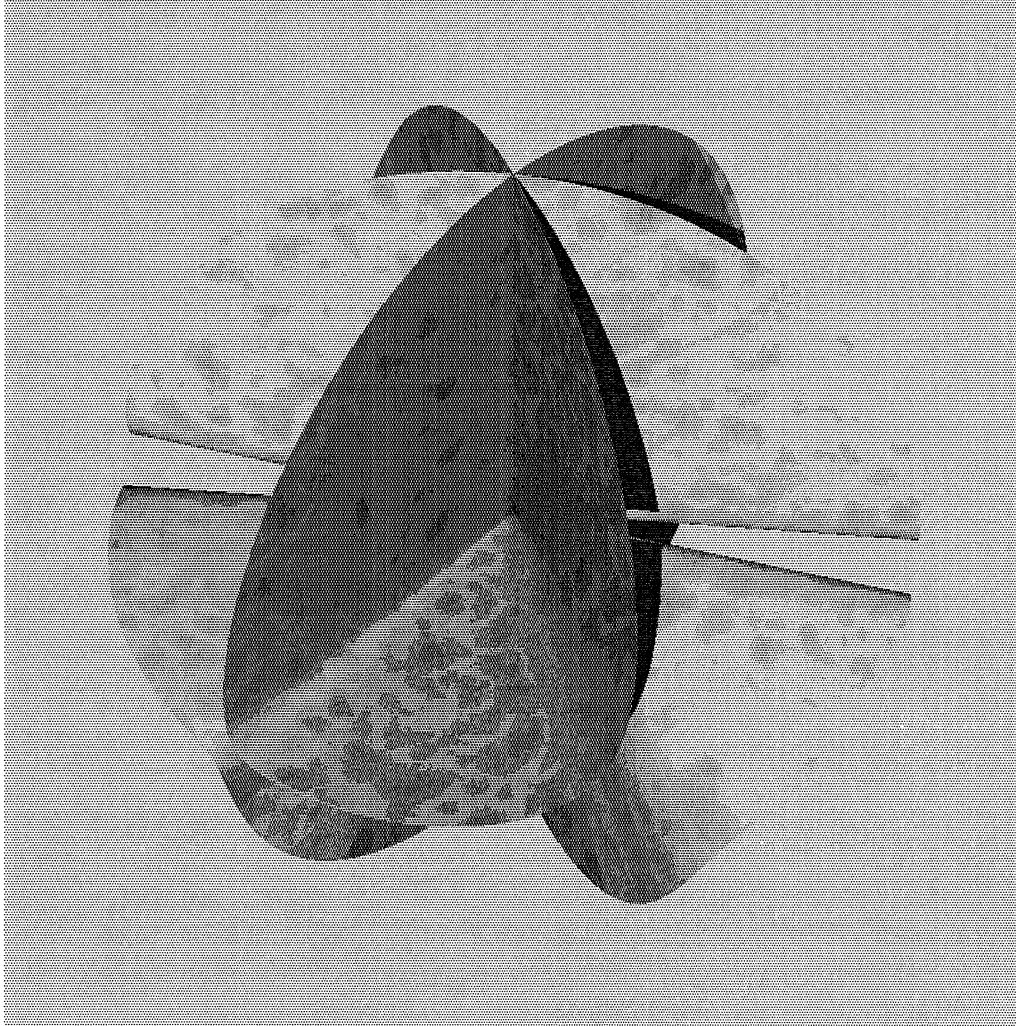
Lemma (4.7): A generic smoothing of $X(C, l)$ is realized by a picture deformation of (C, l) . Hence for every picture deformation $P : X_S \rightarrow S$ a well defined smoothing component $\mathcal{C}(P) \in \mathcal{S}(X)$.

proof: By openness of versality, it suffices to show that for any decorated curve (C, l) , there exists a one parameter picture deformation. To see this, use the Scott-deformation of the curve singularity C , see (1.11). Because one has $l(i) \geq m(i)$ for all i one can make a one-parameter deformation of the decorated curve, such that on the general fibre of the deformation of the curve C we have just ordinary m -fold points. Having done this, it is easy to make a one-parameter deformation of the decorated curve, for which on the general fibre the divisor l_s is reduced. \square

We remark the the procedure in the proof of lemma (4.7) defines a preferred smoothing for each sandwiched singularity X . It should not surprise the reader that this smoothing occurs on the Artin-component, as was already stated at the end of the first section.

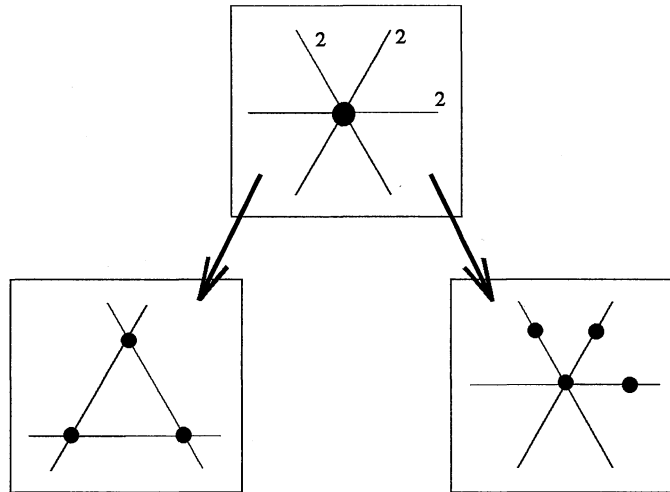
Example (4.8): The cone over the rational normal curve of degree 4 has two different smoothing components, as was discovered by Pinkham, [24]. We will explain how to see this with our method.

As curve C we take three lines in the plane, with the number 2 attached to each branch. The corresponding surface $Y(C, 2, 2, 2)$ in three space looks as follows:

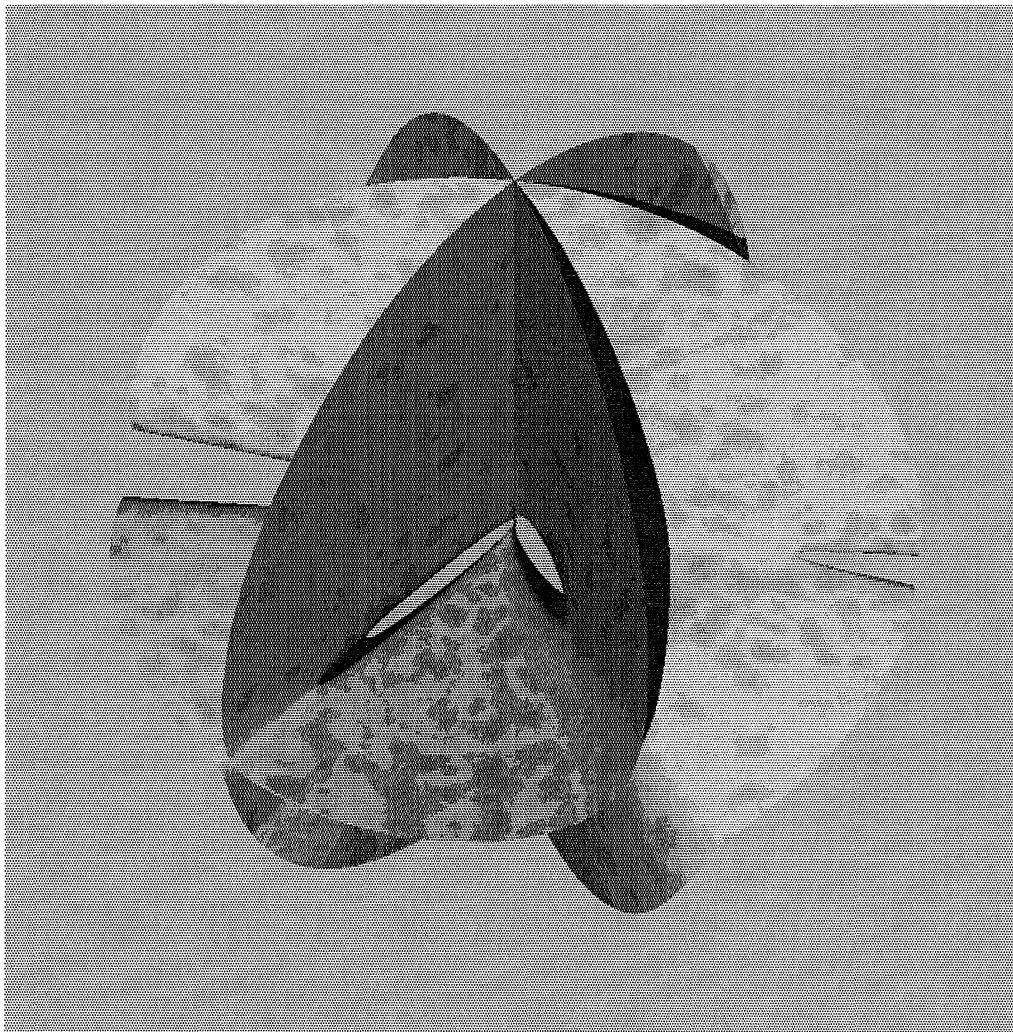


The surface $Y(C, 2, 2, 2) : z(3x^2 - y^2)y + (0.1)x^4 = 0$

There are two possible picture deformations:

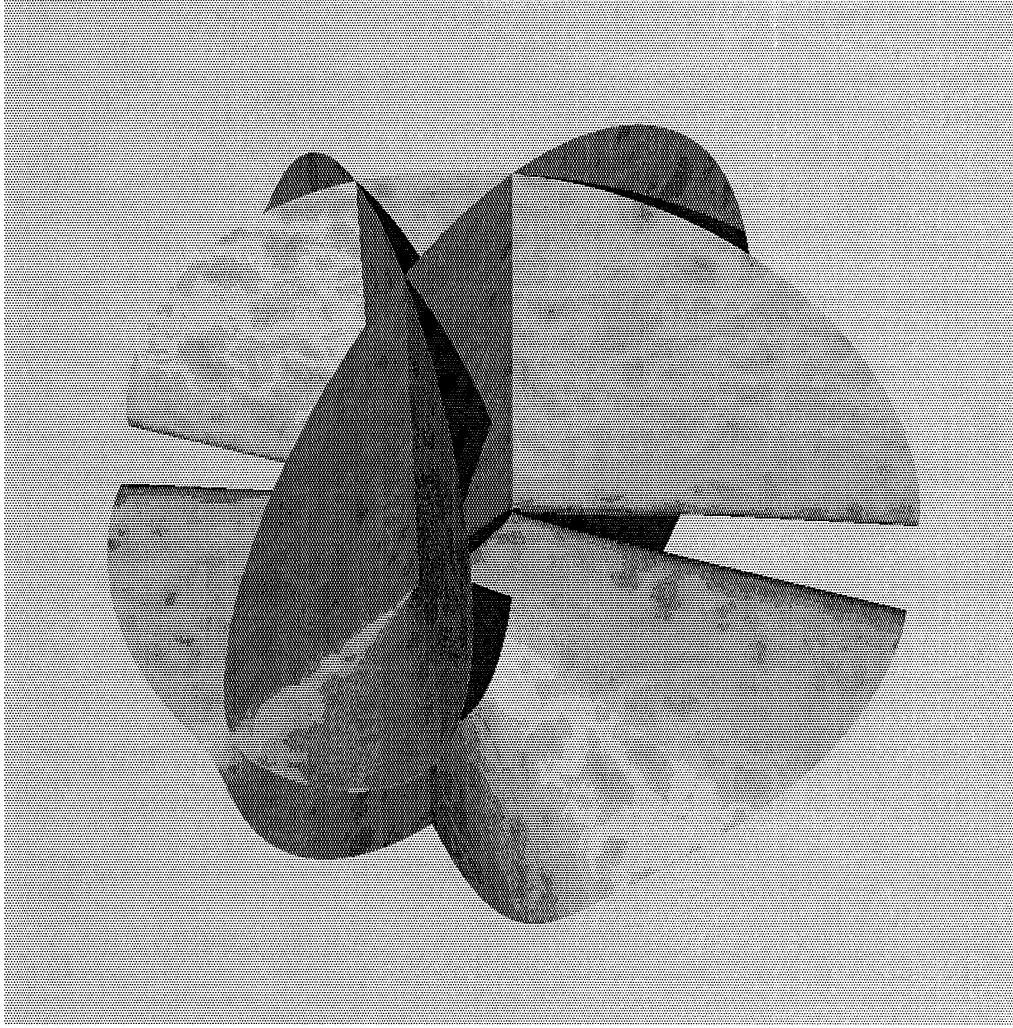


The picture at the right corresponds to the Artin-component, the picture at the left is the qG-smoothing, occurring on the one dimensional component of the base space, The corresponding surfaces in three space look like:



**Deformation of $Y(C, 2, 2, 2)$ corresponding
to smoothing of $X(C, 2, 2, 2)$ over the Artin component**

$$z(3x^2 - y^2)y + (0.1)(x^3)(x - (0.35)) = 0$$



**Deformation of $Y(C, 2, 2, 2)$ corresponding to smoothing
of $X(C, 2, 2, 2)$ over the one-dimensional component**

$$z(3x^2 - y^2)(y + (0.3)) + (0.02)(3x^2 + (0.3)y)^2$$

The map $\phi : \mathcal{S}(X) \longrightarrow \mathcal{I}(C, l)$

Using the picture interpretation of smoothing components, we can define a discrete invariant that contains a lot of interesting information.

Definition (4.9):

Consider a picture deformation $P : X_S \longrightarrow S$. Let P_s be the set of points of C_s obtained as the image of l_s , counted without multiplicity.

Let \mathbf{P} be the free \mathbb{Z} -module on the set P_s , and \mathbf{L} be the free \mathbb{Z} -module on the branches C_i of C (or the branches $C_{i,s}$ of C_s). Then we define the incidence map of a picture deformation

$$\mathbf{I} : \mathbf{P} \longrightarrow \mathbf{L}$$

by defining it for the basis elements $p \in P_s$ by:

$$\mathbf{I}(p) = \sum_{p \in C_i} m(C_{i,s,p}) C_i$$

Here $m(C_{i,s,p})$ denotes the multiplicity of the branch $C_{i,s}$ at the point p .

The information of the incidence map is of course the same as that of the incidence *matrix* with respect to the natural bases of the curves and the points. This matrix is determined up to permutation of columns (*utpoc*), because the lattice \mathbf{P} has a natural bases of points which are determined up to an ordering. The branches of the curve C however can be ordered once and for all, so this gives the lattice \mathbf{L} a fixed basis.

We will use the notation $\mathbf{I}_{Artin} : \mathbf{P}_{Artin} \longrightarrow \mathbf{L}$ for the incidence map of the special deformation considered in (1.11).

Example (4.10): In example (4.8) we get for the incidence matrices for the Artin-component and the small component:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

We will define a set $\mathcal{I}(C, l)$ of maps that could a priori occur as incidence maps of picture deformations of $X(C, l)$.

Notation (4.11): For any free \mathbb{Z} -module \mathbf{P}^* with an (unordered) set of basis elements $p \in \mathbf{P}^*$ we define:

1. the *trivial inner product* by $\langle p, p \rangle = -1$, $\langle p, q \rangle = 0$ for $p \neq q$.
2. the *characteristic vector* $K = \sum_{p \in P} p$.
3. The *quadratic function* $Q : \mathbf{P}^* \longrightarrow \mathbb{Z}$; $v \mapsto \frac{1}{2}(\langle v, v \rangle + \langle K, v \rangle)$
4. Every linear map $\mathbf{I} : \mathbf{P} \longrightarrow \mathbf{L}$ induces by composition with its transpose a quadratic function

$$Q_{\mathbf{I}} : \mathbf{L}^* \longrightarrow \mathbb{Z}; \quad v \mapsto Q(\mathbf{I}^*(v))$$

5. In particular, we become a quadratic function Q_{Artin} belonging to $\mathbf{I}_{Artin} : \mathbf{P}_{Artin} \longrightarrow \mathbf{L}$.

Definition (4.12): Let (C, l) be a decorated curve, \mathbf{L} the free module spanned by its branches. We define:

$$\mathcal{I}(C, l) = \{\mathbf{I} : \mathbf{P} \longrightarrow \mathbf{L} \mid Q_{\mathbf{I}} = Q_{Artin}\}$$

Here \mathbf{P} runs over all possible free \mathbb{Z} -modules with an unordered set P of basis elements, and \mathbf{I} over all possible linear maps with $\mathbf{I}(p) > 0$ for all $p \in P$. We call $\mathcal{I}(C, l)$ the set of combinatorial smoothing components of $X(C, l)$, or of (C, l) .

In more down to earth terms, elements of $\mathcal{I}(C, l)$ can be represented by incidence matrices \mathbf{I} consisting of r row vectors $v_i = (v_{i1}, v_{i2}, \dots)$. The condition on the quadratic function $Q_{\mathbf{I}}$ is translated into the following properties:

1. $\sum_j v_{ij}(v_{ij} - 1) = 2\delta(C_i)$ for all i .
2. $\langle v_i, v_j \rangle = (C_i \cdot C_j)$ for all $i \neq j$.
3. $\sum_j v_{ij} = l(i)$ for all i .

We see that the first two conditions express numerically the δ -constancy of the deformation of C , whereas the third expresses the flatness of l , that is, the conservation

of the number of points on each branch of \tilde{C} .

Therefore, every picture deformation gives us an element of $\mathcal{I}(C, l)$. As result we get a well-defined map

$$\phi : \mathcal{S}(X(C, l)) \longrightarrow \mathcal{I}(C, l)$$

This map ϕ is our first approximation of a combinatorial description of $\mathcal{S}(X)$. In ideal situations one would have that this map is an isomorphism. This happens for instance for cyclic quotient singularities in their standard sandwiched representations, see (6.18). The determination of the *image* of ϕ is equivalent to the problem of realising a combinatorial possibility by a picture deformation of the curve. This can be a very difficult and delicate problem, and depends in general on the *moduli* of the singularity $X(C, l)$, see (6.4). The *fibres* of the map ϕ correspond to irreducible strata in the δ -constant base space of C which realise on their generic point the given incidence map, forgetting about the free points. We do not have any example where injectivity fails, and *hope* that ϕ is always injective, or at least if the C_i are smooth.

In some cases we know the irreducibility of these strata:

Cases (4.13):

1. *There is one component corresponding to $\mathbf{I}_{\text{Artin}}$, the Artin component.*
2. *In case $l(i) \geq c_i$ for all i , there is one component corresponding to \mathbf{I}_{gen} , the incidence matrix of the generic δ -constant deformation of C , where C_s has only ordinary double points.*
3. *The multiplicity of C less than 4.*

proof: The second case is easy, as the corresponding stratum in the δ -constant base space of C is in fact the base space itself. For 3, the cases that the multiplicity of C is 1 or 2 is trivial, the case that the multiplicity is 3 follows from [17]. To give a proof of the first case it suffices to show that the deformation of $X(C, l)$ has simultaneous resolution after base change, if the incidence matrix is the multiplicity matrix. So suppose one has a one-parameter δ -constant deformation $C_S \rightarrow S$ of the curve C over a small disc S , which has the desired incidence matrix. In particular there is an ordinary m -fold point, where m is the multiplicity of the curve C . After a finite base change one has a section: $\sigma : S \rightarrow \mathbb{C}^2 \times S$, with the property that σ maps all $s \in S - \{0\}$ to the ordinary m -fold point, and such that $\sigma(S)$ is smooth. Let us first look what happens in the limit. The δ -constant deformation induces a one parameter R.C. deformation $Y_S \rightarrow S$ of $Y(C)$ and a one-parameter deformation $X_S \rightarrow S$ of $X(C)$. We blow up $\mathbb{C}^3 \times S$ in $\sigma(S) \times z$ -axis. This induces a modification of \bar{Y}_S of Y_S , and therefore a modification \bar{X}_S of X_S . The special fibre is obtained from the blow up of \mathbb{C}^2 by glueing smooth planes along the branches of the strict transform of C . On the general fibre the same is done at the ordinary m -fold point. Blowing up sections one after another we conclude that the deformation $X_S \rightarrow S$ has *simultaneous improvement*. To conclude the proof for the sandwiched singularity $X(C, l)$ one uses the deformation (2.11) of $X(C)$, which is unobstructed against our deformation $X_S \rightarrow S$. This deformation is realised by a deformation of the improvement so one deduces that the deformation of $X(C, l)$ has, after finite base change, a modification for which on every fibre just rational double points occur. \square

Reduced fundamental cycle

Suppose one has $X = X(C, l) \cong X(C', l')$, two different sandwiched representations of the same singularity. We then have two different maps $\phi : \mathcal{S}(X) \rightarrow \mathcal{I}(C, l)$, $\phi' : \mathcal{S}(X) \rightarrow \mathcal{I}(C', l')$. The question arises how to relate the incidence matrix of a

smoothing of X in the sandwiched representation (C, l) to the incidence matrix of the *same* smoothing of X in the sandwiched representation (C', l') .

If C and C' have different numbers of branches, it is not clear at all how to relate combinatorial solutions for (C, l) and (C', l') . In fact, it may very well happen that $\mathcal{I}(C, l)$ and $\mathcal{I}(C', l')$ have different number of elements, see (4.20). The complete combinatorial information contained in *all* different sandwiched representations seems to require some new information. We hope to come back to this matter in a future paper.

There is however a simple answer if both curves C and C' only have *smooth* branches. Then $X = X(C, l) \cong X(C', l')$ is a rational surface singularity with reduced fundamental cycle. There are at most $\text{mult}(X)$ sandwiched representations, with curves just having smooth branches, see (1.5). In this case it is true that each representation gives equivalent information, and that the combinatorial structure can be related *directly* to the resolution graph Γ . In order to formulate the result we need some notions from [18].

Let X be a rational surface singularity of multiplicity $m = \text{mult}(X)$ and with reduced fundamental cycle, and resolution graph Γ . A general hypersurface section of X is isomorphic to the m coordinate axes in \mathbb{C}^m . We denote by $\{H_p : p \in \mathcal{H}\}$ the set of irreducible components of this general hypersurface section. We can lift the general hypersurface section to the minimal resolution of X . Then the strict transform \tilde{H}_p of H_p intersects exactly one exceptional curve which we call E_p .

Definition (4.14):

1. For $p, q \in \mathcal{H}$, $p \neq q$, one puts $l(p, q)$ to be one plus the number of exceptional curves in the chain from E_p to E_q .
2. For $p, q, r \in \mathcal{H}$, p, q, r all different, one puts $\rho(p, q; r)$ to be the number of exceptional curves in the intersection of the chain from E_p to E_r and the chain from E_q to E_r .
3. We call the $l(p, q)$ and $\rho(p, q; r)$ the length and overlap functions of the graph Γ .

Definition (4.15): Let X be a rational surface singularity with reduced fundamental cycle. A Γ -representation of X consist of vectors

$$v_{pq} = v_{qp} \in \{0, 1\}^n \text{ for some } n; \quad p \neq q \in \mathcal{H}$$

with the conditions:

1. The number of non-zero entries in v_{pq} is $l(p, q)$
2. $v_{pq} + v_{qr} + v_{rp} = 0$ modulo 2
3. $\langle v_{pq}, v_{pr} \rangle = \rho(q, r; p)$.

Here $\langle \cdot, \cdot \rangle$ denotes the ordinary inner product.

So in a Γ -representation, the vectors v_{pq} represent the chains in Γ , and inner products represent lengths and overlaps of chains. Of course, Γ -representations are considered utpoc, that is, up to permutation of columns.

Recall from (1.5) that for every choice $p \in \mathcal{H}$ we get a sandwiched representation (C, l) . A combinatorial solution of the smoothing problem for the (C, l) gives us vectors v_{pq} with exactly $l(p, q)$ non-zero entries, for this fixed chosen p and arbitrary $q \in \mathcal{H}$, $q \neq p$. Moreover, one has $\langle v_{pq}, v_{pr} \rangle = \rho(q, r; p)$. Therefore, it is trivial that any Γ -representation of $X = X(C, l)$ gives a combinatorial smoothing of (C, l) . The main result on Γ -representations is that the converse also holds.

Theorem (4.16): Let $X(C, l)$ be a sandwiched representation with smooth

branches of X , corresponding to a $p \in \mathcal{H}$. Then any solution of the combinatorial smoothing problem for (C, l) gives rise to a Γ -representation of X , and vice versa.

It follows that if one has a decorated curve (C', l') , such that C' has smooth branches and $X \cong X(C', l')$, then there is a bijection: $\psi : \mathcal{I}(C, l) \rightarrow \mathcal{I}(C', l')$

proof: As described above, we have already v_{pq} for some fixed $p \in \mathcal{H}$, satisfying the statement about the intersection product. The second condition for a Γ -representation for X gives that one can define $v_{rq} = v_{pr} + v_{pq} \bmod 2$. Having done that, one has $v_{rq} + v_{sr} + v_{qs} = v_{pr} + v_{pq} + v_{ps} + v_{pr} + v_{pq} + v_{ps} = 0 \bmod 2$. We have to prove that the number of nonzero entries in v_{qr} is equal to $l(q, r)$. But this is by definition equal to

$$l(p, q) + l(p, r) - \langle v_{pq}, v_{pr} \rangle = l(p, q) + l(p, r) - \rho(q, r; p).$$

It follows from an easy property of trees that this number is equal to $l(q, r)$. Finally, we must show that $\langle v_{qs}, v_{qr} \rangle = \rho(r, s; q)$. We do this for the case $s = p$ first. Because $v_{qr} = v_{pr} + v_{pq} \bmod 2$, this translates to $\langle v_{pq}, v_{pr} \rangle + \langle v_{pq}, v_{pq} \rangle - 2 \langle v_{pr}, v_{pq} \rangle = \rho(r, p; q)$. But the left hand side we know to be $l(p, q) - \rho(r, q; p)$. The general case is similar, as we now also know all $\langle v_{qp}, v_{qr} \rangle$ for all q . \square

We need some results from [18]. A general fibre of a one-parameter smoothing of a rational surface singularity with reduced fundamental cycle X can also be given directly (i.e. without using projections) by the following system of equations:

$$z_{pq}z_{qp} = T_{pq}(x) ; \quad z_{pq} - z_{qr} = \phi(p, q; r)(x)$$

satisfying the so-called *Rim* equations:

$$T_{pq} = \phi(r, q; p)\phi(r, p; q)$$

$$\phi(p, q; s) + \phi(q, r; s) + \phi(r, p; s) = 0$$

We may suppose that all roots of T_{pq} are distinct.

Definition (4.17): One can define vectors v_{pq} in the following way:

Look at the roots of all T_{pq} on the complex line. The total number of them (counted without multiplicity) we call n . Take any numbering p_1, \dots, p_n of these roots. Then entry j of v_{pq} is 1 exactly when p_j is a root of the function T_{pq} .

Lemma (4.18): With these definitions of v_{pq} one gets a Γ -representation of X .

proof: Because of the Rim equations $T_{pq} = \phi(r, q; p)\phi(r, p; q)$ one sees that the roots of $\phi(r, q; p)$ and of $\phi(r, p; q)$ are different and all of multiplicity one. Look at the product $T_{pq}T_{qr}T_{rp}$, which by the Rim equations is equal to

$$\phi(r, p; q)\phi(r, q; p)\phi(p, q; r)\phi(p, r; q)\phi(q, r; p)\phi(q, p; r)$$

which is equal to

$$-\phi(r, p; q)^2\phi(r, q; p)^2\phi(q, p; r)^2.$$

So every root appears twice, from which $v_{pq} + v_{qr} + v_{rp} = 0$ modulo two follows. For the same reason and because the number of roots of $\phi(q, r; p)$ is equal to $\rho(q, r; p)$ the statement $\langle v_{pq}, v_{pr} \rangle = \rho(q, r; p)$ follows. \square

Theorem (4.19): Suppose $X = X(C, l) \cong X(C', l')$, with both C and C' having smooth branches. Then the diagram:

$$\begin{array}{ccc} \phi : \mathcal{S}(X) & \longrightarrow & \mathcal{I}(C, l) \\ \downarrow & & \downarrow \psi \\ \phi' : \mathcal{S}(X) & \longrightarrow & \mathcal{I}(C', l') \end{array}$$

is commutative, with ψ as in (4.16).

proof: Fix a $p \in \mathcal{H}$, and some $s \in \mathcal{H}$, different from p . To make notation simpler, we put:

$$z_q = z_{qp}; \quad T_{pq} = T_q; \quad z_{ps} = y; \quad \phi(q, s; p) = \phi(q)$$

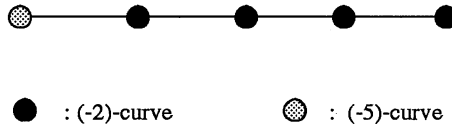
in the above equations.

One can eliminate variables in the equations of the smoothing, as to get the following system of equations:

$$\begin{aligned} z_q(y - \phi_q(x)) &= T_q \\ z_q z_r &= \frac{T_r}{\phi_q(x) - \phi_r(x)} z_q - \frac{T_q}{\phi_q(x) - \phi_r(x)} z_r. \end{aligned}$$

We can interpret the equations $\prod (y - \phi_q)$ as a δ -constant deformation of a curve C , and the T_q as divisors on the branch defined by $(y - \phi_q) = 0$. In this way one gets a pair (C, l) , and it is left to the reader to show that the equations just given gives the space $X(C, l)$ of (2.5). \square

Example (4.20): It is not true in general that $\mathcal{I}(C, l) = \mathcal{I}(C', l')$, for decorated curves such that $X(C, l) \cong X(C', l')$. For example, take (C, l) the decorated curve $(C, 9)$, with C the curve given by the equation $y^5 + x^4 = 0$. The sandwiched singularity $X(C, l)$ is a cyclic quotient singularity with dual graph of resolution:



In a sandwiched representation $X(C', l')$ with smooth branches as considered in the subsection on cyclic quotient singularities, we have that the set $\mathcal{I}(C', l')$ consists of two elements. However, the set $\mathcal{I}(C, l)$ consists of the elements $(4, 1, 1, 1, 1, 1)$, $(3, 2, 2, 2)$ and $(3, 3, 1, 1, 1)$. As one can realise the first two incidence matrices, it follows that there exist no deformation of the curve $\{y^5 + x^4 = 0\}$ to a curve with two D_4 singularities. An elementary argument for the non-existence of such an adjacency runs for example as follows: consider the line L between the two D_4 -singularities. So L intersects the curve with multiplicity ≥ 6 . If we degenerate to $\{y^5 + x^4 = 0\}$, L becomes a smooth curve with contact ≥ 6 . But the maximal contact with $\{y^5 + x^4 = 0\}$ is only 4.

This example clearly shows that in case C has non-smooth branches the set $\mathcal{I}(C, l)$ does not contain all combinatorial information of the situation.

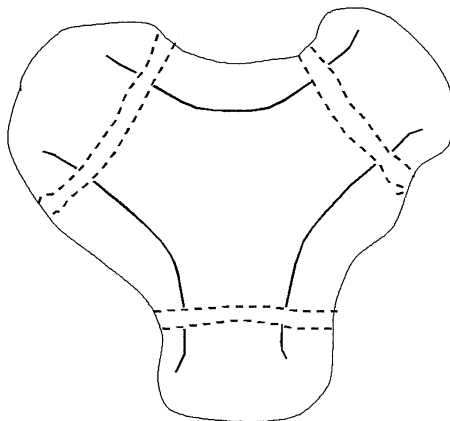
§5 Topological Aspects

In this section we will study in more detail the topology of the smoothing obtained from a picture deformation as explained in section 4.

Let as always $X = X(C, l)$ be a sandwiched singularity. The first remark is that a picture deformation $P : X_S \rightarrow S$ gives us a precise description of the Milnor fibre $F := X_s$ occuring on the component $C(P) \in S(X)$. One has:

Proposition (5.1): *Choose a small contractible disc D representing the smooth space Z . Let P_s and C_s the points and the curves in D of the general fibre of the picture deformation $Y_S \rightarrow S$. Then the Milnor fibre $F := X_s$ is diffeomorphic to the complement of the strict transform of C_s on D blown up in P_s .*

proof: This is just a special case of the construction in (4.5). The remark is that because we have a picture deformation, the curve C_s has only ordinary d -fold points, which are resolved by one blow up. \square



**Model of the Milnor fibre
over the small component of the (-4) .**

The Homology of the Milnor fibre

From this model one readily obtains a description of the homology of the Milnor fibre F in terms of the incidence map $\mathbf{I} : \mathbf{P} \rightarrow \mathbf{L}$. (In the sequel, all (co)homology is with *integer* coefficients.)

Theorem (5.2): *There is an exact sequence:*

$$0 \rightarrow H_2(F) \rightarrow \mathbf{P} \xrightarrow{\mathbf{I}} \mathbf{L} \rightarrow H_1(F) \rightarrow 0$$

proof: This is an easy application of the Mayer-Vietoris sequence. Nevertheless, let us spell it out: let \tilde{D} be the space D blown up in the points P_s and denote by $\tilde{C} = \coprod \tilde{C}_i \subset \tilde{D}$ the strict transform of C . The space \tilde{D}^* obtained by removing the interior of a small tubular neighbourhood $\tilde{T} = \coprod \tilde{T}_i$ around \tilde{C} , is diffeomorphic to the Milnor fibre. Note that $\tilde{T} \cap \tilde{D}^* = \partial \tilde{T}$, the boundary of the tubular neighbourhood. We write the Mayer-Vietoris sequence for the pair (\tilde{T}, \tilde{D}^*) :

$$\dots H_2(\tilde{D}^* \cap \tilde{T}) \rightarrow H_2(\tilde{D}^*) \oplus H_2(\tilde{T}) \rightarrow H_2(\tilde{D}) \rightarrow H_1(\tilde{D}^* \cap \tilde{T}) \dots$$

which reduces to:

$$0 \rightarrow H_2(\tilde{D}^*) \rightarrow H_2(\tilde{D}) \rightarrow H_1(\partial \tilde{T}) \rightarrow H_1(\tilde{D}^*) \rightarrow 0$$

One has an isomorphism $\mathbf{P} \cong H_2(\tilde{D})$, by taking a point $p \in P_s$ to the fundamental class $[E_p]$ of the exceptional \mathbb{P}^1 over p . Furthermore, $\mathbf{L} \cong H_1(\partial \tilde{T})$, via the map that

associates to a branch of C a small loop running in the positive direction on the boundary $\tilde{D}^* \cap \tilde{T} = \partial\tilde{T}$ of the \tilde{T} . From the definition of the boundary map in the Mayer-Vietoris sequence one gets that the geometrical description of the resulting map $\mathbf{P} \longrightarrow \mathbf{L}$ is as follows: take $p \in P_s$, look at E_p , intersect this with \tilde{T} , and take its boundary; this is a collection of circles around \tilde{C} , hence an element of \mathbf{L} . This means, the resulting map $\mathbf{P} \longrightarrow \mathbf{L}$ is exactly the incidence map \mathbf{I} . \square

With the same easy we get the cohomology of the Milnor fibre:

Corollary (5.3): *Applying $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ to $\mathbf{P} \longrightarrow \mathbf{L}$ we get the map*

$$\mathbf{L}^* \xrightarrow{\mathbf{I}^*} \mathbf{P}^*$$

with kernel $H^1(F) = 0$ and cokernel $H^2(F)$.

We note that the vanishing of $H^1(F)$ for general normal surface singularities is due Greuel and Steenbrink, [10]. Equivalently, the group $H_1(F)$ is finite. It is not clear to us how to see directly that the incidence map has maximal rank. Also note that we have a $\mu = 0$ -smoothing, meaning that $\mu(F) := \text{rk}(H_2(F)) = 0$ exactly when $\text{rk}(\mathbf{L}) = \text{rk}(\mathbf{P})$, that is, if the incidence matrix is a square matrix.

Example (5.4): For the two components in Pinkhams example we find immediately from the incidence matrices (4.10):

Artin-component: $H_1(F) = 0$; $H_2(F) = \mathbb{Z}$

Small component: $H_1(F) = \mathbb{Z}/2$; $H_2(F) = 0$

Remark also that in case there is a free point on *every* branch, $H_1(F) = 0$. It turns out that even the fundamental group is zero. For this we refer to the discussion on the fundamental group at the end of this section.

The Intersection form

Recall that we have an intersection form on the Milnor fibre. To be more precise, we have a natural map

$$H_2(F) \xrightarrow{j} H_2(F, \partial F) \cong H^2(F),$$

which by transposition gives us the intersection form

$$\langle -, - \rangle: H_2(F) \times H_2(F) \longrightarrow \mathbb{Z}$$

This intersection form is very easy to describe in terms of the incidence matrix. For this, we put on \mathbf{P} the trivial inner product, $\langle p, q \rangle = 0$ if $p \neq q$; $p, q \in P_s$ and $\langle p, p \rangle = -1$ for all $p \in P_s$ as in (4.9). Using $\langle -, - \rangle$ we get an identification $\mathbf{P} \longrightarrow \mathbf{P}^*$.

Theorem (5.5): *The intersection form on $H_2(F) \subset \mathbf{P}$ is the restriction of the trivial inner product on \mathbf{P} .*

proof: It is clear that $\mathbf{P} \approx H_2(\tilde{D})$ is an isomorphism of inner product spaces. The statement is a formal consequence of how homology and cohomology are related, but let us give a geometrical argument. According to (5.2) a cycle $c = \sum c_p \cdot p \in H_2(F) \subset \mathbf{P}$ is represented by closed surfaces that consist of parts of two types: 1) some E_p 's, with discs removed around the intersection points $\tilde{C} \cap E_p$ and 2) some cylinders running inside the boundary $\partial\tilde{T}$ of the tubular neighbourhood. Now the self-intersection of a cycle can be computed by shifting the cycle to one that is

transverse to the original one, and then count the number of intersection points. We can shift the cylinders in and out without introducing intersection points by varying the radius of the tubular neighbourhood, so the self intersection is as if the cylinders just were not there. So we see that the self-intersection of a cycle is computed as the self-intersection of $\sum c_p \cdot [E_p] \in \mathbf{P}$ as stated in the proposition. \square

Corollary (5.6): *The intersection form on the Milnor fibre is negative definite.* Of course, this is well known for any smoothing of a rational surface singularity, but here we see it really happen.

On the Monodromy Group

The Milnor fibres form a fibration over $\mathcal{C}(P)^*$, the complement of the discriminant in $\mathcal{C}(P)$. If we fix a base point s , we get a monodromy representation

$$\pi_1(S(P)^*, s) \longrightarrow \text{Aut}(H_2(F), < -, - >)$$

whose image we denote by $G(P)$. If we look at what happens to our picture (P_s, C_s) when we move with s over the whole smoothing component, we see that when we return, our points P_s undergo some permutation, whereas the curves are not permuted. But obviously, the incidence structure must remain intact. As a consequence we see:

Proposition (5.7):

$$G(P) \subseteq \overline{G}(P) := \{M : \mathbf{P} \longrightarrow \mathbf{P} \text{ permutation matrix} \mid \mathbf{I} \cdot M = \mathbf{I} \}$$

Conjecture (5.8): *We claim that in fact one has*

$$G(P) = \overline{G}(P).$$

We think it is true in all the examples we have studied. It is true for the Artin-component, and furthermore it is true for all components of a cyclic quotient singularity, by a result of K. Behnke and J. Christophersen, [5]. It is *not* obvious to us that the group $\overline{G}(P)$ is independent of the chosen sandwiched representation. However, in case of reduced fundamental cycle \overline{G} can be characterized in terms of the Γ -representation as the group of permutations of $\{0, 1\}^n$ that operate trivial on all vectors v_{pq} .

Divisors on smoothings

One can give a complete description of divisor classes on X and the total space $\mathcal{X} := X_S$ of a smoothing in terms of the incidence matrix.

Recall that on any normal singularity one defines the *class group* as

$$Cl(X) := \{\text{Weil divisors on } X\} / \{\text{Principal divisors}\}$$

One has for a rational surface singularity $Cl(X) \cong H_1(X - \{0\}) \cong H_1(L)$, where L is the link of the singularity. In general one has for rational singularities $Cl(\mathcal{X}) = H^2(\mathcal{X} - \{0\})$, by a theorem of Flenner. By a result of Looijenga-Wahl one has furthermore that

$$H^2(\mathcal{X} - \{0\}) \cong H^2(F)^\pi$$

where $H^2(F)^\pi \subset H^2(F)$ denotes that part of the cohomology that is invariant under the monodromy of the family $X_S \rightarrow S$. Because in our case the monodromy is of finite order, we can always make a finite base-change to arrive at the situation that the monodromy is trivial, so that we have

$$Cl(\mathcal{X}) \cong H^2(\mathcal{X} - \{0\}) \cong H^2(F)$$

From now on we always will assume we have done this. The link L is isomorphic to the boundary ∂F of the Milnor fibre. The specialisation map

$$Cl(\mathcal{X}) \rightarrow Cl(X)$$

can be identified with a map

$$H^2(F) \rightarrow H_1(\partial F)$$

This map is part of the long exact homology sequence of the pair $(\partial F, F)$ when we use the isomorphism $H^2(F) \cong H_2(F, \partial F)$. This can now be put in a big exact diagram, describing in a combinatorial way the specialisation of divisors:

Diagram (5.9):

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & \mathbf{L}^* & \xrightarrow{=} & \mathbf{L}^* & & \\
& & \mathbf{I}^* \downarrow & & \mathbf{J} \downarrow & & \\
0 & \rightarrow & \ker(\mathbf{I}) & \rightarrow & \mathbf{P} & \xrightarrow{\mathbf{I}} & \mathbf{L} \rightarrow \text{Coker}(\mathbf{I}) \rightarrow 0 \\
& & \cong \downarrow & & \downarrow & & \cong \downarrow \\
0 & \rightarrow & H_2(F) & \rightarrow & H^2(F) & \rightarrow & H_1(\partial F) \rightarrow H_1(F) \rightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & \rightarrow & 0
\end{array}$$

The map $\mathbf{J} : \mathbf{L}^* \rightarrow \mathbf{L}$ is defined as the composition $\mathbf{I}\mathbf{I}^*$. It is easy to see that the associated quadratic form $\mathbf{L}^* \times \mathbf{L}^* \rightarrow \mathbb{Z}$ is just the same as the quadratic form belonging to Q_{Artin} of (4.11).

So we see that a sandwiched representation gives rise to a particular realisation of the class group as the discriminant of a quadratic form on a preferred set of generators.

Corollary (5.10): [23]. *For any (combinatorial) $\mu = 0$ smoothing $|H_1(L)|$ is a square.*

proof: This number is $\det(\mathbf{J}) = \det(\mathbf{I}\mathbf{I}^*) = \det(\mathbf{I})^2$. □

The Canonical Class

The relative canonical class of a smoothing restricts to the canonical class $K \in H^2(F)$ of the Milnor fibre. There is a natural lift of this class to an element of $\mathbf{P}^* = \mathbf{P}$.

Theorem (5.11): *The canonical class is represented by*

$$K = \sum_{p \in P_s} p,$$

that is by the vector $(1, 1, \dots, 1)$ in \mathbf{P}^* .

proof: The Milnor fibre is just an open part of \tilde{D} , the blow up of Z in the points $p \in P_s$. Hence, its canonical class is the restriction of the canonical of \tilde{D} , which clearly is as stated. \square

Corollary (5.12): *A smoothing $P : \mathcal{X} \rightarrow S$ is a qG-smoothing in the sense of [20], if and only if*

$$K \in \text{Im}(\mathbf{L}^* \otimes \mathbb{Q} \rightarrow \mathbf{P}^* \otimes \mathbb{Q}),$$

that is, if and only if $(1, 1, \dots, 1)$ is a rational linear combination of the rows of the incidence matrix.

The proof of the following theorem illustrates our lack of insight into combinatorial matters:

Theorem (5.13): *A gG-smoothing of a sandwiched singularity $X(C, l)$ has no free points.*

proof: Suppose there is a free point on, say, the first branch. We claim that then also the singularity $X(C, l_k)$, defined by the *same* curve C , and with $l_k(i) = l(i)$ for all $i \neq 1$ and $l_k(1) = l(1) + k$, will have a qG-smoothing. It suffices to show the claim for $k = 1$. Having a picture deformation of $X(C, l)$, one gets a picture deformation of $X(C, l_1)$, by plotting some extra point somewhere, not at one of the points of the picture deformation of $X(C, l)$. So the incidence matrix \mathbf{I}_1 of the picture deformation of $X(C, l_1)$ is obtained from the incidence matrix \mathbf{I} of $X(C, l)$ by adding the column $(1, 0, \dots, 0)^t$. Because we have a qG-smoothing for $X(C, l)$ one has:

$$K = (1, \dots, 1) = \sum \alpha_i I^*(C_i)$$

for some $\alpha_i \in \mathbb{Q}$ by (5.11). But then it follows that

$$(K, 1) = \sum \alpha_i I_1^*(C_i),$$

hence one has a qG-smoothing for $X(C, l_1)$ by (5.11). This shows the claim.

We arrive at a contradiction as follows. According to Kollár [20] (6.3) one has the following Theorem:

Let $(M, F) \subset (\tilde{X}, E)$, where \tilde{X} is the minimal resolution of a rational surface singularity X . Suppose that M is the resolution of a qG-singularity. Then, contracting F and all (-2) curves of E not intersecting F , one gets a P -modification of X , giving rise to a smoothing component of X .

Applying this to $X(C, l_k)$ one deduces that the number of smoothing components of $X(C, l_k)$ is *unbounded* if $k \rightarrow \infty$. This is in contradiction with the stability result (3.9). \square

On the fundamental group of the Milnor fibre

As the Milnor fibre is the complement of \tilde{C} in the smooth surface \tilde{D} , its fundamental group can be described in a manner similar to the Lefschetz-van Kampen-Zariski method for the fundamental group of a plane curve complement. It turns out that again in the case that all branches of the curve C are smooth, so in the case that X has reduced fundamental cycle, there is a really simple presentation for $\pi_1(F)$.

Consider as before a picture deformation $P : X_S \rightarrow S$ of $X = X(C, l)$. We fix a generic projection to a line with coordinate x . We may assume that for all $s \neq 0$ the

curve $C_{i,s}$ has precisely $m - r$ vertical tangents, or, ramification points Q_1, \dots, Q_{m-r} (m =multiplicity of C , r number of branches). Also we can assume that each of these points is distinct from each of the points P_s of the picture, and all the points have distinct projections on the x -axis. We fix a value for s . We will choose a representative of Z of the form $U \times V$, where U is a disc in the y -axis, and V a disc in the x -axis in such a way that:

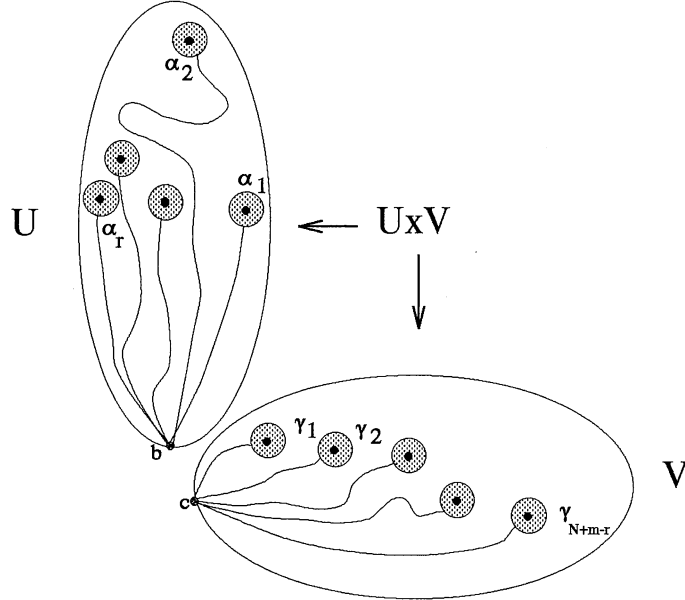
1. P_s and $Q_i \subset \text{Interior}(U \times V)$
2. $\partial(C_{i,s}) \cap U \times V \subset U \times \partial V$.

We choose a base point $a \in \partial U \times \partial V$. Denote by x_1, x_2, \dots, x_N , the x -coordinates of all the points of the picture, by $x_{N+1}, \dots, x_{N+m-r}$ the x -coordinates of the ramification points. Then for all $x \in V - \{x_1, x_2, \dots, x_{N+m-r}\}$ the intersection of $U \times x$ consists of a finite number of points, equal to the number r of branches of C . We choose non-intersecting paths $\alpha_1, \alpha_2, \dots, \alpha_r$ running from $b \in U$ to each of the r intersection point $C \cap U \times \{c\}$, and back again, in the usual way, such that the product

$$\alpha_1 \cdot \alpha_2 \cdots \alpha_r$$

is homotopic to the loop consisting of ∂U , with positive direction. This implies a certain *ordering* of the components C_i .

Also we choose a system of non-intersecting paths $\gamma_1, \dots, \gamma_N, \dots, \gamma_{N+m-r}$ running to the points $x_1, x_2, \dots, x_{N+m-r}$ and back.



The usual systems of paths

It is clear that the paths $\alpha_1, \alpha_2, \dots, \alpha_r$ form a system of generators for $\pi_1(U - U \cap C, c) \rightarrow \pi_1(F, a)$. By van Kampen's theorem, all the relations between these generators arise from the identifications that occur above the points x_i . To see what these relations are, we only have to analyse what happens over the preimage of that paths γ_i . There are two cases:

1. For $i = 1, 2, \dots, N$, we have that above γ_i some of the generators "come together" at a point $p \in P_s$. If these generators are $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}$, with $i_1 < i_2 < \dots < i_k$,

then one gets a relation

$$\alpha_{i_1} \cdot \alpha_{i_2} \cdots \alpha_{i_k} = 1$$

because in the blow-up at p one introduces precisely a \mathbb{P}^1 with holes, the boundary circles of which correspond to the α 's.

2. For $i = N + 1, \dots, N + m - r$, something different happens. When we make a detour around γ_i , the set of non-intersecting curves α_j gets conjugated to some system α'_j . One then gets the relations as usual:

$$\alpha_j^{-1} \cdot \alpha'_j = 1$$

These are usually very difficult to determine. In the case where $m = r$, i.e. if all the branches are smooth, we only have relations of type 1, and therefore an nice answer:

Theorem (5.14): *Let $X = X(C, l)$ a sandwiched singularity, with all branches of C smooth. Let $P : X_S \rightarrow S$ be a picture deformation, with incidence matrix \mathbf{I} . Choose an ordering of the branches of C . A presentation of the fundamental group of the Milnor fibre $\pi_1(F)$ of the Milnor fibre F is given by*

$$\langle \alpha_1, \alpha_2, \dots, \alpha_r \mid R \rangle$$

where R is the set of relations generated by

$$\alpha_{i_1} \cdot \alpha_{i_2} \cdots \alpha_{i_k}$$

for each $p \in P_s$, where $C_{i_1}, C_{i_2}, \dots, C_{i_k}$ are the branches of C that come together at p , and $i_1 < i_2 < \dots < i_r$.

Corollary (5.15): *If there are free points on each branch, then $\pi(F) = \{1\}$.*

Because of this we see that topologically the most interesting things will happen if there are few points, that is in the unstable range.

§6 Examples and Applications

In this section we collect some interesting examples of smoothings that can be obtained from the picture method as explained in section 4.

Line Configurations

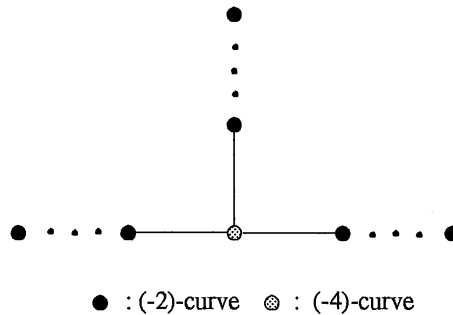
The study of line configurations is a classical field of research. Most of the literature is concerned with special configurations that arise from geometrical constructions. Furthermore, there is the book of Grünbaum, [11] that is mainly concerned with real line configurations, and also configurations involving "bended" lines, which he calls pseudolines. In any case, it will be clear that anything interesting on line configurations will have a bearing on smoothing components of certain rational surface singularities. We will discuss this now in some more detail.

One to Six lines

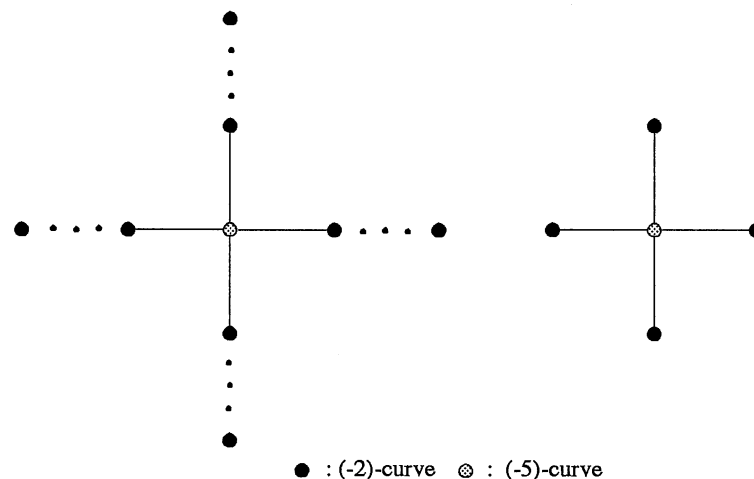
We start with curves C that consist of at most 6 lines (or smooth branches). Of course, the case of 1 line is trivial. We just get an A_k singularity, see (1.5). The case

of two lines is hardly more interesting, as such a curve has no non-trivial δ -constant deformation. The corresponding singularity is a rational triple point, and it is well known that the base space of such a singularity is smooth, as it is Cohen-Macaulay of codimension two.

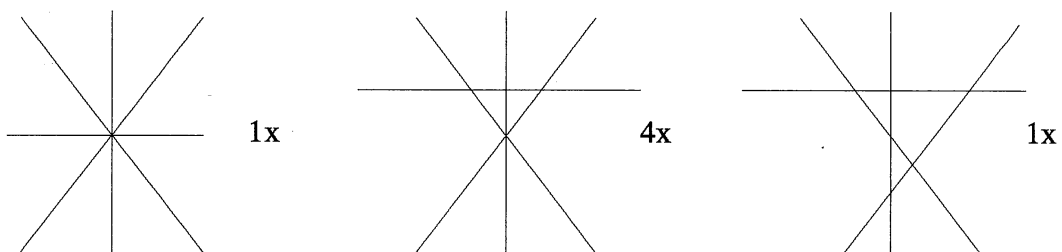
Example(6.1): The first interesting example occurs with three lines: this is Pinkhams example, see (4.8). All singularities with resolution graph as indicated below have two smoothing components, corresponding to the two line configurations of (4.8).



Example(6.2): Next we come to four lines. This is already more interesting. We have a four parameter series of singularities, with resolution graph as in the left hand side of the picture:

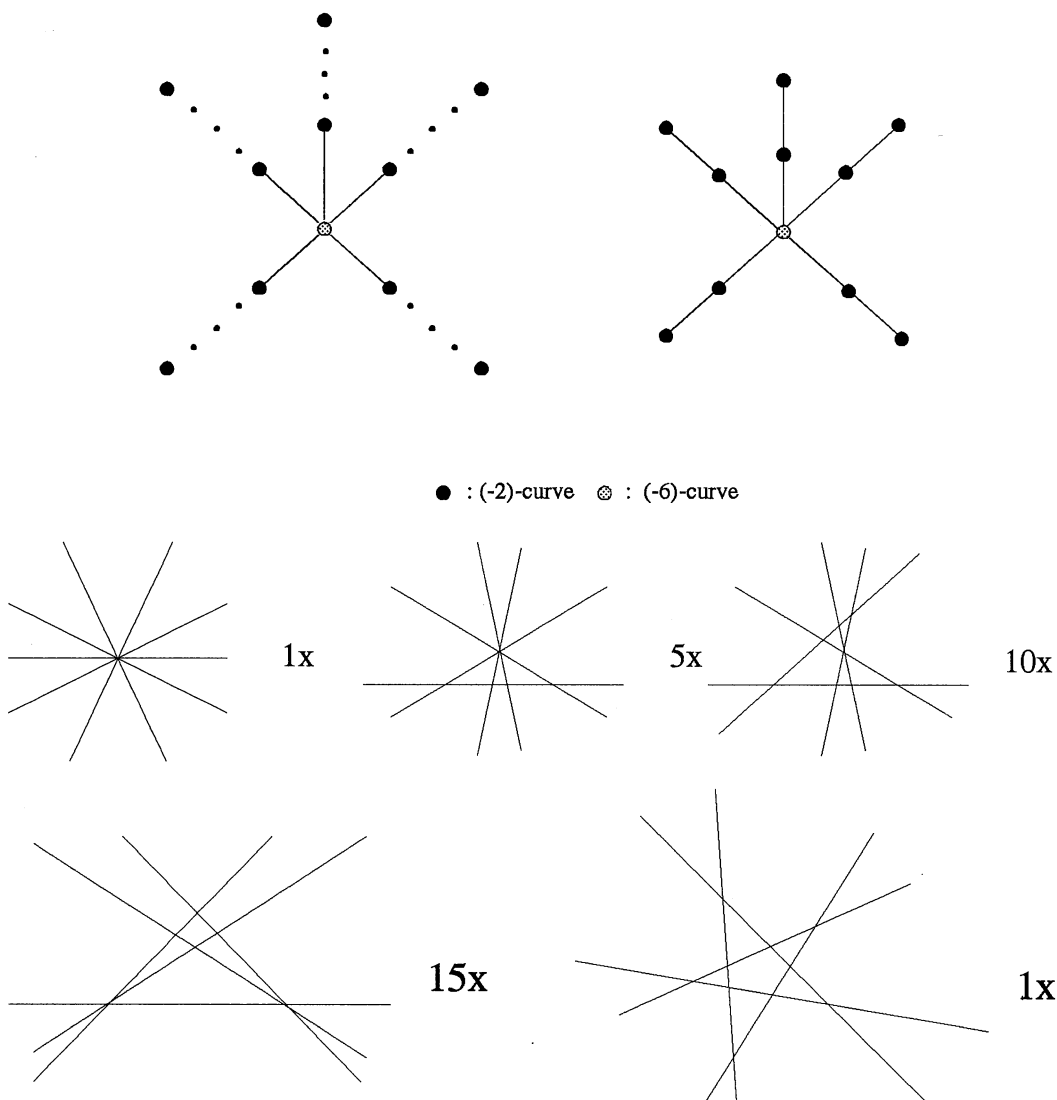


The *stable range*, see (3.9), starts with the graph with five curves at the right hand side. This singularity has six smoothing components, corresponding to the following 3 different line configurations.



Note that for the second configuration we had to pick out one of the four lines and shift it away. So this line configuration gives us four smoothing components. This is indicated by the 4 in the picture. It is also interesting to see what happens in the sub-stable range. For example, for (-5) we get only one component. If there are one, two or three (-2) 's around the (-5) , then one has two, three or four components corresponding to the second line configuration. Only if all four (-2) 's are present, we find all six smoothing components.

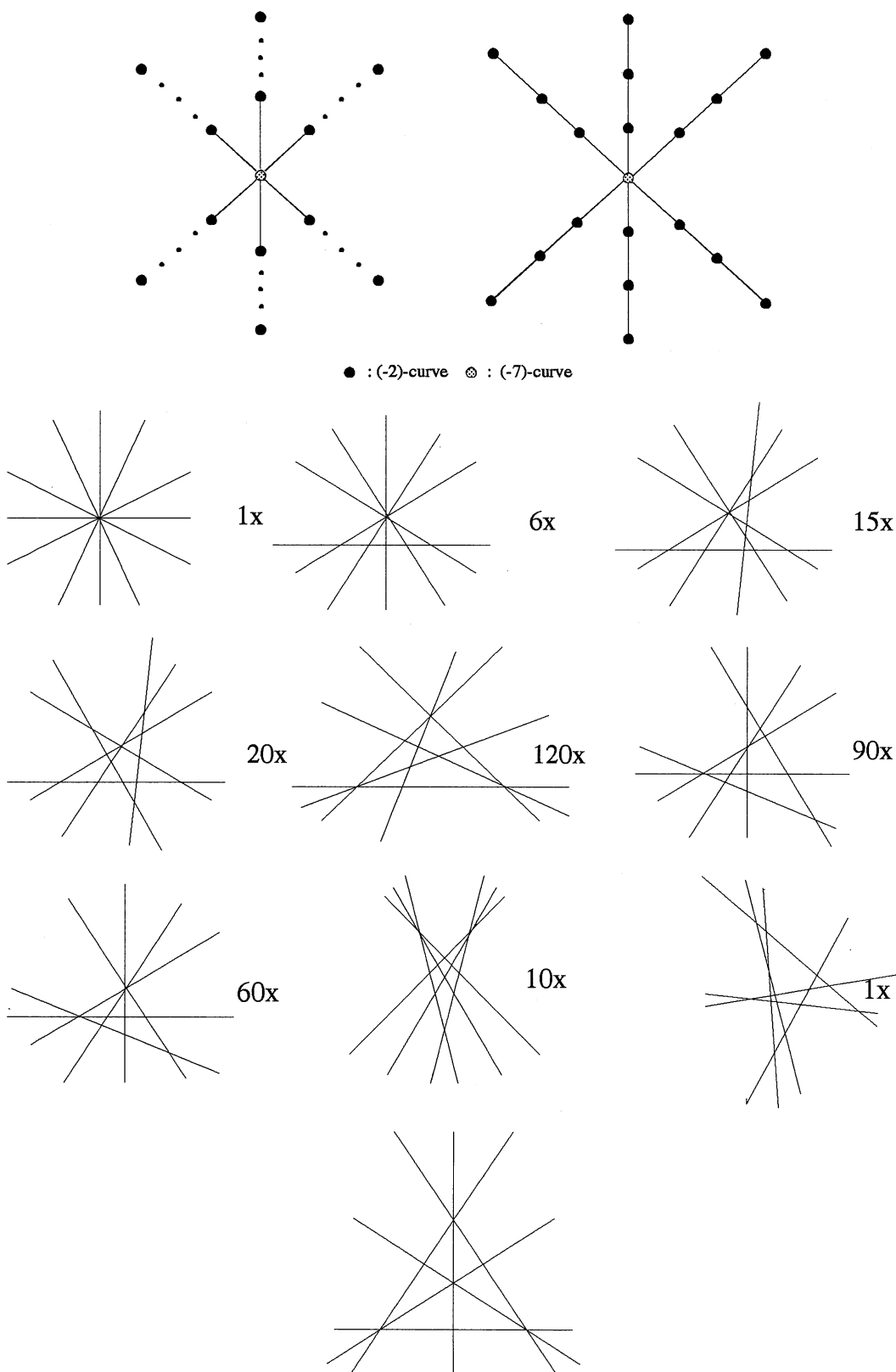
Example (6.3): The case of five lines are treated in the same way. Below we indicate the dual resolution graphs and the beginning of the stable range. As before, the numbers indicate how many smoothing components correspond to the given line configuration.



So in total we have 32 components. We remark that the line configuration with weight 15 occurs as picture deformation of the singularity with dual resolution graph consisting of a (-6) surrounded by four (-2) 's. As was shown by Stevens [31], this singularity has three qG-smoothing components. Here we can also see how it happens: the line connecting the two triple points in the configuration is distinguished.

The four others have to be paired in two groups. This is possible in three ways. From (5.12) we see that these smoothings indeed are qG.

Example (6.4): The case of six lines:

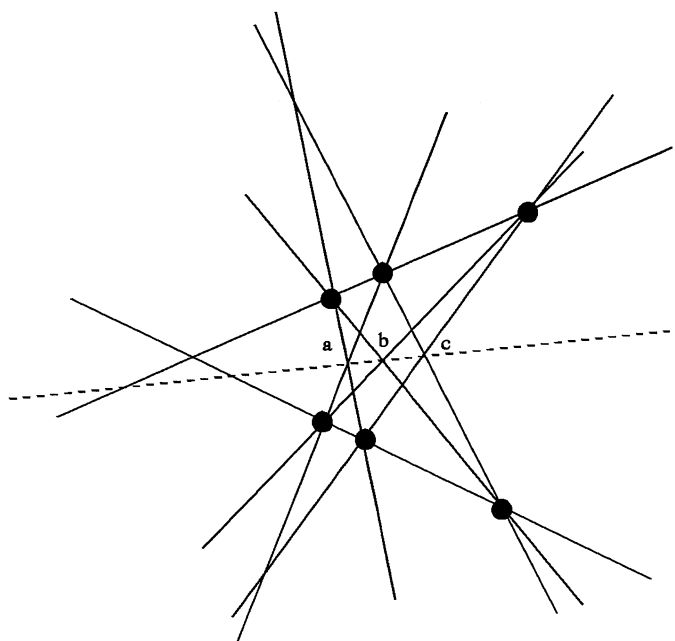


This case leads to a new phenomenon: the curve singularity consisting of six *general* lines through one point, does not have an adjacency to the last curve configuration. For such an adjacency to exist, the tangent directions of the six lines have to satisfy a certain relation. In fact, the six lines correspond to six points on \mathbb{P}^1 , and the condition is that these points are *paired in an involution*, i.e. are inverse image of three points under a $2 : 1$ map $\mathbb{P}^1 \rightarrow \mathbb{P}^1$. This is a divisor in the moduli space of six points on \mathbb{P}^1 . As a consequence, we see that a *general* singularity with exceptional set a (-7) curve, with 6 (-2) curves intersecting this (-7) curve, does not have a smoothing component corresponding to the last line configuration, but that for a divisor in moduli space there is such an extra component. An example of this type was known to exist by J. Wahl (unpublished) but this one has reduced fundamental cycle.

In principle one can go on with this game with more lines. There arise more and more special configurations, in an ever increasing complexity. The only thing one might hope for is, that it would be possible to say something about what happens for generic moduli. This generic number of line configurations seems to be unknown, and we gave up after listing configurations with 9 lines.

Some special configurations

It is of some interest to look at *special configurations*, and to see if their existence leads to interesting smoothing components. We have seen the first special configuration with six lines. But what to say about the following? **Example (6.5):**



This is the well-known Pappos-configuration. This is an incidence theorem in the sense that due to the incidence structure of the lines, the constructed points a, b, c are on the dashed line. So we have 9 lines, the 8 we started with, and the dashed ninth line. We can degenerate the configuration to 9 lines through one point by parallel shifting. The resulting curve singularity has an interesting property: If we only apply deformations of negative weight, we get several components, including the one corresponding to the Pappos-configuration. But there is something we do not

obtain in this way: the *pseudo* Pappos-configuration where the dashed line is slightly bended as to go through a and b , but misses c . To obtain the adjacency to this curve configuration, and hence a corresponding smoothing of the singularity, we need to include in our deformation terms of positive weight.

Example (6.6): Another thing we can look at is the fundamental group. If all branches are smooth, a presentation of this group was described in (5.14). If one computes the fundamental groups in the examples we discussed before, one will find that these groups are *abelian*. In general however, there is no reason for these fundamental groups to be abelian, and indeed J. Wahl has found an example of a smoothing of a rational surface singularity whose Milnor fibre has a non-abelian, but finite, fundamental group (unpublished). We will give an example of a Milnor fibre whose fundamental group is *infinite*.

Consider the nine flex points of a smooth cubic curve in \mathbf{P}^2 . It is well known that the line connecting two flexes intersects the cubic in another flex point. In total one gets in this way 12 flex-lines, each containing three of the flex points. We take the dual of this configuration. So we get 9 lines and 12 points, and through each of the 12 points there are 3 of the nine lines, and there are no further intersection points. If denote the lines by $1, 2, \dots, 9$, then the 12 intersecting triples are:

$$\begin{array}{cccc} 123 & 147 & 159 & 168 \\ 456 & 258 & 267 & 249 \\ 789 & 369 & 348 & 357 \end{array}$$

We denote the generators of the fundamental group by the same numbers $1, \dots, 9$. The above 12 products are, according to (5.14), precisely the defining relations for the fundamental group. We consider the quotient G of this group by putting:

$$x := 1 = 2 = 3; y := 4 = 5 = 6; z := 7 = 8 = 9$$

It is immediate from the relations of the fundamental group that a presentation for G is:

$$\langle x, y, z : x^3 = y^3 = z^3 = xyz = e \rangle$$

This is the well-known triangle group $(3, 3, 3)$, corresponding to the tessellation of the Euclidean plane by equilateral triangles, see [8] p. 25. In particular it is an *infinite* group.

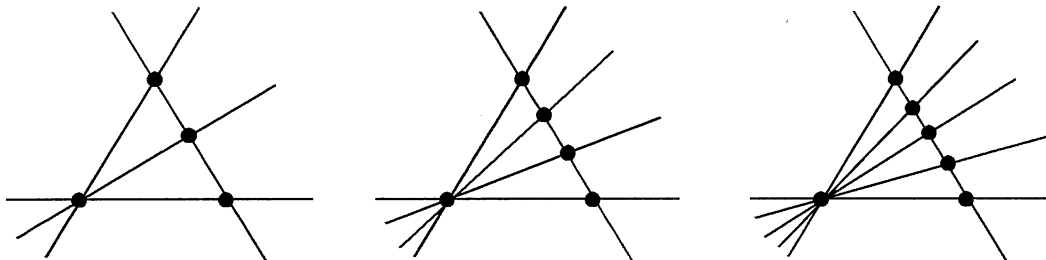
The fundamental group has other interesting quotients. For example, by putting the element 3 equal to the identity one gets a group with presentation:

$$\langle x, y : x^3 = e; x^{-1}yx = y^{-2} \rangle$$

One deduces that $y^9 = e$, hence we get a finite non-abelian group of order 27. This group itself is the fundamental group of the Milnor fibre of a smoothing of the sandwiched singularity obtained by forgetting the third line.

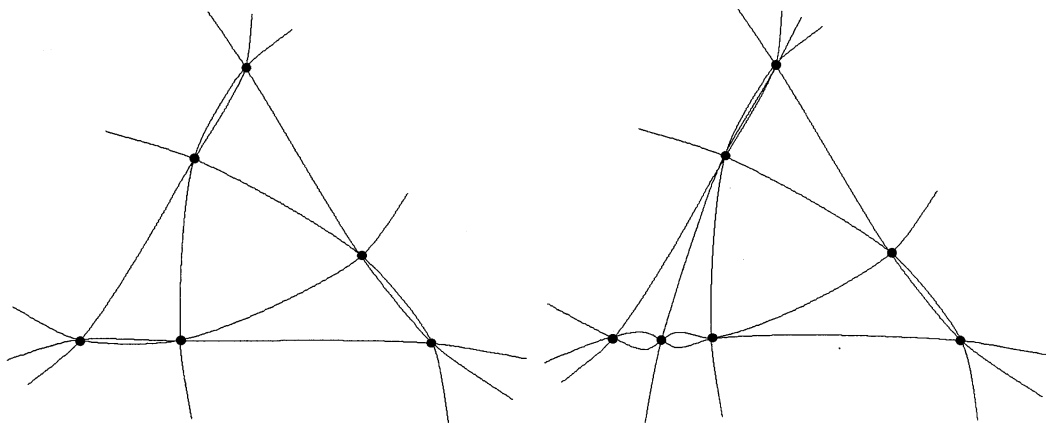
$\mu = 0$ -smoothings

We have seen that for the small component of Pinkham's example (4.8) one gets a smoothing with $\mu = 0$. Equivalently, the incidence matrix is a square matrix; there are as many points as curves in the configuration. It is interesting to see what other singularities admit such a $\mu = 0$ -smoothing. An obvious way to generalise the triangle is as follows:



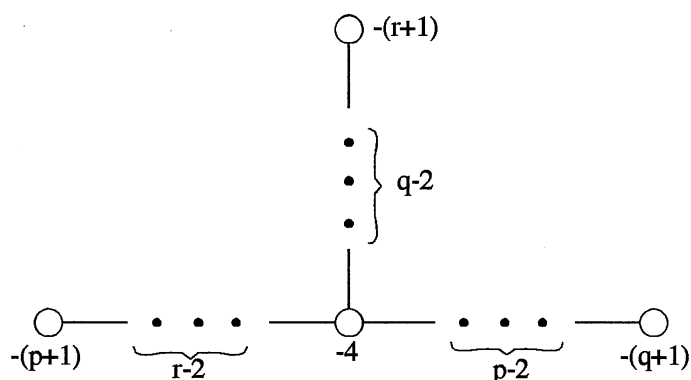
These configurations are called *near pencils* in [11]. These are the only line configurations with as many lines as points.

This is another way to make configurations with as many points as curves:



If one looks for a moment, one will realise that in fact here we have a three parameter family of such curves. The curve $C(p, q, r)$ roughly has the shape of a triangle, where the sides consist of *bundles* of p , q and r curves respectively, which are nearly straight lines.

The dual resolution graph of the corresponding sandwiched singularity is:



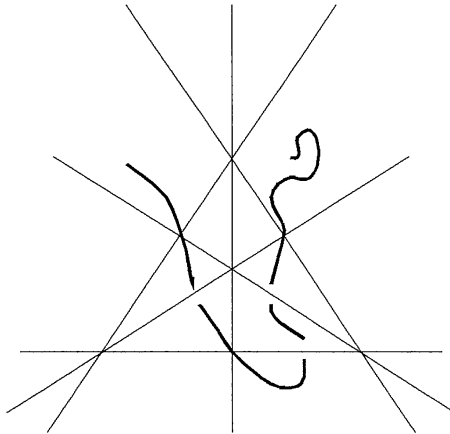
This series was discovered by Wahl [34]. These are the simplest examples of $\mu = 0$ smoothings. Note that in case the $p = q = r$ there are *two distinct* $\mu = 0$ smoothings, as we have a choice of rotation to the left or to the right. Are there any more $\mu = 0$ smoothings? The answer is: yes, but not too many. It is quite hard to produce such examples. There is a secret list compiled by J. Wahl, and the singularities in his list are sandwiched. We did the exercise of writing down the corresponding incidence matrices, but it is much harder to see that the deformation of the curve singularity really exists. One can try to construct other examples with the picture

method, and sometimes one gets interesting candidates, but in all cases there was no new $\mu = 0$ -smoothing for some reason. The question is:

Is Wahl's list complete?

To conclude the discussion, what about the following?

Example (6.7):



The heavy line has intersection one with all the other lines. But such a configuration can not occur in a small deformation of any curve singularity consisting of seven lines: the adjacency is forbidden for example by the semi-continuity of the singularity spectrum, [29]. On the other hand, such a configuration *is* possible in characteristic 2: the incidence structure is just \mathbf{P}^2 over the field of two elements. So it seems that in characteristic 2 the singularity has an extra smoothing component, which will have $\mu = 0$. Of course, this is just the tip of the iceberg.

Cyclic Quotient Singularities

The deformation theory of cyclic quotient singularities (CQS) has been investigated by several authors, see [25], [21] [7], [30] [5] and [2]. One of the beautiful results is the correspondence between the components of a versal deformation and chains representing zero, or, what is the same, triangulations of this m -gon. (m is the multiplicity of the singularity.) This correspondence was discovered by Christophersen, who constructed smoothings for every chain representing zero. Subsequently Stevens proved that all smoothings can be constructed this way.

Let us be more precise. The dual graph of the resolution of a CQS is a chain. We take as first blown up curve in the sandwiched representation an endpoint of the resolution graph, and such that the curve C has just *smooth* branches, see (1.5). For the incidence matrix of a smoothing of a CQS, one gets then a matrix with the following properties, which follow from the fact that the dual resolution graph is a chain:

Definition (6.8): A matrix M with $r \geq 2$ rows

$$v_i = (v_{ij}) ; v_{ij} \in \{0, 1\}$$

is called a CQS-matrix if the formula:

$$\langle v_i, v_j \rangle = \langle v_i, v_i \rangle - 1 \quad \text{for all } 1 \leq i < j \leq r$$

holds. Here $\langle v_i, v_j \rangle = \sum_k v_{ik} v_{jk}$.

Let $|v_i|$ be the number of non-zero entries in v_i . We define numbers:

$$a_1 = |v_1| ; a_i = |v_i| - |v_{i-1}|$$

A cyclic quotient singularity can also be labelled by those numbers, thus we may write $X(a_1, \dots, a_r)$ for a cyclic quotient.

Let

$$K_r = \{[k_1, \dots, k_r] : k_1 - \frac{1}{k_2 - \frac{1}{\ddots - \frac{1}{k_r}}} = 0\}$$

be the set of chains representing zero of length r . The central result about deformations of cyclic quotient singularities is the following:

Theorem (6.9): [7], [30]. *Let $X(a_1, \dots, a_r)$ be a CQS. Then there is a bijection between $S(X(a_1, \dots, a_r))$ and $\{[k_1, \dots, k_r] \in K_r \text{ with } k_i \leq a_i \text{ for all } i\}$.*

The purpose of this subsection is to discuss this Theorem with the picture method. The first thing to do is to discuss the combinatorial components.

Lemma/Definition (6.10): *A CQS-matrix M has the following structure:*

$$M = (M_{red} | M_{triv}),$$

where M_{triv} consist of columns of the type:

$$(0, \dots, 0, 1, \dots, 1)^t.$$

We call M reduced if there are no such columns.

proof: This is completely trivial. □

So this is a very trivial way to make new CQS-matrices out of old one. But there is another, more interesting, way to produce CQS matrices of bigger size out of an old one:

Lemma (6.11): *Let M be a CQS-matrix with r rows v_1, \dots, v_r , and $0 \leq k \leq r$ be a number. Then we produce a new CQS-matrix with $r+1$ rows w_1, \dots, w_{r+1} by putting for the case $k=0$:*

$$w_1 = (1, 0, \dots, 0)$$

$$w_i = (0, v_{i-1}) \text{ for } i \geq 1$$

In case $k \geq 1$, we define:

$$w_i = (0, v_i) \text{ for } i \leq k-1$$

$$w_k = (1, v_k) ; w_{k+1} = (0, v_k)$$

$$w_i = (1, v_{i-1}) \text{ for } i \geq k+2$$

Then this gives us a new CQS-matrix, and all CQS-matrices with $r+1$ rows can be obtained for an CQS-matrix with r rows by this procedure.

proof: The fact that the constructed matrix is a CQS-matrix is trivial. It is obvious that it suffices to prove the second statement for *reduced* matrices, and this is the content of the following proposition. \square

Proposition (6.12): *Let M be a reduced CQS-matrix with r rows. Then M is a matrix of size r by $r-1$. Then we have one of the following cases:*

A *There is a column of type $(1, 0, \dots, 0)^t$ and the first row is equal to $(1, 0, \dots, 0)$*

B *There exists a column (which we can and will suppose to be the first one) of type*

$$\begin{pmatrix} 0 & \dots & 1 & 0 & 1 & \dots & 1 \end{pmatrix}^t$$

\uparrow
 k

(The case of a column of type $(0, \dots, 0, 1, 0)^t$ is also allowed). Moreover, all entries of v_k and v_{k+1} are identical, except for the first entry (where they are different).

proof: We proceed by proving by induction on r the above statements plus the fact that by adding a row v_{r+1} to a reduced CQS-matrix never leads to a CQS-matrix. If $r = 2$, there is just one reduced CQS-matrix:

$$M = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Consider a reduced CQS-matrix M with r rows. Deleting the last row v_r we get a CQS-matrix, which by induction cannot be reduced. Because M is reduced itself, we conclude that there exists a column of type

$$a := (0, \dots, 0, 1, \dots, 1, 0)^t$$

By induction we also have a column of type

$$b := \begin{pmatrix} 0 & \dots & 1 & 0 & 1 & \dots & 1 & * \end{pmatrix}^t$$

\uparrow
 k

or of type:

$$(1, 0, \dots, 0, *)^t$$

in M . Here $*$ denotes the entry of v_r in that column. The second case is easy. From $\langle v_1, v_r \rangle = \langle v_1, v_1 \rangle - 1$ one concludes that $* = 0$. The last entry in v_1 also has to be zero, (all others are by induction) for the same reason. For the first case, we have either $a = b$, or $a \neq b$. If $a = b$, delete the first column to get a CQS-matrix. Some easy arguments show that $v_r = v_{r-1}$ up to the entry in the first column n . If $a \neq b$ it follows from the assumption $\langle v_r, v_{r-1} \rangle = \langle v_{r-1}, v_{r-1} \rangle - 1$ that entry $*$ is equal to 1. (In case that the entry of column b is 0 for v_{r-1} , use v_{r-2} instead.) This shows the existence of the claimed column. But the induction hypothesis gives

us more. It also says that there exist two consecutive rows, say the k -th and $k+1$ -st row, of type (utpoc):

$$\begin{pmatrix} 1 & w & *_1 \\ 0 & w & *_2 \end{pmatrix}$$

The first column corresponds column b the last column to the column $a = (0, \dots, 0, 1, \dots, 1, 0)^t$. We claim that the case $*_1 = 0$; $*_2 = 1$ cannot occur. Suppose it does. Then the number of nonzero entries in v_k and v_{k+1} are equal. We already proved that v_r has a 1 in the first column, and a 0 in the last column. Then we get a contradiction by using

$$\langle v_k, v_r \rangle = \langle v_k, v_k \rangle - 1 = \langle v_{k+1}, v_{k+1} \rangle - 1 = \langle v_{k+1}, v_r \rangle.$$

We still have to prove that the matrix M cannot be extended by adding a row v_{r+1} to M to get a CQS-matrix. Suppose it were possible. By using

$$\langle v_k, v_r \rangle + 1 = \langle v_k, v_k \rangle = \langle v_{k+1}, v_{k+1} \rangle - 1 = \langle v_{k+1}, v_r \rangle.$$

we deduce that this v_{r+1} has a 1 in the first column. Therefore one can delete the first column and the $k+1$ -st row to get a CQS-matrix of smaller size. Now use induction. \square

In order to establish the correspondence with triangulation we will introduce the *Difference Matrix* ΔM of a CQS-matrix M .

Definition (6.13): Let M be a CQS matrix with r rows v_1, \dots, v_r . Then the difference matrix ΔM has r rows $\delta_1, \dots, \delta_r$ with entries in $\{0, 1\}$ defined by:

$$\delta_1 = v_1; \delta_i = v_i - v_{i-1} \text{ for } i \geq 2$$

(These calculations are done modulo 2).

A reduced CQS-matrix M gives rise to a reduced difference matrix ΔM , which is a matrix without a column which has just one non-zero entry. We can no describe the correspondence between triangulations of the $(r+1)$ -gon and reduced difference matrices ΔM obtained from CQS-matrices:

Theorem (6.14): Consider a triangulation of the $r+1$ -gon, with distinguished vertex $*$. Number the vertices $\delta_1, \dots, \delta_r$ anti-clockwise beginning at the vertex closest to $*$. The triangulation consist of $r-1$ triangles P_1, \dots, P_{r-1} , with we list in any order we like. Consider the matrix $\Delta M = (m_{ij})$ of size r by $r-1$ with entries in $\{0, 1\}$ by the condition:

$$m_{ij} = 1 \text{ if } \delta_i \text{ is a vertex of triangle } P_j$$

$$m_{ij} = 0 \text{ otherwise.}$$

Then ΔM is the difference matrix of a reduced CQS-matrix M , and all reduced difference matrices of CQS-matrices can be obtained this way.

proof: It is known that any triangulation of the $(r+2)$ -gon can be obtained from that of an $(r+1)$ -gon by the following procedure. Place a new vertex between two vertices of the $(r+1)$ -gon, and connect the new vertex to both vertices, where it is placed between, by a line. This construction corresponds exactly to the difference of the construction of a new reduced CQS-matrix out of an old one (6.11). \square

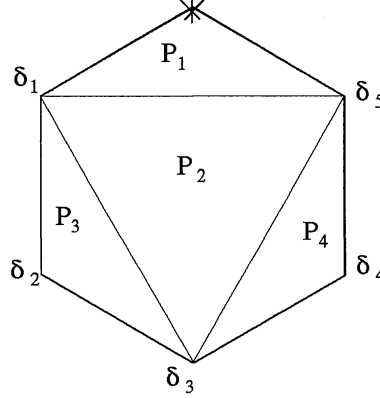
Lemma (6.15): [30] *There is a 1-1 correspondence between chains representing zero of length and triangulations of the $r + 1$ -gon, by defining*

$$k_i = \#\{\text{triangles of which } \delta_i \text{ is a vertex}\}$$

So we see that we can write down a difference matrix (and hence an incidence matrix for the cyclic quotient singularity $X(a_1, \dots, a_r)$ exactly when $k_i \leq a_i$ for all i . So we proved:

Theorem (6.16): *Let $X(a_1, \dots, a_r)$ be a CQS. Then there is a 1-1 correspondence between the combinatorial components of a semi-universal deformation of $X(a_1, \dots, a_r)$ and the $[k_1, \dots, k_r] \in K_r$ with $k_i \leq a_i$ for all i .*

Example (6.17): Consider the following triangulation of the 6-gon:



The k -chain is $[3, 1, 3, 1, 3]$. Moreover the difference matrix ΔM , and the CQS-matrix M belonging to the triangulation are

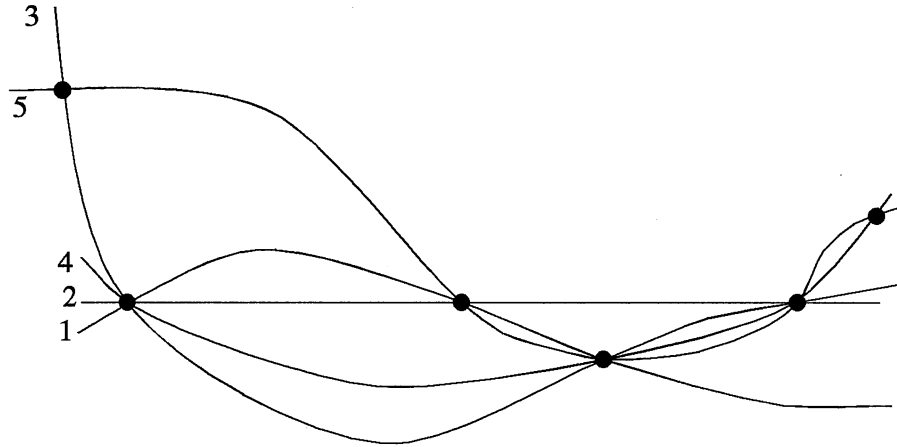
$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Theorem (6.18:) *For a CQS $X = X(C, l)$, with C having smoothing branches, the map $\phi : \mathcal{S}(X) \rightarrow \mathcal{I}(C, l)$ is bijective.*

proof: Because one knows already that the number of smoothing components of a CQS by (6.9), it suffices to show that by the previous Theorem that every combinatorial smoothing can in fact be realised. This will be done by induction on the number of branches, the case of two branches being trivial. According to lemma (6.14) we know how combinatorially the smoothings are realised inductively. We consider the second case only, the first one being even easier. So in the first case we have a distinguished row, the $(k+1)^{st}$. One knows by induction that there is a delta constant deformation of the curve C' , obtained from C by throwing away branch C_{k+1} , with incidence matrix obtained from the CQS matrix by throwing away column $k+1$. Similarly for the curve C'' , obtained from C by throwing away branch C_k . We may even assume by induction that those deformations are compatible in the sense that they induce the *same* deformation of the curve obtained from C by throwing away both branch C_k and C_{k+1} . So we can glue these two deformations, as to realise a δ -constant deformation of the curve C . It is not so difficult to show

that this δ -constant deformation of C has the desired incidence matrix. \square

These curves belonging to smoothing components of cyclic quotients look rather strange. For example, the curve of example (6.19) looks like:



This discussion on smoothings of cyclic quotient singularities gives evidence for the conjecture on the monodromy group, see (5.8), because Behnke and Christophersen [5] proved that for a CQS $X(a_1, \dots, a_r)$ the monodromy group on the component corresponding to the chain representing zero $[k_1, \dots, k_r]$ is precisely $\prod_i S_{a_i - k_i}$.

Corollary (6.19): *The Milnor fibre of a smoothing of a cyclic quotient singularity has cyclic fundamental group.*

proof: Theorem (5.14) gives a presentation of the fundamental group of the Milnor fibre in terms of the incidence matrix, in case the curve C has smooth branches. It suffices to show that the group, constructed in a way analogous to (5.14) for a *reduced* CQS-matrix is isomorphic to \mathbb{Z} . But this group is in fact equal to the group constructed in this way by using the difference matrix ΔM . The proof is now easy using induction. \square

On a conjecture of Kollár

It might be obvious to the reader by now, that if for a sandwiched singularity $X(C, l)$, the function l is *big*, there will be a lot of components. If, on the contrary, l is *small* we have difficulties occupying the inverse images of the singular points on the normalisation. This is related to the following conjecture of Kollár.

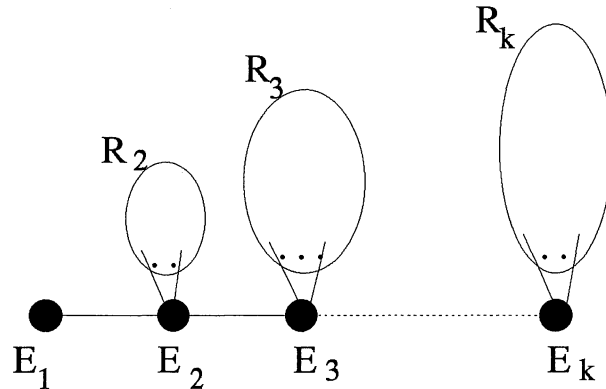
Conjecture (6.22): *Let X be a rational surface singularity. Suppose that all exceptional curves on the minimal resolution have self-intersection at most -5 . Then the base space of a semi-universal deformation of X has just one component, the Artin component.*

The conjecture is sharp in the sense that if there is an exceptional (-4) curve, then there are at least two components, as was proved by Kollár [20]. Not all singularities as in Conjecture (6.22) are sandwiched, but the simplest counterexample we could find has more than 100 exceptional curves on the minimal resolution.

Proving Conjecture (6.22) for sandwiched singularities in general turned out to be too difficult for us. As usual however, the case of reduced fundamental cycle is easier:

Theorem (6.23): *Conjecture (6.22) is true if X has reduced fundamental cycle.*

proof: We put the dual resolution graph Γ in the following schematic form:



The E_j 's and R_j 's have the property that $l(E_j, F) \leq j$ for all curves F from R_j . This can always be done: one chooses E_1 to be an end-point of a longest chain in the resolution graph, etc.

We suppose that the self intersections of all curves are at most -5 , except maybe E_k , which might have self-intersection -4 . We take a sandwiched representation as in (1.5), and choose E_1 as first blown up curve. The branches of C are all smooth, and correspond to the chains running from the chosen hyperplane branch at E_1 to the other hyperplane branches.

The proof goes by (double) induction over k and the number of exceptional curves in the resolution. The induction hypothesis is that if the self-intersection of E_k is less than -4 there is one combinatorial smoothing, and that if the self-intersection of E_k is -4 there will be two combinatorial smoothings, together with statements about the structure of the incidence matrices. The case $k = 1$ is easy, the case $k = 2$ is left as a (boring) exercise to the reader. The induction hypothesis about the structure

of the incidence matrix is that there is a submatrix of one of the following types.

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 & 1 & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 & 1 & 1 \end{pmatrix}$$

Moreover, we have just ones in the first column of the incidence matrix. The vectors in the first three rows of the matrix are part of vectors v_1^i , which belong to chains at E_1 . Those of the second three rows are part of vectors v_2^i , $i = 1, 2, 3$, belonging to chains running to hyperplane branches at E_2 or in R_2 , etc, the vectors v_j^i correspond to chains running to a hyperplane branch at E_j or an exceptional curve of R_j . They are chosen in such a way that $\langle v_i^*, v_j^* \rangle = \min(i, j)$. (* denotes any index). The fact that there are *at least three vectors* v_j^i for fixed j follows from the assumption that the selfintersections of the exceptional curve is at most -5 , or -4 for E_k . Consider a resolution graph as above with $k + 1$ curves E_i in the chain. In the incidence matrix of a smoothing we have vectors v_{k+1}^j , corresponding to chains running to a hyperplane branch at E_{k+1} or an exceptional curve belonging to R_{k+1} . The number of non-zero entries in such a v_{k+1}^j is *at most* $2k + 2$, because the length of a chain with endpoint E_{k+1} in R_{k+1} is at most $k + 1$. Hence the length of a chain with endpoint E_1 is at most $2k + 1$. Deleting all rows belonging to v_{k+1}^j we get an incidence matrix of an other singularity, which has (by induction) a submatrix as described above. Let us now take a vector v out of these v_{k+1}^j . We claim that it must have a one in the first row. Suppose not. Consider the case that the submatrix is of the first type. The way to get at least $\langle v, v_j^i \rangle = j - 1$ with the least number of ones used by v is by putting ones in the second through $k - th$ row (in the case of the first submatrix). But then the intersection is still not ok. Because of the condition $\langle v_i^*, v_j^* \rangle = \min(i, j)$, we have to use at least $3k$ extra ones in the vector v to get the desired intersections. So in total we need $k + 3k$ ones in the vector v , but we have at most $2k + 2$ ones at our disposal. So the claim follows if $k \geq 2$. The case of a submatrix of the second type is similar, only somewhat more complicated.

At this moment we conclude that in the incidence matrix there is a columns of ones. Deleting this column, and all rows where the number of nonzero entries is two, we have the solution of the combinatorial smoothing problem for the singularity obtained from the original singularity by deleting the exceptional curve E_1 . We now have the possibility of taking another sandwiched representation, with an endpoint of the resolution graph as first blown-up curve. Because the combinatorial solutions to the deformation problem is independent of the sandwiched representation if the curves have smooth branches only, see (4.16), we can deduce (by double induction on k and the number of exceptional curves) that there is only one solution to the combinatorial smoothing problem.

This solution must be the Artin component combinatorial solution, as this solution always exists, and any smoothing with this incidence matrix can indeed simultaneously be resolved after base change, see (4.13). This completes the proof. \square

7 Appendix: R.C. Deformations

We review some facts on R.C. deformations from [14] and [15]. Consider a (multi-) germ X of a Cohen-Macaulay space, and a map $X \rightarrow Y$ which is finite, surjective and generically 1-1. Consider the conductor

$$I := \text{Hom}_Y(\mathcal{O}_X, \mathcal{O}_Y),$$

and let Σ be the space defined by I .

Suppose Y is Gorenstein. Then the conductor I satisfies the *Ring Condition* (R.C.):

$$\text{Hom}_Y(I, I) = \text{Hom}_Y(I, \mathcal{O}_Y).$$

Indeed, by duality for finite maps we have that $\mathcal{O}_X = \text{Hom}_Y(I, \mathcal{O}_Y)$, and the ring condition says exactly that the \mathcal{O}_Y -module \mathcal{O}_X in fact has a ring structure. Conversely, starting with Y and any R.C. ideal I , we can construct an X mapping to Y whose conductor is exactly I .

Equations for X

In case that Y is a hypersurface in $\mathcal{O}_{\mathbb{C}^n}$ one can obtain equations for X as follows: The $\mathcal{O}_{\mathbb{C}^n}$ -module I is Cohen-Macaulay, as it is the dual of \mathcal{O}_X . As such, I is isomorphic to the cokernel of a $(t+1) \times (t)$ matrix M^* with entries in $\mathcal{O}_{\mathbb{C}^n}$. Furthermore, the maximal minors of M^* give generators of the ideal I , so that $I = (\Delta_0, \Delta_1, \dots, \Delta_t)$. Consider the transposed matrix M . Because $f \in I$ we can, by adding a (upper) row to M , make a matrix \widetilde{M} , such that $f = \det(\widetilde{M})$. This matrix defines a map $\widetilde{M} : \mathcal{O}_Y^{t+1} \rightarrow \mathcal{O}_Y^{t+1}$, whose cokernel is isomorphic to $\text{Hom}_Y(I, I) = \text{Hom}_Y(I, \mathcal{O}_Y) \cong \mathcal{O}_X$.

Therefore, the i -th row of \widetilde{M} corresponds to a certain element $u_i \in \mathcal{O}_X$, $i = 0, 1, \dots, t$. Note that $u_0 = 1$.

One can embed X in $\mathbb{C}^t \times \mathbb{C}^n$ with coordinates $u_1, \dots, u_t, x_1, \dots, x_n$. If we write $\widetilde{M} = (M_{ij})$, then each column gives us an equation:

Linear Equations (7.1):

$$\sum_{i=0}^t M_{ij} u_i = 0$$

which we refer to as the *linear* equations. Because $\text{Hom}_Y(I, I)$ is a ring we have that $u_k u_l$ is in $\text{Hom}_Y(I, I)$. As such there must exist g_{kl}^i in \mathcal{O}_Y such that:

Quadratic Equations (7.2):

$$u_k u_l = \sum_{i=0}^t g_{kl}^i u_i \text{ for } k, l \geq 1$$

which we refer to as the *quadratic equations*. Take lifts of g_{kl}^i to $\mathcal{O}_{\mathbb{C}^n}$. The quadratic equations are now uniquely determined up to the linear equations. These linear and quadratic equations give an embedding of X in \mathbb{C}^{n+t} . In fact, starting from the matrix M one can construct a projective resolution, see [19].

R.C. Deformations

The R.C. condition makes sense in a relative situation over any ring. Therefore, one can talk about R.C. deformations.

Definition (7.3):

An *R.C. Deformation* of (Σ, Y) over a germ of an analytic space S is given by a flat deformation $\Sigma_S \hookrightarrow Y_S$ of $\Sigma \hookrightarrow Y$ over S , such that the ideal $I_S \subset \mathcal{O}_{Y_S}$ of Σ_S satisfies R.C.:

$$\text{Hom}_{Y_S}(I_S, I_S) = \text{Hom}_{Y_S}(I_S, \mathcal{O}_{Y_S}).$$

We denote by $\text{Def}(\Sigma, Y)$ the functor of R.C. deformations of (Σ, Y) .

The main theorems to be applied in this paper are:

Theorem (7.4): ([15], (1.1))

In the above situation, there is a natural equivalence of functors:

$$\text{Def}(\Sigma, Y) \xrightarrow{\sim} \text{Def}(X \rightarrow Y)$$

This theorem is particularly useful for Y a hypersurface singularity, because of:

Theorem (7.5): ([15], (1.16)) *Suppose moreover that Y is a hypersurface singularity. Then the forgetful functor*

$$\text{Def}(X \rightarrow Y) \longrightarrow \text{Def}(X)$$

is smooth.

In case that Y is a hypersurface singularity in \mathbb{C}^n , given by $f = 0$, then the R.C. condition can be expressed in terms of the *evaluation map*. For this, consider the normal module $N_\Sigma = \text{Hom}_\Sigma(I, \mathcal{O}_\Sigma)$. Over a base S we just add everywhere a suffix S , so for example $N_{\Sigma_S} = \text{Hom}_{\Sigma_S}(I_S, \mathcal{O}_{\Sigma_S})$, etc. Suppose $\Sigma_S \rightarrow Y_S$ is a deformation of $\Sigma \rightarrow Y$. Then one has:

Lemma (7.6): ([15] (1.12)). $(\Sigma_S \rightarrow Y_S) \in \text{Def}(\Sigma, Y)(S)$ if and only if:

$$\text{ev}_{f_S} : N_{\Sigma_S} \rightarrow \mathcal{O}_{\Sigma_S}$$

$$n \mapsto n(f_S)$$

is the zero map. Here $f_S = 0$ is an equation of Y_S .

Infinitesimal considerations

It follows from the above lemma that first order R.C. deformations are represented by *admissible pairs*:

Definition (7.7): *In the above situation, we define R.C. admissible pairs by:*
 $\mathcal{A}(I, f) = \{(n, g) \in N_\Sigma \oplus \mathcal{O} \mid f + \epsilon g \in (\Delta_1 + \epsilon n(\Delta_1), \dots, \Delta_p + \epsilon n(\Delta_p)) \text{ such that } ev_{f+\epsilon g} = 0\}$
Furthermore, for an ideal I we denote by $I^{ev} = \{g \in I : ev_g = 0\}$

There is an obvious map $\mathcal{A} \rightarrow N_\Sigma$, whose kernel is seen to be exactly I^{ev} . The image of the map can be computed as the kernel of *Hessian map*

$$h_f : N_\Sigma \rightarrow N^*/I,$$

as defined in ([14] (3.7)). The latter module is the cokernel of the double duality map

$$I/I^2 \rightarrow N_\Sigma^* = Hom(N_\Sigma, \mathcal{O}_\Sigma).$$

So we have an exact sequence of the form:

$$0 \longrightarrow I^{ev} \longrightarrow \mathcal{A}(I, f) \longrightarrow N_\Sigma \xrightarrow{h_f} N^*/I \longrightarrow N^*/(I + h_f(N_\Sigma)) \longrightarrow 0$$

The module written at the right plays the rôle as obstruction space: the obstruction to extend a given deformation over S to one over a small extension $0 \longrightarrow V \longrightarrow S' \longrightarrow S \longrightarrow 0$ turns out to be exactly the class of

$$[ev_{f_{S'}} : N_{S'} \longrightarrow \mathcal{O}_{\Sigma_{S'}}] \in (N^*/(I + h_f(N_\Sigma))) \otimes V$$

where we have chosen arbitrary lifts for f_S and Σ_S to S' , see [14].

First order R.C. deformations are obtained from these admissible pairs, by dividing out the action of the coordinate transformations and multiplication of the equation by a unit.

Proposition (7.8) (see [14]) *Let $J_\Sigma(f) = \{\theta(f) : \theta(I) \subset I\}$. Then there is an exact sequence:*

$$0 \rightarrow I^{ev}/(f, J_\Sigma(f)) \longrightarrow T^1(\Sigma, Y) \longrightarrow Ker(T^1(\Sigma) \rightarrow N^*/I) \rightarrow 0$$

Here, of course, $T^1(\Sigma, Y)$ describes the infinitesimal R.C. deformations of (Σ, Y) .

The map $T^1(\Sigma, Y) \longrightarrow T^1(X)$

As we have a smooth map $Def(\Sigma, Y) \longrightarrow Def(X)$ there is an induced surjection on the level of tangent spaces. To describe this, one has to recall the embedding of X as described in (7.1), (7.2), and see which admissible pairs arise from coordinate transformations in this bigger space.

An R.C. admissible pair (n, g) can be given by a perturbation of the matrix \tilde{M} . Indeed n gives a deformation of Σ , which by well known facts on deformations of Cohen-Macaulay codimension two spaces, can be given by a deformation, say $\tilde{M} + \epsilon N$, of the matrix \tilde{M} . The upper row of \tilde{M} is also deformed, to give a matrix $\tilde{M} + \epsilon \tilde{N}$ such that $f + \epsilon g = det(\tilde{M} + \epsilon \tilde{N})$. Consider a vectorfield of the form $u_k \theta$ on \mathbb{C}^{n+t} ,

where θ is a vectorfield on \mathbb{C}^n . We can let it act on the linear equations defining the embedding of $\text{Hom}_Y(I, I)$ in \mathbb{C}^{n+t-1} . Of course, in general, quadratic terms appear, but they can and will be removed by using the quadratic equations. In this way, one gets a perturbation of the matrix \tilde{M} , which will give an R.C. admissible pair. This procedure gives us a map: $\mathcal{O}_X \otimes \Theta_{\mathbb{C}^n} \longrightarrow \mathcal{A}$.

Theorem (7.9) (see [15]).

$$T^1(X) = \text{Coker}(\mathcal{O}_X \otimes \Theta_{\mathbb{C}^n} \longrightarrow \mathcal{A})$$

δ -constant deformations of plane curves

Let $Y = C \subset \mathbb{C}^2$ be an isolated plane curve singularity, $X = \tilde{C} \rightarrow C$ be the normalization. It is well-known that the deforming C in a δ -constant way is the “same” as studying deformations of C that admit simultaneous normalisation. So the tangent space to semi-universal δ -constant deformation will be

$$T^1(\tilde{C} \longrightarrow \mathbb{C}^2) = T^1(\tilde{C} \longrightarrow C)$$

If the map $\tilde{C} \longrightarrow \mathbb{C}^2$ is given by $t \longrightarrow (x(t), y(t))$, then an arbitrary perturbation to $(x(t) + \epsilon\xi(t), y(t) + \epsilon\eta(t))$ will give rise to a δ -constant deformation. The coordinate changes in source and target will divide out $J := \mathcal{O}_{\tilde{C}}(\partial x/\partial t, \partial y/\partial t)$ resp. $(\mathcal{O}_C \oplus \mathcal{O}_C)$, so that

$$T^1(\tilde{C} \longrightarrow C) \approx (\mathcal{O}_{\tilde{C}}/\mathcal{O}_C \oplus \mathcal{O}_{\tilde{C}}/\mathcal{O}_C)/J$$

In this article we use what we call the *R.C. description* of δ -constant deformations of plane curves. Let I be as usual the conductor, and Σ the fat point defined by it. Because the deformation functors are equivalent, we have an isomorphism

$$T^1(\tilde{C} \longrightarrow C) \cong T^1(\Sigma, C)$$

On the level of representatives, this map is given by

$$(\xi, \eta) \mapsto (n, g) = (\xi\partial_x(\Delta) + \eta\partial_y(\Delta), \xi\partial_x(f) + \eta\partial_y(f))$$

for Δ an element of I , as an explicit computation learns. Remark that, in general, the forgetful functor $\text{Def}(\Sigma, C) \rightarrow \text{Def}(C)$ is **not** injective. Indeed, for plane curve singularities C , with normalisation \tilde{C} , Buchweitz [3] proved that the kernel of the map:

$$T^1(\tilde{C} \rightarrow C) \longrightarrow T^1(C)$$

has dimension $m - r$, where m is the multiplicity, and r the number of branches of C . For example the deformation $y^2 - x^3 + \epsilon x^2$ gives a trivial deformation of the cusp C but is not a trivial deformation of the diagram $\tilde{C} \rightarrow C$. This is in contrast with the theory of admissible deformations. In ([14] (1.11)) it is proved that under reasonable circumstances the corresponding forgetful functor for admissible deformations is injective.

Example (7.10): We consider Y , the E_6 singularity given by $y^3 - x^4 = 0$, and Σ the space defined by the conductor of the normalization. The conductor is given by

the ideal $I = (x^2, xy, y^2)$. We first determine equations describing \tilde{C} in \mathbb{C}^4 . For the matrix \tilde{M} one can take:

$$\begin{pmatrix} y & 0 & -x^2 \\ x & y & 0 \\ 0 & x & y \end{pmatrix}$$

Calling $u_1 = u$ and $u_2 = v$, we therefore have as linear equations

$$y + ux = 0; \quad uy + vx = 0; \quad -x^2 + vy = 0$$

From these one can compute the quadratic equations, and they turn out to be:

$$u^2 = v; \quad uv = -x; \quad v^2 = y$$

Remark that v can be eliminated to give the parametrization $x = -u^3; y = u^4$ of the E_6 singularity.

We describe the vectorspace $T^1(\Sigma, C)$. It is tedious to check $\text{Ker}(h_f : T^1(\Sigma) \rightarrow N^*/I)$ is represented by the normal module element $(x^2, xy, y^2) \rightarrow (y, 0, 0)$. Furthermore, $I^{ev} = (x, y)^3$, and $J_\Sigma(f) = (y^3, y^2x, x^3y, x^4)$. One concludes that $T^1(\Sigma, Y)$ is three dimensional.

To see the vectorspace $T^1(\tilde{C})$, we still have to divide out the vector fields $u\partial_x, v\partial_x, u\partial_y$ and $v\partial_y$. We will just do $v\partial_y$ leaving the others to the reader. The action is:

$$(y + ux) \rightarrow v; \quad (uy + vx) \rightarrow uv; \quad (x^2 + vy) \rightarrow v^2$$

Because $uv = -x$ and $v^2 = y$ we see that this is the same as:

$$(y + ux) \rightarrow v; \quad (uy + vx) \rightarrow -x; \quad (x^2 + vy) \rightarrow y$$

Therefore, the deformation of the matrix is:

$$\begin{pmatrix} y & -\epsilon x & -x^2 + \epsilon y \\ x & y & 0 \\ \epsilon & x & y \end{pmatrix}$$

Hence we see that the R.C. admissible pair maps to the unique element in $\text{Ker}(h_f : T^1(\Sigma) \rightarrow N^*/I)$. After dividing out all elements one sees that $T^1(\tilde{C}) = 0$, as it should be, because \tilde{C} is the normalization of the E_6 singularity, which is a smooth space.

The following Theorem is of crucial importance in this paper:

Theorem (7.11):

$$I^{ev} = \{g \in \mathbb{C}\{x, y\} : \text{ord}(g_i) \geq c_i + m(i)\}$$

Here $m(i)$ is the sum of multiplicities as defined in (1.2).

Proof: The lemma is easy for a curve consisting of m smooth branches intersecting mutually transverse. In that case the conductor is $I = \mathfrak{m}^{m-1}$, where \mathfrak{m} is the maximal ideal. Every $n \in \text{Hom}(I, \mathcal{O}/I)$ has values in \mathfrak{m}^{m-2} and conversely, every

assignment of values in \mathfrak{m}^{m-2} to a minimal set of generators of I defines an element in $\text{Hom}(I, \mathcal{O}/I)$. (Easy to check, and left to the reader.) Hence for $g \in I$ to satisfy $ev_g = 0$, it is necessary and sufficient that $g \in \mathfrak{m}^m$. This gives the lemma for this case.

In the general case we use the one-parameter deformation of Scott, see (1.11), and induction. So we have the deformed curve C_s , defined by $f_s = 0$. It has branches C_{is} , which has (possibly) two singular points:

A: m_i branches passing through the m -fold point. (Here we put m_i to be the multiplicity of branch C_i .)

B: the singularity of the first blow up of C which is on the branch C_i .

We let c'_i be the conductor of this singularity described under **B**.

Every $g \in I^{ev}$ defines an infinitesimal δ -constant deformation of C by the formula $f + \epsilon g = 0$. As the functor of δ -constant deformations of a plane curve singularity is *smooth*, we can find a lift g to a g_s such that $f_s + \epsilon g_s = 0$ defines a relative δ -constant deformation for *any* $g \in I^{ev}$. It is necessary that the following two conditions hold for restriction g_{is} of g_i to C_{is} , in order for g to be in I^{ev} :

- 1) It vanishes with order m at any of the m_i points of $C_{i,s}$ mapping to A.
- 2) (By induction) It vanishes with order $m(i) - m_i + c'_i$ at B.

Because the conductor I is flat over the parameter space, we have

$$c_i = c'_i + m_i(m - 1)$$

If we now let s go to zero, we see that g_i has to vanish with order

$$m(i) - m_i + c'_i + m_i m = m(i) + c'_i + m_i(m - 1) = m(i) + c_i$$

On the other hand, if the vanishing order of g_i is $\geq m(i) + c_i$ we can lift g_i to g_{is} etc. \square

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