

Total objects in inductively defined types*

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1 Introduction

In Kristiansen and Normann [7] we defined two hierarchies of "types". In both hierarchies we started with the set of natural numbers and the set of boolean values as base types, and we closed under certain dependent sums and dependent products.

The difference between the two hierarchies in [7] is that one is based on coherence spaces and stability (see Girard [4]), while the other is based on Scott-Ershov domains. Stoltenberg-Hansen, Lindström and Griffor [15] gives a good introduction to the theory of domains.

Common for both hierarchies is that the types in the hierarchy will be represented by elements s in a domain S , (coherence space or Scott-Ershov domain) with an interpretation $I(s)$ as a subdomain of one universal domain D .

The parameterisations will be stable or continuous maps $F : I(s) \rightarrow S$.

By induction we define the well formed type expressions S_{wf} and the total elements in $I(s)$ for each $s \in S_{\text{wf}}$.

One of the main results in [7] is that for $s \in S_{\text{wf}}$, the total objects in $I(s)$ are dense in $I(s)$. This holds for both hierarchies.

In Normann [10] we suggested an axiomatisation of an abstract notion of totality. Kristiansen [6] adapted this suggestion to coherence spaces and qualitative domains in general. In [7] we showed that all the coherence spaces

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with totality constructed in the hierarchy will, in a uniform way, satisfy these axioms.

Berger [3] suggested an alternative way of defining totality in Scott-Ershov domains. In [7] we also showed that the Scott-Ershov domains in that hierarchy in a uniform way satisfy the Berger axioms.

In Normann [13], the hierarchy of domains from [7] is extended to one also including the empty type as a base type. The empty type is represented by a domain with one bottom element and no total objects. It is shown that in this hierarchy, all parameterisations and functions will respect extensional equality.

In this paper we will investigate interpretations of inductively defined types. We will do so both in the setting of coherence spaces and of Scott-Ershov domains. One reason for this is that the nature of the results obtained in the two settings differ. This means that we have revealed a different set of aspects about induction for the two directions of investigation.

The results in section 2 are due to the first author, and appeared first in her thesis [6]. The results in section 3 and the appendix are due to the second author. Some of these results appeared in a setting of Kleene associate representation of types and objects in Normann [11].

2 Inductively defined coherence spaces

2.1 Coherence spaces with totality

Following Girard [4] we let a coherence space be a pair $(X, |X|)$ where $|X|$ is a set and X is a family of subsets of $|X|$ satisfying:

If $x \subseteq |X|$, then $x \in X$ if and only if for every two-point subset $A \subseteq |X|$ we have that $A \in X$.

We see that the empty set and every singleton will be in X , and we will have

- i) $x \in X \wedge y \subseteq x \Rightarrow y \in X$.
- ii) The union of a directed set from X will also be in X .

If a, b are in $|X|$, we say that a and b are *coherent* if $\{a, b\} \in X$. In general, a set of elements in X is coherent if its union is in X .

If X and Y are two coherence spaces, we let $F : X \rightarrow Y$ be *stable* if F commutes with directed unions and satisfies

$$x \cup y \in X \Rightarrow F(x \cap y) = F(x) \cap F(y).$$

Girard [4] showed that the set of stable functions can be represented as a coherence space on the set of *trace elements*

$$\{(A, a) \mid A \in X \text{ is finite and } a \in |Y|\}.$$

where (A, a) and (B, b) are coherent if $A \cup B \notin X$ or if $A \cup B \in X$, $\{a, b\} \in Y$ and $a \neq b$ or $A = B$.

In [7] we defined what we mean by a coherence space with totality, and we constructed a hierarchy of coherence spaces with totality closing the base types \mathbb{N} and Boole under products and sums of stable parameterisations of coherence spaces with totality. We will give the main definitions here, but for some of the results we will rely on knowledge to [7].

Definition 1 Let $(X, |X|)$ be a coherence space.

- a) A *chain* in X is a set C of finite sets A in X such that they are pairwise incoherent.
- b) If $x \in X$ and C is a chain, we say that x *meets* C if x contains an element of C as a subset.

Remark

Girard [5] used coherence spaces to give a model for linear logic. The set of chains will be exactly $?(X^\perp)$ as defined in [5], i.e. a coherence space itself.

Definition 2 a) A *coherence space with totality* is a tripple

$$(X, Q, \{C_q\}_{q \in Q})$$

where X is a coherence space, Q is a set and C_q is a chain in X for each $q \in Q$.

- b) If $(X, C, \{C_q\}_{q \in Q})$ is a coherence space with totality, then $x \in X$ is *total* in X if x meets C_q for all $q \in Q$.

This definition is taken from Normann [10] where we give a general conceptual analysis of totality. Basic fragments of this analysis are also found in [7]. We see Q as a set of questions q , C_q as the set of possible ways to process an atomic answer to q . Coherency in X will then mean that we have given a coherent way of answering various questions. An object is total if it answers all questions.

There has been other ways of defining totality in coherence spaces. In the original [4] all subsets of X can be accepted as the set of total objects when we interpret a type variable as a coherence space with totality. Then each closed type term is interpreted as one particular coherence space with one fixed set of total elements. The main information obtained from this construction is that a closed term t of a closed type T will be interpreted as a total object in the interpretation of T .

Kristiansen [6] shows (in a slightly different setting) how the total objects in Girard's interpretation of a closed type term can be defined via chains. The representation is, however, not effective.

Loader [8] use coherence spaces with totality to give an interpretation of fragments of linear logic. Loader works with dual pairs

$$(X, X_{\text{TOT}}), (X^\perp, X_{\text{TOT}}^\perp)$$

where $X_{\text{TOT}}^\perp \subseteq X^\perp$ and

$$\begin{aligned} X_{\text{TOT}} &= \{x \in X \mid \forall y \in X_{\text{TOT}}^\perp (x \cap y \neq \emptyset)\} \\ X_{\text{TOT}}^\perp &= \{y \in X^\perp \mid \forall x \in X_{\text{TOT}} (x \cap y \neq \emptyset)\}. \end{aligned}$$

The sets in X_{TOT}^\perp can be seen as unary chains, i.e. chains of one point sets.

One reason why we cannot follow Loader's definition of totality is that our interpretation of Σ -types as coherence spaces with totality does not fit into his approach.

Loader use a symmetry between X_{TOT} and X_{TOT}^\perp . We might use a similar symmetry by introducing

$$X_{\text{CHAIN}} = \{C \mid C \text{ is a chain and } C \text{ meets } x \text{ for all total } x\}.$$

However, we see a coherence space with totality as given by an explicit set of questions, and then the total objects as given implicitly. Technically this is seen by the explicit way we construct the questions and chains for strictly positive operators. If we represent totality by dual pairs as above, we need

double-negations to get the set of total chains, and we will loose some of the constructive aspects of these constructions.

To explain our view further, let us consider the construction of $\mathbf{X} \rightarrow \mathbf{Y}$ where \mathbf{X} and $\mathbf{Y} = (Y, Q, \{C_q\}_{q \in Q})$ are two coherence spaces with totality.

Let $S = \{(x, q) \mid x \in X_{\text{TOT}} \wedge q \in Q\}$.

Let $\{(A_1, a_1), \dots, (A_n, a_n)\} \in D_{(x, q)}$ if A_1, \dots, A_n are finite subsets of x and $\{a_1, \dots, a_n\} \in C_q$.

(We assume that $\{a_1, \dots, a_n\}$ is without repetition.)

We see that the chains in $\mathbf{X} \rightarrow \mathbf{Y}$ are explicitly given from the total elements in \mathbf{X} and the chains in \mathbf{Y} .

$(X \rightarrow Y)_{\text{TOT}}$ will normally be of higher complexity, and $(X \rightarrow Y)_{\text{CHAIN}}$ may be of even higher complexity, than the chains in $\mathbf{X} \rightarrow \mathbf{Y}$.

In order to define fixpoints of inductive definitions, we need suitable categories in which we can take direct limits.

Definition 3 Let $\mathbf{X} = (X, Q, \{C_q\}_{q \in Q})$ and $\mathbf{Y} = (Y, P, \{D_p\}_{p \in P})$ be two coherence spaces with totality.

- a) Following Girard [4] we let f be a morphism from X to Y if $f : |X| \rightarrow |Y|$ is 1-1 and for all $a, b \in |X|$ we have that a and b are coherent in X if and only if $f(a)$ and $f(b)$ are coherent in Y .
- b) A morphism from \mathbf{X} to \mathbf{Y} is a pair (f, π) where f is a morphism from X to Y and $\pi : P \rightarrow Q$ such that for all $p \in P$ and all finite $A \in X$ we have

$$A \in C_{\pi(p)} \Leftrightarrow f[A] \in D_p.$$

This defines the category K^+ of coherence spaces with totality.

Remark

In Normann [10] an analogue definition is discussed and justified. In brief we can say that the questions in P represent refinements of some questions in Q . If

$$X = \mathbb{N} \oplus \emptyset \text{ and } Y = \mathbb{N} \oplus (\mathbb{N} \rightarrow \mathbb{N})$$

we see that in order to always obtain an atomic answer in X we only need to ask

What is the object

while in Y we must have infinitely many refinements to

What is the object, and in case it is of the form $\text{right}(f)$, what is $f(n)$?

Lemma 1 *Let \mathbf{X} and \mathbf{Y} be as above, (f, π) a morphism from \mathbf{X} to \mathbf{Y} . If $x \in X_{\text{TOT}}$ then $f[x] \in Y_{\text{TOT}}$.*

The proof is simple and is left for the reader.

The category K^+ will contain direct limits. Since we will need the explicit construction of the direct limit, we will give it here:

If $\{(f_{i,j}, \pi_{ij})\}_{i \leq j}$ is a system of morphisms on the directed set

$$(X_i, P_i, \{C_{p,i}\}_{p \in P_i})_{i \in I},$$

we let $X, \{f_i\}_{i \in I}$ be the limit of the X_i in the standard way,

$$P = \{p \in \prod_{i \in I} P_i \mid \pi_{ij}(p(j)) = p(i) \text{ whenever } i \leq j\}$$

with $\pi_i(p) = p(i)$.

We let $C_p = \lim_{i \in I} C_{p(i)}$ in the usual way.

It is easy to show that this construction gives a direct limit.

Lemma 2 *Let Z be any coherence space with a selected set Z_{TOT} of total objects.*

The operator $\Gamma(X) = Z \rightarrow X$ can be extended to a functor on K^+ that commutes with direct limits.

Proof:

Let $\mathbf{X} = (X, Q, \{C_q\}_{q \in Q})$ and $\mathbf{Y} = (Y, P, \{D_p\}_{p \in P})$.

Following Girard [4], if $f : |X| \rightarrow |Y|$ is a morphism, we let

$$\Gamma(f) : |\Gamma(X)| \rightarrow |\Gamma(Y)|$$

be defined by $\Gamma(f)(A, a) = (A, f(a))$.

Moreover, if $\pi : P \rightarrow Q$ we let $\Gamma(\pi)(x, p) = (x, \pi(p))$.

Now let $\{(X_i, P_i, \{C_{p,i}\}_{p \in P_i})\}_{i \in I}, \{(f_{ij}, \pi_{ij})\}_{i \leq j}$ be a directed system, and let $(X, P, \{C_p\}_{p \in P}), \{f_i\}_{i \in I}, \{\pi_i\}_{i \in I}$ be the direct limit as constructed above. Girard [4] showed that Γ commutes with direct limits when restricted to the

coherence spaces, and by the explicit construction of the questions and chains it is easy to see that the construction of $\Gamma(\pi)$ and the chain $C_{(x,p)}$ for $p \in P$ also commutes with the limit.

Remark

If we want to construct a canonical category of coherence spaces with just a set of total objects or with a set X_{TOT} with a dual set X_{CHAIN} , the only natural choice is to say that $f : |X| \rightarrow |Y|$ is a morphism if f in addition to being a morphism from X to Y also maps total elements to total elements. Then direct limits will exist, but the functors $\Gamma_Z(X) = Z \rightarrow X$ will in general not commute with direct limits.

If we restrict ourselves to Loader's coherence spaces with totality, direct limits will still exist, and when Z is a K^+ -object, the functor Γ_Z will commute with direct limits in the Loader category. The direct limit in this category and in the category K^+ will not always contain the same total objects, even when each $(X_i)_{\text{TOT}}$ are the same. In the appendix we prove that the functors Γ_Z will preserve direct limits in the Loader category. We do not need this result for the rest of the paper, but it indicates that our treatment of strictly positive induction in the next section can be carried out in the Loader category.

2.2 Strictly positive inductive operators

In this section we will show how to interpret the least fixpoint of strictly positive operators as coherence spaces with totality. In addition to the function space construction of the previous section, we will use disjoint sums and cartesian products of coherence spaces with totality:

Definition 4 Let $\mathbf{X} = (X, Q, \{C_q\}_{q \in Q})$ and $\mathbf{Y} = (Y, P, \{D_p\}_{p \in P})$ be two coherence spaces with totality.

We define $\mathbf{Z} = \mathbf{X} \oplus \mathbf{Y}$ by

$$|X \oplus Y| = \{l, l(a), r, r(b) \mid a \in |X| \wedge b \in |Y|\}.$$

We let

- l be coherent with $l(a)$
- r be coherent with $r(b)$

- $l(a)$ is coherent with $l(a')$ when a is coherent with a' in X
- $r(b)$ is coherent with $r(b')$ when b is coherent with b' in Y .

We let $R = \{(l, q) \mid q \in Q\} \cup \{(r, p) \mid p \in P\} \cup \{0\}$.

We let

$$\begin{aligned} E_{(l,q)} &= \{r\} \cup \{l[A] \mid A \in C_q\}, \\ E_{(r,p)} &= \{l\} \cup \{r[B] \mid B \in D_p\}, \\ E_0 &= \{l, r\}. \end{aligned}$$

Then $\mathbf{Z} = (Z, R, \{E_r\}_{r \in R})$.

Remark

$z \in Z$ is total if

$$z = \{l\} \cup \{l(a) \mid a \in x\} \text{ for some } x \in X_{\text{TOT}}$$

or

$$z = \{r\} \cup \{r(b) \mid b \in y\} \text{ for some } y \in Y_{\text{TOT}}.$$

This is not the same definition of sum as used in [4], but $\mathbf{X}_0 \oplus \mathbf{X}_1$ is isomorphic to $\Sigma(i \in \{0, 1\})\mathbf{X}_i$ as defined in [7]

Lemma 3 *The operator \oplus has a canonical extension to a functor $\Gamma_{\oplus}(\mathbf{X}, \mathbf{Y})$ on K^+ that commutes with direct limits.*

The proof is trivial and is left for the reader.

Definition 5 If \mathbf{X} and \mathbf{Y} are two coherence spaces with totality as in the previous definition, we define $\mathbf{Z} = \mathbf{X} \times \mathbf{Y}$ as follows:

$\mathbf{Z} = (Z, R, \{E_r\}_{r \in R})$ where

- $|Z| = \{l(a) \mid a \in |X|\} \cup \{r(b) \mid b \in |Y|\}$.
- $l(a)$ is coherent with $l(b)$ when a and b are coherent in X .
- $r(a)$ is coherent with $r(b)$ when a and b are coherent in Y .
- $l(a)$ and $r(b)$ are always coherent.
- $R = \{(l, q) \mid q \in Q\} \cup \{(r, p) \mid p \in P\}$.

- $E_{(l,q)} = \{l[A] \mid A \in C_q\}$.
- $E_{(r,p)} = \{r[A] \mid A \in D_p\}$.

Remark

This corresponds to the definition from [4].

We do not see \times and \oplus as dual constructions. We actually have that $\mathbf{X}_0 \times \mathbf{X}_1$ is isomorphic to $\Pi(i \in \{0, 1\})\mathbf{X}_i$ as defined in [7].

Lemma 4 *The operator \times has a canonical extension to a functor Γ_\times on K^+ that commutes with direct limits.*

The proof is trivial and is left for the reader.

Definition 6 An operator $\Gamma(X_1, \dots, X_n)$ is strictly positive if it is defined from coherence spaces \mathbf{U} with totality and the variables X_1, \dots, X_n using \oplus , \times and \rightarrow , and where no variable occurs on the lefthandside in a subexpression $\Gamma_1(X_1, \dots, X_n) \rightarrow \Gamma_2(X_1, \dots, X_n)$.

A strictly positive operator can be extended to a functor on K^+ commuting with direct limits.

The category K^+ contains a minimal stucture m with

$$|m| = \emptyset, Q_m = \{\star\} \text{ and } C_\star = \emptyset.$$

For any $(X, Q, \{C_q\}_{q \in Q})$, let f be the empty embedding and let $\pi(q) = \star$. It is easy to see that (f, π) is the unique morphism from m to $(X, Q, \{C_q\}_{q \in Q})$. The consequence is that if $\Gamma(X)$ is a strictly positive operator, then we have a least fixpoint in K^+ , namely

$$\mathbf{X} = \lim_{i \rightarrow \infty} \mathbf{X}_i,$$

where $\mathbf{X}_0 = m$ and $\mathbf{X}_{i+1} = \Gamma(\mathbf{X}_i)$.

A strictly positive operator Γ can also be seen as a functor Γ^w on the category K^w of pairs (X, X_{TOT}) where X is a coherence space and X_{TOT} is a set of total objects. In this category the morphisms will be morphisms in the category of coherence spaces that preserves totality.

The functors Γ^w will not in general commute with direct limits. We still have the following result:

Theorem 1 *Let $\Gamma(X)$ be a strictly positive operator.*

Then the least fixpoint of Γ in the category K^+ of coherence spaces with totality will have the same total elements as the least fixpoint of Γ^w .

This theorem was proved in a slightly different setting in Normann [10], where we gave an abstract definition of a strictly positive functor and proved the corresponding theorem for this general class of functors.

Kristiansen [6] transcribed the proof to the setting of qualitative domains, of which coherence spaces is a special case.

We will give an outline of the proof.

First we observe that if Γ is a strictly positive operator in the variables X_1, \dots, X_n , and if $\mathbf{Y} = (Y, D, \{D_p\}_{p \in P}) = \Gamma(\mathbf{X}_1, \dots, \mathbf{X}_n)$ where

$$\mathbf{X}_i = (X_i, Q_i, \{C_q^i\}_{q \in Q_i})$$

for $i = 1, \dots, n$, then the chains in \mathbf{Y} will be defined in a uniform way from the chains in the \mathbf{X}_i via an index-set $J \cup I_1 \cup \dots \cup I_n$ where

- for each $j \in J$ there is a $p_j \in P$ where D_{p_j} is independent of $\mathbf{X}_1, \dots, \mathbf{X}_n$
- for each k , each $i \in I_k$ and each $q \in Q_k$ there is a $p_{(i,q)} \in P$ where the chain $D_{p_{(i,q)}}$ only depends on C_q^i

and P contains nothing more than these p_j and $p_{(i,q)}$.

The intuition is that every $z \in \Gamma(X_1, \dots, X_n)$ will have an evaluation tree. For $z \in \mathbf{U} \rightarrow \Gamma'(X_1, \dots, X_n)$ there is one initial evaluation $z(x)$ for each total $x \in U$, and the rest of the tree will be the evaluation trees of $z(x)$ in $\Gamma'(X_1, \dots, X_n)$.

In $\Gamma = \Gamma_1 \times \Gamma_2$ there is one initial evaluation for each coordinate.

In $\Gamma = \Gamma_1 \oplus \Gamma_2$ there are two initial evaluations answering the questions: Is z a rightwinger? Is z a leftwinger.

If we get the answer NO!, the evaluation stops, otherwise it continues.

Thus $j \in J$ represents an explicit evaluation down to a parameter used strictly positively in Γ , and a particular index p for a chain C_p in that parameter. z will meet D_j if and only if this explicit evaluation of z halts, or if the evaluated value meets C_p .

$i \in I_k$ represents an explicit evaluation down to the variable X_k and z will meet $D_{p_{(i,q)}}$ if and only if the corresponding evaluation of z either stops in a \oplus -suboperator, or the evaluated value of z meets C_q^i .

Thus, if $\Gamma(X)$ is a unary operator, then $P \approx J + (I \times Q)$ uniformly in Q . This "denotation system" for questions is functorial in the sense that if $\pi : Q_1 \rightarrow Q_2$ then $\Gamma(\pi) : P_1 \rightarrow P_2$ by $\Gamma(\pi)(j) = j$ for $j \in J$ and $\Gamma(\pi)(i, q) = (i, \pi(q))$ for $i \in I$ and $q \in Q$.

It follows that in the limit of the sequence $m, \Gamma(m), \Gamma(\Gamma(m)), \dots$ the chains will be given by either a finite sequence i_1, \dots, i_k, j with $i_1, \dots, i_k \in I$ and $j \in J$, or by an infinite sequence i_1, i_2, \dots from I .

The chain connected with i_1, i_2, \dots will be the direct limit of the chains connected with i_1, \dots, i_k, \star . The only way we can get elements into this limit chain is via suboperators of the \oplus -form. Thus z meets the limit chain connected with i_1, i_2, \dots if and only if evaluation of z along the path connected with i_1, i_2, \dots will stop. Thus z is total in the limit if and only if the evaluation tree of z is wellfounded, and always ends up in a total element of one of the parameter types. But this is exactly the definition of the least fixpoint of Γ^w .

This ends our sketch of the proof of the theorem.

Remark

A more detailed and technical proof is given in Normann [10]. There we also show that the operator

$$\Gamma(X_1, \dots, X_n) = \mu X (X = \Gamma_0(X, X_1, \dots, X_n))$$

commutes with direct limits, by showing that this operator is a strictly positive operator in the abstract sense.

The main significance of this result is not that it is possible to define the fixpoint via some set of chains, but that we have a representation of the chains that is natural and represents the fact that totality just means that the evaluation tree is well founded. In the next subsection we will show that also for general positive inductive definitions, the least fixpoint in K^w coincides with the least fixpoint in K^+ . We see this as an evidence to the fact that the total elements of these fixpoints do not just happen to have a dual set of chains, but are actual total because they answer a set of questions constructed from the operator and process itself.

First we will, however, see how our inductively defined coherence spaces fit in with the hierarchy in [7]:

In [7] we constructed a hierarchy $\{I(s)\}_{s \in S_{\text{wf}}}$ of coherence spaces, where S_{wf} are the well founded objects in a coherence space S . The base types are the natural coherence spaces for the natural numbers and for the set of boolean values.

The constructions of \oplus , \times and \rightarrow can be seen as special cases of the constructions in [7]. If we do this, we see that every strictly positive operator with parameters from S_{wf} can be seen as a stable function $\gamma : S \rightarrow S$.

γ has a least fixpoint s , and $I(s)$ will be isomorphic to the fixpoint of Γ as a coherence space.

Thus every type defined by strictly positive induction and well founded types as parameters can be represented in S .

We can of course repeat this construction, using inductively defined types as parameters, and we can close under sums and products of stable parameterisations. One problem is that we in the interpretation of dependent sums use what we call an E-structure on the parameter space even to define the sum as a coherence space with totality. Thus we must be careful if we try to construct a hierarchy with inductively defined types as an integrated part. In section 2.4 we will define E-structures and give criteria for when an inductively defined coherence space with totality has an E-structure.

2.3 Positive inductive operators

If $\Gamma(X_1, \dots, X_n)$ is an operator on coherence spaces defined from some parameters and using \oplus , \times and \rightarrow , each occurrence of a variable X_i will be positive with signature $+$, or negative with signature $-$, in the usual sense. In this section we will consider positive operators $\Gamma(X)$ and show:

1. Γ can be considered to be a functor on the category K^+ of coherence spaces with totality.
2. Γ can be considered to be a functor on the category K^w of coherence spaces with weak totality.
- 3 The least fixpoint of Γ in K^+ and in K^w will have the same underlying domain (trivial) and the same set of total elements.

In order to handle this, we need to operate on three categories.

The positive category K^+ is defined in section 2.1 and the weakly positive category K^w is defined in section 2.2.

Definition 7 Let K^- be the *negative* category of coherence spaces with totality, where the objects are pairs (X, X_{TOT}) and an embedding $f : |X| \rightarrow |Y|$ is a morphism if $f^{-1}(y) \in X_{\text{TOT}}$ whenever $y \in Y_{\text{TOT}}$.

For all three categories we use the notation that \mathbf{X} etc. is an object with underlying coherence space X and total objects X_{TOT} .

Lemma 5 Let $\Gamma(X_1, \dots, X_n)$ be an operator where for each $i = 1, \dots, n$, all occurrences of X_i have the same signature $\sigma(i)$. Then Γ can be extended to a positive functor

$$\Gamma^+ : K^{\sigma(1)} \times \dots \times K^{\sigma(n)} \rightarrow K^+$$

and one negative functor

$$\Gamma^- : K^{\sim\sigma(1)} \times \dots \times K^{\sim\sigma(n)} \rightarrow K^-$$

where $\sim\sigma(i)$ is the opposite signature of $\sigma(i)$.

Proof:

We already proved that Γ_{\oplus} and Γ_{\times} can be extended to Γ_{\oplus}^+ and Γ_{\times}^+ .

It is easy to see that they also extend to Γ_{\oplus}^- and Γ_{\times}^- .

Let $\Gamma_{\rightarrow}(X_1, X_2) = X_1 \rightarrow X_2$.

Let $f : \mathbf{X}_1 \rightarrow \mathbf{Y}_1$ be a K^- -morphism and let $(g, \pi) : \mathbf{X}_2 \rightarrow \mathbf{Y}_2$ be a K^+ -morphism.

Let $(f \rightarrow g) : (X_1 \rightarrow X_2) \rightarrow (Y_1 \rightarrow Y_2)$ be the coherence space morphism.

A typical question in $\mathbf{Y}_1 \rightarrow \mathbf{Y}_2$ is $p_{(y,p)}$.

Let $\delta(p_{(y,q)}) = p_{(f^{-1}(y), \pi(q))}$.

Then $(f \rightarrow g, \delta)$ is a K^+ -morphism, and when we let

$$\Gamma_{\rightarrow}^+(f, (g, \pi)) = (f \rightarrow g, \delta)$$

we get a functor.

We let $\Gamma_{\rightarrow}^-((f, \pi), g) = f \rightarrow g$.

It is easy to see that this defines a functor.

Corollary 1 If $\Gamma(X)$ is a positive operator, then Γ can be extended to a functor Γ^+ on K^+ .

Γ^+ will in general not commute with direct limits, but will still have a least fixpoint in the category K^+ . In order to avoid too much notation, we will use the same letter m for the least elements in K^+ and K^w resp.

The inductive definitions of the least fixpoints in the two categories K^+ and K^w will trivially give the same coherence space at all levels, and thus the underlying coherence space of the two fixpoints will be the same, and that fixpoint is reached at level ω .

Since m is a substructure of all other coherence spaces with totality, the inductive definition gives us an increasing sequence in both categories, i.e. we have that the inclusion maps between the pointsets of each coherence space will be the coherence space morphism.

We let \preceq^+ and \preceq^w be the two substructure-relations connected with the two categories.

Lemma 6 *If X_i occurs positively, but never strictly positively in Γ , we may extend the functor*

$$\Gamma^+ : K^{\sigma(1)} \times \dots \times K^+ \times \dots \times K^{\sigma(n)} \rightarrow K^+$$

to a functor

$$\Gamma^+ : K^{\sigma(1)} \times \dots \times K^w \times \dots \times K^{\sigma(n)} \rightarrow K^+.$$

Proof:

The chains in X_i only contributes to the chains in $\Gamma(X_1, \dots, X_n)$ when X_i has a strictly positive occurrence.

From now on we let $\Gamma(X)$ be a fixed positive operator. By separating the strictly positive occurrences of X from the other occurrences, we find an operator $\Delta(Y, Z)$ where:

$$\Gamma(X) = \Delta(X, X).$$

Z is strictly positive in Δ .

Y is positive, but nowhere strictly positive in Δ .

Let $\mathbf{U} = (U, U_{\text{TOT}})$ be the least fixpoint of Γ in K^w .

Let $\Delta_0(Z) = \Delta(\mathbf{U}, Z)$.

Δ_0 is a strictly positive operator. Let \mathbf{V} be the least fixpoint of Δ_0 in K^+ .

We want to show that \mathbf{V} is the least fixpoint of Γ in K^+ , and that $V_{\text{TOT}} = U_{\text{TOT}}$.

Lemma 7 $V_{\text{TOT}} = U_{\text{TOT}}$

Proof:

We define \mathbf{U}_α by recursion on the ordinal number α as follows:

Let $\mathbf{U}_0 = m$, $\mathbf{U}_{\alpha+1} = \Gamma^w(\mathbf{U}_\alpha)$ and for limit ordinals λ let

$\mathbf{U}_\lambda = \lim_w \{\mathbf{U}_\alpha \mid \alpha < \lambda\}$.

By construction we have $\mathbf{U}_\alpha \preceq^w \mathbf{U}$.

Since $m \preceq^w \mathbf{V}$ we can prove by induction on α that $\mathbf{U}_\alpha \preceq^w \mathbf{V}$. The induction step is

$$\mathbf{U}_{\alpha+1} = \Delta^w(\mathbf{U}_\alpha, \mathbf{U}_\alpha) \preceq^w \Delta^w(\mathbf{U}, \mathbf{V}) = \mathbf{V}$$

since \mathbf{V} by Theorem 1 is the least fixpoint of Δ_0^w .

This shows that $\mathbf{U} \preceq^w \mathbf{V}$, i.e. that $U_{\text{TOT}} \subseteq V_{\text{TOT}}$.

On the other hand, \mathbf{U} is a fixpoint of Δ_0^w , and since (V, V_{TOT}) is the least fixpoint of Δ_0^w , we have $V_{\text{TOT}} \subseteq U_{\text{TOT}}$.

Lemma 8 \mathbf{V} is the least fixpoint for Γ^+ in K^+ .

Proof:

By Lemmas 6 and 7 we have

$$\Gamma^+(\mathbf{V}) = \Delta^+(\mathbf{V}, \mathbf{V}) = \Delta^+(\mathbf{U}, \mathbf{V}) = \Delta_0^+(\mathbf{V}) = \mathbf{V}.$$

Thus \mathbf{V} is a fixpoint of Γ in K^+ .

Let \mathbf{W} be the least fixpoint of Γ^+ in K^+ . In particular we have $U_{\text{TOT}} \subseteq W_{\text{TOT}}$ since \mathbf{U} is the least fixpoint in K^w . Then

$$U_{\text{TOT}} \subseteq W_{\text{TOT}} \subseteq V_{\text{TOT}} = U_{\text{TOT}},$$

so $V_{\text{TOT}} = W_{\text{TOT}}$.

Let $\mathbf{V}_0 = m$, $\mathbf{V}_{n+1} = \Delta^+(\mathbf{U}, \mathbf{V}_n)$.

By induction we see that $\Delta^+(\mathbf{U}, \mathbf{V}_n) \preceq^+ \mathbf{W}$ so $\mathbf{V} \preceq^+ \mathbf{W}$.

It follows that $\mathbf{W} = \mathbf{V}$.

We have then proved

Theorem 2 *Let $\Gamma(X)$ be a positive operator. Then the least fixpoint of Γ^+ in K^+ and the least fixpoint of Γ^w in K^w will have the same underlying coherence space and the same total objects.*

2.4 Density in inductively defined coherence-spaces with totality

When we constructed our hierarchy of coherence spaces with totality in [7], the density theorem for that hierarchy was one of the main results. In fact we proved that the spaces of that structure satisfies even stronger requirements, the set of total objects and the set of total chains are both uniformly dense.

The density theorem cannot be expected to be satisfied in general. The equation

$$X = X \times X$$

has a least solution with no total elements. We will give a simple criterion for when the methods of [7] can be extended to an inductively defined coherence space with totality. In order to make this precise, we must recollect a few definitions and constructions from [7]

Definition 8 Let X be a coherence space, X_{fin} the finite elements of X .

a) An *E-structure* on X is

- a set $\{E_A\}_{A \in X_{\text{fin}}}$ of elements in X such that $A \subseteq E_A$
- a set $\{C_a\}_{a \in |X|}$ of chains in X such that $\{a\} \in C_a$

such that each E_A will meet each C_a .

b) If X_{TOT} is the set of total elements in \mathbf{X} for some coherence space with totality, then an E-structure on X will *match* \mathbf{X} if each E_A is total and each C_a meets all total $x \in X$.

c) If $(X, \{E_A\}_{A \in X_{\text{fin}}}, \{C_a\}_{a \in |X|})$ and $(Y, \{F_B\}_{B \in Y_{\text{fin}}}, \{D_b\}_{b \in |Y|})$ are two coherence spaces with E-structures, then a morphism $f : X \rightarrow Y$ is an *E-structure morphism* if $f[E_A] \subseteq F_{f[A]}$ for all $A \in X_{\text{fin}}$ and $f[a] \in D_{f(a)}$ whenever $A \in C_a$ for all $a \in |X|$.

d) A *weak E-structure* is like an E-structure, except that we permit E_A or C_a to be undefined for certain $A \in X_{\text{fin}}$ and $a \in |X|$.

Whenever they both are defined, they shall meet, and the concept of matching is the same.

For f to be a morphism on weak E-structures we in addition demand:

E_A defined $\Rightarrow F_{f[A]}$ defined.

C_a defined $\Rightarrow D_{f(a)}$ defined.

The two categories K^+ and K^- can easily be extended to categories of coherence spaces with totality and matching E-structures, or with matching weak E-structures. We will let K^+ , K^w and K^- denote these extended categories here.

Remark

In [7] we let the E-structure be a part of the concept 'qualitative domain with totality'.

In [7] we showed that the operators \oplus , \times and \rightarrow can be extended to coherence spaces with E-structures. We will not repeat the construction here.

There are two ways to define an E-structure on $\mathbf{X} \oplus \mathbf{Y}$; extending the empty set to the left, i.e. in the X -part, or to the right. We let \oplus_l and \oplus_r be the two variants.

A *decoration* of an operator Γ is an assignment of l or r to any occurrence of \oplus in Γ .

From now on $\Gamma(X)$ will be a decorated operator, and Γ^E the canonical extension to E-structures and weak E-structures. We will also assume that all occurrences of X in Γ will have the same signature.

If X is positive in Γ , then Γ will have a least fixpoint with a matching weak E-structure that is a fixpoint of Γ^E . We will characterise when this actually is an E-structure. As a part of the proof, we will see how to find an optimal decoration in order to obtain an E-structure in the limit.

Definition 9 Let $M = (\emptyset, \{\emptyset\})$ be the coherence space with empty domain and with the empty set as total.

Lemma 9 a) *For every coherence space \mathbf{X} with weak totality there is a unique K^- -morphism $f : M \rightarrow \mathbf{X}$.*

b) *M has a matching E-structure.*

Proof:

a) is trivial. To prove b) we let $E(\emptyset) = \emptyset$, and there are no a 's for which we need to define C_a .

Definition 10 Let $\Gamma(X)$ be as above.

We define the *reduced value* $\Gamma^r(m)$ by iterating the transformations \mapsto defined below. Here \mathbf{X} will denote a coherence space with totality and a matching E-structure, and we will assume that all parameters used in Γ will be such coherence spaces.

$$\begin{aligned} m \times \mathbf{X} &\mapsto m & \mathbf{X} \times m &\mapsto m \\ m \times m &\mapsto m \\ m \oplus_l \mathbf{X} &\mapsto m & \mathbf{X} \oplus_l m &\mapsto \mathbf{X} \\ m \oplus_r \mathbf{X} &\mapsto \mathbf{X} & \mathbf{X} \oplus_r m &\mapsto m \\ m \oplus_r m &\mapsto m & m \oplus_l &\mapsto m \end{aligned}$$

Below, we will let $\mathbf{X} \preceq^* \mathbf{Y}$ mean that there is a morphism from \mathbf{X} to \mathbf{Y} in the category K^* .

Lemma 10 a) *If X is positive in Γ , then $\Gamma^r(m) = m$ or $\Gamma^r(m)$ is a coherence space with totality and a matching E-structure.*

In both cases $\Gamma^r(m) \preceq^w \Gamma(m)$

b) *If X is negative in Γ , then $\Gamma^r(m)$ is a coherence space with totality and a matching E-structure.*

Moreover $\Gamma^r(m) \preceq^- \Gamma(m)$.

Proof:

a) and b) are proved simultaneously by induction on Γ , and the proof is simple.

Lemma 11 *Let Γ be a positive operator as above.*

If $\Gamma^r(m) \neq m$, then $\Gamma_\infty(m)$ has a matching E-structure that is a fixpoint of Γ^E .

Proof:

We have

$$m \preceq \Gamma^E(m) \preceq \Gamma^E(\Gamma^E(m)).$$

We also have

$$m \preceq \Gamma^r(m) \preceq \Gamma^E(m).$$

It follows that for all n

$$\Gamma_n^E(m) \preceq \Gamma_n^E(\Gamma^r(m)) \preceq \Gamma_{n+1}^E(m).$$

Then

$$\lim_{n \rightarrow \infty} \Gamma_n^E(\Gamma^r(m)) = \lim_{n \rightarrow \infty} \Gamma_n^E(m)$$

so the latter is an E-structure.

It is then easy to see that this E-structure will match the limit coherence space with totality.

An optimal decoration will be one that avoids getting $\Gamma^r(m) = m$ whenever possible.

Lemma 12 *Let Γ be a positive operator.*

If $E(\emptyset)$ is defined in $\Gamma^E(m)$, then $\Gamma^r(m) \neq m$.

Proof:

The proof is by a simple induction on Γ . Because of Lemma 10 b), we do not need to consider suboperators where X occurs negatively. In the \oplus_l -case, $E(\emptyset)$ is defined if and only if it is defined in the left hand side suboperator.

Theorem 3 *Let Γ be a decorated, positive operator, where all parameters are coherence spaces with totality and matching E-structures. Then the least fixpoint of Γ^+ has a matching E-structure that is a fixpoint for Γ^E if and only if $\Gamma^r(m) \neq m$.*

Proof:

Lemma 11 states the if-part.

If we have a matching E-structure, then $E(\emptyset)$ must be defined in some $\Gamma_n(m)$, where Γ_n is the n 'th iteration of Γ . By Lemma 12, we must have that $(\Gamma_n)^r(m) \neq m$.

But this is only possible when $\Gamma^r(m) \neq m$.

This ends the proof of the theorem.

3 Inductively defined Scott-Ershov domains

3.1 Domains with totality

In section 2 we showed how inductively defined types could be interpreted as coherence spaces with totality. In this section we will see how we may use

Scott-Ershov domains for the same purpose. We will assume that the reader is familiar with the general theory of domains, as given e.g. in Stoltenberg-Hansen, Lindström and Griffor [15]. In this section we will use the term *domain* for a Scott-Ershov domain.

One possible representation of a domain is as the set of ideals in a finitely complete partial ordering. (A partial ordering is finitely complete if all finite, bounded subsets have least upper bounds.)

Without loss of generality we may use finitely complete preorderings instead. This was observed by Palmgren and Stoltenberg-Hansen in [14], where they introduced the *cusl*, a finitely complete preordering with an explicit least upper bound operator for bounded pairs.

The representation of domains used in [7] and [12] was *cusl*'s with one extra property: There are only finitely many ways to organise one element p of the preordering as the explicit least upper bound of other elements. This property was used technically in both [7] and [12]. In [13] we called this an *iei-structure*.

We will be interested in domains X with a selected set X_{TOT} of total objects. Berger [2, 3] was the first to analyse a general concept of totality for domains. He defines a set $Y \subseteq X$ to be a set of total elements if there is a separating set of partial continuous Boolean valued tests that works on every $y \in Y$. The family of domains with totality satisfying Berger's axioms and density, will be closed under the formation of sums, products and function spaces. In our view, Berger's axioms are reflecting the underlying assumption that the total input material is dense in its underlying domain.

In [13] we suggested a pragmatic axiomatisation of totality:

Definition 11 $\mathbf{X} = (X, X_{\text{TOT}})$ is a *domain with totality* if X is a domain, $X_{\text{TOT}} \subseteq X$ and the following are satisfied:

- i) $x \sqsubseteq y \sqsubseteq z \wedge x, y \in X_{\text{TOT}} \Rightarrow y \in X_{\text{TOT}}$.
- ii) The relation

$$x \approx y \Leftrightarrow x \sqcap y \in X_{\text{TOT}}$$

is an equivalence relation on X_{TOT} .

Remark

Berger's axioms will imply that the total objects are upwards closed, so in particular i) is satisfied.

ii) is not automatically satisfied, but will be a consequence of the axioms for KLS-totality from [2, 3] used to prove a general version of the Kreisel-Lacombe-Shoenfield theorem for domains.

In [13] we constructed a hierarchy of domains with totality, where we introduced base domains for the empty type, the type of natural numbers and the type of boolean values, and where we closed under dependent sums and products of continuous parameterisations of domains with totality. We used a domain S of syntactic forms, and each $s \in S$ will have an interpretation $I(s)$ as a subdomain of a universal domain D . In the rest of section 3 we will show how inductively defined types can be represented as elements of S with a canonical set of total elements. This work is a direct continuation of [13], and for a detailed understanding we will have to assume intimate knowledge to [13]. On the other hand, we will mostly indicate how the arguments from [13] can be adjusted to give the desired properties of the extended semantics, and the casual reader should be able to follow the main lines of ideas.

Convention

In section 3.3 we will need one extra base type, with one single total element. Thus, without altering the notation, all constructions referred to are assumed to be extended with one extra atomic element C in $|S|$ and one extra atomic element c in $|D|$ with $|I(C)| = \{\perp, c\}$ and $I([C])_{\text{TOT}} = \{[c]\}$.

In [13] we isolate the set S_{wf} of well founded syntactic forms, and by simultaneous recursion we define the set $I(s)_{\text{TOT}}$ for each $s \in S_{\text{wf}}$. It is shown that these are domains with totality in the technical sense, but it is also shown that the equivalence relation on the total elements will correspond to extensional equality. This is done by isolating a relation E on S_{wf} and a relation R on the set of pairs (s, x) where $s \in S_{\text{wf}}$ and $x \in I(s)_{\text{TOT}}$, and then proving that they correspond to the \approx -relation on the two underlying sets. As a result, we may divide the whole hierarchy out by the equivalence relations, and we get a well founded set-theoretical hierarchy with genuine functions and ordered pairs.

The types of the induced set-theoretical hierarchy will have inherited topologies. In order to be able to represent functions that are continuous with

respect to these topologies as functions in the hierarchy, we have introduced multivalued partial functions, but we require that a total object is single valued on total input. In order to handle this we extend any domain X to the domain mX of multiobjects. The details are given in [13].

3.2 Typestreams

In this section we will extend the hierarchy S_{wf} and $\{I(s)\}_{\text{TOT}}$ to a hierarchy TS , $\{I(s)_{\text{TOT}}\}_{s \in TS}$ of *typestreams* with total objects.

The idea is that certain types defined by strictly positive induction can be viewed as elements of S . Typestreams represent a generalisation of types defined by strictly positive induction. They first appeared in Normann [11] in the setting of associates.

Definition 12 By recursion on the ordinal number α we define TS_α and $\{I(s)_{\text{TOT}}\}_{s \in TS_\alpha}$ as follows

- a) TS_α is the largest subset of S satisfying
 - i) $[O]$, $[B]$, $[C]$ and $[N]$ are in TS_α .
 - ii) If $s = (\Pi, s_1, F)$ or $s = (\Sigma, s_1, F)$, then $s \in TS_\alpha$ if $s_1 \in TS_\beta$ for some $\beta < \alpha$ and for all $x \in I(s_1)_{\text{TOT}}$ we have $F(x) \in TS_\alpha$.
- b) By a standard least fixpoint induction, and assuming that $I(t)_{\text{TOT}}$ is defined when $t \in TS_\beta$ for some $\beta < \alpha$, we define $\{I(s)_{\text{TOT}}\}_{s \in TS_\alpha}$ as the least solution to the defining equations from [13], i.e.:
 - The total elements of $I([B])$, $I([N])$ and $I([C])$ are representatives for the boolean values, the natural numbers and the constant value resp., while $I([O])$ has no total objects.
 - The total elements of $I(\Sigma, s_1, F_s)$ are essentially the set of pairs (x, y) with x total in $I(s_1)$ and y total in $I(F_s(x))$.
 - The total elements of (Π, s_1, F_s) are essentially the set of functions z such that $z(x)$ is total in $I(F_s(x))$ whenever x is total in $I(s_1)$.

Lemma 13 *If $s \in TS_\alpha$ and $\alpha < \beta$ then $s \in TS_\beta$ and $I(s)_{\text{TOT}}$ is independent of the β .*

The proof is trivial.

Lemma 14 *Let $r \notin TS_\alpha$. Then at least one of two hold:*

i) $s \subseteq r \Rightarrow s \notin TS_\alpha$.

ii) $r \subseteq t \Rightarrow t \notin TS_\alpha$.

Proof:

For $\alpha = 0$ this is trivial.

Assume that the statement holds for all $\beta < \alpha$.

We have essentially defined $\neg TS_\alpha$ by positive induction, and we deal with the cases of this definition.

If $r = \{\perp\}$, then i) holds.

If $r = (\Pi, r_1, F_r)$ we have several subcases:

If $r_1 \notin TS_\beta$ for any $\beta < \alpha$, we see by Lemma 13 that either i) or ii) holds for r_1 for all $\beta < \alpha$. The same property will hold for r at level α .

If $r_1 \in TS_\beta$ for some $\beta < \alpha$, we must have an $x \in I(s_1)_{\text{TOT}}$ with $F_r(x) \notin TS_\alpha$.

If i) holds for $F_r(x)$ then i) holds for r , and if ii) holds for $F_r(x)$, or if $F_r(x)$ is not single valued, then ii) holds for r .

The case $r = (\Sigma, r_1, F)$ is treated in exactly the same way.

Lemma 15 *Let $x \in I(s)$, $y \in I(r)$ and $z \in I(t)$ where $s, r \in TS_\alpha$, $x \in I(s)_{\text{TOT}}$ and $z \in I(t)_{\text{TOT}}$, then $r \in TS_\alpha$ and $y \in I(r)_{\text{TOT}}$.*

Proof:

$r \in TS_\alpha$ by Lemma 14.

The rest of the lemma follows by induction on the rank of x as a total object in $I(s)_{\text{TOT}}$.

We may now define the relations E and R of extentional equality by a simultaneous definition by recursion on α .

Assume that E and R is defined on TS_β for $\beta < \alpha$. We then let

E be the largest relation on TS_α satisfying the defining equations in [13],

i.e. if $s E t$ then one of the following holds:

$s = t = [O]$, $s = t = [B]$, $s = t = [C]$, $s = t = [N]$,

$s = (\Sigma, s_1, F_s)$ and $t = (\Sigma, t_1, F_t)$, $s_1 E t_1$ and for all $x \in I(s_1)_{\text{TOT}}$ and

all $y \in I(t_1)_{\text{TOT}}$, if $(s_1, x) R (t_1, y)$, then $F_s(x) E F_t(y)$.

or finally

$s = (\Pi, s_1, F_s)$ and $t = (\Pi, t_1, F_t)$, $s_1 E t_1$ and for all $x \in I(s_1)_{\text{TOT}}$ and all $y \in I(t_1)_{\text{TOT}}$, if $(s_1, x) R (t_1, y)$, then $F_s(x) E F_t(y)$.

R is defined on the set of pairs (s, x) with $s \in TS_\alpha$ and $x \in I(s)_{\text{TOT}}$ exactly as in [13], i.e. $(s, x) R (t, y)$ if $s E t$ and x and y are hereditarily extensionally equal.

Lemma 16 a) *For each α and $s, t \in TS_\alpha$ we have*

$$s E t \Leftrightarrow s \cap t \in TS_\alpha$$

b) *For each $s, t \in TS_\alpha$, $x \in I(s)_{\text{TOT}}$ and $y \in I(t)_{\text{TOT}}$ we have*

$$(s, x) R (t, y) \Leftrightarrow s \cap t \in TS_\alpha \wedge x \cap y \in I(s \cap t)_{\text{TOT}}.$$

Proof:

Both arguments are essentially given in [13].

The lemma must be proved simultaneously by induction on α .

To prove a) we must use a contrapositive argument in both directions, while in order to prove b), we use the proof from [13] directly.

We let TS be the union of all the TS_α . In [13] we introduced the map $[x]_t$ extending x to an element of $I(t)$ and the map $y \cap |I(s)|$ when $s \subseteq t \in S$, and we proved that they map total objects on total objects for s and t in S_{wf} . This result easily extends to TS , and as a consequence we get that E and R still are equivalence relations when extended to TS .

We call the elements of TS for *typestreams*. Typestreams is a general way of modelling types defined by some strictly positive induction. As a general example, let us consider types of wellfounded trees:

Let $s \in S_{\text{wf}}$ and let $F : I(s)_{\text{TOT}} \rightarrow S_{\text{wf}}$ be continuous.

We may form the type $W = W_{s,F}$ as the least solution to the equation

$$W = \Sigma(x \in I(s))(I_{F(x)} \rightarrow W).$$

We may find an element t in S representing this set by taking the least solution in S to the domain equation

$$t = (\Sigma, s, \lambda x.(\Pi, F(x), \lambda y.t)).$$

Then t will be a typestream, and the total elements will essentially be the wellfounded trees of sequences from $I(s)_{\text{TOT}}$ where each $x \in I(s)_{\text{TOT}}$ leads to a branching over $I(F(x))_{\text{TOT}}$. This is close to the W -type in Martin-Löf systems.

3.3 Recursion on typestreams

Typestreams are generalisations of types defined by strictly positive induction. One important aspect of strictly positive induction is that we may use these types as a basis for recursive constructions. In this section we will formulate two semantical recursion schemes for recursion in type-streams, and prove that these schemes leads to continuous well typed functions. We may have similar schemes for recursive definitions of types, and we will then get continuous parameterisations of types indexed over the total objects in a typestream.

For a strictly positive operator, we have a clear concept of the immediate predecessors, and thus by iteration we may define the predecessors of an object. One problem in formulating general recursion schemes is that the set of immediate predecessors, or the set of all predecessors, will in general not be a type in the hierarchy. The best we can do is to define a type of indices for the predecessors and the map sending an index to the corresponding predecessor.

For every $s \in TS$ and $x \in I(s)_{\text{TOT}}$ we will define the *support type* $\text{sup}(s, x)$ of indices for the predecessors, together with the map $\rho_{s,x}$ sending $y \in \text{sup}(s, x)_{\text{TOT}}$ to the corresponding predecessor, and the map $R_{s,x}$ giving the type of the corresponding predecessor.

Definition 13 Let $P_\alpha = \{(s, x) \mid s \in TS_\alpha \wedge x \in I(s)_{\text{TOT}}\}$.
Let P be the union of all the P_α 's.

Remark

P can be seen as the total elements of the domain generated by the set of pairs (σ, p) such that $\sigma \in |S|$ and $p \in |I(\sigma)|$ with the pairwise ordering.

We will see the operators \oplus and \times as special cases of dependent sum and dependent product, where $[B]$ will be the index domain.

Definition 14 Let $(s, x) \in P$. We define $\text{sup}(s, x)$ as the least solution in S to the following set of equations:

If s represents a base type, we let $\text{sup}(s, x) = [O]$.

If $s = (\Pi, s_1, F_s)$ we let

$$\text{sup}(s, x) = s_1 \oplus (\Sigma, s_1, \lambda y. \text{sup}(F_s(y), x(y)))$$

with the obvious interpretation of \oplus as an operator on S .

If $s = (\Sigma, s_1, F_s)$ we let

$$\text{sup}(s, x) = [C] \oplus \text{sup}(F_s(\pi_0(x), \pi_1(x))).$$

Lemma 17 If $(s, x) \in P_\alpha$, then $\text{sup}(s, x) \in TS_\alpha$.

The proof is trivial. We actually have that if s is a tpestream where all parameters are taken from S_{wf} , then $\text{sup}(s, x)$ will be an element of S_{wf} .

We now introduce the maps ρ and R :

Definition 15 Let $(s, x) \in P$ and let $y \in \text{sup}(s, x)_{\text{TOT}}$

By recursion on the rank of (s, x) we define $R_{s,x}(y) \in S$ and

$\rho_{s,x}(y) \in I(R_{s,x}(y))_{\text{TOT}}$ as follows:

If s represents a base type, there is nothing to construct.

Case 1: $s = (\Pi, s_1, F_s)$

If $y = ([tt], y')$, let $R_{s,x}(y) = F_s(y')$, and let $\rho_{s,x}(y) = x(y')$.

If $y = ([ff], (y', z))$ let $R_{s,x}(y) = R_{F_s(y'), x(y')}(z)$ and let

$$\rho_{s,x}(y) = \rho_{F_s(y'), x(y')}(z).$$

Case 2: $s = (\Sigma, s_1, F_s)$

If $y = ([tt], [c])$, we let $R_{s,x}(y) = F_s(\pi_0(x))$ and $\rho_{s,x}(y) = \pi_1(x)$.

If $y = ([ff], z)$, let $R_{s,x}(y) = R_{F_s(\pi_0(x)), \pi_1(x)}(z)$ and let

$$\rho_{s,x}(y) = \rho_{F_s(\pi_0(x)), \pi_1(x)}(z).$$

Lemma 18 If $(s, x) \in P_\alpha$ and $y \in \text{sup}(s, x)_{\text{TOT}}$ then $R_{s,x}(y) \in TS_\alpha$ and $\rho_{s,x}(y) \in R_{s,x}(y)_{\text{TOT}}$.

The proof is trivial by induction on the rank of (s, x) as an element of P_α . Both R and ρ will be continuous by the fixpoint theorem for domains.

Definition 16 Let $F : P_\alpha \rightarrow TS_\alpha$ be continuous.

Let $h : D^* \rightarrow D^*$ be continuous, where $D^* = \bigcup \{I(s) \mid s \in S\}$.

Let $(s, x) \in P_\alpha$.

We say that h matches F on $\text{sup}(s, x)$ if

$$h(y) \in I(F(R_{s,x}(y), \rho_{s,x}(y)))_{\text{TOT}}$$

for all $y \in \text{sup}(s, x)$.

Remark

The intuition behind this definition is that F maps any predecessor of (s, x) to a type, and h will map the index of that predecessor to an element of the type given by F . We need this concept to formulate what we mean by a correctly typed function defined on the predecessors.

Theorem 4 (Semantical recursion scheme)

Let $F : P_\alpha \rightarrow TS_\alpha$ be continuous.

Assume that $G : S \times D^* \times (D^* \rightarrow D^*) \rightarrow D^*$ is continuous such that if $(s, x) \in P_\alpha$ and h matches F on $\text{sup}(s, x)$, then $G(s, x, h) \in I(F(s, x))_{\text{TOT}}$.

Then there is a unique continuous function H defined on P_α such that for each $(s, x) \in P_\alpha$:

$$H(s, x) \in I(F(s, x))_{\text{TOT}}$$

and

$$H(s, x) = G(s, x, H'(s, x))$$

where

$$H'(s, x)(y) = H(R_{s,x}(y), \rho_{s,x}(y)).$$

Proof:

H' is defined recursively from H , so the equation

$$H(s, x) = G(s, x, H'(s, x))$$

has a solution by the fixpoint theorem for domains. By induction on the rank of $(s, x) \in P_\alpha$ we then show that $H(s, x) \in I(F(s, x))_{\text{TOT}}$ and that $H'(s, x)$ matches F on $\text{sup}(s, x)$, a consequence of the induction hypothesis.

Finally by a standard induction on the rank of $(s, x) \in P_\alpha$ we show that $H(s, x)$ is unique. The details are routine.

Theorem 4 is general in the sense that we may define H by induction not just knowing H on the immediate predecessors, but on any set of predecessors we might choose. The disadvantage of the result is that it aims at defining H on all of P_α , while for many applications of recursion this will be requiring too much. We will now prove a slightly more general result.

Definition 17 a) Let \mathcal{X} be a subset of P_α , let $K : \mathcal{X} \rightarrow TS_\alpha$ be continuous, and let $T : \{(s, x, y) \mid (s, x) \in \mathcal{X} \wedge y \in I(K(s, x))_{\text{TOT}}\} \rightarrow D^*$ be continuous such that

$$T_{s,x} = \lambda y. T(s, x, y)$$

is a 1-1 map from $I(K(s, x))_{\text{TOT}}$ to $I(\text{sup}(s, x))_{\text{TOT}}$.

We say that (\mathcal{X}, K, T) is an *inductive system* if

$$(R_{s,x}(T_{s,x}(y)), \rho_{s,x}(T_{s,x}(y))) \in \mathcal{X}$$

for all $(s, x) \in \mathcal{X}$ and $y \in I(K(s, x))_{\text{TOT}}$.

b) If (\mathcal{X}, K, T) is an inductive system, $F : \mathcal{X} \rightarrow TS_\alpha$ is continuous, $h : D^* \rightarrow D^*$ is continuous and $(s, x) \in X$, we say that h *matches* F on $K(s, x)$ if

$$h(y) \in I(F(R_{s,x}(T_{s,x}(y)), \rho_{s,x}(T_{s,x}(y))))_{\text{TOT}}$$

for all $y \in K(s, x)$.

An inductive system is in a sense a substructure of $(P, \{\text{sup}(s, x)\}_{(s,x) \in P})$, where we replace P by a subset \mathcal{X} and we permit a restricted set of predecessors of (s, x) indexed by the type $K(s, x)$. Via $T_{s,x}$ the type $K(s, x)$ can be embedded into the full set of predecessors of x as an element of $I(s)_{\text{TOT}}$. The predecessors of x in view of the system will then both be in \mathcal{X} and be true predecessors.

Theorem 5 (Extended semantical recursion scheme)

Let (\mathcal{X}, K, T) be an inductive system, $F : \mathcal{X} \rightarrow ST_\alpha$ be continuous.

Assume that G is continuous such that if $(s, x) \in X$ and h matches F on $K(s, x)$, then $G(s, x, h) \in I(F(s, x))_{\text{TOT}}$.

Then there is a unique continuous function H defined on \mathcal{X} such that for each $(s, x) \in \mathcal{X}$ we have

$$H(s, x) \in I(F(s, x))_{\text{TOT}}$$

and

$$H(s, x) = G(s, x, H'(s, x))$$

where

$$H'(s, x) = \lambda y H(R_{s,x}(T_{s,x}(y)), \rho_{s,x}(T_{s,x}(y))).$$

Proof:

The proof is almost the same as the proof of Theorem 4, and is left for the reader.

We call these theorems *semantical recursion schemes* because they represent valid forms of recursive definitions for the particular semantics we develop here. In type theory we will also find logical recursion schemes; that is axioms for the existence of functions defined by recursion and rules for evaluation of such functions. We expect that our semantical recursion schemes will be useful if our methods are extended to the construction of a model for type theory with strictly positive induction.

We will give two important examples on how to use Theorem 5, both concerned with strictly positive inductive definitions. If $\mathbf{X} = \Gamma(\mathbf{X})$ is the least fixpoint of a strictly positive inductive definition on domains with totality with parameters represented as tpestreams in TS , we have shown that there is an $s \in TS$ such that $\mathbf{X} = (I(s), I(s)_{\text{TOT}})$.

Example 1

If Γ is a strictly positive inductive operator, we define the canonical index-type $K_\Gamma(x)$ as an element of S for the immediate predecessors of $x \in \Gamma(X)$ by recursion on Γ :

$$\Gamma(X) = I(s)_{\text{TOT}}: K_\Gamma(x) = [O].$$

$$\Gamma(X) = X: K_\Gamma(x) = [C].$$

$$\Gamma(X) = (\Sigma, s_1, \lambda y \Gamma_0(y)(X)): K_\Gamma(x) = K_{\Gamma_0(\pi_0(x))}(\pi_1(x)).$$

$$\Gamma(X) = (\Pi, s_1, \lambda y \Gamma_0(y)(X)): K_\Gamma(x) = (\Sigma, s_1, \lambda y K_{\Gamma_0(y)}(x(y))).$$

Let $(I(s), I(s)_{\text{TOT}})$ be the least fixpoint of Γ . We construct an inductive system with $\mathcal{X} = \{(s, x) \mid x \in I(s)_{\text{TOT}}\}$ and the function K_Γ as K . It is trivial to define the map T .

This example shows that standard recursion via the immediate predecessors leads to well typed, continuous functions.

Example 2

We let Γ and s be as in Example 1. By a similar construction as the one in Example 1 we may isolate the type of all predecessors of x in $I(s)_{\text{TOT}}$, indexed by K'_Γ . The definition is as in Example 1 with the following changes:

We define $K'_{\Gamma, \Gamma'}$ for all subformulas Γ' of Γ .

In the case $\Gamma'(X) = X$ we let $K'_{\Gamma', \Gamma}(x) = [C] \oplus K'_{\Gamma, \Gamma}(x)$.

For all other case of Γ' we use the defining equation for K'_Γ in Example 1.

Finally we let $K'_\Gamma = K'_{\Gamma, \Gamma}$.

These equations for K'_Γ has a solution that we use as the function K in the inductive system. \mathcal{X} will be as in example 1, and T is easy to define.

This example shows that we may use recursion on the full set of predecessors in order to define continuous and well typed functions.

3.4 Positive types

In section 3.3 we used the fixpoint theorem for domains to show that any type defined by a strictly positive operator can be viewed as an element of S . For this particular result we do not need that the operator is strictly positive, any operator $\Gamma(X)$ defined from the variable X and parameters $I(s)$ for $s \in S$ will have a fixpoint $I(s_\Gamma)$. The problem comes when we want to define the total elements of $I(s_\Gamma)$.

We could define the total elements of typestreams in stages, assuming that we knew the total elements of any $I(s)$ occuring negatively in the construction. In this section we want to find representations of types defined by general positive inductive definitions. It turns out that we cannot do this within S . We have to work within an extended domain T where each $t \in T$ has a signature. Let us first look at one example:

$$\Gamma(X) = (X \rightarrow \mathbb{N}) \rightarrow \mathbb{N}.$$

The corresponding $s \in S$ will be the least solution to the equation

$$s = (\Pi, (\Pi, s, \lambda y.[N]), \lambda y.[N])$$

This will be the same as the least solution to the equation

$$s = (\Pi, s, \lambda y.[N]).$$

Then we have a problem. If we isolate a set of total elements in $I(s)$ that gives a least fixpoint for Γ , what are then the total elements of

$$(I(s), I(s)_{\text{TOT}}) \rightarrow \mathbb{N}?$$

The underlying domain will be the same, but we cannot use the same set of total elements.

In the domain T we will essentially have multirepresentations of the elements in S , by decorating each occurrence of Π and Σ with a signature $+$ or $-$. (We will use \star when we do not want to specify the signature.) T will be divided in a positive part T^+ and a negative part T^- . In principle Π^+ operates from the negative to the positive side, while Π^- operates from the positive to the negative side. We want, however to include the use of parameters, and we will permit positive objects to occur at a location where there should be a negative object. In this case we will assume that totality is already defined at a previous stage in the hierarchy.

In the Σ -construction, the parameter-type is used both positively (in the definition of the total objects) and negatively (in the definition of a total parameterisation). We will thus only permit positive objects to represent the parameters in a Σ -type, and then we will assume that totality is already defined.

Since the elements of T essentially will be decorated elements of S , we can define the interpretation $I(t)$ by first stripping off the decoration, and then use the implementation $I(s)$ from [7, 13].

Definition 18 Let T be the amalgamation of the domains T^+ and T^- as defined below:

$[O]$, $[B]$, $[N]$ and $[C]$ are all in both T^+ and T^- .

If $s_1 \in T$ and $F_s : I(s_1) \rightarrow T$, then

$(\Pi^+, s_1, F_s) \in T^+$ and $(\Pi^-, s_1, F_s) \in T^-$.

If $s_1 \in T^+$ and $F_s : I(s_1) \rightarrow T$, then

$(\Sigma^+, s_1, F_s) \in T^+$ and $(\Sigma^-, s_1, F_s) \in T^-$.

For $s \in T$ we let $I(s)$ be the interpretation we get when all upper indices $+$ and $-$ are removed from the elements of s .

We are now ready to define the hierarchy $\{T_\alpha\}_{\alpha \in \text{ON}}$, and simultaneously the total elements $I(t)_{\text{TOT}}$ for each $t \in T_\alpha$.

Definition 19 Let $T_0 = \{[O], [B], [C], [N]\}$ with the obvious interpretation of the total objects.

For limit ordinals λ we let

$$T_\lambda = \bigcup_{\alpha < \lambda} T_\alpha.$$

Now assume that $T_\alpha \subseteq T^+$ is defined, together with the total objects.

We let U_α be the largest subset of T such that

- i) If $t = (\Sigma^+, t_1, F_t)$ then $t_1 \in T_\alpha$ and for all $x \in I(t_1)_{\text{TOT}}$ we have $F_t(x) \in U_\alpha \cap T^+$.
- ii) If $t = (\Sigma^-, t_1, F_t)$ then $t_1 \in T_\alpha$ and for all $x \in I(t_1)_{\text{TOT}}$ we have $F_t(x) \in U_\alpha \cap T^-$ or $F_t(x) \in T_\alpha$.
- iii) If $t = (\Pi^+, t_1, F_t)$, then either $t_1 \in T_\alpha$ and for all $x \in I(t_1)_{\text{TOT}}$ we have that $F_t(x) \in U_\alpha$, or $t_1 \in T^- \cap U_\alpha$.
- iv) If $t = (\Pi^-, t_1, F_t)$, then $t_1 \in T^+ \cap U_\alpha$.

We let $T_{\alpha+1} = U_\alpha \cap T^+$.

Finally, for $\beta \in ON$ and $t \in U_\alpha$ we define the set $P(\alpha)_\beta^t$ as follows:

If $t \in T_\alpha$, we let $P(\alpha)_\beta^t = I(t)_{\text{TOT}}$ for all β .

If $t \in (T^+ \cap U_\alpha) \setminus T_\alpha$, we let $P(\alpha)_0^t = \emptyset$.

If $t \in (T^- \cap U_\alpha)$, we let $P(\alpha)_0^t = I(t)$.

For all limit ordinals λ , we define $P(\alpha)_\lambda^t$ by unions of previous values for $t \in T^+$ and by intersection of previous values for $t \in T^-$.

We now assume that $P(\alpha)_\beta^t$ is defined for all $t \in U_\alpha$. We have the following cases:

Case 1 $t = (\Pi^+, t_1, F_t)$.

We let $P(\alpha)_{\beta+1}^t = \{x \in I(t) \mid \forall y \in P(\alpha)_\beta^{t_1} (F_t(y) \in U_\alpha \cap T^+) \wedge x(y) \in P(\alpha)_\beta^{F_t(y)}\}$.

Case 2 $t = (\Pi^-, t_1, F_t)$.

We let $P(\alpha)_{\beta+1}^t = \{x \in I(t) \mid \forall y \in P(\alpha)_\beta^{t_1} ((F_t(y) \in T_\alpha \vee F_t(y) \in U_\alpha \cap T^-) \wedge x(y) \in P(\alpha)_\beta^{F_t(y)})\}$.

Case 3 $t = (\Sigma^*, t_1, F_t)$, we let $P(\alpha)_{\beta+1}^t = \{x \in I(t) \mid \pi_0(x) \in I(t_1)_{\text{TOT}} \wedge \pi_1(x) \in P(\alpha)_\beta^{t_1}\}$.

Finally, we let

$$I(t)_{\text{TOT}} = \bigcup_{\beta \in \text{ON}} P(\alpha)_{\beta}^t$$

for $t \in T_{\alpha+1}$

We have to prove that this definition is well behaved.

Lemma 19 *For a fixed α , the set $P(\alpha)_{\beta}^t$ will increase with increasing β when $t \in U_{\alpha} \cap T^+$ and will decrease with increasing β when $t \in U_{\alpha} \cap T^-$.*

The proof is trivial, and is left for the reader.

The lemma shows that at the end we obtain a fixpoint for the family $\{P(\alpha)_{\beta}^t\}_{t \in U_{\alpha}}$.
Let κ_{α} be the least ordinal κ such that for all $t \in U_{\alpha}$ we have that $P(\alpha)_{\kappa}^t = P(\alpha)_{\kappa+1}^t$.

Lemma 20 a) *For each ordinal α , $T_{\alpha} \subseteq T_{\alpha+1}$.*

- b)** *If $\gamma \leq \alpha$ and $t \in U_{\gamma}$, then $P(\gamma)_{\beta}^t \subseteq P(\alpha)_{\beta}^t$ for $t \in T^+$ and $P(\gamma)_{\beta}^t \supseteq P(\alpha)_{\beta}^t$ for $t \in T^-$.*
- c)** *If $\gamma \leq \alpha$ and $t \in U_{\gamma}$, then $P(\alpha)_{\beta}^t \subseteq P(\gamma)_{\kappa_{\gamma}}^t$ for $t \in T^+$ and $P(\alpha)_{\beta}^t \supseteq P(\gamma)_{\kappa_{\gamma}}^t$ for $t \in T^-$.*

Proof:

a), b) and c) are proved simultaneously by induction on α . b) and c) ensures that the total objects in $I(t)$ are independent of the α for which $t \in T_{\alpha}$. For each α , b) and c) are proved by a trivial induction on β .

Lemma 21 a) *If $s \subseteq t \subseteq r$ are elements of T , with $s \in T_{\alpha}$ and $r \in T_{\alpha}$, then $t \in T_{\alpha}$*

- b)** *If $s \subseteq t \subseteq r$ with $s, t \in U_{\alpha}$, if $x \in I(s)$, $y \in I(t)$ and $z \in I(r)$, and if $x \subseteq y \subseteq z$, $x \in P(\alpha)_{\beta}^s$ and $z \in P(\alpha)_{\beta}^r$, then $t \in U_{\alpha}$ and $y \in P(\alpha)_{\beta}^t$*

Proof:

The proof is by a simple induction on α , and subinduction on β .

The property for T_{α} leads to the same property for U_{α} , which again is used to prove the property for $T_{\alpha+1}$.

Lemma 22 Define the relation E on T_α by

$$s E t \Leftrightarrow s \cap t \in T_\alpha$$

and define the relation R on

$$Q_\alpha = \{(s, x) \mid s \in T_\alpha \wedge x \in I(x)_{\text{TOT}}\}$$

by

$$(s, x) R (t, y) \Leftrightarrow s E t \wedge x \cap y \in I(s \cap t)_{\text{TOT}}.$$

Then E and R are equivalence relations.

Proof:

Simultaneously we use induction on α and prove

- i) If $s, t, r \in T_\alpha$, $s \cap t \in T_\alpha$ and $t \cap r \in T_\alpha$, then $s \cap t \cap r \in T_\alpha$
- ii) If s, t and r are as above, if x, y and z are total elements in $I(s)$, $I(t)$ and $I(r)$ resp. and if $x \cap y \in I(s \cap t)_{\text{TOT}}$ and $y \cap z \in I(t \cap r)_{\text{TOT}}$, then $x \cap y \cap z \in I(s \cap t \cap r)_{\text{TOT}}$.

The lemma will follow easily.

ii) is proved by proving the same property for each $P(\alpha)_\beta^t$ for each β . This is trivial by induction on β .

i) is proved by proving this property for U_α :

Let

$$V = \{s \cap t \cap r \mid s, t, r \in U_\alpha \wedge s \cap t \in U_\alpha \wedge t \cap r \in U_\alpha\}$$

We prove that V satisfies the closure properties of U_α .

We use the induction hypothesis and that the decomposition of an element s in T into s_1 and F_s commutes with intersections, and that

$$F_{s \cap t}(x) = F_s([x]_{s_1}) \cap F_t([x]_{t_1})$$

for $x \in I(s_1 \cap t_1)$. The details can be found in [7, 13].

Definition 20 Let T_{IND} be the union of all the T_α 's

In Normann [12, 13] it is shown that the ordinal height of the hierarchy S_{wf} of well founded types is the first ordinal not recursive in 3E and a real. It is not hard to see that this result also holds for the hierarchy of tpestreams of the previous section. For the hierarchy of general positive inductions, this is not obvious, and we conjecture that the closing ordinal of the hierarchy S_{IND} will be the least ordinal not in the closure of the set of hereditarily countable sets HC under set recursion (Normann [9]) relative to the nest admissible set operator, see Barwise, Gandy and Moschovakis [1]

We will give an example on how an inductively defined type can be interpreted as an element of T_{IND} . In section 2 we defined two natural categories of coherence spaces with totality, K^+ and K^w . There is a standard way of organising domains to a category, i.e. via projection pairs, see e.g. [15]. The analogue to K^w will be projection pairs that preserve totality upwards, and it is easy to see that the fixpoints we construct in interpreting types defined by positive induction as elements of T_{IND} will be the least fixpoints of the corresponding functor in this category. We do not have a category of domains with totality corresponding to K^+ . As we see it, this is a weakness. We believe that the conceptual understanding of totality, and the understanding of the nature of the total objects in a type defined by positive induction would be better if one found a natural category of domains with totality resembling the category K^+ .

Example

Let $A = I(t_1)_{\text{TOT}}$ and let $B_x = I(F_t(x))_{\text{TOT}}$.

We seek the least solution to the equation

$$X = \prod_{x \in A} ((X \rightarrow B_x) \rightarrow \mathbb{N})$$

This leads to an equation in T :

$$s = (\Pi^+, t_1, \lambda x \in I(t_1).(\Pi^-, (\Pi^+, s, \lambda y \in I(s).F_t(x)), \lambda z \in I(r_x).[N]))$$

where r_x is an abbreviation for

$$(\Pi^+, s, \lambda y \in I(s).F_t(x)).$$

This equation has a minimal solution s , and it is clear that if $t_1 \in T_\alpha$ and if each $F_t(x) \in T_\alpha$, then $s \in U_\alpha \cap T^+$.

Tracing the definition of $P(\alpha)_{3,\beta}^s$ we see that this corresponds to the β 'th level of the induction induced by the equation. Thus, at the end, we end up with the inductive closure as $I(s)_{\text{TOT}}$.

3.5 Density in positive types

In section 2.4 we gave a characterisation of when a coherence space defined by positive induction was an E-structure. It will be impossible to do a similar thing for our hierarchy T_{IND} of domains, first of all because we do not have any analogue of an E-structure in this case.

The first obstacle is that we have included the empty type as a base type. Then, even in the hierarchy S_{wf} , the question: *Is a given type nonempty?* will be undecidable in the functional 3E . If we restrict our construction to a domain based on B , N and C we are essentially back to the domain S^* from [7]. This case is simpler also because we do not need to consider multivalued functions. In this section we will let S^* be the domain of syntactic forms for types based on the singleton C , the boolean values B and the natural numbers N , and closed under the formation of Π - and Σ -types of single valued parameterisations. We let S_{wf}^* be the simple well founded hierarchy.

From [7] we will, uniformly in each $s \in S^*$ have a partial function $h_{s,x} : \mathbb{N} \rightarrow \mathbb{N}$ such that if $s \in S_{\text{wf}}^*$ and $x, y \in I(s)_{\text{TOT}}$, then $h_{s,x}$ is total, and $h_{s,x} = h_{s,y}$ if and only if x and y are consistent.

Moreover, for each $s \in S^*$ and $p \in |I(s)|$ we have an $E_{s,p} \in I(s)$ with p consistent with $E_{s,p}$ such that if $s \in S_{\text{wf}}^*$, then $E_{s,p} \in I(s)_{\text{TOT}}$.

In order to give an interpretation of domains defined by positive induction in this restricted case, we still have to extend S^* by decorating occurrences of Π and Σ with signatures. Let T^* be the resulting domain. In [7] we defined the substructure S_{lim}^* of syntactic forms where we never reach the bottom element \perp of S^* by evaluating parameterisations along extension maps $E_{s,x}$. We let T_{lim}^* be the corresponding subset of T^* , and the result from [7] that each $h_{s,E_{s,p}}$ will be total for $s \in T_{\text{lim}}^*$ and $p \in |I(s)|$ will still hold (using a suitable enumeration of $|D|$ as in [7]).

If $s \in T_{\text{lim}}^*$ and somehow, $I(s)_{\text{TOT}}$ is a set of total elements, we say that *the E-structure on $I(s)$ is total* when $h_{e,x}$ is total for all $x \in I(s)_{\text{TOT}}$, and $E_{s,p} \in I(s)_{\text{TOT}}$ for all $p \in |I(s)|$.

Our aim in this section is to give a nontrivial sufficient condition for when an inductively defined domain representable in T^* will support that the E-

structure is total. For these types, the density theorem will be a consequence.

Now assume that we have defined a set T_α^* of types with totality, all of positive signature, such that the E-structure is total for each $s \in T_\alpha^*$. We will use the terminology of the previous section, but with a slightly different interpretation.

Definition 21 a) Let U_α^* be the largest subset of T_{lim}^* that has the closure properties of U_α of Definition 19, when we replace T and T_α by T^* and T_α^* .

b) We define $P^*(\alpha)_\beta^s$ in complete analogy with the $P(\alpha)_\beta^s$ of Definition 19.

The h -functions are in principle defined by the effect of the object on an enumerated set of E -functions. Thus the challenge is to prove that the E -functions are total, and the totality of the h -functions will more or less be a consequence. The E -functions in turn are defined inductively, and a main problem is to show that $E_{s,\perp}$ is total, for the rest we may use induction.

Definition 22 Let V_α^* be the least subset of U_α^* inductively defined by

- i) $T_\alpha^* \subseteq V_\alpha^*$
- ii) If $s = (\Sigma^*, s_1, F_s) \in U_\alpha^*$ and $F_s(E_{s_1,\perp}) \in V_\alpha^*$, then $s \in V_\alpha^*$.
- iii) If $s = (\Pi^*, s_1, F_s) \in U_\alpha^*$ and $F_s(x) \in V_\alpha^*$ for all total $x \in I(s_1)$, then $s \in V_\alpha^*$.

Lemma 23 If $s \in V_\alpha^*$, then $E_{s,\perp}$ is total in $I(s)$.

Proof

We use induction on the rank of s in V_α^* .

Lemma 24 If $s \in U_\alpha^*$ is a positive element, and $E_{s,\perp}$ is total in $I(s)$, then $s \in V_\alpha^*$.

Proof:

We use induction on the least β such that $E_{s,\perp} \in P^*(\alpha)_\beta^s$.

Theorem 6 Let $s \in V_\alpha^*$

a) For all $p \in |I(s)|$ we have that $E_{s,p} \in I(s)_{\text{TOT}}$.

b) For $k \in \mathbb{N}$ and $x \in I(s)_{\text{TOT}}$ we have that $h_{s,x}(k) \in \mathbb{N}$.

Proof:

We prove the theorem simultaneously and uniformly for all $s \in V_\alpha^*$ by induction on

- a) the index of p in the preferred enumeration of $|D|$
- b) k

Lemma 23 provides us with the induction start, and the rest of the proof is simple.

We now end the construction of our hierarchy of inductively defined types with density. We let $T_{\alpha+1}^*$ be the positive elements of V_α^* .

Our sufficient condition for the E-structure to be total is of course that $s \in T_\alpha^*$ for some ordinal α . We saw that when T_α^* was defined, we gave a sufficient and necessary condition for the E-structure of an element of U_α^* to be total. But it is plausible that the E-structure of some $t \in T_{\text{IND}}$ is both defined and total, without $t \in T_\alpha^*$ for any α . It is just not hereditarily total, i.e total in all subtypes.

4 Appendix

In this appendix we will see that in the natural category connected with Loader's notion of a coherence space with totality (see [8]), the strictly positive operators will commute with direct limits. In order to do so, we must give a precise definition of the category and of the limit. For the rest of the concepts, see the discussion in section 2.1 about Loader's approach to totality.

Let (X, X_{TOT}) and $(X^\perp, X_{\text{TOT}}^\perp)$ be a dual pair of coherence spaces with totality, and let (Y, Y_{TOT}) and $(Y^\perp, Y_{\text{TOT}}^\perp)$ be another dual pair. We let $f : |X| \rightarrow |Y|$ be a morphism if f defines an embedding from X to Y such that $f[x] \in Y_{\text{TOT}}$ whenever $x \in X_{\text{TOT}}$.

We observe that this category contains direct limits as follows: If $\{X^i, X_{\text{TOT}}^i\}$ is a directed system under $\{f_{ij}\}_{i \leq j}$, we let

$$X, \{f_i\}_{i \in I} = \lim_{i \in I} \{X^i, f_{ij}\}_{i \leq j}$$

in the usual way.

Let

$$X_{\lim} = \{f_i[x] \mid i \in I \text{ and } x \in X_{\text{TOT}}^i\}$$

and let

$$X_{\text{TOT}} = (X_{\lim})^{\perp\perp}.$$

Then (X, X_{TOT}) , $\{f_i\}_{i \in I}$ is the direct limit of the directed system.

We also observe that

$$C \in X_{\text{TOT}}^\perp \Leftrightarrow f_i^{-1}[C] \in (X_i)_{\text{TOT}}^\perp \text{ for all } i.$$

This follows by a simple calculation, using $X_{\text{TOT}}^\perp = X_{\lim}^\perp$.

Theorem 7 *If \mathbf{Z} is a coherence-space with totality, then the functor*

$$\Gamma_Z(X) = \mathbf{Z} \rightarrow X$$

commutes with direct limits in the Loader category.

Proof:

Let

$$\{X_i, (X_i)_{\text{TOT}}\}_{i \in I}, \{f_{ij}\}_{i \leq j}$$

be a directed system with limit

$$(X, X_{\text{TOT}}), \{f_i\}_{i \in I}.$$

Since $\lim_{i \in I} (\mathbf{Z} \rightarrow \mathbf{X}_i)$ clearly is a substructure of $\mathbf{Z} \rightarrow \mathbf{X}$, we have for every $C \in (Z \rightarrow X)_{\text{TOT}}^\perp$ that $\Gamma(f_i)^{-1}[C] \in (Z \rightarrow X_i)_{\text{TOT}}^\perp$ for all $i \in I$.

We will prove the converse, and thereby show that $(Z \rightarrow X, (Z \rightarrow X)_{\text{TOT}})$ is indeed the limit.

Without loss of generality we may assume that $(X_i)_{\text{TOT}} \neq \emptyset$ for at least one i , i.e. that $X_{\lim} \neq \emptyset$.

Now let $C \in (Z \rightarrow X)^\perp$ be such that $\Gamma_Z(f_i)^{-1}(C) \in (Z \rightarrow X_i)_{\text{TOT}}^\perp$ for all i .

Claim 1

Let $x = \bigcup \{A \mid \exists b((A, b) \in C)\}$. Then $x \in Z_{\text{TOT}}$.

Proof

It is easy to see that $x \in Z$.

Let $y \in (X_i)_{\text{TOT}}$ and let K be a total chain in Z .

For $z \in Z$, let $F_{K,y}(z) = y$ if z meets K ; while $F_{K,y}(z) = \emptyset$ otherwise.

$F_{K,y} : Z \rightarrow X_i$ is stable and total, so $\Gamma_Z(f_i)^{-1}(C) \cap F_{K,y} \neq \emptyset$.

Let $(A, b) \in \Gamma_Z(f_i)^{-1}(C) \cap F_{K,y}$.

Then $A \in K$ and $A \subseteq x$ because $(A, f_i(b)) \in C$.

Thus x meets K . The arbitrary choice of K shows that x is total.

Now let $F \in (Z \rightarrow X)_{\text{TOT}}$. We have to show that F meets C .

Claim 2

$B = \{b \mid (A, b) \in C \text{ for some } A\}$ is in X_{TOT}^\perp .

Proof

It is easy to see that $B \in X^\perp$. We have to show that B is total.

Let $y \in (X_i)_{\text{TOT}}$. Let $G(z) = y$ for all $z \in Z$. Then G meets $\Gamma(f_i)^{-1}(C)$ in some (\emptyset, b) , and thus $f_i[y]$ meets B in $f_i(b)$.

Since B meets all $f_i[y]$ for all $y \in (X_i)_{\text{TOT}}$, we have that $B \in (X)_{\text{TOT}}^\perp$.

Claim 3

$K = \{A \mid (A, a) \in F \text{ for some } a \in B\}$ is a total chain in \mathbf{Z} .

Proof

It is easy to see that K is a chain. Totality follows from claim 2 and the totality of F .

Now let $F(x)$ meet B in b .

Claim 4

b can be extended to a y in X_{lim} .

Proof

If this is not the case, $B \setminus \{b\} \in X_{\text{lim}}^\perp$ so $B \setminus \{b\} \in X_{\text{TOT}}^\perp$ and $F(x)$ does not meet $B \setminus \{b\}$. But $F(x) \in X_{\text{TOT}}$.

This proves the claim.

Let y be as in Claim 4, and let $y = f_i[y']$, where $y' \in [X_1]_{\text{TOT}}$.

For $z \in Z$, let $G(z) = y'$ if z meets K , $G(z) = \emptyset$ otherwise, where K is the total chain from Claim 3.

Then $G \in (Z \rightarrow X_i)_{\text{TOT}}$, so G meets $\Gamma(f_i)^{-1}(C)$ in a point (A, b') where $A \in K$.

Since $(A, f_i(b')) \in C$ we have that $f_i(b') \in y$. Then $b = f_i(b')$ since $b \in B \cap y$ too.

Since $A \in K$ we have $(A, b'') \in F$ for some $b'' \in B$. But then $b'' \in F(x)$ so b'' is coherent with b . $B \in X^\perp$ so $b'' = b$. It follows that $(A, b) \in F \cap C$.

F was an arbitrary stable function, so $C \in (Z \rightarrow X)_{\text{TOT}}^\perp$.

The equivalence

$$C \in (Z \rightarrow X)_{\text{TOT}}^\perp \Leftrightarrow \forall i \Gamma(f_i)^{-1}(C) \in (Z \rightarrow X_i)_{\text{TOT}}^\perp$$

shows that $\lim_{i \in I} (Z \rightarrow \mathbf{X}_i) = (Z \rightarrow \lim_{i \in I} \mathbf{X}_i)$ as totality spaces.

This ends the proof of the theorem.

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