Vector fields and deformations of isotropic super-Grassmannians of maximal type

by

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VECTOR FIELDS AND DEFORMATIONS 
OF ISOTROPIC SUPER-GRASSMANNIANS 
OF MAXIMAL TYPE

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ABSTRACT. One determines the holomorphic vector fields and the deformations of 
the isotropic super-Grassmannians of maximal type $F^{o} \text{Gr}_{2r|2s,r\mid s}$ associated with the 
complex or-hosymplectic Lie superalgebras.

1. Preliminaries

In [2,6,7] the holomorphic vector fields and the deformations of complex super-
Grassmannians were studied. It was proved, in particular, that, for a wide class
of super-Grassmannians, all holomorphic vector fields are induced by linear trans-
formations and the tangent sheaf 1-cohomology vanishes. Here we want to apply
the same methods in order to get similar results for isotropic super-Grassmannians
of maximal type associated with orthosymplectic Lie superalgebras. It turns out
that the super-Grassmannian of maximal type associated with the Lie superalge-
bra $\mathfrak{osp}_{2r-1|2s}(\mathbb{C})$ is isomorphic to a connected component of that associated with
$\mathfrak{osp}_{2r|2s}(\mathbb{C})$ (which is well known in the classical situation), and so we shall study
only the latter case.

Let us denote by $\text{IGr}_{2r|2s,r\mid s}$ the isotropic super-Grassmannian of maximal type
associated with the classical Lie superalgebra $\mathfrak{osp}_{2r|2s}(\mathbb{C})$ (see [4]). Its reduction
is the product of two isotropic complex Grassmannians $\text{IGr}_{2r,r}^{s,s} \times \text{IGr}_{2s,s}^{s,s}$, where
the first factor is the Grassmannian of isotropic r-planes in the vector space $\mathbb{C}^{2r}$
endowed with a non-degenerate symmetric bilinear form, while the second one is
that of isotropic s-planes in $\mathbb{C}^{2s}$ endowed with a non-degenerate skew-symmetric
bilinear form. The supermanifold $\text{IGr}_{2r|2s,r\mid s}$ admits a natural transitive action
of the orthosymplectic Lie supergroup $\text{OSP}_{2r|2s}(\mathbb{C})$, inducing on its reduction the
standard transitive action of the Lie group $\text{O}_{2r}(\mathbb{C}) \times \text{Sp}_{2s}(\mathbb{C})$.

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Let \((e_1, \ldots, e_{2r}), (f_1, \ldots, f_{2s})\) be the standard bases of \(\mathbb{C}^{2r}, \mathbb{C}^{2s}\) respectively. We suppose that the orthosymplectic Lie supergroup leaves invariant the bilinear form in \(\mathbb{C}^{2r|2s}\) given in the basis \((e_1, \ldots, e_{2r}, f_1, \ldots, f_{2s})\) by the matrix

\[
\begin{pmatrix}
0 & 1_r & 0 & 0 \\
1_r & 0 & 0 & 0 \\
0 & 0 & 0 & 1_s \\
0 & 0 & -1_s & 0
\end{pmatrix}
\]

We denote by \(o\) the graded isotropy subspace of maximal dimension

\[o = (e_{r+1}, \ldots, e_{2r}, f_{s+1}, \ldots, f_{2s})\]

of \(\mathbb{C}^{2r|2s}\). It is well known that the manifold \(\text{IGr}^a_{2r,r}\) has two connected components, while \(\text{IGr}^a_{2s,s}\) is connected. We choose the connected component

\[M = \text{I}^a\text{Gr}^a_{2r, r} \times \text{IGr}^a_{2s, s}\]

of \(\text{IGr}^a_{2r, r} \times \text{IGr}^a_{2s, s}\), containing the point \(o\), and denote by \(\text{I}^a\text{Gr}^a_{2r|2s, r|s}\) the corresponding connected component of \(\text{IGr}_{2r|2s, r|s}\). Sometimes we will denote this supermanifold by \((M, \mathcal{O})\), where \(\mathcal{O}\) is its structure sheaf.

The natural action of the Lie supergroup \(\text{OSp}_{2r|2s}(\mathbb{C})\) induces the transitive action of its identity component \(\text{SOSp}_{2r|2s}(\mathbb{C})\) on \((M, \mathcal{O})\). The reduction of the latter supergroup is

\[G = G_0 \times G_1,\]

where

\[G_0 = \text{SO}_{2r}(\mathbb{C}), \quad G_1 = \text{Sp}_{2s}(\mathbb{C}).\]

Let \(P\) denote the stabilizer \(G_o\) of the point \(o \in M\) in \(G\); we have

\[P = P_0 \times P_1,\]

where \(P_0 \subset G_0, P_1 \subset G_1\). The subgroup

\[R = R_0 \times R_1,\]

where

\[R_0 \simeq \text{GL}_r(\mathbb{C}), \quad R_1 \simeq \text{GL}_s(\mathbb{C}),\]

leaving invariant the subspaces

\[(e_1, \ldots, e_r), (e_{r+1}, \ldots, e_{2r}), (f_1, \ldots, f_s), (f_{s+1}, \ldots, f_{2s}),\]

is the reductive part of \(P\). The matrices from \(R\) are of the form

\[
\begin{pmatrix}
A & 0 & 0 & 0 \\
0 & (A^t)^{-1} & 0 & 0 \\
0 & 0 & B & 0 \\
0 & 0 & 0 & (B^t)^{-1}
\end{pmatrix},
\]
where \( A \in \text{GL}_r(\mathbb{C}), \ B \in \text{GL}_s(\mathbb{C}), \) while those from \( P \) have the form

\[
\begin{pmatrix}
A & 0 & 0 & 0 \\
U & (A^t)^{-1} & 0 & 0 \\
0 & 0 & B & 0 \\
0 & 0 & V & (B^t)^{-1}
\end{pmatrix}.
\]

The tangent Lie algebras and Lie superalgebras of Lie groups and Lie supergroups will be denoted, as usually, by the corresponding Gothic lower case letters. We have

\[
g = g_0 \oplus g_1, \quad g_0 = \mathfrak{so}_{2r}(\mathbb{C}), \quad g_1 = \mathfrak{sp}_{2s}(\mathbb{C}).
\]

The Lie algebra \( p \) of \( P \) admits the semi-direct decomposition

\[
p = r + n,
\]

where \( n \) is the nil-radical of \( p \). We have

\[
n = n_0 \oplus n_1,
\]

where \( n_0 \subset g_0, \ n_1 \subset g_1 \) consist of the matrices

\[
u = \begin{pmatrix}
0 & 0 \\
U & 0
\end{pmatrix}, \quad v = \begin{pmatrix}
0 & 0 \\
V & 0
\end{pmatrix},
\]

\( U \) and \( V \) being a skew-symmetric \( r \times r \) and a symmetric \( s \times s \)-matrix respectively. The subalgebra \( n \) is commutative.

We shall use the standard coordinate system on \( \text{IGr}_{2r|2s}, \) in a neighborhood of \( o \) introduced in [4, Ch. 5, Sec. 6], changing slightly the notation; more precisely, transposing the coordinate matrix. This matrix will have the form

\[
\begin{pmatrix}
X & \Xi \\
1_r & 0 \\
-\Xi^t & Y \\
0 & 1_s
\end{pmatrix},
\]

where \( X = (x_{\alpha \beta}) \) and \( Y = (y_{ij}) \) are a \( r \times r \)-matrix and a \( s \times s \)-matrix of even coordinates, \( X^t = -X, \ Y^t = -Y, \) and \( \Xi = (\xi_{as}) \) is a \( r \times s \)-matrix of odd entries. At the point \( o \) we have \( x_{\alpha \beta} = y_{ij} = 0. \) The natural action of \( \text{OSp}_{2r|2s}(\mathbb{C}) \) on \( \text{IGr}_{2r|2s}, \) is given by the matrix multiplication from the left.

Let \( \rho_0, \rho_1 \) be the standard representations of \( \text{GL}_r(\mathbb{C}), \text{GL}_s(\mathbb{C}) \) and \( \sigma_0, \sigma_1 \) their adjoint representations in the corresponding derived algebras \( \mathfrak{sl}_r(\mathbb{C}), \ p = r, s. \) The trivial 1-dimensional representation of any group will be denoted by 1. In what follows, we shall omit for simplicity the trivial factors 1 in the notation of the representations.

As in [6], we exploit the theory of homogeneous vector bundles. Let \( E = E_\psi \) be a finite-dimensional \( P \)-module determined by a holomorphic linear representation \( \psi \) of \( P. \) We denote by \( E = E_\psi \) the corresponding homogeneous vector bundle over \( M \) and by \( \mathcal{E} = \mathcal{E}_\psi \) the sheaf of its holomorphic sections. As is well known, the
tangent sheaf $\Theta$ on $M$ is isomorphic to $\mathcal{E}_\tau$, where the isotropy representation $\tau$ of $P$ is completely reducible and satisfies the condition

$$(3) \quad \tau|_R = \bigwedge^2 \rho_0 + S^2 \rho_1.$$ 

The supermanifold $(M, \mathcal{O})$ is, in general, non-split. As usually, we associate with it the split supermanifold $(M, \mathrm{gr} \mathcal{O})$. Its structure sheaf is the graded sheaf associated with the filtration

$$(4) \quad \mathcal{O} = \mathcal{J}^0 \supset \mathcal{J}^1 \supset \mathcal{J}^2 \supset \ldots,$$

where $\mathcal{J} = (\mathcal{O}_1)$. We have $\mathrm{gr} \mathcal{O} \simeq \bigwedge \mathcal{E}$, where $\mathcal{E} = \mathcal{J}/\mathcal{J}^2$. The holomorphic vector bundle $\mathcal{E}$ over $M$ associated with $\mathcal{E}$ has the fibers $\mathcal{E}_x = \mathcal{J}_x/m_x \mathcal{J}_x$, $x \in M$, where $m_x$ is the maximal ideal of $\mathcal{O}_x$.

Clearly, the action of $\mathrm{OSp}_{2|2} (\mathbb{C})$ on the super-Grassmannian induces actions of $G$ on the sheaves $\mathcal{O}$, $\mathcal{J}$, $\mathcal{E}$ and on the vector bundle $\mathcal{E}$, covering the standard action of $G$ on $M$. Thus, $\mathcal{E}$ is a homogeneous vector bundle over $M$.

**Proposition 1.** We have

$$\mathrm{gr} \mathcal{O} \simeq \bigwedge \mathcal{E}_\varphi,$$

where $\varphi$ is the irreducible representation of $P$ such that

$$\varphi|_R = \rho_0^* \otimes \rho_1^*.$$ 

**Proof.** Clearly, $\mathcal{J}/\mathcal{J}^2 = \mathcal{E}_\varphi$, where $\varphi$ is the representation of $P$ induced in the fibre $\mathcal{E}_o = \mathcal{J}_o/m_0 \mathcal{J}_o$. To calculate it, we use the coordinate matrix (2). The action of $P$ on $(M, \mathcal{O})$ is expressed by means of the coordinates in the following way:

$$\tilde{Z} = \left( \begin{array}{cccc} A & 0 & 0 & 0 \\ U & (A^t)^{-1} & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & V & (B^t)^{-1} \end{array} \right) \left( \begin{array}{cc} X & \Xi \\ 1_x & 0 \\ -\Xi^t & Y \\ 0 & 1_s \end{array} \right),$$

$$(5) \quad = \left( \begin{array}{cc} AX & A\Xi \\ (A^t)^{-1} + UX & U\Xi \\ -B\Xi^t & BY \\ -V\Xi^t & (B^t)^{-1} + VY \end{array} \right).$$

We must reduce the result to the form (2) by multiplying from the right by the matrix $\left( \begin{array}{cc} (A^t)^{-1} + UX & U\Xi \\ -V\Xi^t & (B^t)^{-1} + VY \end{array} \right)^{-1}$. We may set $X = 0, Y = 0$ which simplifies the calculation. Then

$$\left( \begin{array}{cc} (A^t)^{-1} & U\Xi \\ -V\Xi^t & (B^t)^{-1} \end{array} \right)^{-1} \equiv \left( \begin{array}{cc} A^t & -A^t U\Xi B^t \\ B^t V\Xi^t A^t & B^t \end{array} \right)$$
modulo $\mathcal{J}_o^2$. Hence,
\[
\tilde{Z} \equiv \begin{pmatrix}
0 & A^2B^t \\
1_r & 0 \\
-B^tA^t & 0 \\
0 & 1_s
\end{pmatrix}
\]
modulo $m_o\mathcal{J}_o^2$. Since the entries of $\Xi$ determine a basis of $E_o$, this implies our assertion.

Our goal is to calculate the 0- and 1-cohomology of the tangent sheaf $\mathcal{T} = \text{Der} \mathcal{O}$ of $\text{IGr}_{2r|2s,r|s}$. As in [6], we consider first the $\mathbb{Z}$-graded sheaf $\hat{\mathcal{T}} = \text{Der} \text{ gr} \mathcal{O}$. It is known (see [4]) that for any $q \geq -1$ there exists a natural exact sequence of sheaves
\[
0 \to \mathcal{T}_{(q+1)} \to \mathcal{T}_{(q)} \to \hat{T}_q \to 0,
\]
where $\mathcal{T}_{(q)}$ are the subsheaves of $\mathcal{T}$ forming a filtration of this sheaf and defined by
\[
\mathcal{T}_{(-1)} = \mathcal{T},
\]
\[
\mathcal{T}_{(q)} = \{ \delta \in \mathcal{T} | \delta \mathcal{O} \subset \mathcal{J}^q, \delta \mathcal{J} \subset \mathcal{J}^{q+1} \}, \quad q \geq 0.
\]
The sequence (6) will permit us to relate the cohomology of $\mathcal{T}$ to that of $\hat{T}$. To calculate the cohomology of the latter sheaf, one uses the exact sequence
\[
0 \to \mathcal{A}_{q+1} \overset{\alpha}{\to} \hat{T}_q \overset{\beta}{\to} \mathcal{B}_q \to 0.
\]
Here
\[
\mathcal{A}_q = \mathcal{E}_\varphi^* \otimes \bigwedge^q \mathcal{E}_\varphi = \mathcal{E}_{\Phi_q}
\]
with
\[
\Phi_q = \varphi^* \otimes \bigwedge^q \varphi,
\]
and
\[
\mathcal{B}_q = \Theta \otimes \bigwedge^q \mathcal{E}_\varphi = \mathcal{E}_{T_q},
\]
with
\[
T_q = \tau \otimes \bigwedge^q \varphi.
\]
The mapping $\beta$ is the restriction of a derivation of degree $q$ onto the structure sheaf $\mathcal{F}$ of $M$, and $\alpha$ identifies any sheaf homomorphism $\mathcal{E}_\varphi \to \bigwedge^{p+1} \mathcal{E}_\varphi$ with its extension which is a derivation of degree $q$ and is zero on $\mathcal{F}$. In particular,
\[
\mathcal{T}_{(-1)} \simeq \mathcal{A}_0 = \mathcal{E}_{\Phi}^* = \mathcal{E}_{\varphi^*}.
\]

Now we make some remarks concerning the action of the group $G$ on the sheaves involved. Clearly, the action of $G$ on the structure sheaf $\mathcal{O}$ induces an action of
$G$ on $T$, preserving the parities. It follows that $G$ preserves the filtrations $(4)$ and $(7)$, inducing an action on the sheaf $\mathcal{T}$. Thus, $\mathcal{T}_q$ for any $q$ is a locally free analytic sheaf on $M$ which is homogeneous with respect to $G$. One sees easily that the homomorphisms in the exact sequences $(6)$ and $(8)$ are $G$-equivariant.

To conclude these preliminaries, we shall write explicitly certain fundamental vector fields on $(M, \mathcal{O})$ associated with the action of $G$, using the local coordinates from $(2)$. Let us denote by $X \mapsto X^*$ the Lie superalgebra homomorphism $\mathfrak{osp}_{2r|2s}(\mathbb{C}) \to H^0(M, T)$ induced by the action of $\text{SOSp}_{2r|2s}(\mathbb{C})$ on $(M, \mathcal{O})$.

Let

$$H = \text{diag}(\lambda_1, \ldots, \lambda_r, -\lambda_1, \ldots, -\lambda_r, \mu_1, \ldots, \mu_s, -\mu_1, \ldots, -\mu_s)$$

be the general diagonal matrix lying in $\mathfrak{g}$. Using $(5)$, we get

$$(11) \quad H^* = \sum_{\alpha < \beta} (\lambda_\alpha + \lambda_\beta)x_{\alpha\beta} \frac{\partial}{\partial x_{\alpha\beta}} + \sum_{i \leq j} (\mu_i + \mu_j)y_{ij} \frac{\partial}{\partial y_{ij}} + \sum_{\alpha, i} (\lambda_\alpha + \mu_i)\xi_{\alpha i} \frac{\partial}{\partial \xi_{\alpha i}}.$$ 

Now, for the elements $u, v \in n$ given by $(1)$, we get, using $(5)$ again:

$$u^* = \sum_{\alpha, \beta} (\mathcal{X} U X)_{\alpha\beta} \frac{\partial}{\partial x_{\alpha\beta}} - \sum_{i, j} (\Xi^t U \Xi)_{ij} \frac{\partial}{\partial y_{ij}} + \sum_{\alpha, k} (\mathcal{X} U \Xi)_{\alpha k} \frac{\partial}{\partial \xi_{\alpha k}},$$

$$v^* = -\sum_{\alpha, \beta} (\Xi V \Xi^t)_{\alpha\beta} \frac{\partial}{\partial x_{\alpha\beta}} + \sum_{i, j} (\mathcal{Y} V Y)_{ij} \frac{\partial}{\partial y_{ij}} + \sum_{\alpha, k} (\Xi V \Xi)_{\alpha k} \frac{\partial}{\partial \xi_{\alpha k}}.$$ 

Let us choose the basis $X_{\alpha\beta} (\alpha < \beta)$, $Y_{ij} (i < j)$ of $n$ given by

$$X_{\alpha\beta} = \frac{1}{2}(E_{\alpha\beta} - E_{\beta\alpha}),$$

$$Y_{ij} = \frac{1}{2}(F_{ij} + F_{ji}) (i \neq j),$$

$$Y_{ii} = F_{ii},$$

where $E_{\alpha\beta}$ and $F_{ij}$ are the natural bases of the vector spaces of matrices $M_r(\mathbb{C})$ and $M_s(\mathbb{C})$ respectively. Then, in particular, we have

$$(13) \quad X_{\gamma\delta}^* = \sum_{\gamma, \delta} x_{\gamma\delta} x_{\gamma\delta} \frac{\partial}{\partial x_{\gamma\delta}} - \sum_{i, j} \xi_{\alpha i} \xi_{\beta j} \frac{\partial}{\partial y_{ij}}$$

$$+ \frac{1}{2} \sum_{\gamma, k} (x_{\gamma\alpha} \xi_{\beta k} - x_{\gamma\beta} \xi_{\alpha k}) \frac{\partial}{\partial \xi_{\gamma k}},$$

$$Y_{ij}^* = -\sum_{\alpha, \beta} \xi_{\alpha i} \xi_{\beta j} \frac{\partial}{\partial x_{\alpha\beta}} + \sum_{k, l} y_{ki} y_{lj} \frac{\partial}{\partial y_{kl}}$$

$$+ \frac{1}{2} \sum_{\gamma, k} (y_{jk \gamma} \xi_{\gamma i} + y_{ik} \xi_{\gamma j}) \frac{\partial}{\partial \xi_{\gamma k}} (i \neq j),$$

$$Y_{ii}^* = -\sum_{\alpha, \beta} \xi_{\alpha i} \xi_{\beta i} \frac{\partial}{\partial x_{\alpha\beta}} + \sum_{k, l} y_{ki} y_{li} \frac{\partial}{\partial y_{kl}}$$

$$+ \sum_{\gamma, k} y_{ik \gamma} \xi_{\gamma i} \frac{\partial}{\partial \xi_{\gamma k}}.$$
Let now $n^-$ be the nilpotent subalgebra of $\mathfrak{g}$ complementary to $\mathfrak{p}$; it has the form

$$n^- = n^-_0 + n^-_1,$$

where $n^-_0 \subset \mathfrak{g}_0$, $n^-_1 \subset \mathfrak{g}_1$ consist of the matrices

$$u = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix},$$

$U$ and $V$ being a skew-symmetric $r \times r$ and a symmetric $s \times s$ matrix respectively (cf. (1)). Consider the basis of $n^-$ formed by the elements $U_{\alpha \beta}$ ($\alpha < \beta$), $V_{ij}$ ($i < j$), $V_{ii}$ corresponding to the matrices $U = E_{\alpha \beta} - E_{\beta \alpha}$, $V = E_{ij} + E_{ji}$ ($i < j$); $E_{ii}$ respectively. One sees easily that

(14)

$$U^*_{\alpha \beta} = \frac{\partial}{\partial x_{\alpha \beta}}, \quad V^*_{ij} = \frac{\partial}{\partial y_{ij}}.$$

2. The cohomology of $A_q$ and $B_q$

In this section we shall calculate the 0- and 1-cohomology of the sheaves $A_q$ and $B_q$. As in [6,7], we use the theorem of Bott (see [1], Theorem IV') permitting to calculate the cohomology of the homogeneous sheaf $\mathcal{E}_\psi$ on $M$ defined by a completely reducible representation $\psi$ of $P$. More precisely, this theorem gives an algorithm for determining the highest weights of the $G$-modules $H^p(M, \mathcal{E}_\psi)$ in terms of the highest weights of $\psi$. To apply it, we have to introduce some notation related to weights and roots of $G$.

We choose the Cartan subalgebra $t = t_0 \oplus t_1$ in the tangent Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of $G$ such that $t_0$ and $t_1$ are the Cartan subalgebras of $\mathfrak{g}_0$ and $\mathfrak{g}_1$, respectively, formed by all diagonal matrices

$$H_0 = \text{diag}(\lambda_1, \ldots, \lambda_r, -\lambda_1, \ldots, -\lambda_r),$$

$$H_1 = \text{diag}(\mu_1, \ldots, \mu_s, -\mu_1, \ldots, -\mu_s).$$

We consider the following system of positive roots:

$$\Delta^+ = \Delta^+_0 \cup \Delta^+_1,$$

where

$$\Delta^+_0 = \{\lambda_i - \lambda_j, \lambda_i + \lambda_j (i < j)\},$$

$$\Delta^+_1 = \{\mu_p - \mu_q (p < q), \mu_p + \mu_q (p < q)\}.$$

The half of the sum of all positive roots of $\mathfrak{g}_0$, $\mathfrak{g}_1$, $\mathfrak{g}$ will be denoted by $\gamma_0$, $\gamma_1$, $\gamma$ respectively; we have $\gamma = \gamma_0 + \gamma_1$. The corresponding system of simple roots of $\mathfrak{g}$ is

$$\Pi = \Pi_0 \cup \Pi_1,$$

where

$$\Pi_0 = \{\alpha_1, \ldots, \alpha_r\}, \quad \Pi_1 = \{\beta_1, \ldots, \beta_s\}$$
are the systems of simple roots of $\mathfrak{g}_0$, $\mathfrak{g}_1$ respectively; here we denote
\[
\alpha_1 = \lambda_1 - \lambda_2, \ldots, \alpha_{r-1} = \lambda_{r-1} - \lambda_r, \alpha_r = \lambda_{r-1} + \lambda_r;
\beta_1 = \mu_1 - \mu_2, \ldots, \beta_{s-1} = \mu_{s-1} - \mu_s, \beta_s = 2\mu_s.
\]
We denote by $t^*(\mathbb{R})$ the real subspace of $t^*$ spanned by all $\lambda_i$, $\mu_p$, and define the scalar product on $t^*(\mathbb{R})$ such that $\lambda_i$, $\mu_p$ form its orthonormal basis. As usually, $\lambda \in t^*(\mathbb{R})$ is called dominant if $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Delta^+$ or, equivalently, for all $\alpha \in \Pi$. Following Bott [1], we say that $\lambda$ has index 1 if $(\lambda, \alpha) > 0$ for all $\alpha \in \Delta^+$ except of one root $\beta \in \Delta^+$, for which $(\lambda, \beta) < 0$. Now, $\lambda$ is called singular if $(\lambda, \alpha) = 0$ for a certain $\alpha \in \Delta$. These definitions will be used with respect to $\mathfrak{g}_0$, $\mathfrak{g}_1$ as well.

Clearly, the subgroup $P = G_o$ defined above is a parabolic subgroup of $G$ containing the Borel subgroup $B^-$ corresponding to $-\Delta^+$. The system of simple roots of its reductive part $R$ is $\Sigma = \Pi - \{\alpha_r, \beta_s\}$. An element $\lambda \in t^*(\mathbb{R})$ is called $R$-dominant if $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Sigma$.

It is convenient to characterize an element $\lambda \in t^*(\mathbb{R})$ by the numbers $\lambda_\alpha = 2(\lambda, \alpha)/(\alpha, \alpha)$, $\alpha \in \Pi$, which are actually the coordinates of $\lambda$ in the basis of the so-called fundamental weights. We have $\gamma_\alpha = 1$ for all $\alpha \in \Pi$. An element $\lambda$ is dominant if and only if $\lambda_\alpha \geq 0$ for all $\alpha \in \Pi$.

The following proposition is well known and very easy to verify:

**Proposition 2.** An element

$$
\lambda = \sum_{i=1}^{r} k_i \lambda_i, \ k_i \in \mathbb{R},
$$

is dominant if and only if $k_1 \geq k_2 \geq \ldots \geq |k_r|$. It is $R$-dominant if and only if $k_1 \geq k_2 \geq \ldots \geq k_r$.

An element

$$
\lambda = \sum_{j=1}^{s} l_i \mu_j, \ l_j \in \mathbb{R},
$$

is dominant if and only if $l_1 \geq l_2 \geq \ldots \geq l_s \geq 0$. It is $R$-dominant if and only if $l_1 \geq l_2 \geq \ldots \geq l_s$.

We have to study the highest weights of the representations $\Phi_q$ and $T_q$ of $P$ defined by (9) and (10), respectively. It follows from Proposition 1 that

$$
\Phi_q|R = (\rho_0 \otimes \rho_1)^q \bigwedge (\rho_0^* \otimes \rho_1^*).
$$

Denote by $i$, $i_a$ indices running over $1, \ldots, r$, and by $j$, $j_\beta$ those running over $1, \ldots, s$. The weights of $\Phi_q$ have the form

$$
\Lambda = \Lambda_0 + \Lambda_1,
$$

where

$$
\Lambda_0 = \lambda_i - \lambda_{i_1} - \ldots - \lambda_{i_q},
\Lambda_1 = \mu_j - \mu_{j_1} - \ldots - \mu_{j_\beta}.
$$
Similarly, (3) implies that
\[ T_q = T'_q + T''_q, \]
where
\[
T'_q | R = \bigwedge^2 (\mathcal{R}_0) \bigwedge^q (\mathcal{R}_0^* \otimes \mathcal{R}_1^*),
\]
\[
T''_q | R = (S^2 \rho_1) \bigwedge^q (\mathcal{R}_0^* \otimes \mathcal{R}_1^*).
\]
The weights of \( T'_q, T''_q \) have the form
\begin{equation}
\Lambda = \Lambda_0 + \Lambda_1,
\end{equation}
where for \( T'_q \) we have
\begin{equation}
\begin{aligned}
\Lambda_0 &= \lambda_i + \lambda_k - \lambda_{i_1} - \ldots - \lambda_{i_q}, \quad i < k, \\
\Lambda_1 &= -\mu_{j_1} - \ldots - \mu_{j_q},
\end{aligned}
\end{equation}
and for \( T''_q \)
\begin{equation}
\begin{aligned}
\Lambda_0 &= -\lambda_{i_1} - \ldots - \lambda_{i_q}, \\
\Lambda_1 &= \mu_j + \mu_k - \mu_{j_1} - \ldots - \mu_{j_q}, \quad j \leq l.
\end{aligned}
\end{equation}

We denote by \( I_0, I_1 \) the standard representations and by \( A_0, A_1 \) the adjoint representations of \( G_0, G_1 \) respectively. Remark that in the case \( r = 1 \) we have \( G_0 = R_0 \simeq \text{GL}_1(\mathbb{C}) \), and \( I_0 = \rho_0 + \rho_0^* \).

**Proposition 3.** Suppose that \( r \geq 2, s \geq 1 \). Then the \( G \)-module \( H^0(M, \mathcal{A}_0) \simeq \mathbb{C}^{2r} \otimes \mathbb{C}^{2s} \) is irreducible with the representation \( I_0 \otimes I_1 \). For \( r = 1, s \geq 1 \), the \( G \)-module \( H^0(M, \mathcal{A}_0) \simeq \mathbb{C}^{2s} \) is irreducible with the representation \( \rho_0 \otimes I_1 \).

We have
\[
H^p(M, \mathcal{A}_0) = 0
\]
for any \( p \geq 1 \).

**Proof.** The highest weight of \( \Phi_0 = \varphi^* \) is \( \lambda_1 + \mu_1 \). It is dominant and is the highest weight of the representation \( I_0 \otimes I_1 \) (for \( r \geq 2 \)) or \( \rho_0 \otimes I_1 \) (for \( r = 1 \)) of \( G \). Our assertions follow from the theorem of Bott.

**Proposition 4.** Suppose that \( r \geq 1, r \neq 2, s \geq 1 \). Then
\[
H^0(M, \mathcal{A}_1) \simeq \mathbb{C}
\]
(the trivial \( G \)-module). In the case \( r = 2, s \geq 1 \) we have
\[
H^0(M, \mathcal{A}_1) \simeq \mathbb{C} \oplus \mathfrak{sl}_2(\mathbb{C}),
\]
where the first summand is the trivial \( G \)-module and the second one is the irreducible \( G \)-module with highest weight \( \lambda_1 - \lambda_2 \). In both cases we have
\[
H^p(M, \mathcal{A}_1) = 0, \ p \geq 1.
\]
Proof. Clearly, for \( r \geq 2, s \geq 2 \) we have
\[
\Phi_1 \mid R = (\rho_0 \rho_0^*) \otimes (\rho_1 \rho_1^*)
= (1 + \sigma_0) \otimes (1 + \sigma_1) = 1 + \sigma_0 + \sigma_1 + \sigma_0 \otimes \sigma_1.
\]
The trivial component gives the 1-dimensional trivial \( G \)-module. The highest weights of the non-trivial components are
\[
\Lambda_0 = \lambda_1 - \lambda_r, \quad \Lambda_1 = \mu_1 - \mu_s, \quad \Lambda_0 + \Lambda_1.
\]
The weight \( \Lambda_0 + \gamma \) is singular for \( r \geq 3 \), since
\[
(\Lambda_0 + \gamma)_{\alpha_r} = (\Lambda_0 + \gamma_0)_{\alpha_r} = -1.
\]
In the case when \( r = 2 \) the weight \( \Lambda_0 = \lambda_1 - \lambda_2 \) is dominant and determines the restriction of \( \text{Ad}_0 \) onto one of the simple ideals of \( \mathfrak{g}_0 \simeq \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \) (which coincides actually with \([t_0, t_0] \)). Now, \( \Lambda_1 + \gamma \) is singular for \( s \geq 2 \), since
\[
(\Lambda_1 + \gamma)_{\beta_s} = (\Lambda_1 + \gamma_1)_{\beta_s} = -1.
\]
Therefore, \( \Lambda_0 + \Lambda_1 + \gamma \) is singular, too.

Thus, the proposition follows from the theorem of Bott. In the cases \( r = 1 \) or \( s = 1 \) the corresponding adjoint representation does not enter into the expression of \( \Phi_1 \), and we get the same result.

**Proposition 5.** For any \( r \geq 1, s \geq 1 \) we have
\[
H^0(M, \mathcal{A}_q) = H^1(M, \mathcal{A}_q) = 0, \quad q \geq 2.
\]

**Proof.** Let \( \Lambda \) be a highest weight of \( \Phi_q \). Using its expression given by (15) and (16), we easily see from Proposition 2 that \( \Lambda_0 \) and \( \Lambda_1 \) can not be dominant. Therefore the situation when \( \Lambda \) is dominant or \( \Lambda + \gamma \) has index 1 is impossible.

**Proposition 6.** For \( r \geq 3, s \geq 1 \), the \( G \)-module
\[
H^0(M, \mathcal{B}_0) \simeq \mathfrak{so}_{2r}(\mathbb{C}) \oplus \mathfrak{sp}_{2s}(\mathbb{C})
\]
splits into the sum of two irreducible components with the representations \( \text{Ad}_0 \), \( \text{Ad}_1 \). In the case \( r = 2 \), \( s \geq 1 \) the \( G \)-module
\[
H^0(M, \mathcal{B}_0) \simeq \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sp}_{2s}(\mathbb{C})
\]
splits into the sum of two irreducible components the first of which has the highest weight \( \lambda_1 + \lambda_2 \) while the second one is \( \text{Ad}_1 \). In the case \( r = 1, s \geq 1 \) we have the irreducible \( G \)-module
\[
H^0(M, \mathcal{B}_0) \simeq \mathfrak{sp}_{2s}(\mathbb{C})
\]
with the representation \( \text{Ad}_1 \).

We have
\[
H^p(M, \mathcal{B}_0) = 0
\]
for any \( p \geq 1 \) and all \( r \geq 1, s \geq 1 \).

**Proof.** By (3), the highest weights of \( T_0 = \tau \) are \( \lambda_1 + \lambda_2 \) (for \( r \geq 2 \)) and \( 2\mu_1 \). These are the highest weights of \( \text{Ad}_0 \) (if \( r \geq 3 \)) and \( \text{Ad}_1 \). If \( r = 2 \), then \( \lambda_1 + \lambda_2 \) is the highest weight of the restriction of \( \text{Ad}_0 \) onto a simple ideal of \( \mathfrak{g}_0 \) (the complement to the ideal considered in Proposition 4).
Proposition 7. If \( r \geq 2, s \geq 1 \), then we have
\[
H^p(M, B_1) = 0
\]
for any \( p \geq 0 \). If \( r = 1, s \geq 1 \), then
\[
H^0(M, B_1) \simeq \mathbb{C}^{2s}
\]
is the irreducible \( G \)-module with the representation \( \rho_0^* \otimes Id_1 \) and
\[
H^p(M, B_1) = 0
\]
for any \( p \geq 1 \).

Proof. One see easily that, for \( r \geq 2 \),
\[
T_1 | R = (\bigwedge^2 \rho_0 \rho_0^*) \otimes \rho_1^* + \rho_0^* \otimes (S^2 \rho_1) \rho_1^*.
\]
Clearly, \( \lambda_r + \gamma \) and \( \mu_s + \gamma \) are singular, and hence \( \Lambda + \gamma \) is singular for any weight of \( T_1 \). The theorem of Bott implies our assertion.

In the case \( r = 1 \) we have
\[
T_1 | R = \rho_0^* \otimes (S^2 \rho_1) \rho_1^*.
\]
The highest weights of this representation are \(-\lambda_1 + \mu_1\) and (for \( s \geq 2 \)) \( 2\mu_1 - \mu_s \).
The first weight is dominant and gives the representation \( \rho_0^* \otimes Id_1 \), while the sum of the second one with \( \gamma \) is singular.

Proposition 8. Suppose that \( r \geq 2, s \geq 1 \). Then
\[
H^0(M, B_2) = 0, \; H^1(M, B_2) \simeq \mathbb{C}^2
\]
(the trivial \( G \)-module). If \( r = 1, s \geq 1 \), then
\[
H^p(M, B_2) = 0, \; p = 0, 1.
\]

Proof. By (3) we have
\[
T_2 | R = (\bigwedge^2 \rho_0 + S^2 \rho_1) \bigwedge (\rho_0^* \otimes \rho_1^*)
\]
\[
= (\bigwedge^2 \rho_0 + S^2 \rho_1)(\bigwedge^2 \rho_0^* \otimes S^2 \rho_1^* + S^2 \rho_1^* \otimes \bigwedge^2 \rho_1^*)
\]
\[
= (\bigwedge^2 \rho_0)(\bigwedge^2 \rho_0^*) \otimes S^2 \rho_1^* + (\bigwedge^2 \rho_0)(S^2 \rho_0^*) \otimes \bigwedge^2 \rho_1^*
\]
\[
+ (\bigwedge^2 \rho_0) \otimes (S^2 \rho_1)(S^2 \rho_1^*) + (S^2 \rho_0^*) \otimes (S^2 \rho_1)(\bigwedge^2 \rho_1^*).
\]
The first three of these four summands exist only when \( r \geq 2 \). For the first one, any highest weight has the form (see (17),(18),(19))

\[
\Lambda = \Lambda_0 + \Lambda_1,
\]

where

\[
\Lambda_0 = \lambda_i + \lambda_j - \lambda_k - \lambda_l, \quad \Lambda_1 = -2\mu_s.
\]

Clearly,

\[
r_{\beta_+}(\Lambda_1 + \gamma_1) = r_{\beta_+}(-\beta_+ + \gamma_1) = \beta_+ + \gamma_1 - \beta_+ = \gamma_1.
\]

Hence, \( \Lambda_1 + \gamma_1 \) has index 1. Therefore, we have interest only in the case when \( \Lambda_0 \) is dominant. Using Proposition 2, one sees easily that this is possible only for \( \Lambda_0 = 0 \) (which is a highest weight indeed!). Then \( \Lambda + \gamma \) has index 1. By the algorithm of Bott, there corresponds to \( \Lambda \) an irreducible component of the \( G \)-module \( H^1(M, B_2) \) with highest weight \( r_{\beta_+}(\Lambda + \gamma) - \gamma = 0 \). Quite similarly, the third summand gives (if \( r \geq 2 \)) only the 1-dimensional trivial component of \( H^1(M, B_2) \).

Now let \( \Lambda = \Lambda_0 + \Lambda_1 \) be a highest weight of one of two remaining summands. One sees easily from Proposition 2 that neither \( \Lambda_0 \), nor \( \Lambda_1 \) is dominant (\( \Lambda_0 = 0 \) is not a highest weight in these cases!). Therefore \( \Lambda \) can not be dominant, nor can \( \Lambda + \gamma \) have index 1.

**Proposition 9.** Suppose that \( r \geq 1, s \geq 1 \). Then

\[
H^0(M, B_q) = H^1(M, B_q) = 0
\]

for any \( q \geq 3 \).

**Proof.** Let \( \Lambda \) be a weight of \( T'_q \). Using (18), we see, by Proposition 3, that \( \Lambda_0 \) can not be dominant if \( q \geq 3 \) and that \( \Lambda_1 \) can not be dominant if \( q \geq 1 \). Quite similarly, for any weight \( \Lambda \) of \( T'_q \) we see, using (19), that \( \Lambda_0 \) can not be dominant if \( q \geq 1 \) and that \( \Lambda_1 \) can not be dominant if \( q \geq 3 \). Thus, \( \Lambda \) can not be dominant, nor can \( \Lambda + \gamma \) have index 1. The proposition follows now from the theorem of Bott.

**3. The cohomology of \( \tilde{T} \)**

As in [6], we shall use here some further results of Bott’s paper [1]. Let \( E \) be a holomorphic \( P \)-module. Then (see [1], Theorem I and Corollary 2 of Theorem W2) we have an isomorphism

\[
H^p(M, E)^G \simeq H^p(n, E)^r
\]

between the \( G \) invariants and the \( r \) invariants of the corresponding cohomology groups. This isomorphism is compatible with the homomorphisms induced by homomorphisms of \( P \)-modules.

These considerations can be applied to calculate the cohomology of \( \mathcal{A}_q \) and \( \mathcal{B}_q \) by expressing explicitly the cocycles which represent the basic cohomology classes. We need such an expression for the group \( H^1(M, B_2) \).

We shall use the standard coordinate system on \( \text{IGr}_{2r|r,2s|s} \) in a neighborhood of \( o \) given by (2). As in [6], we note that the adjoint action of \( \mathfrak{p} \) on \( \mathfrak{n} \) coincides with \( \tau^* \); hence \( \mathfrak{n} \), as a \( \mathfrak{p} \)-module, is isomorphic to the cotangent space \( T_o(M)^* \) of \( M \). By
this isomorphism, the basis $dx_{\alpha\beta}$ ($\alpha < \beta$), $dy_{ij}$ ($i \leq j$) of $T_0(M)^*$ corresponds to the basis (12) of $n$.

The result of Bott mentioned above gives the identification

$$H^1(M, B_2) = H^1(n, T_0(M) \otimes \bigwedge^2 E_0)^\tau.$$  

Since $\tau$ and $\phi$ are completely reducible, $n$ acts on the coefficients trivially, and hence the coboundary $\delta$ of the cochain complex $C(n, T_0(M) \otimes \bigwedge^2 E_0)$ is zero. It follows that

$$(20) \quad H^1(n, T_0(M) \otimes \bigwedge^2 E_0)^\tau = C^1(n, T_0(M) \otimes \bigwedge^2 E_0)^\tau \simeq (T_0(M) \otimes T_0(M) \otimes \bigwedge^2 E_0)^\tau.$$  

We are going to describe this vector space explicitly in terms of 1-cochains.

**Proposition 10.** The following two cochains $c_0, c_1$ form a basis of $C^1(n, T_0(M) \otimes \bigwedge^2 E_0)^\tau$:

$$c_0(X_{\alpha\beta}) = \sum_{i,j} \frac{\partial}{\partial y_{ij}} \otimes \xi_{\alpha i} \xi_{\beta j} + \sum_i \frac{\partial}{\partial y_{ii}} \otimes \xi_{\alpha i} \xi_{\beta i}, \quad c_0(Y_{ij}) = 0;$$  

$$c_1(Y_{ij}) = \sum_{\alpha,\beta} \frac{\partial}{\partial x_{\alpha\beta}} \otimes \xi_{\alpha i} \xi_{\beta j}, \quad c_1(X_{\alpha\beta}) = 0.$$  

**Proof.** By Proposition 1, the $P$-module $E_0$ is identified with $(C^r)^* \otimes (C^s)^*$ in such a way that $\xi_{\alpha i} = x_{\alpha} \otimes y_i$, where $x_{\alpha}, y_i$ are the standard coordinates. Then $\bigwedge^2 E_0 = \bigwedge^2((C^r)^* \otimes (C^s)^*)$ will contain an irreducible $P$-submodule isomorphic to $\bigwedge^2(C^r)^* \otimes S^2(C^s)^*$ which is spanned by the elements

$$(x_{\alpha} \otimes x_{\beta} - x_{\beta} \otimes x_{\alpha}) \otimes (y_i \otimes y_j + y_j \otimes y_i) =$$  

$$\xi_{\alpha i} \otimes \xi_{\beta j} - \xi_{\beta i} \otimes \xi_{\alpha j} + \xi_{\alpha j} \otimes \xi_{\beta i} - \xi_{\beta i} \otimes \xi_{\alpha j} = 2(\xi_{\alpha i} \xi_{\beta j} - \xi_{\beta j} \xi_{\alpha i}).$$

Then, by (20), $H^1(n, T_0(M) \otimes \bigwedge^2 E_0)^\tau$ contains the invariants of the submodule $T_0(M) \otimes T_0(M) \otimes \bigwedge^2(C^r)^* \otimes S^2(C^s)^*$. Using (3), we see that precisely two linearly independent invariants lie there, while the complementary submodule does not contain any non-zero invariant. Since the basis $\frac{\partial}{\partial x_{\alpha\beta}}$ ($\alpha < \beta$), $\frac{\partial}{\partial y_{ij}}$ ($i \leq j$) is dual to (12), we get the basic invariants $c_0, c_1$ given by:

$$c_0(X_{\alpha\beta}) = \sum_{i<j} \frac{\partial}{\partial y_{ij}} \otimes (\xi_{\alpha i} \xi_{\beta j} + \xi_{\alpha j} \xi_{\beta i}) + 2 \sum_i \frac{\partial}{\partial y_{ii}} \otimes \xi_{\alpha i} \xi_{\beta i}$$  

$$= \sum_{i,j} \frac{\partial}{\partial y_{ij}} \otimes \xi_{\alpha i} \xi_{\beta j} + \sum_i \frac{\partial}{\partial y_{ii}} \otimes \xi_{\alpha i} \xi_{\beta i},$$  

$$c_0(Y_{ij}) = 0;$$  

$$c_1(Y_{ij}) = \sum_{\alpha < \beta} \frac{\partial}{\partial x_{\alpha\beta}} \otimes (\xi_{\alpha i} \xi_{\beta j} + \xi_{\alpha j} \xi_{\beta i}) = \sum_{\alpha,\beta} \frac{\partial}{\partial x_{\alpha\beta}} \otimes \xi_{\alpha i} \xi_{\beta j},$$  

$$c_1(Y_{ii}) = 2 \sum_{\alpha < \beta} \frac{\partial}{\partial x_{\alpha\beta}} \otimes \xi_{\alpha i} \xi_{\beta i} = \sum_{\alpha,\beta} \frac{\partial}{\partial x_{\alpha\beta}} \otimes \xi_{\alpha i} \xi_{\beta i},$$  

$$c_1(X_{\alpha\beta}) = 0.$$
We are now able to calculate $H^p(M, \tilde{T}), \ p = 0, 1$.

**Theorem 1.** Suppose that $r \geq 2$, $s \geq 2$ or $r \geq 3$, $s \geq 1$. Then the $G$-modules $H^p(M, \tilde{T}_q), \ p = 0, 1; q \geq -1$, are indicated in the following table:

\[
\begin{array}{cccc}
q = & -1 & 0 & 1 & 2 & \geq 3 \\
p = 0 & \mathfrak{osp}_{2r|2s}(\mathbb{C})_1 & \mathfrak{osp}_{2r|2s}(\mathbb{C})_0 & \mathbb{C} & 0 & 0 \\
p = 1 & 0 & 0 & 0 & \mathbb{C}^2 & 0 \\
\end{array}
\]

Here $\mathfrak{osp}_{2r|2s}(\mathbb{C})_0$ and $\mathfrak{osp}_{2r|2s}(\mathbb{C})_1$ are endowed with the adjoint representation of $G$, and $\mathbb{C}$ is the trivial $G$-module.

If $r = 2$, $s = 1$, then the table has the form

\[
\begin{array}{cccc}
q = & -1 & 0 & 1 & 2 & \geq 3 \\
p = 0 & \mathfrak{osp}_{4|2}(\mathbb{C})_1 & \mathfrak{osp}_{4|2}(\mathbb{C})_0 & \mathbb{C}^2 & 0 & 0 \\
p = 1 & 0 & 0 & 0 & \mathbb{C}^2 & 0 \\
\end{array}
\]

Here $\mathbb{C}^2$ is the trivial $G$-module.

If $r = 1$, $s \geq 1$, then the corresponding table is as follows:

\[
\begin{array}{cccc}
q = & -1 & 0 & 1 & 2 & \geq 3 \\
p = 0 & \mathbb{C}^{2s} & \mathfrak{sp}_{2s}(\mathbb{C}) & \mathbb{C}^{2s} & 0 & 0 \\
p = 1 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Here $\mathfrak{sp}_{2s}(\mathbb{C})$ is endowed with the adjoint representation of $G$, $\mathbb{C}$ is the trivial $G$-module and $\mathbb{C}^{2s}$ for $q = -1, 1$ is endowed with the representation $\rho_0 \otimes \text{Id}_1$ or $\rho_0^* \otimes \text{Id}_1$ respectively.

**Proof.** We use the cohomology exact sequences associated with (8). Almost in all cases the mappings in these sequences are determined uniquely. The only difficulty occurs when we try to calculate $H^1(M, \tilde{T}_2)$ with the help of the exact sequence

\[0 \to \mathcal{A}_3 \xrightarrow{\alpha} \tilde{T}_2 \xrightarrow{\beta} B_2 \to 0.\]

By Proposition 5, we have the exact sequence

\[0 \to H^1(M, \tilde{T}_2) \xrightarrow{\beta^*} H^1(M, B_2).\]

If $r = 1$ then, by Proposition 8, we have $H^1(M, B_2) = 0$. Hence, $H^1(M, \tilde{T}_2) = 0$ in this case. In what follows we assume that $r \geq 2$.

By Proposition 8, $H^1(M, B_2) \cong \mathbb{C}^2$ (the trivial $G$-module). The sheaves $\tilde{T}_2$ and $B_2$ are the sheaves of holomorphic sections of homogeneous vector bundles $\tilde{T}_2$ and $B_2 = T(M) \otimes \mathfrak{g}^2 \mathbf{E}_\phi$, and $\beta$ is induced by a homomorphism of these bundles. As we have seen in the beginning of this section, $\beta^*$ is interpreted as the homomorphism of the invariant 1-cohomology of the Lie algebra $\mathfrak{n}$:

\[H^1(\mathfrak{n}, (\tilde{T}_2)_o)^\dagger \to H^1(\mathfrak{n}, T_o(M) \otimes \bigwedge^2 \mathbf{E}_\phi)^\dagger,\]
where \((\tilde{T}_2)_o\) is the fibre of \(\tilde{T}_2\) at the point \(o\) endowed with a natural structure of the \(p\)-module. The group \(H^1(n,(\tilde{T}_2)_o)^r\) coincides with the 1-cohomology of the complex \(C(n,(\tilde{T}_2)_o)^r\) of \(r\)-invariant cochains. Since \(H^1(M,A_3) = 0\) by Proposition 5, the vector space \(C^1(n,(\tilde{T}_2)_o)^r\) is mapped isomorphically onto \(C^1(n,T_o(M) \otimes \Lambda^2 E_0)^r\).

It follows from Proposition 10 that the cochains \(c \in C^1(n,(\tilde{T}_2)_o)^r\) have the form

\[
\begin{align*}
c(X_{\alpha \beta}) &= a \left( \sum_{i,j} \xi_{\alpha i} \xi_{\beta j} \frac{\partial}{\partial y_{ij}} + \sum_i \frac{\partial}{\partial y_{ii}} \right), \\
c(Y_{ij}) &= b \sum_{\alpha, \beta} \xi_{\alpha i} \xi_{\beta j} \frac{\partial}{\partial x_{\alpha \beta}},
\end{align*}
\]

where \(a, b \in \mathbb{C}\). Clearly,

\[
H^1(n,(\tilde{T}_2)_o)^r \simeq \{ c \in C^1(n,(\tilde{T}_2)_o)^r | \delta c = 0 \}.
\]

By the definition of \(\delta\) we have

\[
(\delta c)(x,y) = xc(y) - yc(x), \quad x, y \in n.
\]

The action of \(r\) on \((\tilde{T}_2)_o\) is induced by commuting the fundamental vector fields of the action of \(G\) on \(IGr_{2r|r,2s|s}\) with the elements of \(\tilde{T}_2\), followed by evaluating the commutator at \(X = 0, Y = 0\). It follows from (13) that

\[
(\delta c)(X_{\alpha \beta}, X_{\gamma \delta}) = (\delta c)(Y_{ij}, Y_{kl}) = 0
\]

and that

\[
(\delta c)(X_{\alpha \beta}, Y_{ij}) = (b - a) \sum_{\gamma, k} (\xi_{\alpha j} \xi_{\beta k} \xi_{\gamma i} + \xi_{\alpha k} \xi_{\beta j} \xi_{\gamma i} + \xi_{\alpha i} \xi_{\beta j} \xi_{\gamma i} + \xi_{\alpha k} \xi_{\beta j} \xi_{\gamma i}) \frac{\partial}{\partial \xi_{\gamma i}}.
\]

One sees easily that if \(r \geq 2, s \geq 2\) then \(\delta c = 0\) is equivalent to \(a = b\). The same is true if \(s = 1, r \geq 3\). In the remaining case \(r = 2, s = 1\) we have \(\delta c = 0\) for any invariant cochain \(c\). Thus,

\[
H^1(M,\tilde{T}_2) \simeq H^1(n,(\tilde{T}_2)_o)^r \simeq \begin{cases}
\mathbb{C} & \text{if } r \geq 2, s \geq 2 \text{ or } r \geq 3, s = 1 \\
\mathbb{C}^2 & \text{if } r = 2, s = 1.
\end{cases}
\]

4. The cohomology of \(T\)

In this section, we prove our main theorem about 0- and 1-cohomology of the isotropic super-Grassmannian with values in the tangent sheaf. The proof repeats that of Theorem 2 of [6]. First we state a proposition that will play the main part in it.

It is clear that on the split supermanifold \((M, \text{gr} O)\) there exists a vector field \(\varepsilon \in H^0(M, \tilde{T}_0)\) such that \(\varepsilon(f) = qf\) for any \(f \in \text{gr}_q O\). This vector field commutes with any \(X^*, X \in \mathfrak{g}\), and hence is a basic element of the trivial \(G\)-submodule \(\mathbb{C} \subset H^0(M, \tilde{T}_0)\) (see Theorem 1).
Proposition 11. If \( r \geq 2 \), then \( \varepsilon \) does not lie in the image of the canonical mapping \( H^0(M, \mathcal{T}_0) \to H^0(M, \mathcal{T}_0) \).

Proof. We take as odd coordinates in a neighborhood of \( o \) in \((M, \mathfrak{g}_0 \mathcal{O})\) the elements \( \xi_{\alpha i} = \xi_{\alpha i} + J^2 \). Then, clearly, \( \varepsilon \) is expressed in this neighborhood as

\[
\varepsilon = \sum_{\alpha, i} \xi_{\alpha i} \frac{\partial}{\partial \xi_{\alpha i}}.
\]

Suppose that there exists \( \hat{\varepsilon} \in H^0(M, \mathcal{T}_0) \) inducing the vector field \( \varepsilon \). One may suppose that \( \hat{\varepsilon} \in (H^0(M, \mathcal{T}_0)_\beta)^G \). Then \( [\hat{\varepsilon}, X^*] = 0 \) for any \( X \in \mathfrak{g} \). Consider the action of the derivation \( \hat{\varepsilon} \) in \( \mathcal{O}_o \). The mapping \( X \to X^* \) is a linear representation of the Cartan subalgebra \( \mathfrak{t} \) of \( \mathfrak{g} \), commuting with \( \hat{\varepsilon} \). We see from (11) that \( x_{\alpha \beta}, y_{ij}, \xi_{\alpha i} \) lie in the weight subspaces of this representation, corresponding to the weights \( \lambda_\alpha + \lambda_\beta, \mu_i + \mu_j, \lambda_\alpha + \mu_i \) respectively. It is clear that all these weight subspaces have dimension 1. Since \( \hat{\varepsilon} \) maps any weight subspace into itself, we have

\[
\hat{\varepsilon} = \sum_{\alpha, i} \xi_{\alpha i} \frac{\partial}{\partial \xi_{\alpha i}} + \sum_{\alpha < \beta} a_{\alpha \beta} x_{\alpha \beta} \frac{\partial}{\partial x_{\alpha \beta}} + \sum_{i \leq j} b_{ij} y_{ij} \frac{\partial}{\partial y_{ij}},
\]

where \( a_{\alpha \beta}, b_{ij} \in \mathbb{C} \). Now, we have \( [\hat{\varepsilon}, U_{\alpha \beta}^*] = [\hat{\varepsilon}, V_{ij}^*] = 0 \) which, by (14), implies that \( a_{\alpha \beta} = b_{ij} = 0 \) for all \( \alpha < \beta, i \leq j \). Thus,

\[
\hat{\varepsilon} = \sum_{\alpha, i} \xi_{\alpha i} \frac{\partial}{\partial \xi_{\alpha i}}.
\]

Now, by (13) we see that

\[
[\hat{\varepsilon}, X_{\alpha \beta}^*](y_{ij}) = 2\xi_{\alpha i} \xi_{\beta j}.
\]

This cannot be 0 if \( r \geq 2 \), giving a contradiction.

As a corollary, we want to characterize the split isotropic super-Grassmannians.

Corollary. The super-Grassmannian \( \Gamma^{0}\mathfrak{Gr}_2[r, 2s]_s \) is split if and only if \( r = 1 \).

Proof. Proposition 11 shows that the super-Grassmannian is non-split if \( r \geq 2 \). Now, for \( r = 1 \) we have \( H^1(M, B_q) = 0 \) for all \( q \geq 2 \), by Propositions 8 and 9. Thus, all the obstructions to the splitness are 0 (see [4], Ch.4, Sec. 2), and hence \( \Gamma^{0}\mathfrak{Gr}_2[1, 2s]_s \) is split.

Theorem 2. We have, for any \( r \geq 1, s \geq 1 \),

\[
H^0(M, T) \simeq \mathfrak{osp}_{2r|2s}(\mathbb{C})
\]

as Lie superalgebras, isomorphism being defined by the standard action of \( \mathbb{OSp}_{2r|2s}(\mathbb{C}) \). Also

\[
H^1(M, T) = \begin{cases} 
0 & \text{if } (r, s) \neq (2, 1) \\
\mathbb{C}^{1|0} & \text{if } r = 2, s = 1.
\end{cases}
\]
**Proof.** Suppose first that \((r, s) \neq (1, s)\) and \(\neq (2, 1)\). Then the proof goes precisely as in [6]. Using Theorem 1 and the cohomology exact sequence corresponding to (6), we see that \(H^0(M, T_{(q)}) = H^1(M, T_{(q)}) = 0\) for \(q \geq 3\). For \(q = 2\) this exact sequence shows that \(H^0(M, T_{(2)}) = 0\) and that \(H^1(M, T_{(2)})\) is mapped injectively into \(H^1(M, \tilde{T}_2) \simeq \mathbb{C}^{11}\). Thus, \(H^1(M, T_{(2)}) \simeq \mathbb{C}^{k|0}\), \(k \leq 1\). For \(q = 1\) the exact sequence shows that \(H^0(M, T_{(1)}) = 0\) and that \(H^1(M, T_{(1)}) \simeq \mathbb{C}^{k|0}\). For \(q = 0\) we get the exact sequence

\[
0 \to H^0(M, T_{(1)}) \to H^0(M, T_{(0)}) \to H^0(M, \tilde{T}_0) \\
\to H^1(M, T_{(1)}) \to H^1(M, T_{(0)}) \to H^1(M, \tilde{T}_0).
\]

(21)

This implies that \(H^0(M, T_{(0)})\) is mapped injectively into \(H^0(M, \tilde{T}_0)\). By Proposition 11, the trivial submodule \(\mathbb{C}\) does not lie in the image. Therefore \(H^1(M, T_{(1)}) \neq 0\), and hence \(H^1(M, T_{(1)}) \simeq \mathbb{C}^{1|0}\), \(H^1(M, T_{(0)}) = 0\). Also, \(H^0(M, T_{(0)}) \simeq \mathfrak{osp}_{2r|2s}(\mathbb{C})\). Now, for \(q = -1\) we get the exact sequence

\[
0 \to H^0(M, T_{(0)}) \to H^0(M, T) \to H^0(M, \tilde{T}_{-1}) \\
\to H^1(M, T_{(0)}) \to H^1(M, T) \to H^1(M, \tilde{T}_{-1}).
\]

It implies that

\[
H^0(M, T) \simeq H^0(M, T_{(0)}) \oplus H^0(M, \tilde{T}_{-1}) \simeq \mathfrak{osp}_{2r|2s}(\mathbb{C}), \\
H^1(M, T) = 0.
\]

For the 0-cohomology we mean here an isomorphism of \(G\)-modules. Since \(\mathfrak{osp}_{2r|2s}(\mathbb{C})\) is simple [3], the homomorphism \(X \leadsto X^*\) of this superalgebra into \(H^0(M, T)\) is injective. Therefore it is an isomorphism of Lie superalgebras.

Suppose that \(r = 2, s = 1\). Then the super-Grassmannian has dimension \(2|2\). Using Theorem 1, we see that \(H^1(M, T_{(1)}) \simeq H^1(M, T_{(2)}) \simeq \mathbb{C}^{2|0}\). Then the exact sequence (21) and Proposition 11 give that \(H^1(M, T_{(0)}) \simeq \mathbb{C}^{1|0}\). It follows that \(H^1(M, T) \simeq \mathbb{C}^{1|0}\).

The case \(r = 1\) is the simplest one, and we omit the proof.

It follows from Theorem 2 that the supermanifold \(\Pi^c\text{Gr}^{2r|2s}_{2r|2s}\) is rigid if \((r, s) \neq (2, 1)\) (see [8]). The remaining case \(r = 2, s = 1\) was actually studied before. It is easy to see that \(\Pi^c\text{Gr}_{4|2,2|1}\) is precisely the supermanifold \(G(1, 1)\) from the family \(G(t_1, t_2)\) constructed in [2], where the corresponding part of Theorem 2 was proved. By Theorem 4 of [2], this family is a versal deformation of \(\Pi^c\text{Gr}^{2r|2s}_{2r|2s}\). Thus, we get

**Corollary.** The super-Grassmannian \(\Pi^c\text{Gr}^{2r|2s}_{2r|2s}\) is a rigid supermanifold if and only if \((r, s) \neq (2, 1)\).

**References**


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