Integrals in The Hida Distribution Space ($S'$)

by

Fred Espen Benth

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Integrals in The Hida Distribution Space \((S)^*\).

Fred Espen Benth
Department of Mathematics
University of Oslo
Box 1053 Blindern, N-0316 Oslo
Norway

Abstract

We give sufficient conditions for \((S)^*\)-integrability. The results will be applied on Skorohod integrable processes, where we show the equality between the Skorohod integral and an \((S)^*\)-integral involving white noise and Wick product, only using the Skorohod integrability requirement.

1 Introduction.

This article considers integrals in \((S)^*\), and the connection between an \((S)^*\)-integral and the Skorohod integral. Moreover we will show that

\[
\int_0^t Y_s \circ W_s ds = \int_0^t Y_s \delta B_s
\]

without any significant restrictions on the process \(Y_s\), except the natural requirement of Skorohod integrability. The left hand side of this equation is to be understood as a Lebesgue integral in the \((S)^*\)-sense. The right hand side is the familiar Skorohod integral. The symbol \(\circ\) denotes the Wick-product and \(\delta\) the white noise process. All these notions are discussed in detail.

In [LØU], Th.(3.3) there is an elegant proof of (1). However, the authors have put severe restrictions on the process \(Y_s\). The restrictions are given in the following way: Define \(\lambda\) as the measure on the product-\(\sigma\)-algebra on \(\mathbb{R}^n\) such that

\[
\int f(y) d\lambda(y) = \int \cdots \left( \int \cdots \int f(y_1, \ldots, y_n) e^{-1/2y_1^2} \frac{dy_1}{\sqrt{2\pi}} \cdots e^{-1/2y_n^2} \frac{dy_n}{\sqrt{2\pi}} \right)
\]

Let \(z^\alpha = z_1^\alpha \cdots z_m^\alpha\) where \(z_j \in \mathbb{C}\). If \(Y_t\) has the Wiener-Ito chaos expansion (to be defined later, see (13))

\[
Y_t = \sum_\alpha c_\alpha(t) H_\alpha(\omega)
\]

then we define the Hermite transform of \(Y_t\) to be

\[
\tilde{Y}_t = \sum_\alpha c_\alpha(t) z^\alpha
\]

To obtain (1), [LØU] requires that

\[
\int_0^t \left( \int |\tilde{Y}_s \tilde{W}_s|^2 d\lambda(x) d\lambda(y) \right) ds < \infty
\]
This restriction ensures the existence of the inverse Hermite transform (for more information about the Hermite transform, see [LOU]). In this article however, we prove that Skorohod integrability of $Y_t$ is sufficient to ensure the existence of the left hand side of (1). The proof of the equality goes by a direct calculation, without using any Hermite transforms.

2 The Spaces $(S')$ and $(S')^*$.

We start by recalling some of the basic definitions and features of the white noise probability space. This brief introduction is mostly taken from [GHLØUZ]. For a more complete account, see [HKPS].

As usual, let $S' = S'(\mathbb{R}^d)$ denote the space of tempered distributions on $\mathbb{R}^d$, which is the dual of the well-known Schwartz space $S(\mathbb{R}^d)$. By the Bochner-Minlos theorem there exists a measure $\mu$ on $S'$ such that

$$\int_{S'} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = e^{-\frac{1}{2}||\phi||^2}, \phi \in S(\mathbb{R}^d)$$

where $||.||$ is the $L^2(\mathbb{R}^d)$-norm. This measure corresponds to the bilinear form

$$E(\phi, \psi) = \int_{\mathbb{R}^d} \phi \psi dx; \phi, \psi \in S(\mathbb{R}^d)$$

Let $\mathcal{B}$ denote the Borel sets on $S'$ (equipped with the weak star topology). Then the triple $(S'(\mathbb{R}^d), \mathcal{B}, \mu)$ is called the white noise probability space.

**Definition 1** The white noise process is a map

$$W: S \times S' \to \mathbb{R}$$

given by

$$W(\phi, \omega) = W_\phi(\omega) = \langle \omega, \phi \rangle, \omega \in S', \phi \in S$$

Since $S$ is dense in $L^2$, we can define $\langle \omega, \phi \rangle$ for $\phi \in L^2$ by

$$\langle \omega, \phi \rangle = \lim_{n \to \infty} \langle \omega, \phi_n \rangle$$

where $\phi_n \in S'$ is a sequence converging to $\phi \in L^2$. In particular, if we define

$$\tilde{B}_x(\omega) := \tilde{B}_{x_1, \ldots, x_d}(\omega) := \langle \omega, X_{[0,x_1]} \ldots \omega_{[0,x_d]}(\cdot) \rangle$$

then $\tilde{B}_x$ has an x-continuous version $B_x$ which then becomes a d-parameter Brownian motion.

The d-parameter Wiener-Ito integral of $\phi \in L^2$ is defined by

$$\int_{\mathbb{R}^d} \phi(y) dB_x(\omega) = \langle \omega, \phi \rangle$$

The left hand side coincides with the Ito integral if $\text{supp}(\phi) \subset [0, \infty)$. (See [LOU], p.4). Of special interest now will be the space $L^2(S'(\mathbb{R}^d), \mu)$ or $L^2(\mu)$ for short. The Wiener-Ito chaos expansion theorem says that every $F \in L^2(\mu)$ has the form

$$F(\omega) = \sum_{n=0}^{\infty} \int_{(\mathbb{R}^d)^n} f_n(u) dB_\mu^\otimes n(\omega)$$

where $f_n \in L^2(\mathbb{R}^{nd})$ and $f_n$ is symmetric in all its nd variables (in the sense that $f_n(u_{\sigma_1}, \ldots, u_{\sigma_{nd}}) = f_n(u_1, \ldots, u_{nd})$ for all permutations $\sigma$.) The right hand side of (6) are the multiple Ito integrals.
With $F, f_n$ as in (6) we have

$$||F||_{L^2(\mu)}^2 = \sum_{n=0}^{\infty} n! ||f_n||_{L^2(\mathbb{R}^n)}^2$$  \hspace{1cm} (7)

There is an equivalent expansion of $F \in L^2(\mu)$ in terms of the Hermite polynomials

$$h_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}); n = 0, 1, 2, \ldots$$

We now explain this more closely. Define the *Hermite function* of order $n$ as $\xi_n(x)$

$$\xi_n(x) = \Re^{-1/4}((n-1)!)^{-1/2} e^{-x^2} h_{n-1}(\sqrt{2}x)$$  \hspace{1cm} (8)

where $x \in \mathbb{R}, n = 0, 1, 2, \ldots$  \hspace{1cm} $\{\xi_n\}_{n=1}^{\infty}$ forms an orthonormal basis for $L^2(\mathbb{R})$. Therefore the family $\{\xi_n\}$ of tensor products

$$e_\alpha := e_{\alpha_1, \ldots, \alpha_m} := \xi_{\alpha_1} \otimes \ldots \otimes \xi_{\alpha_m}$$  \hspace{1cm} (9)

(where $\alpha$ denotes the multi-index $(\alpha_1, \ldots, \alpha_m)$) forms an orthonormal basis for $L^2(\mathbb{R}^d)$. Assume that the family of all multi-indices $\beta = (\beta_1, \ldots, \beta_d)$ is given a fixed ordering

$$(\beta^{(1)}, \beta^{(2)}, \ldots, \beta^{(n)}, \ldots)$$

where $\beta^{(k)} = (\beta_1^{(k)}, \ldots, \beta_d^{(k)})$. Put

$$e_n = e_{\beta^{(n)}}, n = 1, 2, \ldots$$

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a multi-index. It was shown by Itô that

$$\int_{\mathbb{R}^d} e_1^{\hat{\beta}^{(1)}} \otimes \ldots \otimes e_m^{\hat{\beta}^{(m)}} dB = \prod_{j=1}^{m} h_{\alpha_j}(\theta_j)$$  \hspace{1cm} (10)

where $\theta_j(\omega) = \int_{\mathbb{R}^d} e_j(x) dB_x(\omega), n = |\alpha|$ and $\otimes$ denotes the *symmetrized tensor product*, so that, e.g., $f \otimes g(x, y) = \frac{1}{2}[f(x)g(y) + f(y)g(x)]$ if $x, y \in \mathbb{R}$ and similarly for more than two variables.

If we define, for each multiindex $\alpha = (\alpha_1, \ldots, \alpha_m)$,

$$H_\alpha(\omega) = \prod_{j=1}^{m} h_{\alpha_j}(\theta_j)$$  \hspace{1cm} (11)

then we see that (10) can be written

$$\int_{\mathbb{R}^d} e_\alpha dB^{[\alpha]} = H_\alpha(\omega)$$  \hspace{1cm} (12)

using multi-index notation: $e_\alpha = e_1^{\hat{\alpha}^{(1)}} \otimes \ldots \otimes e_m^{\hat{\alpha}^{(m)}}$ if $e = (e_1, e_2, \ldots)$. Since the family $\{e_\alpha; |\alpha| = n\}$ forms an orthonormal basis for the symmetric functions in $L^2((\mathbb{R}^d)^n)$, we see by combining (6) and (12) that we have the representation

$$F(\omega) = \sum_{\alpha} c_\alpha H_\alpha(\omega)$$  \hspace{1cm} (13)

(the sum being taken over all multi-indices $\alpha$ of nonnegative integers). Moreover, it can be proved that

$$||F||_{L^2(\mu)}^2 = \sum_{\alpha} \alpha! c_\alpha^2$$  \hspace{1cm} (14)

where $\alpha! = \alpha_1! \ldots \alpha_m!$.

There is a subspace of $L^2(\mu)$ which in some sense corresponds to the Schwartz subspace $S(\mathbb{R}^d)$ of $L^2(\mathbb{R}^d)$. This space is called the *Hida test function space* and is denoted $(\mathcal{S})$. Using the characterization due to Zhang in [Z], a simple description of $(\mathcal{S})$ can be given as follows:
Definition 2 Let $F \in L^2(\mu)$ have the chaos expansion

$$F(\omega) = \sum_{\alpha} c_{\alpha} H_{\alpha}(\omega)$$

Then $F$ is a Hida test function, i.e. $F \in (\mathcal{S})$, if

$$\sup_{\alpha} c_{\alpha}^2 \alpha!(2\mathbb{N})^{\alpha k} < \infty \text{ for all natural numbers } k < \infty \quad (15)$$

where

$$(2\mathbb{N})^\alpha := \prod_{j=1}^{m} (2^d \beta_{1}^{(j)} \cdots \beta_{d}^{(j)})^{\alpha_{j}} \text{ if } \alpha = (\alpha_{1}, \ldots, \alpha_{m}) \quad (16)$$

In this article, the dual of $(\mathcal{S})$, denoted $(\mathcal{S})^*$, will be studied. It is therefore of great importance to have a nice characterization of this space, which is called the Hida distribution space. Another theorem in [Z] states the following:

Theorem 3 A Hida distribution $G$ is a formal series

$$G = \sum_{\alpha} b_{\alpha} H_{\alpha} \quad (17)$$

where

$$\sup_{\alpha} b_{\alpha}^2 \alpha!(2\mathbb{N})^{-\alpha q} < \infty \text{ for some } q > 0 \quad (18)$$

If $G \in (\mathcal{S})^*$ is given by (17) and $F \in (\mathcal{S})$ is given by (13), the action of $G$ on $F$ is given by

$$\langle G, F \rangle = \sum_{\alpha} \alpha! b_{\alpha} c_{\alpha} \quad (19)$$

Note that no assumptions are made regarding the convergence of the formal series in (17).

We can in a natural way regard $L^2(\mu)$ as a subspace of $(\mathcal{S})^*$. In particular, if $X \in L^2(\mu)$ then by (19) the action of $X$ on $F \in (\mathcal{S})$ is given by

$$\langle X, F \rangle = E[X \cdot F]$$

Before we look at an example, we define the important Wick product of two Hida distributions $F, G$:

Definition 4 Let $F = \sum_{\alpha} a_{\alpha} H_{\alpha}, G = \sum_{\beta} b_{\beta} H_{\beta}$ be two elements of $(\mathcal{S})^*$. Then the Wick product of $F$ and $G$ is the element $F \circ G$ in $(\mathcal{S})^*$ given by

$$F \circ G = \sum_{\alpha, \beta} a_{\alpha} b_{\beta} H_{\alpha+\beta} \quad (20)$$

We will in the rest of the article only consider $d = 1$, i.e. $\mathcal{S} = \mathcal{S}(\mathbb{R})$ and $\mathcal{S}' = \mathcal{S}'(\mathbb{R})$. Now, we turn the attention to an important element in $(\mathcal{S})^*$, namely the white noise, $\xi_t(\omega)$. This element is defined as

$$W_t(\omega) = \sum_{k=1}^{\infty} \xi_k(t) H_k(\omega) = \sum_{k=1}^{\infty} \xi_k(t) h_k(\theta_k) \quad (21)$$

where $\epsilon_k = (0, \ldots, 0, 1)$ with 1 on the $k$'th place, $k = 1, 2, \ldots$. We show that the white noise is an $(\mathcal{S})^*$-element: By (16)

$$(2\mathbb{N})^k = 2k$$

and (18) becomes

$$\sup_{\alpha} b_{\alpha}^2 \alpha!(2\mathbb{N})^{-\alpha q} = \sup_{k} \xi_k(t)^2 1(2k)^{-q} < \infty$$

for some $q > 0$, since $sup_t |\xi_k(t)| = O(k^{-1/12})$. (See [HP], p. 571).

After this rather brief introduction to the white noise theory, we will have a look at integrals in $(\mathcal{S})^*$.\[4\]
3 Integrals in \((S)^*\).

The two concepts of integrability of stochastic processes we are going to study, are the following:

**Definition 5** A process \(Y_s \in (S)^*\) for \(s \in [0, t]\) is called \((S)^*\)-integrable if
\[
(Y_s, \psi) \in L^1([0, t]), \forall \psi \in (S)
\]
(22)

The \((S)^*\)-integral is then defined as the unique \((S)^*\)-element
\[
\left( \int_0^t Y_s ds, \psi \right) = \int_0^t (Y_s, \psi) ds
\]
(23)

where \(\psi \in (S)\). (See prop (6.1) in [HKPS]).

**Definition 6** A process \(Y_s \in L^2(\mu)\) for \(s \in [0, t]\) is Skorohod integrable if
\[
\int_0^t E|Y_s|^2 ds + \sum_{m=1}^\infty (m + 1)! ||\tilde{f}_m||^2 < \infty
\]
(24)

where \(\tilde{f}_m\) is the symmetrization of \(f_m(\cdot, \mathcal{H}_m(\cdot))\) in the chaos expansion, (6). Moreover
\[
\int_0^t Y_s \delta B_s = \int_0^t f_0(s) dB_s + \sum_{m=1}^\infty \int_{\mathbb{R}^{m+1}} \tilde{f}_m dB^{\otimes m+1}
\]
(25)

If we have chaos expanded an \((S)^*\)-integrable process, what is its \((S)^*\)-integral? The answer to this question is:

**Proposition 7** Assume \(Y_s \in (S)^*, s \in [0, t]\), has the chaos expansion
\[
Y_s = \sum_\alpha c_\alpha(s) H_\alpha
\]

where
\[
\sum_\alpha \alpha!|a_\alpha| \int_0^t |c_\alpha(s)| ds < \infty
\]

\(\forall \psi = \sum a_\alpha H_\alpha \in (S)\). Then \(Y_s\) is \((S)^*\)-integrable, and
\[
\int_0^t Y_s ds = \sum_\alpha (\int_0^t c_\alpha(s) ds) H_\alpha
\]
(26)

**Proof:** Since
\[
\int_0^t \langle Y_s, \psi \rangle ds \leq \sum_\alpha \alpha!|a_\alpha| \int_0^t c_\alpha(s) ds < \infty
\]
by assumption, the \((S)^*\)-integrability follows from (22). By Th.(2.25) in [F] we can change sums and integrals. Now invoking the definition of \((S)^*\)-integrals, we get
\[
\left( \int_0^t Y_s ds, \psi \right) = \int_0^t \langle Y_s, \psi \rangle ds = \int_0^t \left( \sum_\alpha \alpha!a_\alpha c_\alpha(s) \right) ds
\]
\[ = \sum_{\alpha} \alpha a_\alpha \int_0^t c_\alpha ds = \left( \sum_{\alpha} \int_0^t c_\alpha(s)ds \right) \mathcal{H}_{\alpha}, \psi \]

The proposition follows. \[ \square \]

**Example:** Assume \[ f \in L^2((0,t)) \text{ a.e. } s \in [0,t] \]

Then for \( \psi = \sum_{\alpha} a_\alpha H_{\alpha} \in (\mathcal{S}) \) we have

\[
\sum_{k} |a_{e_k}| \int_0^t |f(s)||\xi_k(s)|ds < \infty
\]

This is so because we can define the element

\[ Z = \sum_{k} (\int_0^t |f(s)||\xi_k(s)|ds)H_{e_k} \]

which is in \((\mathcal{S})^*\), since

\[
\sup_k(\int_0^t |f(s)||\xi_k(s)|ds)^2(2k)^{-q} \leq \sup_k(\int_0^t f^2ds)(\int_0^t \xi_k^2ds)(2k)^{-q} \leq ||f||^21 \sup_k(2k)^{-q} < \infty, \forall q > 0
\]

This implies

\[ \infty > \langle Z, \psi \rangle = \sum_{k} |a_{e_k}| \int_0^t |f(s)||\xi_k(s)|ds \]

Here \( \bar{\psi} = \sum_{\alpha} |a_\alpha| H_{\alpha} \). By prop.(7) it follows that \( f(s)W_s \) is \((\mathcal{S})^*\)-integrable on \([0,t]\) and

\[ \int_0^t f(s)W_s ds = \sum_{k} \int_0^t f(s)\xi_k(s)ds H_{e_k} \]

Note the following equality:

\[ \int_0^t f(s)dB_s = \sum_k \left( \int fX(s)ds \right) \int_\mathbb{R} \xi_k(s)dB_s = \sum_k \int_0^t f(s)\xi_k(s)ds H_{e_k} = \int_0^t f(s)W_s ds \quad (27) \]

To proceed, we need a useful lemma:

**Lemma 8** Assume

\[ \sup_{\alpha} \alpha! \int_0^t |c_\alpha(s)|^2ds < \infty \]

Then

\[ X = \sum_{\alpha} \left( \int_0^t |c_\alpha(s)|ds \right) H_{\alpha} \]

will be an element of \((\mathcal{S})^*\). \[ \square \]

**Proof:** We must show that

\[ \sup_{\alpha} \left( \int_0^t |c_\alpha(s)|ds \right)^2 \alpha!(2N)^{-\alpha q} < \infty \]

for \( a > 0 \). By the Hölder inequality, we get

\[ \sup_{\alpha} \left( \int_0^t |c_\alpha(s)|ds \right)^2 \alpha!(2N)^{-\alpha q} \leq t \sup_{\alpha} \left( \int_0^t c^2_\alpha(s)ds \right) \alpha!(2N)^{-\alpha q} \]
\[ \leq t \sup_{\alpha} \left( \int_0^t c_\alpha^2(s)ds \right) < \infty \]

since
\[ (2\pi)^\alpha = 2^{\alpha_1}(1^{\alpha_2}2^{\alpha_3} \ldots m^{\alpha_m}) \geq 1, \forall \alpha \]

To prove (1), we must classify the processes \( Y_s \) which make \( Y_s \circ W_s \) \((S')\)-integrable. The next proposition deals with this:

**Proposition 9** Assume \( Y_s \in (S)' \) for \( s \in [0,t] \) with chaos expansion
\[ Y_s = \sum_{\alpha} c_\alpha(s)H_\alpha \]

such that
\[ \sup_{\alpha} \int_0^t |c_\alpha(s)|^2ds < \infty \]

Then \( Y_s \circ W_s \) is \((S')\)-integrable on \([0,t]\) and
\[ \int_0^t Y_s \circ W_sds = \sum_{\alpha,k} \left( \int_0^t c_\alpha(s)\xi_k(s)ds \right)H_{\alpha+k} \]

(28)

**Proof:** Let \( \psi = \sum_\alpha a_\alpha H_\alpha \). According to (22), the proposition is proved if
\[ \sum_{\alpha,k}(\alpha + \epsilon_k)! \int_0^t |c_\alpha(s)||\xi_k(s)|ds|a_{\alpha+k}| < \infty \]

By the estimate \( \sup_{s \in \mathbb{R}} |\xi_k(s)| = O(k^{-1/12}) \) in [HP], p.571, we have
\[ \sum_{\alpha,k}(\alpha + \epsilon_k)! \int_0^t |c_\alpha(s)||\xi_k(s)|ds|a_{\alpha+k}| \leq \sum_{\alpha,k}(\alpha + \epsilon_k)!Ck^{-1/12}(\int_0^t |c_\alpha(s)|ds)|a_{\alpha+k}| \]

\[ \leq C \sum_{\alpha,k}(\alpha + \epsilon_k)!\left( \int_0^t |c_\alpha(s)|ds \right)|a_{\alpha+k}| \]

Now put
\[ X = \sum_{\alpha} \int_0^t |c_\alpha(s)|dsH_\alpha \]
\[ Z = \sum_k 1 \cdot H_{\epsilon_k} \]

By lemma(8), \( X \in (S)' \). Since
\[ \sup_k (2k)^{-q} < \infty, \forall q > 0 \]

we have that \( Z \in (S)' \). Hence \( X \circ Z \in (S)' \) and
\[ X \circ Z = \sum_{\alpha,k} \int_0^t |c_\alpha(s)|dsH_{\alpha+k} \]

which implies
\[ (X \circ Z, \tilde{\psi}) = \sum_{\alpha,k}(\alpha + \epsilon_k)! \int_0^t |c_\alpha(s)|ds|a_{\alpha+k}| < \infty \]

where \( \tilde{\psi} = \sum_\alpha |a_\alpha|H_\alpha \). Hence the proposition follows.

An important consequence of this proposition is
Corollary 10 Assume $Y_s$ is Skorohod integrable on $[0, t]$. Then $Y_s \circ W_s$ is $(S')^*$-integrable, and

$$
\int_0^t Y_s \circ W_s ds = \sum_{\alpha, k} \left( \int_0^t c_\alpha(s) \xi_k(s) ds \right) H_{\alpha + \epsilon_k}
$$

(29)

\[ \square \]

Proof: By the Skorohod integrability, (24), we have

$$
\int_0^t E[Y_s^2] ds = \int_0^t \left( \sum_\alpha \alpha c_\alpha(s)^2 \right) ds = \sum_\alpha \alpha! \int_0^t c_\alpha^2(s) ds < \infty
$$

and hence

$$
\sup_\alpha \alpha! \int_0^t |c_\alpha(s)|^2 ds \leq \sum_\alpha \alpha! \int_0^t c_\alpha^2(s) ds < \infty
$$

\[ \square \]

We are now ready to prove the main result of this article:

Theorem 11 Assume $Y_s$ Skorohod integrable on $[0, t]$. Then

$$
\int_0^t Y_s \delta B_s = \int_0^t Y_s \circ W_s ds
$$

(30)

\[ \square \]

Proof: In the proof we use the definitions of the Wick product and the Skorohod integral. Direct calculation will then show (30).

The Wiener-ito chaos expansion gives

$$
Y_s = f_0(s) + \sum_{m=1}^{\infty} \int_{\mathbb{H}_m} f_m(s; u) dB_u^m
$$

$$
= f_0(s) + \sum_{m=1}^{\infty} \left( \sum_{|\alpha| = m} \left( \sum_{|\alpha| = m} (f_m(s; \cdot), \xi^{\alpha}) \right) \int_{\mathbb{H}_m} \xi^{\alpha} dB_u^m \right)
$$

$$
= f_0(s) + \sum_{|\alpha| = 1} (f_{|\alpha|}(s; \cdot), \xi^{\alpha}) H_{\alpha}
$$

Hence, taking the Wick product with $W_s$, we get

$$
Y_s \circ W_s = f_0(s) W_s + \sum_{\alpha, k} (f_{|\alpha|}(s; \cdot), \xi^{\alpha}) \xi_k(s) H_{\alpha + \epsilon_k}
$$

By corollary (10):

$$
\int_0^t Y_s \circ W_s ds = \int_0^t f_0(s) W_s ds + \sum_{\alpha, k} \left( \int_0^t (f_{|\alpha|}(s; \cdot), \xi^{\alpha}) \xi_k(s) ds \right) H_{\alpha + \epsilon_k}
$$

By the definition of the Skorohod integral, we have

$$
\int_0^t Y_s \delta B_s = \int_0^t f_0(s) dB_s + \sum_{m=1}^{\infty} \int_{\mathbb{H}_m} \bar{f}_m dB^{m+1}
$$

8
\[
= \int_0^t f_0(s)dB_s + \sum_{m=1}^{\infty} \sum_{|\alpha|=m+1} (\tilde{f}_m, \xi^{\Phi(\alpha)}) \int_{\mathbb{R}^{m+1}} \xi^{\Phi(\alpha)} dB^{m+1}
\]

\[
= \int_0^t f_0(s)dB_s + \sum_{|\alpha|=2} (\tilde{f}_{|\alpha|-1}, \xi^{\Phi(\alpha)}) H_\alpha
\]

From (27)

\[
\int_0^t f_0(s)dB_s = \int_0^t f_0(s)W_s ds
\]

Hence, we must show that

\[
\sum_{\alpha, \beta} \int_0^t (f_{|\alpha|}(s; \cdot), \xi^{\Phi(\alpha)}(s)ds) H_{|\alpha|+\varepsilon_k} = \sum_{|\alpha|=2} (\tilde{f}_{|\alpha|-1}, \xi^{\Phi(\alpha)}) H_\alpha
\]

Considering the right hand side, we find that

\[
\sum_{|\alpha|=2} (\tilde{f}_{|\alpha|-1}, \xi^{\Phi(\alpha)}) H_\alpha = \sum_{\alpha, \beta} \int_0^t (f_{|\alpha|}(s; \cdot), \xi^{\Phi(\alpha)}(s)ds) H_{|\alpha|+\varepsilon_k}
\]

Hence, it is sufficient to show that

\[
\sum_{\alpha, \beta} \int_0^t (f_{|\alpha|}(s; \cdot), \xi^{\Phi(\alpha)}(s)ds) H_{|\alpha|+\varepsilon_k} = \sum_{\alpha, \beta} \int_0^t (f_{|\alpha|}(s; \cdot), \xi^{\Phi(\alpha)}(s)ds) H_{|\alpha|+\varepsilon_k}
\]

Let $|\alpha| = n$. Then we might write $\alpha$ as

\[
\alpha = \varepsilon_i, i_2, \ldots, i_n
\]

where $\varepsilon_i, i_2, \ldots, i_n$ has ones on the coordinates $i_1, \ldots, i_n$, and zeros everywhere else. If $i_j = i_k$, then the multi-index has 2 on the coordinate $i_j$, and so on. In addition we have

\[
\varepsilon_i, i_2, \ldots, i_n + \varepsilon_k = \varepsilon_{i_1, i_2, \ldots, i_k}
\]

If $u \in \mathbb{R}^{n+1}$, we get

\[
\xi^{\Phi(\varepsilon_i, i_2, \ldots, i_n)}(u) = \frac{1}{(n+1)!} \sum_{\sigma} \xi_{i_1}(u_{\sigma_1}) \cdots \xi_{i_n}(u_{\sigma_n}) \xi_k(u_{\sigma_{n+1}})
\]

where the sum is taken over all permutations $\sigma$ of the set $\{1, \ldots, n+1\}$. In addition

\[
\tilde{f}_n(u) = \frac{1}{n+1} \sum_{j=1}^{n+1} X_{(0, t)}(u_j)f_n(u_j, u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_{n+1})
\]

Therefore

\[
(f_n, \xi^{\Phi(\varepsilon_i, i_2, \ldots, i_n+\varepsilon_k)}) = \frac{1}{(n+1)!(n+1)!} \sum_{j=1}^{n+1} \sum_{\sigma} \int_{\mathbb{R}^{n+1}} X_{(0, t)}(u_j)f_n(u_j, u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_{n+1})
\]

\[
\times \xi_{i_1}(u_{\sigma_1}) \cdots \xi_k(u_{\sigma_{n+1}}) du
\]

\[
= \frac{1}{(n+1)!(n+1)!} \sum_{j=1}^{n+1} \int_0^t \{f_n(u_j, \cdot), \xi^{\Phi(\varepsilon_i, i_2, \ldots, i_n)}(u_j)du_j + \ldots
\]

9
\[ + \int_0^t \left( f_n(u_j; \cdot) \xi_{\infty}^{(e_{i_1 \ldots i_{n-1} + \epsilon_h})} \xi_i(u_j) du_j + \int_0^t \left( f_n(u_j; \cdot), \xi^{(e_{i_1 \ldots i_n})} \xi_k(u_j) du_j \right) \right) \]
\[ = (1/((n+1)! (n+1))) \sum_{i_1, \ldots, i_n, k} \int_0^t \left( f_n(s; \cdot), \xi^{(e_{i_1 \ldots i_n})} \xi_k(s) ds + \ldots + \int_0^t \left( f_n(s; \cdot), \xi^{(e_{i_1 \ldots i_n})} \xi_k(s) ds \right) \right) \]
\[ = (1/(n+1)) \sum_{i_1, \ldots, i_n, k} \left( \int_0^t \left( f_n(s; \cdot), \xi^{(e_{i_1 \ldots i_n})} \xi_k(s) ds + \ldots + \int_0^t \left( f_n(s; \cdot), \xi^{(e_{i_1 \ldots i_n})} \xi_k(s) ds \right) \right) \right) \]
\[ = (1/(n+1)) \sum_{i_1, \ldots, i_n, k} \left( \sum_{i_1, \ldots, i_n} \int_0^t \left( f_n(s; \cdot), \xi^{(e_{i_1 \ldots i_n})} \xi_k(s) ds \right) H_{e_{i_1 \ldots i_n}} + \ldots + \sum_{i_1, \ldots, i_n} \left( \int_0^t \left( f_n(s; \cdot), \xi^{(e_{i_1 \ldots i_n})} \xi_k(s) ds \right) H_{e_{i_1 \ldots i_n}} \right) \right) \]
\[ = \sum_{i_1, \ldots, i_n} \left( \int_0^t \left( f_n(s; \cdot), \xi^{(e_{i_1 \ldots i_n})} \xi_k(s) ds \right) H_{e_{i_1 \ldots i_n}} \right) \]

which shows (21), and hence the theorem.

\[ \blacksquare \]

Corollary 12 Assume \( Y_s \in L^2(\mu) \), \( s \in [0, t] \) is Itô integrable then
\[ \int_0^t Y_s dB_s = \int_0^t Y_s \circ W_s ds \] (32)

\[ \blacksquare \]

Proof: Itô integrability implies Skorohod integrability. See [NZ] for more information about this.

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