Conic bundles in projective fourspace

by

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P. Ellia and G. Sacchiero have shown that if $S$ is a smooth surface in $\mathbb{P}^4$ which is ruled in conics, then $S$ has degree 4 or 5 (cf. [ES]). In this paper we give a proof of this result combining the ideas of Ellia and Sacchiero as they are used in the paper of the second author on plane curve fibrations [Ra] and the recent work of G. Fløystad and the first author bounding the degree of smooth surfaces in $\mathbb{P}^4$ not of general type [BF]. Let $S$ be a smooth conic bundle in $\mathbb{P}^4$. Let $V$ denote the hypersurface which is the union of the planes of the conics on $S$. Let $G \subset \mathbb{P}^9$ be the Grassmannian of planes in $\mathbb{P}^4$ in the Plücker embedding. Since the hypersurface $V$ contains a one dimensional family of planes, we may associate a curve $C_V \subset G$ whose points correspond to the planes in $V$. In the natural incidence variety in $G \times \mathbb{P}^4$ between points and planes, there is a $\mathbb{P}^2$-fibration $W$ over $C_V$ whose projection into $\mathbb{P}^4$ is $V$. If $C_V$ is not smooth consider its normalization $\tilde{C}_V$, and the pullback $\tilde{W}$ of $W$ to $\tilde{C}_V$. The strict transform $\tilde{S}$ of $S$ in $\tilde{W}$ is clearly smooth, since the map $\tilde{S} \rightarrow S$ is birational. So on the complement of some possible $(-1)$-curves this map is an isomorphism. But if $C_V$ is singular there are two plane curves, possibly infinitely close, which are mapped into the same plane in $\mathbb{P}^4$, this is a contradiction. Therefore $C_V$ is smooth. Let $g$ be the genus of $C_V$ and let $\delta = \text{deg} C_V$, thus $\delta$ is also the degree of the hypersurface $V$.

The proof is an exploitation of the relations between the invariants of $S$ and $C_V$. On the one hand they combine with the results of [BF] to give the upper bound $d \leq 42$ for the degree of $S$. On the other hand the curve $C_V$ in the Grassmannian inside $\mathbb{P}^9$, has a genus which is high compared to the degree. Thus Castelnuovo bounds show that the span of this curve is a subspace of $\mathbb{P}^9$. We analyse the intersection of the linear span of $C_V$ and the Grassmannian, which is a variety $T$ cut out by quadrics. The lines and conics on this variety give rise to special curves on $S$ which together with the bound for the degree allow us to conclude. We work over an algebraically closed field of characteristic 0.

(1.1) Lemma. If $S$ has degree $d$ and sectional genus $\pi$, then

(a) $3d \geq 4\delta,$

(b) $\pi - 1 = d + 2g - 2 - \delta,$

(c) $d^2 - 9d - 8(2g - 2) + 2\delta = 0.$

(d) $\pi - 1 = \frac{d^2}{8} - \frac{d}{8} - \frac{3\delta}{4}.$
Proof. Part (a) is the fact that there is a positive number of singularities for the map\[ \pi : S \rightarrow C_V: \] If a fibre is nonreduced i.e. a double line, then the line would have nonpositive selfintersection on S, but every fibre is a conic in a plane so computing the arithmetic genus of a fibre in two ways we get a contradiction. Therefore the number\[ c_2(\Omega_S - \pi^*\Omega_{C_V}) = 3d - 4\delta \]
counts a nonnegative finite number of singular points.
Part (b) is a straightforward calculation using adjunction on W. Part (c) is the double point formula for smooth surfaces in \(\mathbb{P}^4\) applied to S (cf.[HR,p.434]). Part (d) follows from (b) and (c). □

(1.2) Remark. From Severis theorem it follows that the projection of W into \(\mathbb{P}^4\) is linearly normal. Thus by Riemann Roch \(\delta \leq 2 + 3g\).

(1.3) Remark. Smooth surfaces on quadrics are well understood and those on cubics are classified recently (cf. [A],[K]) so we only need to worry about \(\delta \geq 4\). In fact there are conic bundles on quadrics, and any conic bundle on a cubic is also on a quadric.

(1.4) Lemma. \(\delta \leq 3\) or\[ g - 1 \geq \frac{1}{9}\delta^2 - \frac{5}{8}\delta.\]
Proof. Since \(\frac{4}{3}\delta - \frac{3}{2} \geq 0\) when \(\delta \geq 4\), the relations (1.1c) and (1.1a) yields
\[0 = a^2 - 9d - 8(2g - 2) + 2\delta\]
\[\geq \frac{16}{9}\delta^2 - 12\delta - 8(2g - 2) + 2\delta\]
from which the lemma follows. □

This inequality beats the genus bound for curves in \(\mathbb{P}^6\) (cf. [HJ]):

(1.5) Proposition. \(C_V\) is contained in a \(\mathbb{P}^5\), and if \(C_V\) spans a \(\mathbb{P}^5\), then it lies on a surface of degree 4.

Proof. If \(C_V\) spans a \(\mathbb{P}^6\), then the genus bound says that
\[\frac{1}{10}(\delta^2 - 7\delta + 12) \geq \frac{1}{9}\delta^2 - \frac{5}{8}\delta + 1,\]
which yields
\[\delta^2 + \frac{27}{4}\delta - 18 \leq 0.\]
i.e. \(\delta \leq 2\).
If \(C_V\) is not contained in a surface of degree 4 in \(\mathbb{P}^5\), then the genus formula (cf.[HJ]) yields
\[\frac{1}{10}(\delta^2 - 5\delta + 10) \geq \frac{1}{9}\delta^2 - \frac{5}{8}\delta + 1.\]
Thus
\[\delta^2 - \frac{45}{4}\delta \leq 0.\]
Thus the proposition follows from...
(1.6) Lemma. If \( C_V \) spans \( \mathbb{P}^5 \) and does not lie on a surface of degree 4, then \( \delta \geq 12 \).

Proof. If \( C_V \) is rational or elliptic, then its degree is at least 5 or 6, while the above Castelnuovo bound says that \( g \leq 7 \) when \( \delta \leq 11 \). It remains only to check that (1.1c) has no integral solutions, which is straightforward. □

2 Bound for the degree of \( S \)

From the results of [BF] we can show

(2.1) Proposition. If \( S \) is not on a cubic hypersurface, then \( d \leq 42 \).

Proof. We distinguish between the cases whether \( S \) lies on a quartic or a quintic hypersurface or not, and apply Roth's theorem [Ro] to study the genus of a general hyperplane section.

Case 1: Assume \( S \) is not contained in a quintic and \( d > 25 \). Then by Roth's theorem a general hyperplane section of \( S \) is also not contained in a quintic (in \( \mathbb{P}^5 \)). Hence the genus bound for space curves (cf. [GP]) gives

\[
\pi - 1 \leq \frac{d^2}{12} + d.
\]

Combined with (1.1d) and using \( \delta \leq \frac{3d}{4} \) from (1.1a) we find \( d \leq 40 \) in this case.

Case 2: Assume \( S \) is contained in a quintic, not contained in a quartic and \( d > 17 \). As in [BF 1.1b] let

\[
\gamma = \frac{d^2}{10} + \frac{d}{2} + 1 - \frac{2r(5-r)}{5} - \pi
\]

where \( 0 \leq r \leq 4 \) and \( d+r \equiv 0 \pmod{5} \). By the genus bound for space curves (cf. [GP]) \( \gamma \) is a non-negative integer satisfying

\[
\pi - 1 \leq \frac{d^2}{10} + \frac{d}{2} - \gamma
\]

and furthermore (cf. [BF 1.1e])

\[
\frac{d^3}{150} - \frac{d}{6} \leq \chi(\mathcal{O}_S) + \frac{\gamma^2}{2} + \gamma(\frac{d}{5} + \frac{5}{2}).
\]

The first inequality combined with (1.1d) leads to

\[
\gamma \leq \frac{d}{80}(95 - 2d).
\]

Hence a priori \( d \leq 47 \). Moreover the maximal value of \( \gamma \) in the range \( 18 \leq d \leq 47 \) is 14. Now (1.1c) combined with (1.1a) yields

\[
\chi(\mathcal{O}_S) = 1 - g = \frac{d^2}{16} + \frac{9d}{16} - \frac{\delta}{8} \leq -\frac{d^2}{16} + \frac{9d}{16}.
\]
Inserting this and $\gamma = 14$ in the second inequality gives

$$\frac{d^3}{150} - \frac{d}{6} \leq -\frac{d^2}{16} + \frac{9d}{16} + \frac{14d}{5} + 133.$$  

Evaluating shows that $d \leq 30$ in this case.

Case 3: Assume $S$ is contained in a quartic and $d > 10$. As in [BF 1.1b] let

$$\gamma = \frac{d^2}{8} + 1 - \frac{3r(4 - r)}{8} - \pi$$

where $0 \leq r \leq 3$ and $d + r \equiv 0 \pmod{4}$. By the genus bound for space curves (cf. [GP]) $\gamma$ is a non-negative integer and

$$\pi - 1 \leq \frac{d^2}{8} - \gamma.$$  

Combined with (1.1d) and (1.1a) this gives

$$\gamma \leq \frac{d}{8} + \frac{3\delta}{4} \leq \frac{11d}{16}.$$  

We have (cf. [BF 1.1e])

$$\frac{d^3}{96} - \frac{d^2}{16} - \frac{d}{24} + \frac{5}{4} \leq \chi(\mathcal{O}_S) + \frac{\gamma^2}{2} + \gamma\left(\frac{d}{4} + \frac{3}{2}\right).$$

Putting things together (as in case 2) leads to

$$\frac{d^3}{96} - \frac{209d^2}{512} - \frac{157d}{96} + \frac{5}{4} \leq 0$$

which yields $d \leq 42$.\[2.2\]

**Corollary.** If $S$ is not on a cubic, then $\delta \leq 31$.

**Proof** Combine (2.1) with (1.1a).\[2.1\]

3 Some geometry of $C_V$

Now $C_V$ is a curve on the Grassmannian G, which is a variety cut out by quadrics. If $L$ is the linear span of $C_V$, and $T$ is the irreducible component of $G \cap L$ which contains $C_V$, then $T$ is a quadric or lies on more than one quadric in $L$. In each possible case we may describe the family of planes in $\mathbb{P}^4$ corresponding to the closed points of $T$.

(3.1) **Lemma.** The closed points on a line in $G$ correspond to the pencil of planes through a line in a $\mathbb{P}^3$.

The closed points on a conic in $G$ whose plane is not contained in $G$, correspond to the planes of one of the pencils of a quadric of rank 4 in $\mathbb{P}^4$.

**Proof.** Easy.\[3.1\]
(3.2) Lemma. The curve \( C_V \) is rational or elliptic, or \( T \) is a plane, or a quadric surface in a \( \mathbb{P}^3 \) or the whole \( \mathbb{P}^3 \), or a cubic scroll or a cone over a quartic curve or a del Pezzo in \( \mathbb{P}^4 \), or a quadric hypersurface in \( \mathbb{P}^4 \), or \( C_V \) spans a \( \mathbb{P}^5 \).

Proof. The linear span \( L \) of \( C_V \) is at most a \( \mathbb{P}^5 \) by (1.4). If \( L \) is a \( \mathbb{P}^4 \) and \( T \) is a curve, then the intersection \( L \cap G \) is proper and \( C_V \) has degree at most 5 and is rational or elliptic. If \( T \) is a surface, then \( T \) is a cubic scroll or a complete intersection of two quadrics. In the latter case \( T \) is a cone or a del Pezzo surface.

If \( T \) is neither a curve nor a surface in a \( \mathbb{P}^4 \), then it must be a quadric hypersurface.

If \( L \) is a \( \mathbb{P}^3 \), then \( T \) is a curve of degree at most 4, or \( T \) is a quadric or \( T \) is all of \( L \).

If \( L \) is contained in a plane, then \( T \) is the whole plane, a conic or a line. \( \square \)

This exhausts the list of possibilities for \( T \). Lemma (3.1) simplifies the analysis of each case. Thus if \( T \) is a quadric, then all the plane conics of \( T \) correspond to quadrics of rank 4. Now if two conics in \( T \) meet in two points, then the corresponding quadrics of rank 4 have a common vertex. By choosing different conics, we may conclude that the planes corresponding to the points on \( T \) all have a common point, i.e. \( V \) is a cone.

If \( T \) is a (possibly degenerate) del Pezzo surface, we may again find a conic on it such that the hyperplanes through it cuts out a pencil of conics on \( T \). The preceding argument shows that \( V \) is a cone also in this case.

If \( T \) is a cone over an elliptic quartic curve, then \( C_V \) has degree \( \delta = 4\alpha + 1 \) or \( \delta = 4\alpha \) for some \( \alpha \) depending on whether \( C_V \) meets the vertex of \( T \) or not. The corresponding genera are given by \( 2g - 2 = 2(2\alpha + 1)(\alpha - 1) \) and \( 2g - 2 = 4\alpha(\alpha - 1) \) respectively. Combined with the inequality \( \delta \leq 31 \) from (2.2) it is easily checked that (1.1c) has a numerical solution only if \( \alpha = 1 \), i.e. when \( C_V \) is elliptic.

If \( T \) is a \( \mathbb{P}^3 \), then the planes corresponding to the points of \( T \) must all lie in a \( \mathbb{P}^3 \), so \( V \) and \( S \) is degenerate.

If \( T \) is a quadric, then the argument above applies to show that \( V \) is a cone.

If \( T \) is a curve in \( \mathbb{P}^3 \), then \( C_V \) is rational or elliptic. If \( T \) is contained in a plane, then \( V \) is either contained in a \( \mathbb{P}^3 \) or it is a cone. Combined with (1.5) we have shown that

(3.3) Lemma. \( C_V \) is rational or elliptic, or it lies on a cubic scroll in a \( \mathbb{P}^4 \) or it lies on a quartic surface in \( \mathbb{P}^5 \) or \( V \) is a cone.

4 The cone case

(4.1) Proposition. If \( V \) is a cone, then it is a quadric or a \( \mathbb{P}^3 \).

Proof. If \( V \) is a cone then there is some curve on \( W \) which is contracted by the projection into \( \mathbb{P}^4 \). Since it is contracted the numerical class of this curve must be a multiple of the class \( h^2 - \delta h \cdot f \), where \( h \) is the class of a hyperplane section and \( f \) is the class of a fibre. Unless \( C_V \) is rational (i.e. of degree \( \delta \leq 2 \)), \( S \) meets this curve in at most one point, so

\[
0 \leq d - 2\delta \leq 1.
\]

If \( d = 2\delta \), then (1.1) yields

\[
2g - 2 = \frac{1}{2} \delta^2 - 2\delta,
\]

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while if \( d = 2\delta + 1 \), then (1.1) yields

\[
2g - 2 = \frac{1}{2}\delta^2 - \frac{3}{2}\delta - 1.
\]

In both cases a comparison with Castelnuovos bound for the genus of space curves and with the genus of plane curves shows that \( \delta \leq 2 \). \( \Box \)

5 The cubic scroll case

Assume that \( C_V \) lies on a cubic scroll. Let \( E \) be a hyperplane section of the scroll and let \( F \) be a member of the ruling, then numerically \( C_V \equiv \alpha E + \beta F \) where \( \alpha \geq 0 \) and \( \beta \geq -2\alpha \) \( \delta \geq -2\alpha \) and \( \beta = 4\alpha + \beta \). Since \( C_V \) is smooth we get by adjunction \( 2g - 2 = 3\alpha^2 - 6\alpha + 2\alpha \beta - 6\beta \) even if \( T \) is singular. With the inequality \( \delta \leq 31 \) of (2.2), a simple program checks the possibilities and allow us to conclude that (1.1c) has no integral solutions in the given range. Thus

(5.1) Proposition. \( C_V \) is not a cubic scroll.

6 The quartic surface in \( \mathbb{P}^5 \) case.

First assume that \( C_V \) lies on a rational quartic scroll. Let \( E \) be a hyperplane section of the scroll and let \( F \) be a member of the ruling, then numerically \( C_V \equiv \alpha E + \beta F \) where \( \alpha \geq 0 \) and \( \beta \geq -2\alpha \) and \( \delta = 4\alpha + \beta \). Since \( C_V \) is smooth we get by adjunction \( 2g - 2 = 4\alpha^2 - 6\alpha + 2\alpha \beta - 2\beta \) even if the scroll is singular. With the inequality \( \delta \leq 31 \) of (2.2), a simple program checks the possibilities and allow us to conclude that (1.1) has only two integral solutions in the given range. Thus

(6.1) Lemma. If \( C_V \) lies on a rational quartic scroll, then its numerical class is \( 3E - F \) or \( 6E + 2F \).

If the quartic surface is not a scroll then it is a Veronese surface. So \( C_V \) is a plane curve of degree \( a \) embedded by conics in \( \mathbb{P}^5 \). Thus \( \delta = 2a \) and \( 2g - 2 = a(a - 3) \). As above the relation (1.1c) is checked for \( \delta \leq 31 \) i.e. \( a \leq 15 \), and there are no integral solutions. We have shown

(6.2) Lemma. If \( C_V \) spans \( \mathbb{P}^5 \), then it does not lie on a Veronese surface.

We want to exclude the possibilities of (6.1) by a geometric argument: First let \( C_V \) be of numerical type \( 3E - F \) on a scroll in the grassmannian. Then the degree of the conic bundle \( S \) is 15 and the sectional genus is 19. Now, by (3.1), for any general line in the ruling of the scroll there is a hyperplane section of \( S \) consisting of three conic sections and a residual curve \( A \). The family of lines is rational so by Bertini the curve \( A \) is irreducible for a general line in the ruling. The degree of \( A \) is 9 while the arithmetic genus is 16. This is impossible. Similarly let \( C_V \) be of numerical type \( 6E + 2F \) on a scroll in the grassmannian. Then the degree of the conic bundle \( S \) is 36 and the sectional genus is 139. For a general line in the ruling of the scroll there is a hyperplane section of \( S \) consisting of six conic sections and an irreducible residual curve \( A \). The degree of \( A \) is 24 while the arithmetic genus is 133. This is impossible, so we may conclude
(6.3) Proposition. $C_V$ does not span $\mathbb{P}^5$.

7 Conclusion

Combining (3.3), (4.1), (5.1) and (6.3) we are left with case that $C_V$ is rational or elliptic. But by (1.4) this means that $2 \leq \delta \leq 5$. The relations in (1.1) leave us with the possibility that $V$ is a quadric and the surface has degree 4 or 5 or that $V$ is a quartic and the surface is a conic bundle of degree 8 over an elliptic curve. The latter possibility was excluded by Okonek (cf. [Ok]). Therefore we have

(7.1) Theorem (Ellia, Sacchiero). The degree of a conic bundle in $\mathbb{P}^4$ is 4 or 5.

(7.2) Remark. There are surfaces with a 1-dimensional family of conic sections which are not conic bundles, these surfaces are easily seen to be rational and are the cubic scrolls and the Veronese surfaces.

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