Some Exponential Moment Inequalities for the Wiener Functionals

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Abstract

In this paper we prove, mainly, three probabilistic inequalities with which we can control the exponential moments of different Wiener functionals. The first one is an exponential inequality for the Wiener functionals with bounded Gross-Sobolev derivative, the second one is for those whose derivative's norm's square is exponentially integrable and the third one is for the functionals of the divergence form.

1 Introduction

In the analysis of Wiener functionals, for example, in the theory of large deviations, in the applications of the degree theorem or the calculation of the Radon-Nikodym density of the image of Wiener measure under the general, non-invertible shifts (cf., [11, 12]) we often need to control the exponential moments of these functionals with those of their derivatives or some powers of the Ornstein-Uhlenbeck operator.

In this paper we give three main inequalities which are quite useful for this purpose. The first one, which is the subject of the third section, can be written in the following form:

\[ \mu\{\phi > c\} \leq E \left[ \exp \frac{(c - P_T \phi)^2}{2||\nabla \phi||_\infty^2(1 - e^{-2T})^2} \right] + \mu\{P_T \phi > c\}, \]

for any \( T > 0 \), where \( \nabla \) denotes the Gross-Sobolev derivative of the real-valued functional \( \phi \), \( P_T \) is the Ornstein-Uhlenbeck semigroup on the Wiener space and we have a similar expression for \( -\phi \).
The second inequality of interest that we prove in the fourth section, is the following
\[ E[\exp(\phi - E[\phi])] \leq E[\exp \frac{\pi^2}{8} |\nabla \phi|^2_H]. \]
The method that we use to derive it gives also Poincaré type inequalities with better constants than those obtained by the Clark’s formula combined with Burkholder-Davis-Gundy inequalities and the convexity inequalities for the dual predictable projections.

The third inequality is studied in the last section and it says that
\[ E[\exp \delta \eta] \leq E[\exp \alpha |(2I + L)^\beta \eta|^2_H], \]
for a smooth random variable \( \eta \) with values in the Cameron-Martin space \( H \), where, \( \alpha > 0, \beta > 1/2 \) and \( \delta \) denotes the divergence operator with respect to the Wiener measure \( \mu \), called also the Skorohod integral in the case of classical Wiener space. This majoration is important in the degree theory of Wiener functionals. For example, if \( \nabla \eta \) satisfies also a similar integrability condition (cf., the Proposition 1), then the degree of the shift \( w \mapsto w + \eta(w) \) is equal to one (cf.,[12]).

2 Notations and Preliminaries

\( (W,H,\mu) \) denotes an abstract Wiener space, i.e., \( H \) is a separable Hilbert space, identified with its continuous dual, \( W \) is a Banach space into which \( H \) is injected continuously and densely. \( \mu \) is the canonical Gaussian measure on \( W \) whose reproducing kernel Hilbert space is \( H \) and we will call it as the Cameron-Martin space. If \( X \) is any separable Hilbert space, \( p > 1, k \in \mathbb{Z} \) we denote with \( D_{p,k}(X) \), the completion of \( X \)-valued polynomials on \( W \) with respect to the norm
\[ ||\xi||_{D_{p,k}(X)} = ||(I + L)^{k/2}\xi||_{L^p(\mu;X)} \]
where \( L \) is the Ornstein-Uhlenbeck operator on \( W \) (cf.,[1]). \( \nabla \) denotes the Gross-Sobolev derivative (cf.,[1]) on \( W \), let us recall \( \nabla : D_{p,k}(X) \rightarrow D_{p,k-1}(X \otimes H) \) continuously, where \( X \otimes H \) denotes the completed tensor product of \( X \) and \( H \) with respect to the Hilbert-Schmidt topology. Consequently \( \delta = \nabla^* \) is a continuous operator from \( D_{p,k}(X \otimes H) \) into \( D_{p,k-1}(X) \) for any \( p > 1, k \in \mathbb{Z} \).
We call $\delta$ the divergence operator on $W$. We denote by $D(X)$ the intersection of the Sobolev spaces $\{D_{p,k}(X); p > 1, k \in \mathbb{Z}\}$, equipped with the intersection (i.e., projective limit) topology. The continuous dual of $D(X)$ is denoted by $D'(X)$ and in case $X = \mathbb{R}$ we write simply $D_{p,k}, D, D'$ for $D_{p,k}(\mathbb{R}), D(\mathbb{R}), D'(\mathbb{R})$ respectively.

3 An exponential tightness inequality

Let us denote by $P_t$ the Ornstein-Uhlenbeck semigroup on $W$ (cf.,[1]). We have

**Theorem 1** Let $\phi$ be a real valued Wiener functional such that $\nabla \phi \in L^\infty(\mu; H)$. Then we have

$$
\mu\{\phi > c\} \leq E \left[ \exp - \frac{(c - P_T \phi)^2}{2\|\nabla \phi\|_\infty^2(1 - e^{-2T})} 1_{\{c \geq P_T \phi\}} \right] + \mu\{P_T \phi > c\},
$$

and

$$
\mu\{-\phi > c\} \leq E \left[ \exp - \frac{(c + P_T \phi)^2}{2\|\nabla \phi\|_\infty^2(1 - e^{-2T})} 1_{\{c \leq -P_T \phi\}} \right] + \mu\{-P_T \phi > c\},
$$

for any $T > 0$, where $E$ denotes the expectation with respect to $\mu$.

**Proof:** Let us denote by $(X_t; t \geq 0)$ the Ornstein-Uhlenbeck process with values in $W$, which accepts $P_t$ as the semi-group of transition probabilities. We can represent this process as the solution of the following $W$-valued stochastic differential equation:

$$
dX_t = -X_t dt + \sqrt{2} dB_t,
$$

where $(B_t; t \geq 0)$ is the $W$-valued Brownian motion and $X_0$ has the law $\mu$. Since $(X_t; t \geq 0)$ is stationary, we have for any $T > 0$,

$$
\mu\{\phi > c\} = IP_\mu\{\phi(X_T) > c\} \\
\leq IP_\mu\{\sup_{t \leq T} E_\omega[\phi(X_T)|\mathcal{F}_t] > c\},
$$

where $IP_\mu$ is the law of the process $(X_t)$ with the initial measure $\mu$ and $E_\omega$ is the expectation with respect to the law of $X_t$ with the initial measure $\epsilon_\omega$. 

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i.e., the Dirac measure at \( w \in W \). From the simple Markov property, the last term above is equal to

\[
P_\mu \{ \sup_{t \leq T} P_{T-t} \phi(X_t) > c \},
\]

which, in turn equals to

\[
P_\mu \{ \sup_{t \leq T} \left[ P_T \phi(w) + \sqrt{2} \int_0^t (\nabla P_{T-s} \phi(X_s), dB_s) \right] > c \},
\]

where we have used the Ito formula for the Ornstein-Uhlenbeck process with values in \( W \) (cf., [3]). Exponentiation of these inequalities gives, for any \( \theta > 0 \),

\[
\mu \{ \phi > c \} \leq P_\mu \{ \sup_{t \leq T} \exp \sqrt{2} \theta \int_0^t (\nabla P_{T-s} \phi(X_s), dB_s) > \exp \theta (c - P_T \phi) \}
\]

\[
\leq P_\mu \{ \sup_{t \leq T} \exp \sqrt{2} \theta \int_0^t (\nabla P_{T-s} \phi(X_s), dB_s) - \theta^2 \int_0^t |\nabla P_{T-s} \phi(X_s)|_H^2 ds \}
\]

\[
> \exp \theta (c - P_T \phi) - \theta^2 \int_0^T |\nabla P_{T-s} \phi(X_s)|_H^2 ds \}
\]

\[
\leq E \left[ \exp -\theta (c - P_T \phi) + \theta^2 \int_0^T e^{-2(T-s)} \| \nabla \phi \|_\infty^2 ds \right],
\]

where we have used the commutation relation \( \nabla P_t \phi = e^{-t} P_t \nabla \phi \) and the Doob inequality at the last line, which is equal to

\[
E[1_{\{c \geq P_T \phi \}} \exp -\theta (c - P_T \phi) + \frac{\theta^2}{2} (1 - e^{-2T}) \| \nabla \phi \|_\infty^2] +
\]

\[
+ E \left[ 1_{\{c < P_T \phi \}} \exp -\theta (c - P_T \phi) + \frac{\theta^2}{2} (1 - e^{-2T}) \| \nabla \phi \|_\infty^2 \right]
\]

Let us choose \( \theta \) as

\[
\theta = \frac{(c - P_T \phi)^+}{(1 - e^{-2T}) \| \nabla \phi \|_\infty^2} + \epsilon,
\]

where \( \epsilon > 0 \) is arbitrary and \( x^+ \) denotes the positive part of \( x \). Letting \( \epsilon \) tend to zero, we obtain

\[
\mu \{ \phi > c \} \leq E \left[ \exp -\frac{(c - P_T \phi)^2}{2 \| \nabla \phi \|_\infty^2 (1 - e^{-2T})} 1_{\{c \geq P_T \phi \}} \right] + \mu \{ P_T \phi > c \}.
\]

For \( \mu \{ -\phi > c \} \) we proceed similarly. \( \|QED \)
Remark 1 Since $P_T \phi \to E[\phi]$ as $T \to \infty$, at the limit, supposing $E[\phi] = 0$, we obtain the inequality

$$\mu\{|\phi| > c\} \leq 2 \exp \frac{-c^2}{2 \|\nabla \phi\|_\infty^2},$$

which has been established by B. Maurey in the finite dimensional case for the ordinary Fréchet differentiation (cf., [7]). For the proof of the case with Gross-Sobolev derivative in the frame of the Malliavin calculus we refer the reader to [9, 10]. However the result given in the Theorem 1 is finer and gives the possibility of iteration to obtain more delicate estimates.

3.1 Some variations and applications

Suppose that $G$ is a finite dimensional, simply connected Lie group with its Lie algebra $\mathcal{G}$ of left-invariant vector fields. We assume that the metric of $\mathcal{G}$ is invariant under the inner automorphisms. Let $(H_1, \ldots, H_n)$ be a basis of $\mathcal{G}$ and $B_t$ be an $n$-dimensional Brownian motion. Define $X_t$ as the solution of the $G$-valued stochastic differential equation

$$dX_t(w) = H_t(X_t(w)) \ast dB_t^i(w)$$

$$X_0(w) = e,$$

where $e$ is the unit element of $G$ and $\ast$ means the Stratonovich integral. The law of $(X_t; t \in [0, 1])$ defines a probability measure $\nu$ on $C(G) = C_e([0, 1], G)$ (i.e., the group of $G$-valued paths on $[0, 1]$, which are equal to $e$ at $t = 0$).

If $h$ is an element of the Cameron-Martin space $H_1 = H_1([0, 1], \mathcal{G})$, the left Gross-Sobolev derivative of a (cylindrical) functional $f$ on $C(G)$ is defined as

$$D_h f(x) = \frac{d}{d\lambda} f(X(\lambda h)x)|_{\lambda = 0},$$

where $X(h)$ is the solution of the differential equation:

$$\frac{dX_t(h)}{dt} = H_t(X_t(h)) \frac{dh(t)}{dt},$$

$$X_0(h) = e.$$

We then have (cf., [8])

$$(D_h f) \circ X = (\nabla(f \circ X), T^{-1}h)_H,$$
where $T(w) : H_1 \to H_1$ is a random isometry defined by

$$T(w)(h)(t) = \int_0^t AdX_s(w) \frac{dh(s)}{ds} ds.$$ 

Consequently, $D$ is a closable operator on $L^p(\nu)$ for any $p \geq 1$. Let $D^*$ be its formal adjoint (under $\nu$) and denote by $\mathcal{K}$ the operator $D^* \circ D$. It is easy to see that

$$\mathcal{K} f \circ X = L(f \circ X),$$

where $L$ is the Ornstein-Uhlenbeck operator on $C_0([0, 1], G)$. If we define the semi-group $Q_t$ on $C(G)$ with

$$Q_t f \circ X = P_t(f \circ X),$$

then we see that $-\mathcal{K}$ is the infinitesimal generator of $Q_t$. In the same way we can define the Sobolev spaces on $C(G)$ (cf., [8]) which we denote with the obvious notations $S_{p,k}(M), S(M), S'(M)$, etc., where $M$ represents a separable Hilbert space. The Theorem 1 can be translated to this setting in the following manner:

**Proposition 1** Let $f$ be a real-valued functional on $C(G)$ such that $Df \in L^\infty(\nu, H_1([0, 1], G))$. Then we have

$$\nu\{f > c\} \leq E_\nu [1_{\{c \leq Q_t f\}} \exp \frac{(c - Q_t f)^2}{2\|Df\|_\infty^2 (1 - e^{-2t})}] + \nu\{Q_t f > c\},$$

for any $t > 0$ where $E_\nu$ is the expectation with respect to $\nu$.

Letting $t \to \infty$ we obtain

**Corollary 1** Under the hypothesis of the proposition, we have

$$\nu\{|f| > c\} \leq 2 \exp \frac{(c - E_\nu[f])^2}{2\|Df\|_\infty^2}.$$ 

Let now $f$ be as in the corollary, $p > 1$ given and choose $\alpha > 0$ such that $\alpha p < 1/2\|Df\|_\infty^{-2}$. Then Hölder inequality shows that $\exp \alpha f^2$ belongs to the Sobolev space $S_{p,1}$. We have
Proposition 2 Let $F : C(G) \rightarrow \mathbb{R}^d$ be a non-degenerate random variable, i.e., $(\det(DF_i, DF_j))_{i,j=1}^n \in \cap_p L^p(\nu)$. If $q > d$ and $\alpha < \frac{1}{2p+\epsilon} \|DF\|_{\infty}^{-2}$, where $\nu > 0$ is arbitrary, $p^{-1} + q^{-1} = 1$, then $|<\varepsilon_y(F), e^{\alpha t^{2}}| < \infty$ and consequently

$$\nu\{ |\hat{f}| > t |F = y \} \leq Ce^{-\alpha t^2},$$

where $C$ is a constant depending on $y, F$ and $d$, $\varepsilon_y(F)$ is defined as an element of $S'$ and $\hat{f}$ is the redefinition of $f$ with respect to the Sobolev scale $\{S_{p,k}, p > 1, k \in \mathbb{Z}\}$.

Proof: Let us indicate first that, because of the nondegeneracy of $F$, $\varepsilon_y(F)$ is a well-defined, positive distribution. Hence it is a Radon measure on $C(G)$. In fact this follows from [6] and from the commutation relation $\mathcal{K}\phi \circ X = L(\phi \circ X)$. Moreover, as a distribution it belongs to $S_{q-1}$ for $q > d$ (cf., [13]). The proof follows then from the Tchebytchev inequality. $\|QED$}

4 An exponential moment inequality

In this section we will use the following lemma which can be found in [7], whose proof is given below for the sake of completeness:

Lemma 1 Let $X$ be a Gaussian random variable with values in $\mathbb{R}^d$, $(d \geq 1)$. Then for any convex function $U$ on $\mathbb{R}$ and any $C^1$-function $V$ on $\mathbb{R}^d$, we have the following inequality:

$$E \left[ U(V(X) - V(Y)) \right] \leq E \left[ U(\frac{\pi}{2}(V'(X), Y)_{\mathbb{R}^d}) \right],$$

where $Y$ is an independent copy of $X$.

Proof: We have

$$V(X) - V(Y) = \int_0^{\pi/2} \frac{d}{d\theta} V(X \sin \theta + Y \cos \theta) d\theta$$

$$= \int_0^{\pi/2} (V'(X \sin \theta + Y \cos \theta), X \cos \theta - Y \sin \theta)_{\mathbb{R}^d} d\theta.$$
Since $U$ is convex, we have

$$U(V(X) - V(Y)) \leq \int_0^{\pi/2} U\left(\frac{\pi}{2} V'(X \sin \theta + Y \cos \theta), X \cos \theta - Y \sin \theta\right) \frac{1}{\pi} d\theta.$$ 

Since $X \cos \theta - Y \sin \theta$ is independent of $X \sin \theta + Y \cos \theta$ and since both have the same law, which is equal to that of $X$, if we take the expectations of both sides, we obtain:

$$E[U(V(X) - V(Y))] \leq \frac{2}{\pi} \int_0^{\pi/2} E\left[U\left(\frac{\pi}{2} (V'(X), Y)\right)\right] d\theta = E\left[U\left(\frac{\pi}{2} (V'(X), Y)\right)\right].$$

\[Q.E.D\]

**Remark 2** Obviously, we have also

$$E[U(V(X) - E[V(Y)])] \leq E\left[U\left(\frac{\pi}{2} (V'(X), Y)_{\mathbb{R}^d}\right)\right].$$

**Theorem 2** Suppose that $\phi \in D_{p,1}$ for some $p > 1$ and let $U$ be any lower bounded, lower semi-continuous, convex function on $\mathbb{R}$. We have

$$E[U(\phi(w) - \phi(z))] \leq E\left[U\left(\frac{\pi}{2} I_1(\nabla \phi(w))(z)\right)\right],$$

where the expectations are taken on $W \times W$ with respect to the product measure $\mu(dw) \times \mu(dz)$, $w$ and $z$ represent two independent paths and $I_1(\nabla(w))(z)$ denotes the divergence or the first order Wiener integral of the vector field $\nabla \phi(w)$ with respect to the second Wiener measure $\mu(dz)$. In particular, on the classical Wiener space, we can express it as:

$$I_1(\nabla \phi(w))(z) = \int_0^1 d\nabla \phi(w, t) dz(t),$$

where $z = (z(t); t \in [0, 1])$ is the second Wiener process independent of the first Wiener process $w = (w(t); t \in [0, 1])$.

**Proof:** Let us first suppose that $\phi$ is a cylindrical function, i.e., a function of the form

$$\phi(w) = f(\delta h_1(w), \ldots, \delta h_n(w)),$$
where $h_i \in H$ with $(h_i, h_j)_H = \delta_{i,j}$ (Kronecker's delta) and $f$ is a smooth function on $\mathbb{R}^n$. Then $X = (\delta h_1, \ldots, \delta h_n)$ is an $\mathbb{R}^n$-valued Gaussian random variable and

$$I_1(\nabla \phi(w))(z) = \sum_{i=1}^n \partial_i f(\delta h_1, \ldots, \delta h_n) \delta h_i(z)$$

$$= (f'(X), Y),$$

where $Y = (\delta h_1(z), \ldots, \delta h_n(z))$. Hence the inequality is trivially true in this case. For the general case, let $(h_i; i \in \mathbb{N}) \subset W^*$ be complete orthonormal basis of $H$, let $V_n = \sigma\{\delta h_1, \ldots, \delta h_n\}$, i.e., the sigma algebra generated by the random variables $\{\delta h_1, \ldots, \delta h_n\}$. Let $\phi_n(w)$ be defined as

$$\phi_n = E[P_{1/n}\phi|V_n],$$

where $P_{1/n}$ is the Ornstein-Uhlenbeck semigroup on $W$. Then $\phi_n$ is a function of the form $f(\delta h_1, \ldots, \delta h_n)$, with smooth $f$. Consequently, we have

$$E[U(\phi_n(w) - \phi_n(z))] \leq E[U(\frac{\pi}{2} I_1(\nabla \phi_n(w))(z))]$$

$$= E[U(\frac{\pi}{2} I_1(E[e^{-1/n}P_{1/n}\pi_n \nabla \phi|V_n])(z))].$$

where $\pi_n$ is the orthogonal projection from $H$ onto the span$\{h_1, \ldots, h_n\}$, i.e., the subspace of $H$ spanned by $\{h_1, \ldots, h_n\}$. Note that we have, for any $h, h_i \in H$,

$$I_1(\pi_n h)(z) = \sum_{i=1}^n (h, h_i)_H I(h_i)(z) = E_z[I_1(h)|\tilde{V}_n],$$

where $E_z[|\tilde{V}_n]$ denotes the conditional expectation with respect to the second Wiener measure $\mu(dx)$ and $\tilde{V}_n$ is the copy of the sigma algebra $V_n$ on the second Wiener space. Using this identity, we obtain:

$$E \left[U(\frac{\pi}{2} I_1(E[e^{-1/n}P_{1/n}\pi_n \nabla \phi|V_n])(z))\right] =$$

$$= E \left[U(\frac{\pi}{2} e^{-1/n} E_w[E_z[I_1(P_{1/n}\nabla \phi)|\tilde{V}_n]|V_n)]\right]$$

$$\leq E \left[U(\frac{\pi}{2} e^{-1/n} I_1(P_{1/n}\nabla \phi)(w))(z))\right]$$

$$= E \left[U(\frac{\pi}{2} e^{1/n} P_{1/n} I_1(\nabla \phi(\tilde{w}))(z))\right]$$

$$\leq E \left[U(\frac{\pi}{2} e^{-1/n} I_1(\nabla \phi(w)))(z))\right],$$
where all the inequalities follow by convexity. We can interpret $e^{-1/n}$ as the multiplicative coming from $P_{1/n}$ (operating now on the second Wiener space), hence again by the convexity we have:

$$E[U(\phi_n(w) - \phi_n(z))] \leq E[U(\frac{\pi}{2} f_1(\nabla \phi(w))(z))]$$

then, an application of the Fatou Lemma completes the proof. \[QED\]

**Corollary 2** Let $\phi$ be in $D_{p,1}$ for some $p > 1$. Then we have the following inequalities:

1. $$E[\exp(\phi - E[\phi])] \leq E[\exp \frac{\pi^2}{8} |\nabla \phi|_H^2].$$

2. $$E[|\phi - E[\phi]|^{2k}] \leq \left(\frac{\pi}{2}\right)^{2k} \frac{(2k)!}{2^k k!} E[|\nabla \phi|_H^{2k}],$$

for any $k = 0, 1, 2, \ldots$

**Proof:** It is sufficient to apply the theorem with $U(x) = \exp x$ and $|x|^{2k}$.

\[QED\]

**Corollary 3** Suppose that $\phi \in D_{p,2}$ for some $p > 1$ and that $\nabla |\nabla \phi|_H \in L^\infty(\mu, H)$. Then there exists some $\lambda > 0$, such that

$$E[\exp \lambda |\phi|] \leq \infty.$$ 

**Remark 3** In particular, the hypothesis of the corollary is satisfied when $\nabla^2 \phi \in L^\infty(\mu, H \otimes H)$. This is due to the following (trivial) inequality:

$$\|\nabla |\nabla \phi|_H\|_{L^\infty(\mu, H)} \leq \|\nabla^2 \phi\|_{L^\infty(\mu, H \otimes H)}.$$ 

**Proof:** From the theorem, it is sufficient to prove that

$$E[\exp \frac{\lambda^2}{2} |\nabla \phi|_H^2] < +\infty,$$
for some \( \lambda > 0 \) and this follows from the Theorem 1 and the remark which follows it. \( \square \) 

In the Lemma 1 if we iterate the formula for \( V(X) - V(Y) \) by taking the second derivative of \( V \), we find the following inequality:

\[
E[U(V(X) - V(Y) - \frac{\pi}{2}(V'(X), Y))] \leq E[U(\frac{\pi^2}{8}[(V''(X), Y \otimes Y) - (V'(X), X)])],
\]

if we denote by \( \phi(w) \) the random variable \( V(X) \) and by \( z \) the independent copy of \( w \) as before, then we can write formally

\[
(V''(X), Y \otimes Y) - (V'(X), X) = I_2(\nabla^2 \phi(w))(z) + \text{trace}\nabla^2 \phi(w) - (\nabla \phi(w), w) = I_2(\nabla^2 \phi(w))(z) - L\phi(w),
\]

where \( I_2(\nabla^2(w))(z) \) represents the second order Wiener-Ito integral of the kernel \( \nabla \phi(w) \) with respect to the independent path \( z \) and we have used the following identity, which holds for the cylindrical functions: \( -L\phi(w) = \text{trace}\nabla^2 \phi(w) - (\nabla \phi(w), w) \). Using exactly the same method as in the proof of the theorem, with the same hypothesis, we obtain

**Proposition 3** The following identity is valid for any \( \phi \in D_{p,2}, p > 1 \):

\[
E[U(\phi(w) - \phi(z) - \frac{\pi}{2} I_1(\nabla \phi(w))(z)))] \leq E[U(\frac{\pi^2}{8} [I_2(\nabla^2 \phi(w))(z) - L\phi(w)])].
\]

From this general inequality we can also obtain particular ones as in the corollaries of the theorem.

## 5 Exponential integrability of the divergence

We begin with two lemmas which are more or less known (cf., [2, 4])

**Lemma 2** Let \( \phi \in L^p(\mu), p > 1 \), then, for any \( h \in H, t > 0 \), we have

\[
\nabla_h P_t \phi(x) = \frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \int_W \phi(e^{-t}x + \sqrt{1 - e^{-2t}}y)\delta_h(y) \mu(dy)
\]

almost surely, where \( \nabla_h P_t \phi \) represents \( (\nabla P_t \phi, h)_H \).
Proof: From Mehler's formula (cf., [1]), we have
\[
\nabla_h P_t \phi(x) = \frac{d}{d\lambda} \bigg|_{\lambda=0} \int_W \phi(e^{-t}(x + \lambda h) + \sqrt{1-e^{-2t}y}) \mu(dy) \\
= \frac{d}{d\lambda} \bigg|_{\lambda=0} \int_W \phi(e^{-t}x + \sqrt{1-e^{-2t}y} + \frac{\lambda e^{-t}}{\sqrt{1-e^{-2t}}} h) \mu(dy) \\
= \frac{d}{d\lambda} \bigg|_{\lambda=0} \int_W \phi(e^{-t}x + \sqrt{1-e^{-2t}y}) \frac{\lambda e^{-t}}{\sqrt{1-e^{-2t}}} \delta h(y) \mu(dy) \\
= \int_W \phi(e^{-t}x + \sqrt{1-e^{-2t}y}) e^{-t} \frac{\delta h(y)}{\sqrt{1-e^{-2t}}} \mu(dy),
\]

where $\delta h$ denotes $\exp(\delta h - 1/2|h|^2_t)$. \textit{\textit{QED}}

Lemma 3 Let $\xi \in L^p(\mu, H)$, $p > 1$ and for $(x,y) \in W \times W$, $t \geq 0$, define $R_t(x,y)$ as $e^{-t}x + (1 - e^{-2t})^{1/2}y$ and $S_t(x,y)$ as $(1 - e^{-2t})^{1/2}x - e^{-t}y$ (note that $S_t(x,y)$ and $R_t(x,y)$ are independant, identically distributed Gaussian random variables on $(W \times W, \mu(dx) \times \mu(dy))$). We have then the following identity:
\[
P_t \delta \xi(x) = \frac{e^{-t}}{\sqrt{1-e^{-2t}}} \int_W I_1(\xi(R_t(x,y)))(S_t(x,y)) \mu(dy),
\]

where $I_1(\xi(R_t(x,y)))(S_t(x,y))$ denotes the first order Wiener integral of $\xi(R_t(x,y))$ with respect to the independent path $S_t(x,y)$ under the product measure $\mu(dx) \times \mu(dy)$.

Proof: For the typographical facility, we shall denote in the sequel by $e(t)$ the function $\exp(-t/(1-exp(-2t))^{1/2}$. Let now $\phi$ be an element of $D = \cap_{p,k} D_{p,k}$, we have, via duality and using the preceding lemma
\[
<P_t \delta \xi, \phi >= <\xi, \nabla P_t \phi>
= \sum_{i=1}^{\infty} <\xi_i, \nabla h_i P_t \phi>
= \sum_{i} e(t) E \left[ \int_W \xi_i(x) \phi(R_t(x,y)) \delta h_i(y) \mu(dy) \right]
\]

where $(h_i; i \in N) \subset W'$ is a complete orthonormal basis in $H$, $\xi_i$ is the component of $\xi$ in the direction of $e_i$ and $<.,.>$ represents the duality
bracket corresponding to the dual pairs \((D, D')\) or \((D(H), D'(H))\), where 
\[ D(H) = \cap_{p,h} D_{p,h}(H) \] (cf., [1]). Let us make the following change of variables, which preserves \(\mu \times \mu\) :
\[
\begin{align*}
  x & \mapsto e^{-t} x + \sqrt{1 - e^{-2t}} y \\
  y & \mapsto \sqrt{1 - e^{-2t}} x - e^{-t} y.
\end{align*}
\]

We then obtain
\[
\langle P_t \delta \xi, \phi \rangle = e(t) \int_W \phi(x) I_1(\xi(R_t(x, y)))(S_t(x, y)) \mu(dx) \mu(dy),
\]
for any \(\phi \in D\) and the lemma follows from the density of \(D\) in all \(L^p\)-spaces.

\[\|QED\]

We are now ready to prove the following

**Theorem 3** Let \(\beta > 1/2\) and suppose that \(\eta \in D_{2,2\beta}(H)\). Then we have
\[
E[\exp \delta \eta] \leq E[\exp \alpha(2I + L)^\beta \eta]^2_H,
\]
for any
\[
\alpha \geq \frac{1}{2} \left( \frac{1}{\Gamma(\beta)} \int_{R^+} \frac{t^{\beta-1} e^{-2t}}{\sqrt{1 - e^{-2t}}} dt \right)^{-2},
\]
where \(L\) denotes the Ornstein-Uhlenbeck or the number operator on \(W\).

**Proof**: Let \(\xi = (2I + L)^\beta \eta\), then the above inequality is equivalent to
\[
E[\exp(I + L)^{-\beta} \delta \xi] \leq E[\exp \alpha |\xi|^2_H],
\]
where we have used the identity
\[
(I + L)^{-\beta} \delta \xi = \delta (2I + L)^{-\beta} \xi.
\]
We have from the resolvent identity (cf.,[1]) and from the Lemma 3,
\[
(I + L)^{-\beta} \delta \xi = \frac{1}{\Gamma(\beta)} \int_{R^+} t^{\beta-1} e^{-t} P_t \delta \xi dt
\]
\[
= \int_{R^+} \int_W \frac{e^{-t}}{\Gamma(\beta) \sqrt{1 - e^{-2t}}} t^{\beta-1} e^{-t} I_1(\xi(R_t(x, y)))(S_t(x, y)) \mu(dy) dt.
\]
Let
\[ \lambda_0 = \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}^+} \frac{t^{\beta-1}e^{-2t}}{\sqrt{1 - e^{-2t}}} dt \]
and
\[ \nu(dt) = 1_{\mathbb{R}^+}(t) \frac{1}{\lambda_0 \Gamma(\beta)} \frac{t^{\beta-1}e^{-2t}}{\sqrt{1 - e^{-2t}}} dt. \]
Then
\[
E[\exp(I + L)^{-\beta}\delta\xi] = E \left[ \exp \lambda_0 \int_{\mathbb{R}^+} \int_{\mathbb{W}} I_1(\xi(R_t(x,y)))(S_t(x,y)) \mu(dx) \nu(dt) \right] 
\leq \int_{\mathbb{R}^+} \int_{\mathbb{W}} \int_{\mathbb{W}} \exp[\lambda_0 I_1(\xi(R_t(x,y)))(S_t(x,y)) \mu(dx) \mu(dy) \nu(dt) 
= E \left[ \exp \frac{\lambda_0^2}{2} |\xi|^2_H \right].
\]

\[ \boxed{QED} \]

In the applications, we need also to control the moments like \( E[\exp |\nabla \eta|^2] \) (cf., [12]), where \(|.|^2\) represents the Hilbert-Schmidt norm. The following result gives an answer to this question:

**Proposition 4** Suppose that \( \beta > 1/2 \) and that \( \eta \in D_{2,2\beta}(H) \). Then we have
\[
E[\exp |\nabla \eta|^2] \leq E[\exp c(I + L)^{\beta}|\xi|^2_H],
\]
for any
\[
c \geq \left[ \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}^+} t^{\beta-1}e^{-2t} (1 - e^{-2t})^{-1/2} dt \right]^2,
\]
for \( \beta = 1 \) we have \( c \geq 1/4 \).

**Proof:** Setting \( \xi = (I + L)^{\beta}\eta \), it is sufficient to show that
\[
E[\exp |\nabla(I + L)^{-\beta}\xi|^2] \leq E[\exp c|\xi|^2_H].
\]
Let \( (E_i; i \in \mathbb{N}) \) be a complete, orthonormal basis of \( H \otimes H \) (recall that \( H \otimes H \) denotes the completion of the tensor product of \( H \) with itself under the Hilbert-Schmidt topology), then
\[
|\nabla \eta|^2 = \sum_i K_i(\nabla \eta, E_i)_2,
\]

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where \((.,.)_2\) is the scalar product in \(H \otimes H\) and \(K_i = (\nabla \eta, E_i)_2\). Let \(\theta(t)\) be the function
\[
\frac{1}{\Gamma(\beta)} t^{\beta-1} e^{-2t(1-e^{-2t})^{-1/2}}
\]
and let \(\gamma_0 = \int_0^\infty \theta(t)\,dt\). From the Lemma 2, we have
\[
|\nabla \eta(x)|_2^2 = |\nabla (I + L)^{-\beta} \xi(x)|_2^2
\]
\[
= \sum_i K_i(x) \int_{R^+} \theta(t) \int_W (I_i(E_i)(y), \xi(R_i(x, y)))_{H\mu}(dy)\,dt
\]
\[
= \int_{R^+ \times W} \theta(t)(I_i(\nabla \eta(x))(y), \xi(R_i(x, y)))_{H\mu}(dy)\,dt
\]
\[
\leq \int_{R^+} \theta(t)
\left((\int_W |I_i(\nabla \eta(x))(y)|_H^2\mu(dy))^{1/2}(\int_W |\xi(R_i(x, y))|^2_H\mu(dy))^{1/2}\,dt\right)
\]
\[
= \int_{R^+} \theta(t)|\nabla \eta(x)|_2 P_t(|\xi|^2_H)^{1/2}\,dt,
\]
where \(I_i(\nabla \eta(x))(y)\) denotes the first order Wiener integral of \(\nabla \eta(x)\) with respect to the independent path (or variable) \(y\). Consequently
\[
|\nabla \eta|_2 \leq \int_{R^+} \theta(t)(P_t(|\xi|^2_H))^{1/2}\,dt.
\]

Then
\[
E[\exp |\nabla \eta|_2^2] \leq E\left[\exp \int_{R^+} \gamma_0^2 P_t(|\xi|^2_H)^{\theta(t)}\,dt\right]
\]
\[
\leq E\int_{R^+} \frac{\theta(t)}{\gamma_0} \exp \gamma_0^2 P_t(|\xi|^2_H)\,dt
\]
\[
\leq E\int_{R^+} \frac{\theta(t)}{\gamma_0} \exp \gamma_0^2 |\xi|^2_H\,dt
\]
\[
= E\left[\exp \gamma_0^2 |\xi|^2_H\right].
\]

\(\blacksquare\)

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References


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