ISBN 82–553–0843–1
Pure Mathematics

No.16
June 1993

PRIMITIVE ACTIONS AND THE SOPHUS LIE PROBLEM

by

B. Komrakov*

PREPRINT SERIES – Matematisk institutt, Universitetet i Oslo

* Submitted under the 1993 cooperation project between the Department of Mathematics, University of Oslo, and the International Sophus Lie Center.

PRIMITIVE ACTIONS AND THE SOPHUS LIE PROBLEM

CONTENTS

INTRODUCTION

In two crucial mathematical problems related to the applications—the study and solution of differential equations and the constructions of mathematical models for physical theories—the ideas of symmetry developed by the great Norwegian mathematician Sophus Lie play a decisive role. The methods and ideas related to the transformation groups now called Lie groups have drastically changed the aspect of modern mathematics and physics. And this circumstance is basically connected with the constructive approach that is introduced into mathematics by the use of Lie groups.

A whole series of problems, mostly having to do with the classification and study of nonlinear differential equations (where, incidentally, continuous groups first appeared), were reduced by Sophus Lie to the description of subalgebras in Lie algebras and their realization in the form of Lie algebras of vector fields on manifolds. Central in this group of ideas was the following problem, set by Sophus Lie and solved by him in small dimensions: the problem of classifying transitive and primitive (i.e., not possessing invariant foliations) actions.

Understood locally, this problem is equivalent to that of classifying (up to conjugation) the maximal subalgebras of Lie algebras over the fields $\mathbb{C}$ and $\mathbb{R}$. From the global viewpoint, this problem is equivalent to classifying (up to conjugation) the maximal subgroups of complex and real lie groups.

In this text, the Sophus Lie problem is solved in its most general formulation, namely, the text gives *the classification (up to conjugation) of maximal nondiscrete* *subgroups of complex and real Lie groups.*

A unified approach to both problem (over $\mathbb{C}$ and over $\mathbb{R}$), based on the author's notion of *almost primitive subalgebra* (examples of which are any Lie algebras of any maximal subgroup of any real or complex Lie group), is presented. In this work all such subalgebras (up to conjugation) are found; among them, all those which actually correspond to maximal subgroups are distinguished.

The main results of the text are published in [23–26].

The problem of describing primitive (i.e., not possessing invariant foliations) transitive and effective actions of connected Lie groups was stated by Sophus Lie and solved by him in dimensions 1,2 and 3 in [1].

It is natural to call primitive all stationary subalgebras corresponding to primitive, transitive and effective actions of connected Lie groups (without any connectivity requirement on their subgroups).

If a Lie algebra $\bar{\mathfrak{g}}$ is not simple or if $\bar{\mathfrak{g}}$ is simple and a subalgebra $\mathfrak{g}$ of $\bar{\mathfrak{g}}$ is not reductive, then it is quite easy to describe all primitive subalgebras $\mathfrak{g}$ of $\bar{\mathfrak{g}}$.

Thus the problem reduces to describing primitive reductive subalgebras of simple Lie algebras over $\mathbb{C}$ and $\mathbb{R}$.

Maximal semisimple subalgebras of simple Lie algebras over the field $\mathbb{C}$ were classified by E.B. Dynkin [6,7], maximal reductive subalgebras of maximal rank were classified by A. Borel and J. de Siebenthal [8]. M. Golubitsky and B. Rothshild [9] classified primitive nonmaximal reductive subalgebras of maximal rank of simple Lie algebras over $\mathbb{C}$, I.V. Chekalov [10] considered the general situation over $\mathbb{C}$ (irrespective of rank).

The classification of maximal subalgebras of simple Lie algebras over $\mathbb{R}$ (the Sophus Lie problem in its local formulation) remained unsolved. Some examples of maximal subalgebras in classical Lie algebras over $\mathbb{R}$ were presented in the papers [11,12,13].

This problem (the description of primitive and maximal subalgebras of simple Lie algebras) is intimately connected with the classification of all reductive subalgebras of simple Lie algebras up to conjugation; this latter problem is also a classical one, and recently has been actively stimulated by numerous new applications in mathematics, mechanics and physics.

This problem was studied over the field of complex numbers by A.I. Maltsev [14] and E.B. Dynkin [6,7]. Over the field $\mathbb{R}$, E. Cartan began its study [15] by classifying irreducible subalgebras of $\mathfrak{gl}(n,\mathbb{R})$, $n \leqslant 12$. These results were strengthened and simplified by N. Ivahori [16]. Simple subalgebras in simple classical real Lie algebras were described by F.I. Karpelevich [17]. B.P. Komrakov [23] generalized these results to arbitrary reductive subalgebras. Real forms of symmetric subalgebras were found by A.S. Fedenko [18] and M. Berger [19].

Among exceptional Lie algebras, only the symmetric ones (M. Berger [19]), fixed points of third order automorphisms (A. Gray [20]), and S-algebras in the sense of E.B. Dynkin (B.P. Komrakov [21]) have been described in the real case.

The complete solutions of these two important problems—the classification of primitive reductive subalgebras and that of arbitrary subalgebras of real simple Lie algebras—turn out to be closely connected.

Namely, there exists a natural class of reductive subalgebras in complex semisimple Lie algebras (which we have called almost primitive) such that a description of all real forms of subalgebras of this class yields, in principle, the description of all real forms of any reductive subalgebra containing all primitive reductive real subalgebras among its real forms.

If the Lie algebra $\bar{\mathfrak{g}}$ is semisimple, then a primitive subalgebra may be characterized by the condition that the subgroup $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g}) \cap \mathrm{Int}(\bar{\mathfrak{g}})$ of inner automorphisms of the Lie algebra $\bar{\mathfrak{g}}$ that preserve the subalgebra $\mathfrak{g}$ is maximal in $\mathrm{Int}(\bar{\mathfrak{g}})$, the group of inner automorphisms of the Lie algebra $\bar{\mathfrak{g}}$. From this point of view almost primitive subalgebras $\mathfrak{g}$ of the Lie algebra $\bar{\mathfrak{g}}$ are characterized by the maximality of the subgroup $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g})$ in the group $\mathrm{Aut}(\bar{\mathfrak{g}})$ of all automorphisms of the Lie algebra $\bar{\mathfrak{g}}$.

The class of almost primitive subalgebras apparently seems more natural than the class of primitive ones. For example, among the symmetric nonsemisimple subalgebras (which are all almost primitive) there are nonprimitive ones:

$$A_k + A_{l-k-1} + \mathbb{C} \subset A_l, \; A_{l-1} + \mathbb{C} \subset D_l \; (l = 2k+1), \; D_5 + \mathbb{C} \subset E_6.$$

This class, been closely connected with the extension of automorphisms, may be conveniently characterized as the class of primitive subalgebras in intrinsic terms.

The subalgebra $\mathfrak{g}$ of the Lie algebra $\bar{\mathfrak{g}}$ will be called primitive (respectively, almost primitive) if it is maximal among all subalgebras that are stable with respect to $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g}) \cap \mathrm{Int}(\mathfrak{g})$ (respectively, $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g})$).

The importance of the class of almost primitive complex subalgebras introduced above can also be explained by the following circumstance. When we know the real forms for these subalgebras, this allows us to describe, in principle, the real forms of arbitrary reductive subalgebras.

The main problem which is entirely solved in this work is the classification of almost primitive subalgebras of real Lie algebras up to conjugation, the primitive and maximal subalgebras being specified (i.e., the Sophus Lie problem in its most complete formulation).

Before achieving this , it is necessary to conclusively carry out the classification of almost primitive subalgebras of complex Lie algebras and answer a series of other important questions, which in our context must be considered auxiliary.

1. Maximality, primitivity, almost primitivity

Suppose $\bar{G}$ is a Lie group which acts on a manifold M. ($\bar{G}$ and M can be thought of as in either the real or complex category).

Definition 1. A *k-foliation* on M is a collection of k-dimensional immersed submanifolds $\{F_m\}_{m \in M}$ such that for any m, $m\prime \in M$, we have

(a) $m \in F_m$,

(b) F_m is connected and has a countable base for its topology,

(c) either $F_m = F_{m\prime}$ or $F_m \cap F_{m\prime} = \emptyset$.

The unique submanifold of the foliation containing the point m is called the *leaf through m*.

Definition 2. Let F be a foliation on M. We say that F is *invariant under the action* of $\bar{G}$, if for any $x \in \bar{G}$, $m \in M$,

$$x.F_m = F_{x.m},$$

i.e., the action of $\bar{G}$ on M preserves the leaves of the foliation.

There exist two trivial foliations on any manifold; namely foliation of the manifold

(i) into points, or

(ii) into connected components.

These foliations are invariant under any Lie group action.

Definition 3. The action of $\bar{G}$ on M is called *primitive*, if the only foliations on M invariant under the action of $\bar{G}$ are the trivial foliations.

The problem posed by Sophus Lie is to classify up to equivalence all of the primitive transitive and effective actions of real Lie groups on manifolds.

By standard results of the theory of homogeneous spaces, the problem is equivalent to determining all of the pairs $(\bar{G},\ G)$, where $\bar{G}$ is a Lie group and G is a closed Lie subgroup of $\bar{G}$ such that

(a) $\bar{G}$ acts primitively on $\bar{G}/G$, and

(b) G contains no proper normal subgroups of $\bar{G}$.

In this case we say that G is a *primitive* subgroup of $\bar{G}$. Thus the Sophus Lie problem is reduced to describing the set of primitive subgroups of real Lie groups.

The following result was obtained by Golubitsky [3].

Statement.

(i) Let G be a closed maximal Lie subgroup of $\bar{G}$ which contains no proper normal subgroups of $\bar{G}$. Then G is primitive.

(ii) Let G be a nondiscrete primitive subgroup of $\bar{G}$, and G° the connected component of the identity in $\bar{G}$. Then

$$\operatorname{Norm}_{\bar{G}} G^\circ = \left\{ x \in \bar{G} \;\middle|\; xG^\circ = G^\circ x \right\}$$

is a closed maximal Lie subgroup of $\bar{G}$. Moreover

$$\dim G^\circ = \dim G = \dim \operatorname{Norm}_{\bar{G}} G^\circ.$$

Definition 4. Let $\mathfrak{g}$ be a proper subalgebra of a Lie algebra $\bar{\mathfrak{g}}$. Then $\mathfrak{g}$ is called *primitive*, if

(a) $\mathfrak{g}$ contains no proper ideals of $\bar{\mathfrak{g}}$,

(b) there exist a Lie group $\bar{G}$ and a closed maximal Lie subgroup G of $\bar{G}$ such that $L(\bar{G}) = \bar{\mathfrak{g}}$ and $L(G) = \mathfrak{g}$.

(For an arbitrary Lie group G, we denote by $L(G)$ the corresponding Lie algebra).

This definition immediately implies that primitive subalgebras are exactly the isotropy subalgebras corresponding to some primitive effective and transitive action.

It is obvious that any maximal subalgebra $\mathfrak{g}$ of $\bar{\mathfrak{g}}$ containing no proper ideals of $\bar{\mathfrak{g}}$ is primitive. Moreover if $\bar{G}$ and G are connected Lie groups, then any primitive subalgebra $\mathfrak{g}$ of $\bar{\mathfrak{g}}$ is maximal and contains no proper ideals of $\bar{\mathfrak{g}}$. However, if we do not require connectedness, this will not true.

For example, let $\bar{G} = SL(2, \mathbb{R})$ and let G be the maximal of all Lie subgroups corresponding to the Cartan subalgebra of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$:

$$G = \left\{ \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^i \middle| a \in \mathbb{R};\ i = 0, 1 \right\}.$$

Then G is a closed maximal Lie subgroup of $\bar{G}$, whereas the Cartan subalgebra is not maximal in $\mathfrak{sl}(2, \mathbb{R})$, since it belongs to the Borel subalgebra:

$$\left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \middle| x \in \mathbb{R} \right\} \subset \left\{ \begin{pmatrix} x & y \\ 0 & -x \end{pmatrix} \middle| x, y \in \mathbb{R} \right\} \subset \mathfrak{sl}(2, \mathbb{R}).$$

Thus, in terms of Lie algebras, the Sophus Lie problem is equivalent to the classification of all primitive subalgebras of real Lie algebras.

Assume $\bar{\mathfrak{g}}$ is not simple. Then any primitive subalgebra $\mathfrak{g}$ of the Lie algebra $\bar{\mathfrak{g}}$ is maximal and has the form given by the following

Statement. *Suppose $\mathfrak{g}$ is a primitive subalgebra of the Lie algebra $\bar{\mathfrak{g}}$.*

(i) If $\bar{\mathfrak{g}}$ is not semisimple, then there exists an abelian ideal $\mathfrak{a}$ such that $\bar{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{a}$ and $\mathfrak{g}$ acts faithfully and irreducibly on $\mathfrak{a}$.

(ii) If $\bar{\mathfrak{g}}$ is semisimple (but not simple), then there exists a simple Lie algebra $\mathfrak{a}$ such that $\bar{\mathfrak{g}} = \mathfrak{a} \oplus \mathfrak{a}$ and $\mathfrak{g} = \{(x, x) | x \in \mathfrak{a}\}$.

This result is true for both $\mathbb{R}$ and $\mathbb{C}$ and was obtained by V.V. Morozov [2]. It was proved by M. Golubitsky [3] that primitivity implies maximality.

Furthermore if $\bar{\mathfrak{g}}$ is simple and $\mathfrak{g}$ is a nonreductive subalgebra of $\bar{\mathfrak{g}}$, then the primitivity of $\mathfrak{g}$ implies that $\mathfrak{g}$ is a maximal parabolic subalgebra of $\bar{\mathfrak{g}}$. (V.V. Morozov [4], F.I. Karpelevich [5], M. Golubitsky [3]).

Thus the problem is reduced to describing of all primitive reductive subalgebras of simple Lie algebras.

Suppose $\mathfrak{g}_1, \dots, \mathfrak{g}_k$ is a set of subalgebras of a Lie algebra $\bar{\mathfrak{g}}$. By $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g}_1, \dots, \mathfrak{g}_k)$ denote the group of all automorphisms of $\bar{\mathfrak{g}}$ preserving all subalgebras $\mathfrak{g}_1, \dots, \mathfrak{g}_k$. By $\mathrm{Int}(\bar{\mathfrak{g}}, \mathfrak{g}_1, \dots, \mathfrak{g}_k)$ denote the connected component of the identity in $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g}_1, \dots, \mathfrak{g}_k)$.

When the Lie algebra $\bar{\mathfrak{g}}$ is semisimple, it is possible to introduce the concept of primitivity of a subalgebra $\mathfrak{g}$ in a different way.

Definition 4′. Let $\bar{\mathfrak{g}}$ be a semisimple Lie algebra and $\mathfrak{g}$ a proper subalgebra of $\bar{\mathfrak{g}}$. The subalgebra $\mathfrak{g}$ is called *primitive*, if

(a) $\mathfrak{g}$ contains no proper ideals of $\bar{\mathfrak{g}}$,

(b) $\mathfrak{g}$ is the maximal of all subalgebras invariant under $\mathrm{Int}(\bar{\mathfrak{g}}, \mathfrak{g})$.

However the class of primitive subalgebras is in many respects inconvenient to deal with. For example, primitivity of a real subalgebra $\mathfrak{g}$ in $\bar{\mathfrak{g}}$ does not necessarily imply primitivity of the subalgebra $\mathfrak{g}^{\mathbb{C}}$ in $\bar{\mathfrak{g}}^{\mathbb{C}}$. In other words, not all primitive subalgebras $\mathfrak{g}$ of a real Lie algebra $\bar{\mathfrak{g}}$ can be obtained as a real form of a primitive subalgebra of a complex Lie algebra.

In this context, we introduce a new class of subalgebras. This class generalizes the class of primitive subalgebras.

Definition 5. Let $\bar{\mathfrak{g}}$ be a semisimple Lie algebra and $\mathfrak{g}$ a proper subalgebra of $\bar{\mathfrak{g}}$. We say that the subalgebra $\mathfrak{g}$ is *almost primitive*, if

(a) $\mathfrak{g}$ contains no proper ideals of $\bar{\mathfrak{g}}$,

(b) $\mathfrak{g}$ is the maximal of all subalgebras invariant under $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g})$.

The classes of primitive and almost primitive subalgebras can be also characterized in terms of maximality of the corresponding groups of automorphisms.

Statement. *Suppose $\bar{\mathfrak{g}}$ a semisimple Lie algebra and $\mathfrak{g}$ is a proper subalgebra of $\bar{\mathfrak{g}}$ containing no proper ideals of $\bar{\mathfrak{g}}$. The subalgebra $\mathfrak{g}$ is primitive (almost primitive) if and only if the subgroup* $\mathrm{Int}(\bar{\mathfrak{g}}, \mathfrak{g})$ $(\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g}))$ *is maximal in* $\mathrm{Int}(\bar{\mathfrak{g}})$ $(\mathrm{Aut}(\bar{\mathfrak{g}}))$.

The main advantage of the class of almost primitive subalgebras is that they are closed under complexification.

Theorem. *Let $\bar{\mathfrak{g}}$ be a real simple Lie algebra and $\mathfrak{g}$ an almost primitive reductive subalgebra of $\bar{\mathfrak{g}}$. Then $\mathfrak{g}^{\mathbb{C}}$ is an almost primitive subalgebra of $\bar{\mathfrak{g}}^{\mathbb{C}}$.*

The theorem allows to solve our problem in the following way:

1° to classify all of the almost primitive reductive subalgebras of complex simple Lie algebras;

2° to find the real forms of the algebras and subalgebras obtained in 1°;

3° to select all primitive subalgebras of real simple Lie algebras from the set of subalgebras obtained in 2°.

In the process of classification of almost primitive reductive subalgebras of complex simple Lie algebras, classical and exceptional Lie algebras are examined separately. But the classical Lie algebra of type D_4 $(\mathfrak{so}(8, \mathbb{C}))$ is examined together with exceptional Lie algebras.

In the sequel we shall always assume that $\bar{\mathfrak{g}}$ is a simple Lie algebra and $\mathfrak{g}$ is a reductive subalgebra of $\mathfrak{g}$.

2. Almost primitive subalgebras of complex Lie algebras

We say that a pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is *maximal, primitive,* or *almost primitive*, if the subalgebra $\mathfrak{g}$ has the corresponding characterictic.

Theorem. *Let $\mathfrak{g}$ be an almost primitive reductive subalgebra of a complex simple Lie algebra $\bar{\mathfrak{g}}$. Then $\mathfrak{g}$ belongs to one of the following disjoint classes of subalgebras of $\bar{\mathfrak{g}}$:*

(i) the class of simply embedded subalgebras (i.e., the class of centralizers of semisimple elements of $\bar{\mathfrak{g}}$);

(ii) the class of semisimple subalgebras coinciding with their normalizers.

The theorem gives a preliminary description of the subalgebras we are interested in. The classical and exceptional will be examined separately.

2.1. Classical case.

Suppose $\bar{\mathfrak{g}}$ is a classical complex Lie algebra of one of the followings types: A_l $(l \geqslant 1)$, B_l $(l \geqslant 3)$, C_l $(l \geqslant 2)$, D_l $(l \geqslant 5)$. These are the Lie algebras $\mathfrak{sl}(l+1, \mathbb{C})$, $\mathfrak{so}(2l+1, \mathbb{C})$, $\mathfrak{sp}(2l, \mathbb{C})$, and $\mathfrak{so}(2l, \mathbb{C})$ respectively.

Let us describe some classes of subalgebras of classical simple Lie algebras:

a) Suppose $V = V_1 \oplus \cdots \oplus V_r$ is a direct sum of vector spaces. Let us identity the Lie algebras $\mathfrak{sl}(V_i)$, $1 \leqslant i \leqslant r$, with the corresponding subalgebras of $\mathfrak{sl}(V)$. By $\mathfrak{z}$ denote the following commutative subalgebra of $\mathfrak{sl}(V)$:

$$\mathfrak{z} = \{t_1 \cdot \mathrm{Id}_{V_1} + \cdots + t_r \cdot \mathrm{Id}_{V_r} \mid t_1 + \cdots + t_r = 0 \}.$$

Then

$$\mathfrak{g} = \mathfrak{z} \oplus \sum_{i=1}^{r} \mathfrak{sl}(V_i)$$

is a simply embedded subalgebra of the Lie algebra $\bar{\mathfrak{g}} = \mathfrak{sl}(V)$. In addition the $\mathfrak{g}$-module V is completely reducible. In matrix form we have:

$$\bar{\mathfrak{g}} = \mathfrak{sl}(n,\ \mathbb{C}),\ \text{where } n = \dim\ V;$$

$$\mathfrak{g} = \left\{ \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_r \end{pmatrix},\ A_i \in \mathfrak{gl}(n_i,\ \mathbb{C}),\ 1 \leqslant i \leqslant r;\ \sum_{i=1}^{r} \mathrm{tr}\ A_i = 0 \right\},$$

where $n_i = \dim V_i$, $1 \leqslant i \leqslant r$.

For both $\mathfrak{so}(n,\ \mathbb{C})$ and $\mathfrak{sp}(2n,\ \mathbb{C})$, there exist similar subalgebras which consist of several "blocks".

b) Suppose $V = V_1 \otimes \cdots \otimes V_r$ is a tensor product of vector spaces. Let us identity the Lie algebras $\mathfrak{sl}(V_i)$, $1 \leqslant i \leqslant r$, with the following subalgebras of $\mathfrak{sl}(V)$:

$$\mathrm{Id}_{V_1} \otimes \cdots \otimes \mathrm{Id}_{V_{i-1}} \otimes \mathfrak{sl}(V_i) \otimes \mathrm{Id}_{V_{i+1}} \otimes \cdots \otimes \mathrm{Id}_{V_r}.$$

Then $\mathfrak{g} = \sum_{i=1}^{r} \mathfrak{sl}(V_i)$ is a semisimple subalgebra of the Lie algebra $\bar{\mathfrak{g}} = \mathfrak{sl}(V)$ such that $\mathfrak{g}$ coincides with its normalizer. In addition the $\mathfrak{g}$ -module V is irreducible. In matrix form we have:

$$\bar{\mathfrak{g}} = \mathfrak{sl}(n,\ \mathbb{C}),\ \text{where } n = \dim\ V;$$

$$\mathfrak{g} = \left\{ \sum_{i=1}^{r} E_{n_1} \otimes \cdots \otimes E_{n_{i-1}} \otimes A_i \otimes E_{n_{i+1}} \otimes \cdots \otimes E_{n_r} \,\middle|\, A_i \in \mathfrak{sl}(n_i,\ \mathbb{C}),\ 1 \leqslant i \leqslant r \right\},$$

where $n_i = \dim\ V_i$, $1 \leqslant i \leqslant r$. (The action of the matrix $X_1 \otimes \cdots \otimes X_r$ on the vector $v_1 \otimes \cdots \otimes v_r$ is defined by

$$(X_1 \otimes \cdots \otimes X_r) \cdot (v_1 \otimes \cdots \otimes v_r) = (X_1 v_1) \otimes \cdots \otimes (X_r v_r) \quad).$$

Similar subalgebras of the Lie algebras $\mathfrak{so}(n,\ \mathbb{C})$ and $\mathfrak{sp}(2n,\ \mathbb{C})$ can be constructed in an analogous way. However in doing so it must be taken into account that if A_i $(1 \leqslant i \leqslant r)$ are symmetric matrices and B_j $(1 \leqslant j \leqslant l)$ are skew-symmetric matrices, then the matrix

$$X = A_1 \otimes \cdots \otimes A_r \otimes B_1 \otimes \cdots \otimes B_l$$

is symmetric whenever $(-1)^l = 1$ and is skew-symmetric whenever $(-1)^l = -1$. This immediately follows from

$$ {}^tX = {}^tA_1 \otimes \cdots \otimes {}^tA_r \otimes {}^tB_1 \otimes \cdots \otimes {}^tB_l = (-1)^l A_1 \otimes \cdots \otimes A_r \otimes B_1 \otimes \cdots \otimes B_l = (-1)^l X. $$

Following E.B. Dynkin, we say that a subalgebra of a classical Lie algebra is *reducible (irreducible)*, if its canonical representation is reducible (irreducible).

If $\mathfrak{g}$ is an almost primitive reductive subalgebra of $\bar{\mathfrak{g}}$, then it belongs to one of the following pairwise disjoint classes:

(a) the class of simply imbedded (reducible) subalgebras;

(b) the class of reducible semisimple subalgebras;

(c) the class of irreducible nonsimple subalgebras;

(d) the class of irreducible simple subalgebras.

Indeed, if the center of $\mathfrak{g}$ is not the zero set, then $\mathfrak{g}$ coincides with the centralizer of its center. If $\mathfrak{g}$ is semisimple, then it is clear that $\mathfrak{g}$ belongs to one of the classes (b), (c), or (d).

We say that a pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is *almost primitive* and *belongs to the class* (a), (b), (c), or (d), if $\mathfrak{g}$ is an almost primitive subalgebra of $\bar{\mathfrak{g}}$ and belongs to the corresponding class.

In the process of classifying of almost primitive pairs belonging to the class (d), we restrict ourselves to the case when $\mathfrak{g}$ is not maximal in $\bar{\mathfrak{g}}$.

Lemma 1. *Any almost primitive pair belonging to the class (a) is equivalent to one of the following pairs:*

$$
\begin{array}{lll}
(1a)\bar{\mathfrak{g}} = \mathfrak{sl}(n+m,\mathbb{C}), & \mathfrak{g} = \mathfrak{sl}(n,\mathbb{C}) \oplus \mathfrak{sl}(m,\mathbb{C}) \oplus \mathbb{C}, & 1 \leqslant n \leqslant m; \\
(2a)\bar{\mathfrak{g}} = \mathfrak{sl}(nr,\mathbb{C}), & \mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{C}^{r-1}, & r \geqslant 3,\ n \geqslant 1; \\
(3a)\bar{\mathfrak{g}} = \mathfrak{so}(2n,\mathbb{C}), & \mathfrak{g} = \mathfrak{gl}(n,\mathbb{C}), & n \geqslant 5; \\
\quad\ \ \bar{\mathfrak{g}} = \mathfrak{sp}(2n,\mathbb{C}), & \mathfrak{g} = \mathfrak{gl}(n,\mathbb{C}), & n \geqslant 2; \\
(4a)\bar{\mathfrak{g}} = \mathfrak{so}(n+2,\mathbb{C}), & \mathfrak{g} = \mathfrak{so}(n,\mathbb{C}) \oplus \mathbb{C}, & n \geqslant 5,\ n \neq 6; \\
(5a)\bar{\mathfrak{g}} = \mathfrak{so}(2n,\mathbb{C}), & \mathfrak{g} = \mathbb{C}^n, & n \geqslant 5.
\end{array}
$$

The subalgebra from (1a) consists of two blocks; the subalgebra from (2a) consists of several (more than two) blocks of the same length; the subalgebra from (4a) consists of two blocks and one of these blocks is of length n (the condition $n \neq 6$ is imposed, since $\mathfrak{so}(8,\mathbb{C})$ is not a classical Lie algebra); in (5a) $\mathfrak{g}$ is a Cartan subalgebra.

Consider in detail the case (3a). If $\bar{\mathfrak{g}} = \mathfrak{sp}(2n,\mathbb{C})$, then it can be assumed that

$$ \bar{\mathfrak{g}} = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \middle| A, B, C \in \mathfrak{gl}(n,\mathbb{C});\ B = {}^tB,\ C = {}^tC \right\}. $$

Then

$$ \mathfrak{g} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix} \middle| A \in \mathfrak{gl}(n,\mathbb{C}) \right\} \cong \mathfrak{gl}(n,\mathbb{C}). $$

Similarly, if $\bar{\mathfrak{g}} = \mathfrak{so}(2n,\mathbb{C})$, then $\bar{\mathfrak{g}}$ can be identified with the set of matrices:

$$ \left\{ A \in \mathfrak{sl}(2n,\mathbb{C}) \,\middle|\, I_n \cdot A + {}^tA \cdot I_n = 0 \right\}, \text{ where } I_n = \begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix} $$

Then

$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \middle| A, B, C \in \mathfrak{gl}(n, \mathbb{C});\ B + {}^tB = 0,\ C + {}^tC = 0 \right\}$$

$$\text{and } \mathfrak{g} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix} \middle| A \in \mathfrak{gl}(n, \mathbb{C}) \right\} \cong \mathfrak{gl}(n, \mathbb{C}).$$

Lemma 2. *Any almost primitive pair belonging to the class (b) is equivalent to one of the following pairs:*

$(1b)\bar{\mathfrak{g}} = \mathfrak{so}(n+m, \mathbb{C})$,	$\mathfrak{g} = \mathfrak{so}(n, \mathbb{C}) \oplus \mathfrak{so}(m, \mathbb{C}) \oplus \mathbb{C}$,	$3 \leqslant n \leqslant m,\ n+m \geqslant 7,$ $n+m \neq 8;$
$\bar{\mathfrak{g}} = \mathfrak{sp}(2(n+m), \mathbb{C})$,	$\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C}) \oplus \mathfrak{sp}(2m, \mathbb{C})$,	$1 \leqslant n \leqslant m;$
$(2b)\bar{\mathfrak{g}} = \mathfrak{so}(nr, \mathbb{C})$,	$\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{so}(n, \mathbb{C})$,	$r \leqslant 3,\ n \leqslant 3;$
$\bar{\mathfrak{g}} = \mathfrak{sp}(2rn, \mathbb{C})$,	$\mathfrak{g} = \bigoplus_{i+1}^{r} \mathfrak{sp}(2n, \mathbb{C})$,	$r \geqslant 3,\ n \geqslant 1.$

The subalgebras from (1b) consist of two blocks (the condition $n + m \neq 8$ is imposed, since $\mathfrak{so}(8, \mathbb{C})$ is not a classical Lie algebra). The subalgebras from (2b) consist of several (more then two) blocks of the same length.

Lemma 3. *Any almost primitive pair belonging to the class (c) is equivalent to one of the following pairs:*

$(1c)\bar{\mathfrak{g}} = \mathfrak{sl}(nm, \mathbb{C})$,	$\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sl}(m, \mathbb{C}) \oplus \mathbb{C}$,	$2 \leqslant n \leqslant m;$
$(2c)\bar{\mathfrak{g}} = \mathfrak{sl}(n^r, \mathbb{C})$,	$\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{sl}(n, \mathbb{C})$,	$r \geqslant 3,\ n \geqslant 3;$

$(3c)\bar{\mathfrak{g}} = \mathfrak{so}(nm, \mathbb{C})$,	$\mathfrak{g} = \mathfrak{so}(n, \mathbb{C}) \oplus \mathfrak{so}(m, \mathbb{C})$,	$3 \leqslant n \leqslant m,$ $n, m \neq 4;$
$\bar{\mathfrak{g}} = \mathfrak{so}(4nm, \mathbb{C})$,	$\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C}) \oplus \mathfrak{sp}(2m, \mathbb{C})$,	$1 \leqslant n \leqslant m,$ $(n, m) \neq (1, 1),$ $(n, m) \neq (1, 2);$
$\bar{\mathfrak{g}} = \mathfrak{sp}(2nm, \mathbb{C})$,	$\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C}) \oplus \mathfrak{sp}(m, \mathbb{C})$,	$n \geqslant 1,\ m \geqslant 3,$ $m \neq 4;$
$(4c)\bar{\mathfrak{g}} = \mathfrak{so}(4n, \mathbb{C})$,	$\mathfrak{g} = \mathfrak{sp}(2, \mathbb{C}) \oplus \mathfrak{sp}(2, \mathbb{C}) \oplus \mathfrak{so}(n, \mathbb{C})$,	$n \geqslant 3,\ n \neq 4;$
$\bar{\mathfrak{g}} = \mathfrak{sp}(8n, \mathbb{C})$,	$\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C}) \oplus \mathfrak{sp}(2, \mathbb{C}) \oplus \mathfrak{sp}(2, \mathbb{C})$,	$n \geqslant 2;$
$(5c)\bar{\mathfrak{g}} = \mathfrak{so}(n^r, \mathbb{C})$,	$\mathfrak{g} = \bigoplus_{i+1}^{r} \mathfrak{so}(n, \mathbb{C})$,	$r \geqslant 3,\ n \geqslant 3,$ $n \neq 4;$
$\bar{\mathfrak{g}} = \mathfrak{sp}((2n)^{2r+1}, \mathbb{C})$,	$\mathfrak{g} = \bigoplus_{i=1}^{2r+1} \mathfrak{sp}(2n, \mathbb{C})$,	$n \geqslant 1,\ r \geqslant 1;$
$\bar{\mathfrak{g}} = \mathfrak{so}((2n)^{2r}, \mathbb{C})$,	$\mathfrak{g} = \bigoplus_{i=1}^{2r} \mathfrak{sp}(2n, \mathbb{C})$,	$n \geqslant 1,\ r \geqslant 2.$

The subalgebra $\mathfrak{g}$ from (1c) can be identified with the set of matrices:

$$\mathfrak{g} = \{\ A \otimes E_m + E_n \otimes B \mid A \in \mathfrak{sl}(n, \mathbb{C}),\ B \in \mathfrak{sl}(m, \mathbb{C})\}.$$

The subalgebras from (3c) have a similar form. In (2c) $\mathfrak{g}$ can be identified with the set of matrices:

$$\mathfrak{g} = \{ A_1 \otimes E_n \otimes \cdots \otimes E_n + E_n \otimes A_2 \otimes \cdots \otimes E_n + \cdots + E_n \otimes \cdots \otimes E_n \otimes A_r \mid A_i \in \mathfrak{sl}(n, \mathbb{C}), \\ 1 \leqslant i \leqslant r \}.$$

The subalgebras from (5c) can be constructed in a similar way.

A number of conditions in (3c) is imposed, since the following classical Lie algebras are isomorphic:

$$\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C});$$

$$\mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{sp}(2, \mathbb{C}).$$

Therefore the pairs

$$\bar{\mathfrak{g}} = \mathfrak{so}(nm, \mathbb{C}), \ \ \mathfrak{g} = \mathfrak{so}(n, \mathbb{C}) \oplus \mathfrak{so}(m, \mathbb{C}), \ \text{ where } n = 3, m = 4 \text{ or } n = 4, m > 4;$$

$$\bar{\mathfrak{g}} = \mathfrak{sp}(2nm, \mathbb{C}), \ \ \mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C}) \oplus \mathfrak{so}(m, \mathbb{C}), \ \text{ where } m = 4,$$

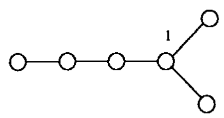
are included into the separate item (4c).

All irreducible pairs ($\bar{\mathfrak{g}}$,$\mathfrak{g}$) are maximal except 19 ones determined by E.B. Dynkin [6]. We are interested in the following question: which of these exceptions are almost primitive? The following lemma gives the answer.

Lemma 4. *Suppose ($\bar{\mathfrak{g}}$, $\mathfrak{g}$)is an almost primitive pair belonging to the class (d) and $\mathfrak{g}$ is not maximal in $\bar{\mathfrak{g}}$. Then the pair ($\bar{\mathfrak{g}}$, $\mathfrak{g}$) is equivalent to the following one:*

$$\bar{\mathfrak{g}} = \mathfrak{so}(495, \mathbb{C}),$$

$\mathfrak{g}$ has type D_6 and the diagram of the $\mathfrak{g}$-module $\mathbb{C}^{495}$ has the form:



Lemmas 1–4 give the complete classification of almost primitive subalgebras of complex classical Lie algebras. In Table 1 it is indicated which of the obtained subalgebras are

maximal;

primitive, but not maximal;

almost primitive, but not primitive.

2.2. Exceptional case (including D_4).

Suppose $\bar{\mathfrak{g}}$ is an exceptional complex Lie algebra or D_4. We shall make use of the following result:

Theorem. *Let $\mathfrak{g}$ be an almost primitive reductive subalgebra of a simple complex Lie algebra $\bar{\mathfrak{g}}$. Then $\mathfrak{g}$ belongs to one and only one of the following classes:*

(i) $\mathfrak{g}$ is a simply imbedded, maximal rank subalgebra;

(ii) $\mathfrak{g}$ is a semisimple, maximal rank subalgebra;

(iii) $\mathfrak{g}$ is a S-subalgebra;

(iv) $\mathfrak{g}$ is a S-subalgebra of a semisimple, maximal rank subalgebra;

Lemma 5. *The classification of simply imbedded almost primitive, maximal rank, reductive subalgebras of exceptional Lie algebras has the form:*
$\bar{\mathfrak{g}} = G_2$, *the mentioned subalgebras do not exist;*
$\bar{\mathfrak{g}} = F_4$, *the mentioned subalgebras do not exist;*
$\bar{\mathfrak{g}} = E_6$, $\mathfrak{g}$ *has type* $D_4 \oplus \mathbb{C}^2,\ D_5 \oplus \mathbb{C},\ \mathbb{C}^6$;
$\bar{\mathfrak{g}} = E_7$, $\mathfrak{g}$ *has type* $E_6 \oplus \mathbb{C},\ \mathbb{C}^7$;
$\bar{\mathfrak{g}} = E_8$, $\mathfrak{g}$ *has type* $\mathbb{C}^8$;
$\bar{\mathfrak{g}} = D_4$, $\mathfrak{g}$ *has type* $A_2 \oplus \mathbb{C}^2,\ A_3 \oplus \mathbb{C},\ \mathbb{C}^4$.

Note that a Cartan subalgebra is almost primitive in $\bar{\mathfrak{g}}$ if and only if the roots of $\bar{\mathfrak{g}}$ are of the same length (this is true for both exceptional and classical Lie algebras).

Lemma 6. *The classification of semisimple almost primitive, maximal rank subalgebras of exceptional Lie algebras has the form:*

$\bar{\mathfrak{g}} = G_2$, $\mathfrak{g}$ has type $A_1 + A_2,\ A_2$;
$\bar{\mathfrak{g}} = F_4$, $\mathfrak{g}$ has type $B_4,\ A_1 + C_3,\ A_2 + A_2,\ D_4$;
$\bar{\mathfrak{g}} = E_6$, $\mathfrak{g}$ has type $A_1 + A_5,\ A_2 + A_2 + A_2$;
$\bar{\mathfrak{g}} = E_7$, $\mathfrak{g}$ has type $A_7,\ A_1 + D_6,\ A_2 + A_5,\ 7A_1,\ D_4 + A_1 + A_1 + A_1$;
$\bar{\mathfrak{g}} = E_8$, $\mathfrak{g}$ has type $D_8,\ E_7 + A_1,\ A_4 + A_4,\ A_8,\ E_6 + A_2,\ D_4 + D_4,\ 8A_1,\ 4A_2$;
$\bar{\mathfrak{g}} = D_4$, $\mathfrak{g}$ has type $4A_1$.

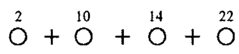
Lemma 7. *The classification of almost primitive S-subalgebras of exceptional Lie algebras has the form:*

$\bar{\mathfrak{g}} = G_2$, $\mathfrak{g}$ has type A_1^{28};
$\bar{\mathfrak{g}} = F_4$, $\mathfrak{g}$ has type $A_1^{156},\ G_2^1 + A_1^8$;
$\bar{\mathfrak{g}} = E_6$, $\mathfrak{g}$ has type $G_2^3,\ G_4^1,\ G_2^1 + A_2^{2''},\ F_4',\ A_2^9$;
$\bar{\mathfrak{g}} = E_7$, $\mathfrak{g}$ has type $A_1^{399},\ A_1^{231},\ G_2^1 + C_3^{1''},\ F_4^1 + A_1^{3''},\ G_2^2 + A_1^7,\ A_2^{21},\ A_1^{24} + A_1^{15}$;
$\bar{\mathfrak{g}} = E_8$, $\mathfrak{g}$ has type $A_1^{1240},\ A_1^{760},\ A_1^{520},\ G_2^1 + F_4^1,\ A_2^{6'} + A_1^{16},\ B_2^{12},\ G_2^1 + G_2^1 + A_1^8$;
$\bar{\mathfrak{g}} = D_4$, $\mathfrak{g}$ has type $B_3^1,\ A_1^2 + B_2^1,\ A_2^3,\ G_2^1$.

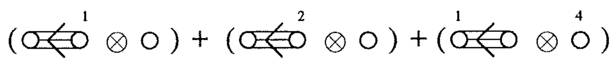
The characteristic representations (i.e., representations on $\bar{\mathfrak{g}}$) of the subalgebras determined in lemma 7 have the form:

$A_1^{28} \subset G_2$:

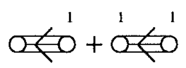
$A_1^{156} \subset F_4$:



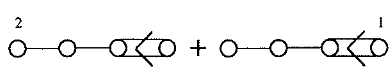
$G_1^2 + A_1^8 \subset F_4$:



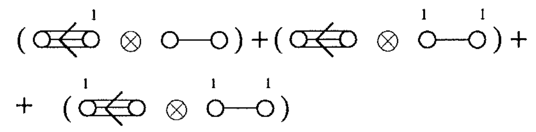
$G_2^3 \subset E_6$:



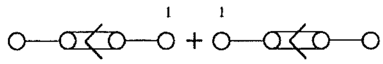
$C_1^4 \subset E_6$:



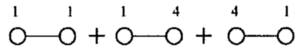
$G_1^2 + A_2^{2''} \subset E_6$:



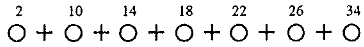
$F_4^1 \subset E_6$:



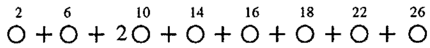
$A_2^9 \subset E_6$:



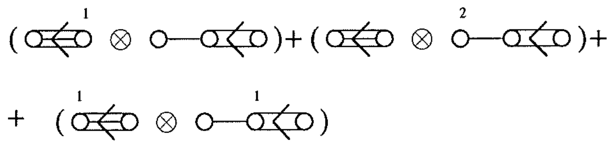
$A_1^{399} \subset E_7$:



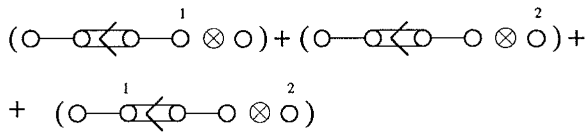
$A_1^{231} \subset E_7$:



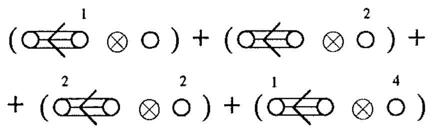
$G_2^1 + C_3^{1''} \subset E_7$:



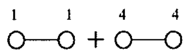
$F_1^4 + A_1^{3''} \subset E_7$:



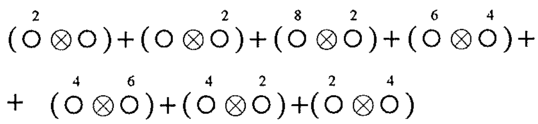
$G_2^2 + A_1^7 \subset E_7$:



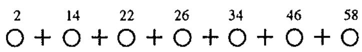
$A_2^{21} \subset E_7$:



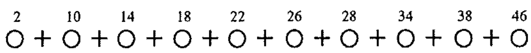
$A_1^{24} + A_1^{15} \subset E_7$:



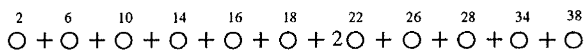
$A_1^{1240} \subset E_8$:



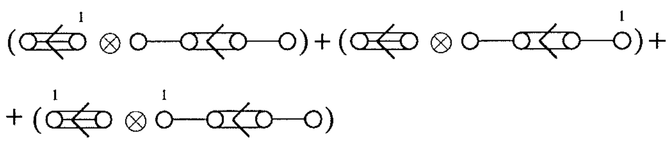
$A_1^{760} \subset E_8$:



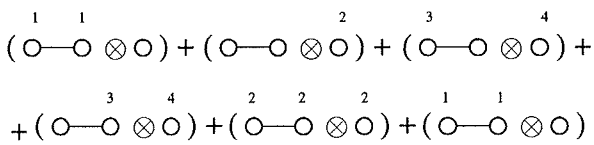
$A_1^{520} \subset E_8$:



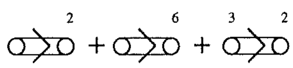
$G_2^1 + F_4^1 \subset E_8$:



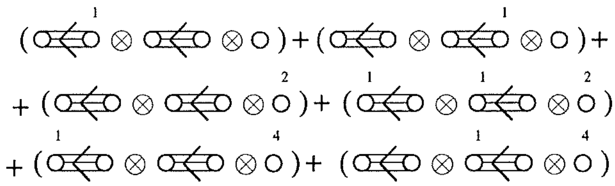
$A_2^{6'} + A_1^{16} \subset E_8$:



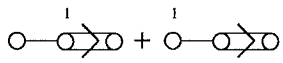
$B_2^{12} \subset E_8$:



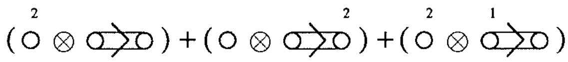
$G_2^1 + G_2^1 + A_1^8 \subset E_8$:



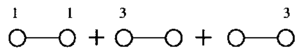
$B_3^1 \subset D_4$:



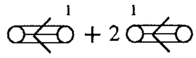
$A_1^2 + B_2^1 \subset D_4$:



$A_2^3 \subset D_4$:



$G_2^1 \subset D_4$:

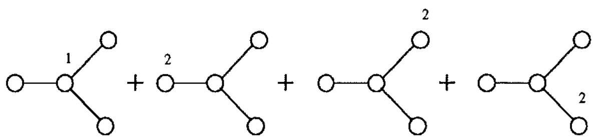


Lemma 8. *The classification of almost primitive S-subalgebras in semisimple, maximal rank subalgebras of exceptional Lie algebras has the form:*

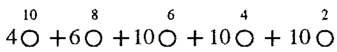
$\bar{\mathfrak{g}} = E_7$, $\mathfrak{g}$ has type D_4^2;
$\bar{\mathfrak{g}} = E_8$, $\mathfrak{g}$ has type A_1^{40}.

The characteristic representations of the subalgebras determined in lemma 8 have the form:

$D_4^2 \subset E_7$:



$A_1^{40} \subset E_8$:



Lemmas 5–8 give the complete classification of almost primitive reductive subalgebras of the exceptional Lie algebras (including D_4). In table 2 it is determined which of the obtained subalgebras are

maximal;
primitive, but not maximal;
almost primitive, but not primitive.

3. The classification of almost primitive subalgebras of real Lie algebras.

Let us recall some constructions relating to real forms of complex Lie algebras.

An *anti-involution of a complex Lie algebra* $\mathfrak{g}$ is an anti-linear mapping $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $\sigma^2 = id$ and $\sigma([x,y]) = [\sigma(x), \sigma(y)]$ for all $x, y \in \mathfrak{g}$. A *real form of complex Lie algebra* $\mathfrak{g}$ is a subalgebra $\mathfrak{g}_{\mathbb{R}}$ such that $\mathfrak{g}_{\mathbb{R}} = \tilde{\mathfrak{g}} \oplus i\tilde{\mathfrak{g}}$. There exists a one-to-one correspondence between the set of all real forms of the Lie algebra $\mathfrak{g}$ and the set of its anti-involutions. Namely, the set of fixed points of an anti-involution σ is a real form of $\mathfrak{g}$ and is denoted by $\mathfrak{g}^\sigma$. Conversely, any real form $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ defines an anti-involution σ of $\mathfrak{g}$ by

$$\sigma(x + iy) = x - iy \text{ for } x, y \in \tilde{\mathfrak{g}}.$$

Here $\tilde{\mathfrak{g}} = \mathfrak{g}^{\sigma}$.

An anti-involution τ of a Lie algebra $\mathfrak{g}$ is called a *Cartan anti-involution*, if it defines a compact real form of $\mathfrak{g}$. For any semisimple complex Lie algebra there exists a Cartan anti-involution. Moreover all its Cartan anti-involutions are conjugate (up to $\operatorname{Int}\mathfrak{g}$) .

Let $\bar{\mathfrak{g}}$ be a semisimple complex Lie algebra. There is a close connection between real forms of $\mathfrak{g}$ and the class of symmetric Lie algebras.

Definition. Let $\bar{\theta}$ be an involution of $\bar{\mathfrak{g}}$ (i.e., an automorphism of second order). The pair $(\bar{\mathfrak{g}}, \bar{\theta})$ is called a *symmetric Lie algebra.*

An *isomorphism of two symmetric Lie algebras* $(\bar{\mathfrak{g}}_1, \bar{\theta}_1)$ and $(\bar{\mathfrak{g}}_2, \bar{\theta}_2)$ is an isomorphism f of the Lie algebras $\bar{\mathfrak{g}}_1$ and $\bar{\mathfrak{g}}_2$ such that $f \circ \bar{\theta}_1 = \bar{\theta}_2 \circ f$.

Suppose $\bar{\mathfrak{g}}^{\bar{\sigma}}$ is the real form of the Lie algebra $\bar{\mathfrak{g}}$ defined by an anti-involution $\bar{\sigma}$; then there exists a Cartan anti-involution τ of $\bar{\mathfrak{g}}$ such that $\bar{\sigma} \circ \tau = \tau \circ \bar{\sigma}$. Then $\bar{\theta} = \bar{\sigma} \circ \tau$ is an involution of $\bar{\mathfrak{g}}$. Thus to any real form $\bar{\mathfrak{g}}^{\bar{\sigma}}$ of $\bar{\mathfrak{g}}$ we can assign a symmetric Lie algebra $(\bar{\mathfrak{g}}, \bar{\theta})$. Generally speaking, the correspondence is nonunique. But all symmetric Lie algebras corresponding to a given real form are isomorphic to each other.

Conversely, if $(\bar{\mathfrak{g}}, \bar{\theta})$ is a symmetric Lie algebra, then there exists a Cartan anti-involution τ of the Lie algebra $\bar{\mathfrak{g}}$ such that $\bar{\theta} \circ \tau = \tau \circ \bar{\theta}$. Then $\bar{\sigma} = \bar{\theta} \circ \tau$ is an involution of $\bar{\mathfrak{g}}$ which defines a certain real form $\bar{\mathfrak{g}}^{\bar{\sigma}}$ of $\bar{\mathfrak{g}}$. This correspondence is also nonunique however all real forms of $\bar{\mathfrak{g}}$ corresponding to a given symmetric Lie algebra $(\bar{\mathfrak{g}}, \bar{\theta})$ are isomorphic to each other.

Thus there exists a one-to-one correspondence (up to isomorphism) between real forms of the Lie algebra $\bar{\mathfrak{g}}$ and symmetric Lie algebras $(\bar{\mathfrak{g}}, \bar{\theta})$.

The constructions described above can be generalized to subalgebra of complex Lie algebras.

Suppose $\bar{\mathfrak{g}}$ is a complex (real) Lie algebra and $\mathfrak{g}$ is a subalgebra of $\bar{\mathfrak{g}}$. Then we say that a *complex (real) pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$ is given. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is called *maximal* (*primitive, almost primitive*), if the subalgebra $\mathfrak{g}$ is maximal (primitive, almost primitive) in $\bar{\mathfrak{g}}$. Two pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are said to be *equivalent*, if there exists an isomorphism f of the Lie algebras $\bar{\mathfrak{g}}_1$ and $\bar{\mathfrak{g}}_2$ such that $f(\mathfrak{g}_1) = \mathfrak{g}_2$.

Definition. A real pair $(\bar{\mathfrak{p}}, \mathfrak{p})$ is called a *real form of a complex pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$, if $\bar{\mathfrak{p}}$ is a real form of the Lie algebra $\bar{\mathfrak{g}}$ and $\mathfrak{p}$ is a real form of the Lie algebra $\mathfrak{g}$.

It is easy to show that there exists a one-to-one correspondence between the set of all real forms of a complex pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ and the set of anti-involutions of $\bar{\mathfrak{g}}$ preserving the subalgebra $\mathfrak{g}$.

Now we introduce the concept of symmetric pair.

Definition.

(*i*) A symmetric Lie algebra $(\mathfrak{g}, \theta)$ is called a *subalgebra of a symmetric Lie algebra* $(\bar{\mathfrak{g}}, \bar{\theta})$, if $\mathfrak{g}$ is a subalgebra of $\bar{\mathfrak{g}}$ such that $\bar{\theta}(\mathfrak{g}) = \mathfrak{g}$ and $\theta = \bar{\theta}|_{\mathfrak{g}}$.

(*ii*) A subalgebra $(\mathfrak{g}, \theta)$ of a symmetric Lie algebra $(\bar{\mathfrak{g}}, \bar{\theta})$ is called an *ideal*, if $\mathfrak{g}$ is an ideal in $\bar{\mathfrak{g}}$.

(*iii*) Let $(\mathfrak{g}, \theta)$ be a subalgebra of a symmetric Lie algebra $(\bar{\mathfrak{g}}, \bar{\theta})$. We say that the pair $((\bar{\mathfrak{g}}, \bar{\theta}), (\mathfrak{g}, \theta))$ is a *symmetric pair.*

Two symmetric pairs $((\bar{\mathfrak{g}}_i, \bar{\theta}_i), (\mathfrak{g}_i, \theta_i))$, $i = 1, 2$, are said to be *equivalent*, if there exists an isomorphism f of the symmetric Lie algebras $(\bar{\mathfrak{g}}_1, \bar{\theta}_1)$ and $(\bar{\mathfrak{g}}_2, \bar{\theta}_2)$ such that $f(\mathfrak{g}_1) = \mathfrak{g}_2$.

Under some additional conditions on a complex pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ there exists a one-to-one correspondence (up to equivalence) between real forms of the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ and symmetric pairs $((\bar{\mathfrak{g}}, \bar{\theta}), (\mathfrak{g}, \theta))$.

Theorem. *Suppose $\bar{\mathfrak{g}}$ is a semisimple complex Lie algebra and $\mathfrak{g}$ is a subalgebra of $\bar{\mathfrak{g}}$ such that $\mathfrak{g}$ coincides with its own normalizer.*

(i) if $\bar{\sigma}$ is an anti-involution of $\bar{\mathfrak{g}}$ such that $\bar{\sigma}(\mathfrak{g}) = \mathfrak{g}$, then there exists a Cartan anti-involution of $\bar{\mathfrak{g}}$ preserving the subalgebra $\mathfrak{g}$ and commuting with $\bar{\sigma}$.

(ii) if $\bar{\theta}$ is an involution of $\bar{\mathfrak{g}}$ such that $\bar{\theta}(\mathfrak{g}) = \mathfrak{g}$, then there exists a Cartan anti-involution of $\bar{\mathfrak{g}}$ preserving the subalgebra $\mathfrak{g}$ and commuting with $\bar{\theta}$.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a complex pair satisfying the assumptions of the theorem. Then there exists a one-to-one correspondence (up to equivalence) between the set of real forms of the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ and the set of symmetric pairs $((\bar{\mathfrak{g}}, \bar{\theta}), (\mathfrak{g}, \theta))$.

Suppose $(\bar{\mathfrak{p}}, \mathfrak{p})$ is a desired real almost primitive pair such that $\bar{\mathfrak{p}}$ is simple and $\mathfrak{p}$ is reductive in $\bar{\mathfrak{p}}$. Then the complex pair $(\bar{\mathfrak{g}}, \mathfrak{g})$, where $\bar{\mathfrak{g}} = \bar{\mathfrak{p}}^{\mathbb{C}}$ and $\mathfrak{g} = \bar{\mathfrak{p}}^{\mathbb{C}}$, is also almost primitive. Moreover, $\mathfrak{g}$ is a reductive subalgebra of the Lie algebra $\bar{\mathfrak{g}}$ coinciding with its own normalizer. However $\bar{\mathfrak{g}}$ is not necessarily simple. This case is described in the following lemma.

Lemma. *Let $(\bar{\mathfrak{p}}, \mathfrak{p})$ be a real almost primitive pair such that $\bar{\mathfrak{p}}$ is simple and $\mathfrak{p}$ is reductive in $\bar{\mathfrak{p}}$. If the Lie algebra $\bar{\mathfrak{p}}^{\mathbb{C}}$ is not simple , then the pair $(\bar{\mathfrak{p}}, \mathfrak{p})$ has one of the following forms:*

(i) $\bar{\mathfrak{p}} = \bar{\mathfrak{a}}_{\mathbb{R}}$, where $\bar{\mathfrak{a}}$ is a complex simple Lie algebra; $\mathfrak{p}$ is a real form of $\bar{\mathfrak{a}}$,

(ii) $\bar{\mathfrak{p}} = \bar{\mathfrak{a}}_{\mathbb{R}}$, where $\bar{\mathfrak{a}}$ is a complex simple Lie algebra; $\mathfrak{p} = \mathfrak{a}_{\mathbb{R}}$, where $\mathfrak{a}$ is an almost primitive reductive subalgebra of $\bar{\mathfrak{a}}$.

The pairs (i) are primitive. The pairs (ii) are primitive if and only if $\mathfrak{a}$ is primitive in $\bar{\mathfrak{a}}$.

Thus in order to classify real almost primitive pairs, we must classify real forms of complex almost primitive pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$, where $\bar{\mathfrak{g}}$ is simple and $\mathfrak{g}$ is reductive in $\bar{\mathfrak{g}}$, and select almost primitive, primitive , and maximal pairs from the obtained real forms.

By virtue of the mentioned correspondence between real forms of a pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ and symmetric pairs $((\bar{\mathfrak{g}}, \bar{\theta}), (\mathfrak{g}, \theta))$, in this paper we first find all symmetric pairs.

Furthermore the concepts of primitivity and almost primitivity can be generalized to symmetric pairs. Suppose $\mathcal{G}$ is a set of subalgebras $\mathfrak{g}_1, \dots, \mathfrak{g}_k$, involutions $\theta_1, \dots, \theta_l$, and anti-involutions $\sigma_1, \dots, \sigma_m$ of a Lie algebra $\bar{\mathfrak{g}}$. By $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathcal{G})$ denote the set of all automorphisms of $\bar{\mathfrak{g}}$ preserving all subalgebras and commuting with all involutions and anti-involutions from $\mathcal{G}$. By $\mathrm{Int}(\bar{\mathfrak{g}}, \mathcal{G})$ denote the connected component of the identity in the group $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathcal{G})$.

Definition. Let $(\mathfrak{g}, \theta)$ be a proper subalgebra of a symmetric Lie algebra $(\bar{\mathfrak{g}}, \bar{\theta})$. The subalgebra $(\mathfrak{g}, \theta)$ is said to be *primitive (almost primitive)*, if

(a) $(\mathfrak{g}, \theta)$ contains no proper ideals of $(\bar{\mathfrak{g}}, \bar{\theta})$,

(b) $(\mathfrak{g}, \theta)$ is the maximal of all subalgebras of $(\bar{\mathfrak{g}}, \bar{\theta})$ invariant under $\mathrm{Int}(\bar{\mathfrak{g}}, \mathfrak{g}, \bar{\theta})$ $(\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g}, \bar{\theta}))$.

Theorem. *Let $\mathfrak{g}$ be a reductive subalgebra of a complex semisimple Lie algebra $\bar{\mathfrak{g}}$ such that $\mathfrak{g}$ coincides with its own normalizer. Let $\bar{\sigma}$ be an involution of $\bar{\mathfrak{g}}$ and τ a Cartan anti-involution of $\bar{\mathfrak{g}}$ such that $\bar{\sigma} \circ \tau = \tau \circ \bar{\sigma}$ and $\bar{\sigma}(\mathfrak{g}) = \tau(\mathfrak{g}) = \mathfrak{g}$. $\bar{\theta} = \tau \circ \bar{\sigma}$ is an involution of $\bar{\mathfrak{g}}$. Put $\theta = \bar{\theta}\,|_{\mathfrak{g}}$ and $\sigma = \bar{\sigma}\,|_{\mathfrak{g}}$.*

(i) If the symmetric pair $((\bar{\mathfrak{g}}, \bar{\theta}), (\mathfrak{g}, \theta))$ is almost primitive, then the corresponding real pair $(\bar{\mathfrak{g}}^{\bar{\sigma}}, \mathfrak{g}^{\sigma})$ is almost primitive.

(ii) If the subalgebra $\mathfrak{g}$ is semisimple, then the real pair $(\bar{\mathfrak{g}}^{\bar{\sigma}}, \mathfrak{g}^{\sigma})$ is primitive (almost primitive) if and only if the corresponding symmetric pair $((\bar{\mathfrak{g}}, \bar{\theta}), (\mathfrak{g}, \theta))$ is primitive (almost primitive).

The real almost primitive pairs obtained as the real forms of the complex almost primitive pairs (determined before) are listed in Tables 3–62. In the classical case, the results are written in classical notation. In the exceptional case, the symmetric pairs corresponding to the obtained real pairs are listed.

4. Examples in small dimensions

4.1. Two-dimensional homogeneous spaces.

Consider the concepts of maximality and primitivity for two-dimensional homogeneous spaces in both complex and real cases.

The problem in the local formulation is equivalent to the classification of all maximal effective subalgebras of codimension 2. In the complex case it was done by Sophus Lie. He obtained the following result.

Any complex maximal effective pair $(\bar{\mathfrak{g}}, \mathfrak{g})$, where $\operatorname{codim}_{\bar{\mathfrak{g}}} \mathfrak{g} = 2$, *is equivalent to one of the following pairs:*

1°. $\bar{\mathfrak{g}} = \mathfrak{sl}(3, \mathbb{C}), \quad \mathfrak{g} = \left\{ \left(\begin{matrix} -x-y & v & w \\ 0 & x & z \\ 0 & u & y \end{matrix} \right) \middle| \, x, y, z, u.v, w \in \mathbb{C} \right\};$

2°. $\bar{\mathfrak{g}} = \mathfrak{gl}(2, \mathbb{C}) \leftthreetimes \mathbb{C}^2, \quad \mathfrak{g} = \mathfrak{gl}(2, \mathbb{C});$

3°. $\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{C}) \leftthreetimes \mathbb{C}^2, \quad \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}).$

In case 1°, the Lie algebra $\bar{\mathfrak{g}}$ is simple and the subalgebra $\mathfrak{g}$ is maximal and parabolic in $\bar{\mathfrak{g}}$. In cases 2° and 3°, the Lie algebra $\bar{\mathfrak{g}}$ is not semisimple and is the semidirect product of the subalgebra $\mathfrak{g}$ and the commutative ideal $\mathfrak{a} = \mathbb{C}^2$. Moreover, $\mathfrak{g}$ acts faithfully and irreducibly on $\mathfrak{a}$.

The corresponding primitive homogeneous spaces are as follows.

1°. The group $SL(3, \mathbb{C})$ acts naturally on $\mathbb{C}P^2$.

2°. The affine group $\mathrm{Aff}(2, \mathbb{C})$ acts on $\mathbb{C}^2$.

3°. The unimodular group acts on $\mathbb{C}^2$.

Considering the problem in the global formulation (i.e., passing form maximal to primitive subalgebras), we obtain one more pair:

4°. $\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{C}), \quad \mathfrak{g} = \left\{ \left(\begin{matrix} x & 0 \\ 0 & -x \end{matrix} \right) \middle| \, x \in \mathbb{R} \right\}.$

In spite of the fact that $\mathfrak{g}$ is not maximal in $\bar{\mathfrak{g}}$, the homogeneous space corresponding to the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is primitive. It can be described as follows.

The manifold M can be embedded into $\mathbb{C}P^1 \times \mathbb{C}P^1$:

$$M = \{ ((x_0 : x_1), (y_0 : y_1)) \in \mathbb{C}P^1 \times \mathbb{C}P^1 \big| \, x_0 y_0 + x_1 y_1 \neq 0\} / \sim,$$

where $((x_0 : x_1), (y_0, y_1)) \sim ((x_0' : x_1'), (y_0', y_1'))$ whenever $x_0 = y_1',\ x_1 = -y_0',\ y_0 = -x_1',\ y_1 = x_0'$.

The action of the group $SL(2,\mathbb{C})$ on M is given by

$$A.((x_0 : x_1), (y_0 : y_1)) = ((x_0' : x_1'), (y_0' : y_1')),$$

where

$$\begin{pmatrix} x_0' \\ x_1' \end{pmatrix} = A \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \text{ and } \begin{pmatrix} y_0' \\ y_1' \end{pmatrix} = {}^tA^{-1} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}.$$

Let us describe maximal and primitive pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ of codimension 2 over the field $\mathbb{R}$.

First we classify all maximal effective pairs. The classification has the form:

1°. $\bar{\mathfrak{g}} = \mathfrak{sl}(3,\mathbb{R}),\quad \mathfrak{g} = \left\{ \begin{pmatrix} -x-y & v & w \\ 0 & x & z \\ 0 & u & y \end{pmatrix} \middle| x, y, z, u.v, w \in \mathbb{R} \right\}$;

2°. $\bar{\mathfrak{g}} = \mathfrak{su}(2),\quad \mathfrak{g} = \left\{ \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} \middle| x \in \mathbb{R} \right\}$;

3°. $\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}),\quad \mathfrak{g} = \left\{ \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix} \middle| x \in \mathbb{R} \right\}$;

4°. $\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}},\quad \mathfrak{g} = \left\{ \begin{pmatrix} x & y \\ 0 & -x \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$;

5°. $\bar{\mathfrak{g}} = \mathfrak{gl}(2,\mathbb{R}) \leftthreetimes \mathbb{R}^2,\quad \mathfrak{g} = \mathfrak{gl}(2,\mathbb{R})$;

6°. $\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \leftthreetimes \mathbb{R}^2,\quad \mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$;

7°. $\bar{\mathfrak{g}} = \mathfrak{g} \leftthreetimes \mathbb{R}^2,\quad \mathfrak{g} = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$;

8°. $\bar{\mathfrak{g}} = \mathfrak{g} \leftthreetimes \mathbb{R}^2,\quad \mathfrak{g} = \left\{ \begin{pmatrix} \alpha x & x \\ -x & \alpha x \end{pmatrix} \middle| x \in \mathbb{R} \right\}, \alpha \in \mathbb{R}_+$.

Note that the only pairs that are real forms of maximal complex pairs are the pairs 1°, 4°, and 5°. In the other cases the subalgebra $\mathfrak{g}^{\mathbb{C}}$ is not maximal in $\bar{\mathfrak{g}}^{\mathbb{C}}$.

The corresponding homogeneous spaces have the form:

1°. The group $SL(2,\mathbb{R})$ acts on $\mathbb{R}P^2$.

2°. The group $SU(2)$ acts naturally on $\mathbb{C}P^1 \approx S^2$.

3°. The group $SL(2,\mathbb{R})$ acts on the Lobachevsky plane ($\approx \mathbb{R}^2$).

4°. The group $SL(2,\mathbb{C})_{\mathbb{R}}$ acts on $\mathbb{C}P^1 \approx S^2$.

5°. The affine group $\mathrm{Aff}(2,\mathbb{R})$ acts on $\mathbb{R}^2$.

6°. The unimodular group acts on $\mathbb{R}^2$.

In cases 7° and 8°, the group $\bar{G}$ such that $\bar{\mathfrak{g}}$ is its Lie algebra can be embedded into $\mathrm{Aff}(2,\mathbb{R})$ and acts on $\mathbb{R}^2$. In particular, in case 8°($\alpha = 0$), we obtain the group of Euclidean transformations of $\mathbb{R}^2$.

Passing from maximal to primitive subalgebras, we obtain one more pair:

9°. $\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}),\quad \mathfrak{g} = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \middle| x \in \mathbb{R} \right\}$.

The description of the corresponding homogeneous space is similar to that of the complex homogeneous space corresponding to the pair 4° in the complex case. Moreover, the real manifold M is diffeomorphic to the Möbius strip.

4.2. Primitive reductive pairs.

Let us describe real primitive pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$, where $\bar{\mathfrak{g}}$ is a simple Lie algebra and $\mathfrak{g}$ is a reductive subalgebra of $\bar{\mathfrak{g}}$ such that $\operatorname{codim}_{\bar{\mathfrak{g}}} \mathfrak{g} \leqslant 4$, and the corresponding homogeneous spaces.

The complexifications of these pairs are primitive and have the form:

1°. $\bar{\mathfrak{g}}^{\mathbb{C}} = \mathfrak{sl}(n+1, \mathbb{C})$, $\quad \mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{C}, n = 1, 2$;

2°. $\bar{\mathfrak{g}}^{\mathbb{C}} = \mathfrak{sp}(4, \mathbb{C})$, $\quad \mathfrak{g}^{\mathbb{C}} = \mathfrak{sp}(2, \mathbb{C}) \oplus \mathfrak{sp}(2, \mathbb{C})$.

First we describe the corresponding complex homogeneous spaces (in 1° this homogeneous space is not primitive for $n = 2$).

1°. Let V be a $(n+1)$-dimensional complex vector space. The action of the group $GL(V)$ on V^* can be defined starting from the equality

$$\langle x.v, x.v^* \rangle = \langle v, v^* \rangle$$

for all $v \in V, v^* \in V^*$, where $x \in GL(V)$. If the matrix of $x \in GL(V)$ in a certain basis of V is equal to A, than the matrix of the action of x on V^* in the dual basis is equal to ${}^tA^{-1}$. Moreover, it is easily proved that $GL(V)$ acts transitively on pairs (v, v^*), where $v \in V, v^* \in V^*$, and $< v, v^* >= 1$. Fix a basis in V. We identify $GL(V)$ with $GL(n+1, \mathbb{C})$.

Projectivizing the spaces V and V^*, we obtain the action of the group $SL(n+1, \mathbb{C})$ on the manifold

$$M = \{ ((x_0 : x_1 : \cdots : x_n), (y_0 : y_1 : \cdots : y_n)) \in \mathbb{C}P^n \times \mathbb{C}P^n | \, x_0 y_0 + x_1 y_1 + \cdots + x_n y_n \neq 0 \}.$$

An element $A \in SL(n+1, \mathbb{C})$ takes the point $((x_0 : x_1 : \cdots : x_n), (y_0 : y_1 : \cdots : y_n))$ to the point $((x_0^1 : x_1^1 : \cdots : x_n^1), (y_0^1 : y_1^1 : \cdots : y_n^1))$, where

$$(x_i^1) = A(x_i), (y_i^1) = {}^tA^{-1}(y_i), \quad 0 \leqslant i \leqslant n.$$

Primitive real forms of $(\bar{\mathfrak{g}}, \mathfrak{g})$ have the following expressions:

(i) $n = 1$:

a) $\bar{\mathfrak{g}} = \mathfrak{u}(2), \mathfrak{g} = \mathbb{T}$;

b) $\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{R}), \mathfrak{g} = \mathbb{R}$;

c) $\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{R}), \mathfrak{g} = \mathbb{T}$.

The corresponding homogeneous spaces are two-dimensional. They are described in section 1.

(ii) $n = 2$:

a) $\bar{\mathfrak{g}} = \mathfrak{su}(3), \mathfrak{g} = \mathbb{T} \oplus \mathfrak{su}(2)$;

b) $\bar{\mathfrak{g}} = \mathfrak{su}(2,1), \mathfrak{g} = \mathbb{T} \oplus \mathfrak{su}(2)$;

c) $\bar{\mathfrak{g}} = \mathfrak{su}(1,2), \mathfrak{g} = \mathbb{T} \oplus \mathfrak{su}(1,1)$.

In all these cases the homogeneous space is as follows: the group $SU(p,q), p+q = 3$, acts naturally on $\mathbb{C}P^2$, and the manifold M is the orbit of the point $(1 : 0 : 0)$ under this action.

In particular, in case *a)* the group $SU(3)$ acts transitively on $\mathbb{C}P^2$. In case *b)* $SU(2,1)$ acts transitively on the following subset of $\mathbb{C}P^2$:

$$M = \left\{ ((x : y : z) \in \mathbb{C}P^2 | \, |x|^2 + |y|^2 > |z|^2 \right\}.$$

In the case c) the group $SU(1,2)$ acts transitively on the following subset of $\mathbb{C}P^2$:

$$M = \left\{ ((x:y:z) \in \mathbb{C}P^2 \,\middle|\, |x|^2 > |y|^2 + |z|^2 \right\}.$$

The condition $|x|^2 > |y|^2 + |z|^2$ implies that $x \neq 0$. This allows to embed M into the affine chart $\{(1 : y/x : z/x) \in \mathbb{C}P^2\}$. Then we see that $SU(1,2)$ acts transitively on the following subset of $\mathbb{C}^2$:

$$M = \left\{ ((x,y) \in \mathbb{C}^2 \,\middle|\, |x|^2 + |y|^2 < 1 \right\}.$$

M as a real manifold is homeomorphic to $\mathbb{R}^4$.

2°. Now describe the complex homogeneous space corresponding to the pair $\bar{\mathfrak{g}} = \mathfrak{sp}(4,\mathbb{C}), \mathfrak{g} = \mathfrak{sp}(2,\mathbb{C}) \oplus \mathfrak{sp}(2,\mathbb{C})$. Let Ω be a nondegenerate skew-symmetric form on a 4-dimensional complex vector space V. There exists a basis of V such that the Gram matrix of the form Ω has the form:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The set of all nondegenerate linear transformations of V preserving the form V is the Lie group $SP(4,\mathbb{C})$. Its Lie algebra coincides with $\bar{\mathfrak{g}}$. The group $SP(4,\mathbb{C})$ acts transitively on the set of two-dimensional subspaces of V such that the restrictions of Ω to them are nondegenerate.

This set can be turned into a smooth manifold. Let us describe the coordinates on this manifold. The set of all two-dimensional subspaces of V is the Grassman manifold $G(4,2)$, which can be embedded into $\mathbb{C}P^5$ in the following way: if $e_1 = (x_1, x_2, x_3, x_4), e_2 = (y_1, y_2, y_3, y_4)$ is a basis of a subspace W in V, than its homogeneous coordinates $(\xi_{12} : \xi_{13} : \xi_{14} : \xi_{23} : \xi_{24} : \xi_{34})$ can be determined from the equality:

$$\xi_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}.$$

These coordinates are called Plücker coordinates. They are independent of the choice of the basis in W and satisfy the condition $\xi_{12}\xi_{34} - \xi_{13}\xi_{24} + \xi_{14}\xi_{23} = 0$. Moreover, it is possible to show that any homogeneous coordinates on $\mathbb{C}P^5$ satisfying this condition define a certain two-dimensional subspace in V. Thus

$$G(4,2) = \left\{ (\xi_{12} : \xi_{13} : \xi_{14} : \xi_{23} : \xi_{24} : \xi_{34}) \in \mathbb{C}P^5 \,\middle|\, \xi_{12}\xi_{34} - \xi_{13}\xi_{24} + \xi_{14}\xi_{23} = 0 \right\}.$$

The restriction of Ω to W is nondegenerate if and only if the following condition holds:

$$\Omega(e_1, e_2) = \xi_{12} + \xi_{34} \neq 0.$$

Put

$$\xi_1 = \xi_{12} + \xi_{34}, \qquad \xi_2 = \xi_{12} - \xi_{34},$$

$$\xi_3 = \xi_{13} + \xi_{24}, \qquad \xi_4 = \xi_{13} - \xi_{24},$$

$$\xi_5 = \xi_{14} + \xi_{23}, \qquad \xi_6 = \xi_{14} - \xi_{23}.$$

Then

$$G(4,2) = \left\{ (\xi_1 : \xi_2 : \xi_3 : \xi_4 : \xi_5 : \xi_6) \in \mathbb{C}P^5 \middle| \xi_1^2 - \xi_2^2 - \xi_3^2 + \xi_4^2 + \xi_5^2 - \xi_6^2 = 0 \right\}$$

and the condition of nondegeneracy of $\Omega|_W$ is equivalent to $\xi_1 \neq 0$.

This allows to embed the considered manifold $M^{\mathbb{C}}$ into $\mathbb{C}^5$:

$$M = \left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{C}^5 \middle| x_1^2 + x_2^2 - x_3^2 - x_4^2 + x_5^2 = 1 \right\}$$

Real forms of the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ have the form:

a) $\bar{\mathfrak{g}} = \mathfrak{sp}(2), \quad \mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2);$
b) $\bar{\mathfrak{g}} = \mathfrak{sp}(1,1), \quad \mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2);$
c) $\bar{\mathfrak{g}} = \mathfrak{sp}(4,\mathbb{R}), \quad \mathfrak{g} = \mathfrak{sp}(2,\mathbb{R}) \oplus \mathfrak{sp}(2,\mathbb{R});$
d) $\bar{\mathfrak{g}} = \mathfrak{sp}(1,1), \quad \mathfrak{g} = \mathfrak{sp}(2,\mathbb{C})_{\mathbb{R}};$
e) $\bar{\mathfrak{g}} = \mathfrak{sp}(4,\mathbb{R}), \quad \mathfrak{g} = \mathfrak{sp}(2,\mathbb{C})_{\mathbb{R}}.$

In this case the corresponding real homogeneous spaces can be constructed in the following way. The group $SP(4,\mathbb{C})$ can be regarded as a real Lie group. There exists a Lie subgroup $\bar{G}$ of $SP(4,\mathbb{C})$ such that $\bar{\mathfrak{g}}$ is the Lie algebra of $\bar{G}$. The action of $\bar{G}$ on $M^{\mathbb{C}}$ is the restriction of the action of $SP(4,\mathbb{C})$. Then M is the orbit of the point $(1,0,0,0,0) \in \mathbb{C}^5$ under the action of $\bar{G}$.

For example, in case *a*), the group $\bar{G}$ has the form:

$$\bar{G} = \{ X \in SP(4,\mathbb{C}) \mid {}^tXX = E \}.$$

The orbit of $(1,0,0,0,0)$ under the action of $\bar{G}$ on $M^{\mathbb{C}}$ is

$$M = \left\{ (x_1, x_2, x_3, x_4, x_5) \in M^{\mathbb{C}} \middle| x_1 = \bar{x}_1, x_2 = \bar{x}_2, -x_3 = \bar{x}_3, -x_4 = \bar{x}_4, x_5 = \bar{x}_5 \right\}.$$

Putting $y_1 = x_1, y_2 = x_2, y_3 = ix_3, y_4 = ix_4, y_5 = x_5$, we obtain

$$M = \left\{ (y_1, y_2, y_3, y_4, y_5) \in \mathbb{R}^5 \middle| y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 = 1 \right\}.$$

Thus the group $\bar{G}$ acts transitively on S^4.

Similarly in cases *b*)-*e*), the corresponding manifold has the form:

b) $\mathbb{R}^4$, c) $S^2 \times \mathbb{R}^2$, d) $S^3 \times \mathbb{R}$, e) $S \times \mathbb{R}^3$.

4.3. Homogeneous spaces and differential equations.

Let $(\bar{G}, M)$ be a real homogeneous space, where the Lie group $\bar{G}$ acts effectively on the manifold M. Let $\bar{\mathfrak{g}}$ be the Lie algebra of $\bar{G}$ and $\mathcal{D}(M)$ the Lie algebra of all vector fields on M. Consider the homomorphism of Lie algebras

$$\rho : \bar{\mathfrak{g}} \to \mathcal{D}(M)$$

corresponding to the action of $\bar{G}$ on M. Since ρ is an injection, it is possible to identify $\bar{\mathfrak{g}}$ with a certain subalgebra of $\mathcal{D}(M)$.

An ordinary differential equation of the first order on the manifold M can be regarded as a smooth mapping $\lambda : \mathbb{R} \to \mathcal{D}(M)$. A solution of this equation is a smooth mapping $\varphi : \mathbb{R} \to M$ such that the tangent vector to the curve φ at any point $t \in \mathbb{R}$ is equal to $\lambda(t)|_{\varphi(t)}$.

Definition. An ordinary differential equation of the first order $\lambda : \mathbb{R} \to \mathcal{D}(M)$ is called *automorphic*, if $\lambda(t) \in \bar{\mathfrak{g}}$ for all $t \in \mathbb{R}$.

Fix a basis $X_1, \dots, X_n$ of $\bar{\mathfrak{g}}$ ($n = \dim \bar{\mathfrak{g}}$). Then any automorphic equation is defined by smooth functions $a_i : \mathbb{R} \to \mathbb{R}, 1 \leqslant i \leqslant n$, such that

$$\lambda(t) = a_1(t)X_1 + \cdots + a_n(t)X_n.$$

The following fact is proved in the theory of differential equations.

Theorem. *For any automorphic equation $\lambda : \mathbb{R} \to \mathcal{D}(M)$ there exists a smooth mapping $g : \mathbb{R} \to \bar{G}$ such that all solutions of the equation have the form:*

$$\varphi(t) = g(t).\varphi(0).$$

Let us explain how to find this mapping g. We shall construct the function of superposition F, which allows to construct the mapping g for an equation $\lambda : \mathbb{R} \to \mathcal{D}(M)$ starting from the number k (known beforehand) of particular solutions of the equation.

Now we introduce the following concept.

Definition. The *stiffness* of the homogeneous space $(\bar{G}, M)$ is the least natural number k for which there exist points $x_1, \dots, x_k$ of M such that the group $\bigcap_{i=1}^{k} \bar{G}_{x_i}$ is discrete.

Let us find the stiffness of some homogeneous spaces described in section 1 and 2.

a) Let $\bar{G} = SL(3, \mathbb{R})$ and $M = \mathbb{R}P^2$. The group $\bar{G}$ acts naturally on M. It is known that the action of $SL(3, \mathbb{R})$ on quadruples of projectively independent points of $\mathbb{R}P^2$ is simply transitive. Thus $k = 4$. For example, for

$$x_1 = (1 : 0 : 0), x_2 = (0 : 1 : 0), x_3 = (0 : 0 : 1), x_4 = (1 : 1 : 1)$$

the group $\bigcap_{i=1}^{4} \bar{G}_{x_i}$ is trivial.

b) The affine group $\mathrm{Aff}(2, \mathbb{R})$ acts on the plane. The action of $\mathrm{Aff}(2, \mathbb{R})$ on triples of affinely independent points of $\mathbb{R}^2$ is simply transitive. Therefore $k = 3$. For example, for $x_1 = (0, 0), x_2 = (1, 0), x_3 = (0, 1)$ the group $\bigcap_{i=1}^{3} \bar{G}_{x_i}$ is trivial.

Assume that the stiffness of $(\bar{G}, M)$ is equal to k and there exist points $x_1, \dots, x_k$ of M such that the group $\bigcap_{i=1}^{k} \bar{G}_{x_i}$ is trivial. Consider the action of $\bar{G}$ on the manifold $M \times M \times \cdot \cdots \times M$($k$ times) defined by

$$g.(m_1, \dots, m_k) = (g.m_1, \dots, g.m_k),$$

where $g \in \bar{G}, m_i \in M, 1 \leqslant i \leqslant k$. Then the action of $\bar{G}$ on the orbit $O(x_1, \dots, x_k)$ of the point $(x_1, \dots, x_k)$ is simply transitive. Consider the *function of superposition*

$$F : O(x_1, \dots, x_k) \to \bar{G},$$

where $F(y_1, \dots, y_k)$ is the element of $\bar{G}$ such that

$$F(y_1, \dots, y_k).(x_1, \dots, x_k) = (y_1, \dots, y_k)$$

In the general case (when the group $\bigcap_{i=1}^k \bar{G}_{x_i}$ is discrete), it is also possible to define a function of superposition but, generally speaking, it will be many-valued.

Now suppose $\lambda : \mathbb{R} \to \mathcal{D}(M)$ is an arbitrary automorphic differential equation. Assume that $\varphi_1, \dots, \varphi_k$ are its particular solutions with initial conditions $\varphi_i(0) = x_i, 1 \leqslant i \leqslant k$. Then it is clear that $(\varphi_1(t), \dots, \varphi_k(t)) \in O(x_1, \dots, x_k)$ for $t \in \mathbb{R}$.

The mapping $g : \mathbb{R} \to \bar{G}$ is defined by $g(t) = F(\varphi_1(t), \dots, \varphi_k(t))$. Thus the knowledge of k particular solutions of the equation λ with definite initial conditions allows to write down the general solution:

$$\varphi(t) = F(\varphi_1(t), \dots, \varphi_k(t)).\varphi(0).$$

Let us find the stiffness k and a function of superposition F for the real homogeneous spaces $1^\circ - 9^\circ$ obtained in section 1.

1°. The group $\bar{G} = SL(3, \mathbb{R})$, $M = \mathbb{R}P^2$. It was mentioned before that the stiffness of this homogeneous space is equal to 4. For example, for $x_1 = (1:0:0), x_2 = (0:1:0), x_3 = (0:0:1), x_4 = (1:1:1)$ we have

$$\bigcap_{i=1}^{4} \bar{G}_{x_i} = \{e\}$$

Suppose $y_i = (a_{1i} : a_{2i} : a_{3i}), 1 \leqslant i \leqslant 4$, is a quadruple of points of $\mathbb{R}P^2$ such that $(y_1, y_2, y_3, y_4) \in O(x_1, x_2, x_3, x_4)$. Direct computation shows that

$$F(y_1, y_2, y_3, y_4) = \lambda A \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{pmatrix},$$

where $A = (a_{ij})_{1 \leqslant i,j \leqslant 3}$, $\begin{pmatrix} p \\ q \\ r \end{pmatrix} = A^{-1} \begin{pmatrix} a_{14} \\ a_{24} \\ a_{34} \end{pmatrix}$, and $\lambda = (pqr \det A)^{-1/3}$.

In the sequel for all homogeneous spaces we write out the stiffness k, points $x_1, \dots, x_k$ such that the group $\bigcap_{i=1}^k \bar{G}_{x_i}$ is discrete, and the function of superposition F.

2°. $\bar{G} = SU(2) = \left\{ \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} \middle| \, |x|^2 + |y|^2 = 1 \right\}$. $M = \mathbb{C}P^1 \approx S^2 = \{(z_0 : z_1)\}$.

The stiffness is $k = 2$.

$$x_1 = (1:0), \ x_2 = (1:1)$$

For $y_1 = (a:b), \ y_2 = (c:d)$, we have

$$F(y_1, y_2) = \begin{pmatrix} \lambda a & -\bar{\lambda}\bar{b} \\ \lambda b & \bar{\lambda}\bar{a} \end{pmatrix},$$

where $\lambda = \sqrt{(\bar{a}c + \bar{b}d)/((|a|^2 + |b|^2)(ad - bc))}$

3°. $\bar{G} = SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ \bar{y} & \bar{x} \end{pmatrix} \middle| \, |x|^2 - |y|^2 = 1 \right\}$. $M = \{z \in \mathbb{C} | |z| < 1\}$.

The action of $\bar{G}$ on M is:

$$\begin{pmatrix} x & y \\ \bar{y} & \bar{x} \end{pmatrix}.(z) = ((\bar{x}z + \bar{y})/(x + yz)).$$

In this case the manifold M can be identified with the Lobachevsky plane. Then $\bar{G}$ is its group of transformations. The stiffness is $k = 2$.

$$x_1 = (0),\ x_2 = (1/2)$$

For $y_1 = (a),\ y_2 = (b)$, we have

$$F(y_1, y_2) = \begin{pmatrix} \lambda & \bar{\lambda}\bar{x} \\ \lambda x & \bar{\lambda} \end{pmatrix},$$

where $\lambda = \sqrt{(\bar{a}b - 1)/(2(1 - |a|^2)(a - b))}$

4°. $\bar{G} = SL(2, \mathbb{C})_{\mathbb{R}}$. $M = \mathbb{C}P^1 \approx S^2$.

The stiffness is $k = 3$.

$$x_1 = (1:0),\ x_2 = (0:1),\ x_3 = (1:1)$$

For $y_i = (a_{1i} : a_{2i}), 1 \leqslant i \leqslant 3$, we have

$$F(y_1, y_2, y_3) = \lambda A \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix},$$

where $A = (a_{ij})_{1 \leqslant i,j \leqslant 3}$, $\begin{pmatrix} p \\ q \end{pmatrix} = A^{-1} \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}$, and $\lambda = (pq \det A)^{-1/2}$.

5°. $\bar{G} = \mathrm{Aff}(2, \mathbb{R}) = GL(2, \mathbb{R}) \leftthreetimes \mathbb{R}^2$, $M = \mathbb{R}^2$.

The stiffness is $k = 3$.

$$x_1 = (0,0),\ x_2 = (1,0),\ x_3 = (0,1)$$

For $y_i = (a_{1i}, a_{2i}), 1 \leqslant i \leqslant 3$, we have

$$F(y_1, y_2, y_3) = \left[\begin{pmatrix} a_{12} - a_{11} & a_{13} - a_{11} \\ a_{22} - a_{21} & a_{23} - a_{21} \end{pmatrix}, \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \right]$$

6°. $\bar{G} = \{(A, v) \in \mathrm{Aff}(2, \mathbb{R}) | \det A = 1\}$, $M = \mathbb{R}^2$.

The stiffness is $k = 3$.

$$x_1 = (0,0),\ x_2 = (1,0),\ x_3 = (0,1)$$

For $y_i = (a_{1i}, a_{2i}),\ 1 \leqslant i \leqslant 3$, we have

$$F(y_1, y_2, y_3) = \left[\begin{pmatrix} a_{12} - a_{11} & a_{13} - a_{11} \\ a_{22} - a_{21} & a_{23} - a_{21} \end{pmatrix}, \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \right]$$

Remark. The examples 5 and 6 are different. In the first case a condition for the point (y_1, y_2, y_3) to belong to the orbit $O(x_1, x_2, x_3)$ has the form:

$$d = \det \begin{pmatrix} a_{12} - a_{11} & a_{13} - a_{11} \\ a_{22} - a_{21} & a_{23} - a_{21} \end{pmatrix} \neq 0.$$

In the second case this condition is $d = 1$.

7°. $\bar{G} = \left\{ (A, v) \in \mathrm{Aff}(2, \mathbb{R}) \;\middle|\; A = e^y \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix},\ y \in \mathbb{R}, x \in [0; 2\pi) \right\}$, $M = \mathbb{R}^2$.

The stiffness is $k = 2$.

$$x_1 = (0, 0),\ x_2 = (1, 0)$$

For $y_1 = (a, b),\ y_2 = (c, d)$, we have

$$F(y_1, y_2) = \left[e^y \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right],$$

where $y = ln\sqrt{(c-a)^2 + (d-b)^2}$ and x can be uniquely determined from the relations:

$$\sin x = e^{-y}(d - b), \quad \cos x = e^{-y}(c - a).$$

8°. $\bar{G} = \left\{ (A,v) \in \mathrm{Aff}(2, \mathbb{R}) \;\middle|\; A = e^{\lambda x} \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}, \begin{matrix} x \in \mathbb{R}, & \text{if } \lambda > 0 \\ x \in [0; 2\pi), & \text{if } \lambda = 0 \end{matrix} \right\}$, $M = \mathbb{R}^2$.

The stiffness is $k = 2$.

$$x_1 = (0, 0),\ x_2 = (1, 0)$$

For $y_1 = (a, b),\ y_2 = (c, d)$, we have

$$F(y_1, y_2) = \left[e^{\lambda x} \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right],$$

where $x = 1/(2\lambda) ln((c-a)^2 + (d-b)^2)$ whenever $\lambda \neq 0$; x can be uniquely determined from the relations:

$$\sin x = d - b, \quad \cos x = c - a$$

whenever $\lambda = 0$.

9°. $\bar{G} = SL(2, \mathbb{R}), M = \{((x_0 : x_1), (y_0 : y_1)) \in \mathbb{R}P^1 \times \mathbb{R}P^1 | x_0 y_0 + x_1 y_1 \neq 0\} / \sim$ (see item 4.1).

The stiffness is $k = 2$.

$$x_1 = ((1 : 0), (1 : 0)),\ x_2 = ((1 : 1), (1 : 0))$$

For $y_1 = ((a_1 : b_1), (c_1 : d_1))$ and $y_2 = ((a_2 : b_2), (c_2 : d_2))$, we have

$$F(y_1, y_2) = \begin{pmatrix} \lambda a_1 & -\mu d_1 \\ \lambda b_1 & \mu c_1 \end{pmatrix},$$

where

$$\lambda = \pm\sqrt{(c_1 a_2 + d_1 b_2)/((a_1 b_2 - b_1 a_2)(a_1 c_1 + b_1 d_1))},$$

$$\mu = \pm\sqrt{(a_1 b_2 - b_1 a_2)/((c_1 a_2 + d_1 b_2)(a_1 c_1 + b_1 d_1))}.$$

Appendix. Classical Lie algebras

Here we describe classical Lie algebras, their notation, their matrix expression, and also a unique realization.

A classical complex Lie algebra is one of the following algebras:

1. $\mathfrak{sl}(n,\mathbb{C})$, the set of square matrices of order n with zero trace. The Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ is simple whenever $n \geqslant 2$ and its dimension is equal to n^2-1.

2. $\mathfrak{so}(n,\mathbb{C})$, the set of skew-symmetric matrices of order n. Since the trace of any skew-symmetric matrix is equal to zero, we see that $\mathfrak{so}(n,\mathbb{C})$ is a subalgebra of the Lie algebra $\mathfrak{sl}(n,\mathbb{C})$. The dimension of $\mathfrak{so}(n,\mathbb{C})$ is equal to $n(n-1)/2$. $\mathfrak{so}(n,\mathbb{C})$ is simple whenever $n=3$ or $n \geqslant 5$. For $n=4$, the following isomorphism exists:

$$\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{C}) \times \mathfrak{so}(3,\mathbb{C}).$$

For $n=2$, the Lie algebra $\mathfrak{so}(2,\mathbb{C})$ is one-dimensional and commutative.

3. $\mathfrak{sp}(2n,\mathbb{C})$, the set of square matrices X of order $2n$ such that the following condition holds:

$$^{t}X J_n + J_n X = 0,$$

where $J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$. Direct calculations show that the Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$ has the form:

$$\left\{ \begin{pmatrix} A & B \\ C & -{}^{t}A \end{pmatrix} \middle| A,B,C \in \mathfrak{gl}(n,\mathbb{C}); B = {}^{t}B, C = {}^{t}C \right\}.$$

Note that for any $X \in \mathfrak{sp}(2n,\mathbb{C})$, we have $\operatorname{tr} X = 0$. Therefore $\mathfrak{sp}(2n,\mathbb{C})$ is a subalgebra of the Lie algebra $\mathfrak{sl}(2n,\mathbb{C})$. The dimension of $\mathfrak{sp}(2n,\mathbb{C})$ is equal to $n(2n+1)$. $\mathfrak{sp}(2n,\mathbb{C})$ is simple for any $n \geqslant 1$. The Lie algebras $\mathfrak{sp}(2,\mathbb{C})$ and $\mathfrak{sl}(2,\mathbb{C})$ coincide.

It should be pointed out that we have the following isomorphism:

$$\mathfrak{so}(3,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) = \mathfrak{sp}(2,\mathbb{C}).$$

Classical real Lie algebras are real forms of classical complex Lie algebras.

Let us recall that there is a one-to-one correspondence between the set of real forms of a complex Lie algebra and the set of anti-involutions of this algebra. Let us classify (up to conjugation) all anti-involutions of classical complex Lie algebras and the corresponding real forms.

1. Any anti-involution of the Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ is conjugate to one and only one of the following anti-involutions:

(1a) $X \mapsto I_{p,q}(-{}^{t}\bar{X})I_{p,q}$, where

$$I_{p,q} = \begin{pmatrix} E_p & 0 \\ 0 & -E_q \end{pmatrix} \text{ and } p+q=n;$$

(1b) $X \mapsto \bar{X}$;

(1c) $n = 2k, \quad X \mapsto J_k \bar{X} J_k^{-1}$, where

$$J_k = \begin{pmatrix} 0 & E_k \\ -E_k & 0 \end{pmatrix}.$$

The set of fixed points of the anti-involution (1a) is denoted by $\mathfrak{su}(p,q)$ or by $\mathfrak{su}(n)$, if $q = 0$. It has the form:

$$\mathfrak{su}(p,q) = \left\{ \begin{pmatrix} X_1 & Y \\ -{}^t\bar{Y} & X_2 \end{pmatrix} \middle| X_1 \in \mathfrak{gl}(p,\mathbb{C}),\ X_2 \in \mathfrak{gl}(q,\mathbb{C}), \right.$$
$$\left. Y \in \mathrm{Mat}_{p\times q}(\mathbb{C}), X_1 + {}^t\bar{X}_1 = X_2 + {}^t\bar{X}_2 = 0,\ \mathrm{tr}(X_1 + X_2) = 0 \right\}.$$

By $\mathfrak{sl}(n,\mathbb{R})$ denote the set of fixed points of the anti-involution (1b). It is the set of all real square matrices of order n with zero trace.

By $\mathfrak{sl}(k,\mathbb{H})$ denote the set of fixed points of the anti-involution (1c). It has the form:

$$\mathfrak{sl}(k,\mathbb{H}) = \left\{ \begin{pmatrix} X & Y \\ -\bar{Y} & \bar{X} \end{pmatrix} \middle| X, Y \in \mathfrak{gl}(k,\mathbb{C}); \Re\,\mathrm{tr}\, X = 0 \right\}.$$

The notation is due to the fact that this Lie algebra can be realized as the set of square matrices of order k over $\mathbb{H}$ with zero trace.

2. Any anti-involution of the Lie algebra $\mathfrak{so}(n,\mathbb{C})$ is conjugate to one and only one of the following anti-involutions:

(2a) $X \mapsto I_{p,q} \bar{X} I_{p,q}$, where $p + q = n$;

(2b) $n = 2k, \quad X \mapsto J_k \bar{X} J_k^{-1}$.

The set of fixed points of the anti-involution (2a) is denoted by $\mathfrak{so}(p,q)$ or by $\mathfrak{so}(n)$, if $q = 0$. It has the form:

$$\mathfrak{so}(p,q) = \left\{ \begin{pmatrix} X_1 & Y \\ {}^t\bar{Y} & X_2 \end{pmatrix} \middle| X_1 \in \mathfrak{gl}(p,\mathbb{R}),\ X_2 \in \mathfrak{gl}(q,\mathbb{R}),\ Y \in \mathrm{Mat}_{p\times q}(\mathbb{R}), \right.$$
$$\left. X_1 + {}^tX_1 = X_2 + {}^tX_2 = 0,\ \mathrm{tr}(X_1 + X_2) = 0 \right\}.$$

The set of fixed points of the anti-involution (2a) is denoted by $\mathfrak{u}^*(k,\mathbb{H})$ and has the form:

$$\mathfrak{u}^*(k,\mathbb{H}) = \left\{ \begin{pmatrix} X & Y \\ -\bar{Y} & \bar{X} \end{pmatrix} \middle| X, Y \in \mathfrak{gl}(k,\mathbb{C}); X + {}^tX = 0,\ {}^t\bar{Y} = Y \right\}.$$

The notation is due to the fact that this Lie algebra can be realized as the set of skew-Hermitian matrices of order k over $\mathbb{H}$.

3. Any anti-involution of the Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$ is conjugate to one and only one of the following anti-involutions:

(3a) $X \mapsto \bar{X}$;

(3b) $X \mapsto K_{p,q}(-{}^t\bar{X})K_{p,q}$, where

$$p + q = n \text{ and } K_{p,q} = \begin{pmatrix} I_{p,q} & 0 \\ 0 & I_{p,q} \end{pmatrix}.$$

The set of fixed points of the anti-involution (3a) is denoted by $\mathfrak{sp}(p,q)$ (note that $p+q$ is equal to n, not $2n$) and has the form:

$$\mathfrak{sp}=\left\{\left(\begin{array}{cccc} X_{11} & X_{12} & X_{13} & X_{14} \\ {}^t\bar{X}_{12} & X_{22} & {}^tX_{14} & X_{24} \\ -\bar{X}_{13} & \bar{X}_{14} & \bar{X}_{11} & -\bar{X}_{12} \\ {}^t\bar{X}_{14} & -\bar{X}_{24} & -{}^tX_{12} & \bar{X}_{22} \end{array}\right)\right| X_{11},X_{13}\in\mathfrak{gl}(p,\mathbb{C}); X_{22},X_{24}\in\mathfrak{gl}(\mathfrak{g},\mathbb{C});$$
$$X_{12},X_{14}\in \mathrm{Mat}_{p\times q}(\mathbb{C});\ {}^t\bar{X}_{11}+X_{11}={}^t\bar{X}_{22}+X_{22}=0;\ {}^tX_{13}=X_{13},{}^tX_{24}=X_{24}\Big\}.$$

It is known that there exists a unique (up to isomorphism) compact real form of a semisimple complex Lie algebra. Compact real forms of classical complex Lie algebras are defined by the Cartan anti-involution: $X\mapsto -{}^t\bar{X}$. They are:

$$\mathfrak{su}(n) \quad \text{for } \mathfrak{sl}(n,\mathbb{C}); \quad \mathfrak{s}\sigma(n) \quad \text{for } \mathfrak{s}\sigma(n,\mathbb{C}); \quad \mathfrak{sp}(n) \quad \text{for } \mathfrak{sp}(2n,\mathbb{C}).$$

Let us describe the general realization of all classical (both complex and real) Lie algebras. Suppose V is a finite-dimensional (right) vector space over the field P, where $P=\mathbb{R},\mathbb{C}$, or $\mathbb{H}$. Fix on arbitrary basis $\{e_1,\dots,e_n\}$ of V ($n=\dim V$). We identify endomorphisms of V with their matrices. Then the set of all endomorphisms of V can be denoted by $\mathfrak{gl}(n,P)$. The set $\mathfrak{gl}(n,P)$ is supplied with the bracket operation $[A,B]=AB-BA$, which turns $\mathfrak{gl}(n,\mathbb{R})$ ($\mathfrak{gl}(n,\mathbb{C})$) into a real (complex) Lie algebra of dimension n^2. Since the field $\mathbb{H}$ is not commutative, the set $\mathfrak{gl}(n,\mathbb{H})$ can be regarded only as a real Lie algebra of dimension $4n^2$.

We have the natural inclusion map of Lie algebras

$$\mathfrak{gl}(n,\mathbb{R})\longrightarrow \mathfrak{gl}(n,\mathbb{C})_{\mathbb{R}}$$

which allows to identify the set $\mathfrak{gl}(n,\mathbb{R})$ with a subspace of $\mathfrak{gl}(n,\mathbb{C})$. Let us construct the inclusion map

$$\pi:\mathfrak{gl}(n,\mathbb{H})\longrightarrow\mathfrak{gl}(2n,\mathbb{C}).$$

We identify the subset $\{a+bi\,|\,a,b\in\mathbb{R}\}$ of $\mathbb{H}$ with the field of complex numbers. Any matrix $A\in\mathfrak{gl}(n,\mathbb{H})$ can be uniquely decomposed: $A=X+Yj$, where X and Y are complex matrices. Put

$$\pi(A)=\begin{pmatrix} X & Y \\ -\bar{Y} & \bar{X}\end{pmatrix}.$$

Then the map π is $\mathbb{R}$-linear and injective. Moreover for any $A,B\in\mathfrak{gl}(n,\mathbb{H})$, the following conditions hold:

$$\pi(AB)=\pi(A)\pi(B) \text{ and } \pi({}^t\bar{A})={}^t\overline{\pi(A)}.$$

This allows to identify the set $\mathfrak{gl}(n,\mathbb{H})$ with the subset

$$\left\{\left.\begin{pmatrix} X & Y \\ -\bar{Y} & \bar{X}\end{pmatrix}\right| X,Y\in\mathfrak{gl}(n,\mathbb{C})\right\}$$

of $\mathfrak{gl}(2n,\mathbb{C})$.

In the sequel, $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{gl}(n, \mathbb{H})$ will mean the corresponding subsets of $\mathfrak{gl}(n, \mathbb{C})$ and $\mathfrak{gl}(2n, \mathbb{C})$ respectively.

Suppose $P = \mathbb{R}$ or $\mathbb{C}$ and f is a bilinear form on V. We say that a matrix $A \in \mathfrak{gl}(n, P)$ *preserves the form* f, if

$$(1) \qquad f(Ax, y) + f(x, Ay) = 0 \text{ for all } x, y \in V.$$

Let $G = (g_{ij})_{1 \leqslant i,j \leqslant n}$ be the Gram matrix of f in the fixed basis ($g_{ij} = f(e_i, e_j)$, $1 \leqslant i, j \leqslant n$). Then condition (1) is equivalent to the matrix equality

$$^tAG + GA = 0.$$

By $\mathrm{Der}(f)$ denote the set of all matrices preserving the form f. It is easily proved that $\mathrm{Der}(f)$ is a Lie algebra over the field P.

The fields $\mathbb{C}$ and $\mathbb{H}$ are supplied with the operation of conjugation $x \to \bar{x}$ which is an automorphism of second order for $\mathbb{C}$ and an anti-automorphism of second order for $\mathbb{H}$ (i.e., $\overline{ab} = \bar{b}\bar{a}$ for all $a, b \in \mathbb{H}$). This allows to consider sesquilinear forms over $\mathbb{C}$ and $\mathbb{H}$, i.e., additive mappings of $V \times V$ into P such that the following condition is satisfied:

$$f(xa, yb) = \bar{a} f(x, y) b \quad \text{for all } a, b \in P, \ x, y \in V.$$

We say that a matrix $A \in \mathfrak{gl}(n, P)$ *preserves a sesquilinear form* f, if

$$(2) \qquad f(Ax, y) + f(x, Ay) = 0 \text{ for all } x, y \in V.$$

It is possible to define the Gram matrix $G = (g_{ij})_{1 \leqslant i,j \leqslant n}$ for a sesquilinear form f by means of the formula

$$g_{ij} = f(e_i, e_j), \quad 1 \leqslant i, j \leqslant n.$$

Then condition (2) is equivalent to the matrix equality

$$^t\bar{A}G + GA = 0.$$

By $\mathrm{Der}(f)$ denote the set of all matrices preserving the sesquilinear form f. The set $\mathrm{Der}(f)$ is a Lie algebra over the field $\mathbb{R}$.

Let us recall that a bilinear form f is called *symmetric (skew-symmetric)*, if

$$f(x, y) = f(y, x) \ (f(x, y) = -f(y, x)) \quad \text{for all } x, y \in V.$$

Similarly, a sesquilinear form f is called *Hermitian (skew-Hermitian)*, if

$$f(x, y) = \overline{f(y, x)} \left(f(x, y) = -\overline{f(y, x)} \right) \quad \text{for all } x, y \in V.$$

Now show that any classical Lie algebra is either a set of matrices of trace 0 over $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ or a Lie algebra $\mathrm{Der}(f)$, for some nondegenerate

symmetric or skew-symmetric form f whenever $P = \mathbb{R}$, $\mathbb{C}$;

Hermitian or skew-Hermitian form f whenever $P = \mathbb{C}$, $\mathbb{H}$.

1. $P = \mathbb{R}$.

The set of real matrices of trace 0 is the classical Lie algebra $\mathfrak{sl}(n, \mathbb{R})$. It is a real form of the classical complex Lie algebra $\mathfrak{sl}(n, \mathbb{C})$.

Suppose f is a nondegenerate symmetric bilinear form. The Gram matrix of f (viewed up to conjugation) has the form:

$$G = \begin{pmatrix} E_n & 0 \\ 0 & -E_q \end{pmatrix}, \text{ where } p + q = n.$$

Then $\mathrm{Der}(f) = \{A \in \mathfrak{gl}(n, \mathbb{R}) | \ {}^tAG + GA = 0\}$ is the Lie algebra $\mathfrak{so}(p, q)$. It is a real form of the Lie algebra $\mathfrak{so}(n, \mathbb{C})$.

Suppose f is a nondegenerate symmetric bilinear form. Then $n = 2k$ and it can be assumed that the Gram matrix of f has the form:

$$G = \begin{pmatrix} 0 & E_k \\ -E_k & 0 \end{pmatrix}.$$

$\mathrm{Der}(f)$ is exactly the Lie algebra $\mathfrak{sp}(2k, \mathbb{R})$. It is a real form of the Lie algebra $\mathfrak{sp}(2k, \mathbb{C})$.

2. Let $P = \mathbb{C}$.

The set of complex matrices of trace 0 is the classical Lie algebra $\mathfrak{sl}(n, \mathbb{C})$.

Suppose f is a nondegenerate symmetric bilinear form. We obtain (up to conjugation) $G = E_n$. Then

$$\mathrm{Der}(f) = \{A \in \mathfrak{gl}(n, \mathbb{C}) | \ {}^tA + A = 0\}$$

is the set of all skew-symmetric matrices. It is the classical Lie algebra $\mathfrak{so}(n, \mathbb{C})$. If $n = 2k$, there exists another matrix representation of the Lie algebra $\mathfrak{so}(n, \mathbb{C})$. In this case it can be assumed that $G = \begin{pmatrix} 0 & E_k \\ E_k & 0 \end{pmatrix}$. Then

$$\mathrm{Der}(f) = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \middle| B + {}^tB = 0, \ C + {}^tC = 0; \ A, B, C \in \mathfrak{gl}(n, \mathbb{C}). \right\}$$

is the Lie algebra isomorphic to $\mathfrak{so}(n, \mathbb{C})$.

Suppose $n = 2k$ and f is a nondegenerate skew-symmetric bilinear form. Then it can be assumed that

$$G = \begin{pmatrix} 0 & E_k \\ -E_k & 0 \end{pmatrix}.$$

The Lie algebra $\mathrm{Der}(f)$ is the classical complex Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$.

Now suppose f is a nondegenerate Hermitian form. Then its Gram matrix is reduced to the form:

$$G = \begin{pmatrix} E_p & 0 \\ 0 & -E_q \end{pmatrix}, \text{ where } p + q = n.$$

In this case the real Lie algebra $\mathrm{Der}(f)$ coincides with the real form $\mathfrak{su}(p, q)$ of the classical complex Lie algebra $\mathfrak{sl}(n, \mathbb{C})$.

Suppose f is a nondegenerate skew-Hermitian form. Then the sesquilinear form g defined by

$$g(x,y) = if(x,y) \quad \text{for all } x, y \in V$$

is Hermitian. It is easily proved that $\mathrm{Der}(f) = \mathrm{Der}(g)$. Thus Lie algebras preserving skew-Hermitian forms and Lie algebras preserving Hermitian forms coincide (over the field $\mathbb{C}$.)

3. Now let $P = \mathbb{H}$.

The set of all quaternion matrices of trace 0 is a real Lie algebra $\mathfrak{sl}(n, \mathbb{H})$ which is a real form of the classical complex Lie algebra $\mathfrak{sl}(2n, \mathbb{C})$.

Suppose f is the Hermitian form with the Gram matrix

$$G = \begin{pmatrix} E_p & 0 \\ 0 & -E_q \end{pmatrix}.$$

Consider the real Lie algebra

$$\mathrm{Der}(f) = \{ A \in \mathfrak{gl}(n, \mathbb{H}) | \ {}^t\bar{A}\bar{G} + \bar{G}A = 0 \}.$$

Direct calculations show that $\mathrm{Der}(f)$ is the Lie algebra $\mathfrak{sp}(p,q)$, that is a real form of the complex Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$.

Finally, suppose f is the skew-Hermitian form with the Gram matrix $G = jE_n$. Then $\mathrm{Der}(f)$ is the real Lie algebra $\mathfrak{u}^*(n, \mathbb{H})$ which is a real form of the classical complex Lie algebra $\mathfrak{so}(2n, \mathbb{C})$.

Tables

In tables below we list complex and real almost primitive pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$, where $\bar{\mathfrak{g}}$ is simple and $\mathfrak{g}$ is a reductive subalgebra of $\bar{\mathfrak{g}}$. When $\bar{\mathfrak{g}}$ is a classical Lie algebra the results are written in classical notation. (See Appendix).

In the column "1" we put the algebra $\bar{\mathfrak{g}}$ and the subalgebra $\mathfrak{g}$ of $\bar{\mathfrak{g}}$.

In the column "2" we put "++", "+", or "−", if the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is maximal, primitive but not maximal, or almost primitive but not primitive respectively.

We say that two subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of a Lie algebra $\bar{\mathfrak{g}}$ are *equivalent* (*conjugate*), if there exists an automorphism (inner automorphism) f of $\bar{\mathfrak{g}}$, such that $f(\mathfrak{g}_1) = \mathfrak{g}_2$. The algebras listed in the tables are classified up to equivalence.

Every class of equivalent subalgebras can be divided into a number of classes of conjugate subalgebras. In the column "3" we put the number of these classes.

Table 1. Complex almost primitive classical pairs

1			2	3
(1a)	$\mathfrak{sl}(n+m,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{C})\oplus\mathfrak{sl}(m,\mathbb{C})\oplus\mathbb{C}$	$n\neq m: -$ $n=m: +$	1
(2a)	$\mathfrak{sl}(nr,\mathbb{C})$	$\overset{r}{\underset{i=1}{\oplus}}\mathfrak{sl}(n,\mathbb{C})\oplus\mathbb{C}^{r-1}$	$+$	1
(3a)	$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{gl}(n,\mathbb{C})$	n is even: $+$ n is odd : $-$	n is even: 2 n is odd : 1
(3a)	$\mathfrak{sp}(2n,\mathbb{C})$	$\mathfrak{gl}(n,\mathbb{C})$	$+$	1
(4a)	$\mathfrak{so}(n+2,\mathbb{C})$	$\mathfrak{so}(n,\mathbb{C})\oplus\mathbb{C}$	$+$	1
(5a)	$\mathfrak{so}(2n,\mathbb{C})$	$\mathbb{C}^n$	$+$	1
(1b)	$\mathfrak{so}(n+m,\mathbb{C})$	$\mathfrak{so}(n,\mathbb{C})\oplus\mathfrak{so}(m,\mathbb{C})$	$++$	1
(1b)	$\mathfrak{sp}(2(n+m),\mathbb{C})$	$\mathfrak{sp}(2n,\mathbb{C})\oplus\mathfrak{sp}(2m,\mathbb{C})$	$++$	1
(2b)	$\mathfrak{so}(nr,\mathbb{C})$	$\overset{r}{\underset{i=1}{\oplus}}\mathfrak{so}(n,\mathbb{C})$	$+$	1
(2b)	$\mathfrak{sp}(2rn,\mathbb{C})$	$\overset{r}{\underset{i+1}{\oplus}}\mathfrak{sp}(2n,\mathbb{C})$	$+$	1
(1c)	$\mathfrak{sl}(nm,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{C})\oplus\mathfrak{sl}(m,\mathbb{C})$	$++$	1
(2c)	$\mathfrak{sl}(n^r,\mathbb{C})$	$\overset{r}{\underset{i=1}{\oplus}}\mathfrak{sl}(n,\mathbb{C})$	$+$	1
(3c)	$\mathfrak{so}(nm,\mathbb{C})$	$\mathfrak{so}(n,\mathbb{C})\oplus\mathfrak{so}(m,\mathbb{C})$	$++$	$n\neq m,\ n,m$ are even; $n=m=4k:2$ $n=m\neq 4k$; $n\neq m,\ n$ or m is odd: 1
(3c)	$\mathfrak{so}(4nm,\mathbb{C})$	$\mathfrak{sp}(2n,\mathbb{C})\oplus\mathfrak{sp}(2m,\mathbb{C})$	$++$	$n=m=2k+1:1$ $n=m=2k;\ n\neq m:2$
(3c)	$\mathfrak{sp}(2nm,\mathbb{C})$	$\mathfrak{sp}(2n,\mathbb{C})\oplus\mathfrak{sp}(m,\mathbb{C})$	$++$	1
(4c)	$\mathfrak{so}(4n,\mathbb{C})$	$\mathfrak{sp}(2,\mathbb{C})\oplus\mathfrak{sp}(2,\mathbb{C})\oplus\mathfrak{so}(n,\mathbb{C})$	n is odd: $-$ n is even: $+$	n is odd: 1 n is even: 2
(4c)	$\mathfrak{sp}(8n,\mathbb{C})$	$\mathfrak{sp}(2n,\mathbb{C})\oplus\mathfrak{sp}(2,\mathbb{C})\oplus\mathfrak{sp}(2,\mathbb{C})$	$+$	1
(5c)	$\mathfrak{so}(n^r,\mathbb{C})$	$\overset{r}{\underset{i+1}{\oplus}}\mathfrak{so}(n,\mathbb{C})$	$+$	n is even: 2 n is odd: 1
(5c)	$\mathfrak{sp}((2n)^{2r+1},\mathbb{C})$	$\overset{2r+1}{\underset{i=1}{\oplus}}\mathfrak{sp}(2n,\mathbb{C})$	$r=n=1:++$ $r>1$ *or* $n>1:+$	1
(5c)	$\mathfrak{so}((2n)^{2r},\mathbb{C})$	$\overset{2r}{\underset{i=1}{\oplus}}\mathfrak{sp}(2n,\mathbb{C})$	$+$	2
(d1)	$\bar{\mathfrak{g}}$ is a classical Lie algebra	$\mathfrak{g}$ is an irreducible subalgebra of $\bar{\mathfrak{g}}$	$++$	n
(d2)	$\mathfrak{so}(495,\mathbb{C})$	$\mathfrak{so}(12,\mathbb{C})$	$+$	1

Remarks.

(1) In irreducible case (d1) we assume that the pair $(\bar{\mathfrak{g}},\mathfrak{g})$ is not one of the 19 exeptions determined by Dynkin [6].

(2) Conditions on the parameters n, m, r are the same as in Lemmas 1–4 (section 2.1).

Table 2. Complex almost primitive exceptional pairs

1		2	3
G_2	$A_1 + A_1$	++	1
	A_2	++	1
	A_1^2	++	1
F_4	B_4	++	1
	$A_1 + C_3$	++	1
	$A_2 + A_2$	++	1
	D_4	+	1
	A_1^{156}	++	1
	$G_2^1 + A_1^8$	++	1
E_6	$D_4 + \mathbb{C}^2$	+	1
	$D_5 + \mathbb{C}$	−	1
	$\mathbb{C}^6$	+	1
	$A_1 + A_5$	++	1
	$A_2 + A_2 + A_2$	++	1
	G_2^3	++	2
	C_4^1	++	1
	$G_2^1 + A_2^{2''}$	++	1
	F_4^1	++	1
	A_2^9	++	2
E_7	$E_6 + \mathbb{C}$	+	1
	$\mathbb{C}^7$	+	1
	A_7	++	1
	$A_1 + D_6$	++	1
	$A_2 + A_5$	++	1
	$7A_1$	+	1
	$D_4 + A_1 + A_1 + A_1$	+	1
	A_1^{399}	++	1
	A_1^{231}	++	1
	$G_2^1 + G_3^{1''}$	++	1
	$F_4^1 + A_1^{3''}$	++	1
	$G_2^2 + A_1^7$	++	1
	A_1^{21}	++	1
	$A_1^{24} + A_1^{15}$	++	1
	D_4^2	+	1

Table 2. (continued)

1		2	3
E_8	$\mathbb{C}^8$	$+$	1
	D_8	$++$	1
	$E_7 + A_1$	$++$	1
	$A_4 + A_4$	$++$	1
	A_8	$++$	1
	$E_6 + A_2$	$++$	1
	$D_4 + D_4$	$+$	1
	$8A_1$	$+$	1
	$4A_2$	$+$	1
	A_1^{1240}	$++$	1
	A_1^{760}	$++$	1
	A_1^{520}	$++$	1
	$G_2^1 + F_4^1$	$++$	1
	$A_2^{6'} + A_1^{16}$	$++$	1
	B_2^{12}	$++$	1
	$G_2^1 + G_2^1 + A_1^8$	$+$	1
	A_1^{40}	$+$	1
D_4	$A_2 + \mathbb{C}^2$	$-$	1
	$A_3 + \mathbb{C}$	$+$	3
	$\mathbb{C}^4$	$+$	1
	$4A_1$	$++$	1
	B_3^1	$++$	3
	$A_1^2 + B_2^1$	$++$	3
	A_2^3	$++$	1
	G_2^1	$-$	1

Table 3. Real almost primitive classical pairs

1		2	3
$\mathfrak{su}(n_1+m_1, n_2+m_2)$	$\mathfrak{su}(n_1,n_2)\oplus\mathfrak{su}(m_1,m_2)\oplus\mathbb{T}$	++	$n_1+m_1=n_2+m_2$, $(n_1,m_1)\neq(n_2,m_2)$, $(n_1,m_1)\neq(m_2,n_2)$: 2 otherwise: 1
$\mathfrak{su}(n,n)$	$\mathfrak{sl}(n,\mathbb{C})_{\mathbb{R}}\oplus\mathbb{R}$	$n\neq m: -$ $n=m: +$	1
$\mathfrak{sl}(n+m,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{R})\oplus\mathfrak{sl}(m,\mathbb{R})\oplus\mathbb{R}$	$n\neq m: -$ $n=m: +$	1
$\mathfrak{sl}(n+m,\mathbb{H})$	$\mathfrak{sl}(n,\mathbb{H})\oplus\mathfrak{sl}(m,\mathbb{H})\oplus\mathbb{R}$	++	1
$\mathfrak{sl}(2n,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{C})_{\mathbb{R}}\oplus\mathbb{T}$	++	n is even:2 n is odd:1
$\mathfrak{sl}(n,\mathbb{H})$	$\mathfrak{sl}(n,\mathbb{C})_{\mathbb{R}}\oplus\mathbb{T}$	++	1
$\mathfrak{su}(r(n-t),rt)$	$\bigoplus_{i=1}^{r}\mathfrak{su}(n-t,t)\oplus\mathbb{T}^{r-1}$	+	1
$\mathfrak{sl}(rn,\mathbb{R})$	$\bigoplus_{i=1}^{r}\mathfrak{sl}(n,\mathbb{R})\oplus\mathbb{R}^{r-1}$	+	1
$\mathfrak{sl}(rn,\mathbb{H})$	$\bigoplus_{i=1}^{r}\mathfrak{sl}(n,\mathbb{H})\oplus\mathbb{R}^{r-1}$	+	1
$\mathfrak{sp}(t,n-t)$	$\mathfrak{su}(t,n-t)\oplus\mathbb{T}$	++	1
$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{su}(t,n-t)\oplus\mathbb{T}$	++	1
$\mathfrak{sp}(n,n)$	$\mathfrak{sl}(n,\mathbb{H})\oplus\mathbb{R}$	+	1
$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{gl}(n,\mathbb{R})$	+	1
$\mathfrak{so}(2t,2(n-t))$	$\mathfrak{su}(t,n-t)\oplus\mathbb{T}$	++	n is odd, $t=0:1$ n and t are even, $t\neq 0:4$ otherwise: 2
$\mathfrak{u}^*(n,\mathbb{H})$	$\mathfrak{su}(t,n-t)\oplus\mathbb{T}$	++	1
$\mathfrak{so}(n,n)$	$\mathfrak{gl}(n,\mathbb{R})$	n is even: $-$ n is odd : $+$	n is even: 2 n is odd : 1
$\mathfrak{u}^*(2n,\mathbb{H})$	$\mathfrak{sl}(n,\mathbb{H})\oplus\mathbb{R}$	+	1
$\mathfrak{so}(t+2,n-t)$	$\mathfrak{so}(t,n-t)\oplus\mathbb{T}$	++	$n=2(t+1):2$ $n\neq 2(t+1):1$
$\mathfrak{so}(t+1,n-t+1)$	$\mathfrak{so}(t,n-t)\oplus\mathbb{R}$	+	1
$\mathfrak{u}^*(n+1,\mathbb{H})$	$\mathfrak{u}^*(n,\mathbb{H})\oplus\mathbb{T}$	++	1
$\mathfrak{so}(2n,\mathbb{R})$	$\mathbb{T}^n$	+	1
$\mathfrak{so}(n,n)$	$\mathbb{R}^n$	+	1
$\mathfrak{sp}(n_1+m_1,n_2+m_2)$	$\mathfrak{sp}(n_1,n_2)\oplus\mathfrak{sp}(m_1,m_2)$	++	$n_1+m_1=n_2+m_2$, $(n_1,m_1)\neq(n_2,m_2)$, $(n_1,m_1)\neq(m_2,n_2)$: 2 otherwise: 1

Table 3. (continued)

1		2	3
$\mathfrak{sp}(2(n+m),\mathbb{R})$	$\mathfrak{sp}(2n,\mathbb{R})\oplus\mathfrak{sp}(2m,\mathbb{R})$	++	1
$\mathfrak{sp}(n,n)$	$\mathfrak{sp}(n,\mathbb{C})_{\mathbb{R}}$	++	1
$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{C})_{\mathbb{R}}$	++	1
$\mathfrak{so}(n_1+m_1,n_2+m_2)$	$\mathfrak{so}(n_1,n_2)\oplus\mathfrak{so}(m_1,m_2)$	++	$n_1+m_1=n_2+m_2$, $(n_1,m_1)\neq(n_2,m_2)$: 2 otherwise: 1
$\mathfrak{u}^*(n+m,\mathbb{H})$	$\mathfrak{u}^*(n,\mathbb{H})\oplus\mathfrak{u}^*(m,\mathbb{H})$	++	1
$\mathfrak{so}(n,n)$	$\mathfrak{so}(n,\mathbb{C})_{\mathbb{R}}$	++	n is even: 2 n is odd : 1
$\mathfrak{u}^*(n,\mathbb{H})$	$\mathfrak{so}(n,\mathbb{C})_{\mathbb{R}}$	++	1
$\mathfrak{sp}(rt,r(n-t))$	$\bigoplus_{i=1}^{r}\mathfrak{sp}(t,n-t)$	+	1
$\mathfrak{sp}(2nr,\mathbb{R})$	$\bigoplus_{i=1}^{r}\mathfrak{sp}(2n,\mathbb{R})$	+	1
$\mathfrak{so}(rn,r(n-t))$	$\bigoplus_{i=1}^{r}\mathfrak{so}(t,n-t)$	+	1
$\mathfrak{u}^*(rn,\mathbb{H})$	$\bigoplus_{i=1}^{r}\mathfrak{u}^*(n,\mathbb{H})$	+	1
$\mathfrak{su}(tm+sn-2st,mn-tm-sn+2st)$	$\mathfrak{su}(t,n-t)\oplus\mathfrak{su}(s,m-s)$	++	1
$\mathfrak{sl}(nm,\mathbb{H})$	$\mathfrak{sl}(n,\mathbb{R})\oplus\mathfrak{sl}(m,\mathbb{H})$	++	1
$\mathfrak{sl}(nm,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{R})\oplus\mathfrak{sl}(m,\mathbb{R})$	++	n or m is odd: 1 $n=m=4k+2:1$ otherwise: 2
$\mathfrak{sl}(4nm,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{H})\oplus\mathfrak{sl}(m,\mathbb{H})$	++	$n=m=2k+1:1$ otherwise: 2
$\mathfrak{su}(n(n+1)/2,n(n-1)/2)$	$\mathfrak{sl}(n,\mathbb{C})_{\mathbb{R}}$	++	1
$\mathfrak{sl}(n^2,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{C})_{\mathbb{R}}$	++	n is even: 2 n is odd : 1
$\mathfrak{su}(\frac{n^r+(n-2t)^r}{2},\frac{n^r-(n-2t)^r}{2})$	$\bigoplus_{i=1}^{r}\mathfrak{su}(t,n-t)$	+	1
$\mathfrak{sl}(2^{2r}n^{2r+1},\mathbb{H})$	$\bigoplus_{i=1}^{2r+1}\mathfrak{sl}(n,\mathbb{H})$	+	1
$\mathfrak{sl}(2n^{2r},\mathbb{R})$	$\bigoplus_{i=1}^{2r}\mathfrak{sl}(n,\mathbb{H})$	+	2
$\mathfrak{sl}(n^r,\mathbb{R})$	$\bigoplus_{i=1}^{r}\mathfrak{sl}(n,\mathbb{R})$	+	n is even: 2 n is odd : 1
$\mathfrak{u}^*(2nm,\mathbb{H})$	$\mathfrak{sp}(2n,\mathbb{R})\oplus\mathfrak{sp}(s,m-s)$	++	1
$\mathfrak{su}(4nm)$	$\mathfrak{sp}(2n,\mathbb{R})\oplus\mathfrak{sp}(2m,\mathbb{R})$	++	$n=m=2k+1:1$ $n\neq m$, n,m are odd: 2 otherwise: 4
$\mathfrak{su}(4ns+4mt-8st,4nm-4ns-4mt+8st)$	$\mathfrak{sp}(t,n-t)\oplus\mathfrak{sp}(s,m-s)$	++	$n=m=2k+1, s=t=0:1$ $n=m=2k+1, s=t\neq0$; $n=m=2k, s=t=0$; $n\neq m, s=t=0:2$ otherwise: 4

Table 3. (continued)

1		2	3
$\mathfrak{su}(2n^2-n, 2n^2+n)$	$\mathfrak{sp}(2n, \mathbb{C})_{\mathbb{R}}$	++	n is even: 4 n is odd : 2
$\mathfrak{sp}(ns+mt-2st, nm-ns-mt+2st)$	$\mathfrak{sp}(t, n-t) \oplus \mathfrak{so}(s, m-s)$	++	1
$\mathfrak{sp}(4nm, \mathbb{R})$	$\mathfrak{sp}(t, n-t) \oplus \mathfrak{sl}(m, \mathbb{H})$	++	1
$\mathfrak{sp}(2nm, \mathbb{R})$	$\mathfrak{sp}(2n, \mathbb{R}) \oplus \mathfrak{so}(s, m-s)$	++	1
$\mathfrak{sp}(4nm)$	$\mathfrak{sp}(2n, \mathbb{R}) \oplus \mathfrak{sl}(m, \mathbb{H})$	++	1
$\mathfrak{so}(ns+mt-2st, nm-ns-mt+2st)$	$\mathfrak{so}(t, n-t) \oplus \mathfrak{so}(s, m-s)$	++	n and m are odd: 1 nm is even, $n+m$ is odd, $t=s=0$ or $ts \neq 0 : 1$ $n=m=4k+2$, $s=t=0 : 1$ $n=m=4k+2$, $s=t=2l+1 : 1$ nm is even, $n+m$ is odd, $t=0$, $s \neq 0$ or $t \neq 0$, $s=0 : 2$ n, m are even, $n \neq m$, $s=t=0 : 2$ n, m are even, $n \neq m$, s or t is odd: 2 $n=m=4k$, $s=t=0 : 2$ $n=m=4k+2$, $s=t=2l \neq 0 : 2$ $n=m$, s or t is odd, $s \neq t : 2$ $n=m=4k$, $s=t=2l+1 : 2$ otherwise: 4
$\mathfrak{u}^*(nm, \mathbb{H})$	$\mathfrak{u}^*(n, \mathbb{H}) \oplus \mathfrak{so}(s, m-s)$	++	1
$\mathfrak{so}(4nm)$	$\mathfrak{u}^*(n, \mathbb{H}) \oplus \mathfrak{u}^*(m, \mathbb{H})$	++	$n=m=2k+1 : 1$ $n \neq m$, n, m are odd: 2 otherwise: 4
$\mathfrak{so}(\frac{n(n+1)}{2}, \frac{n(n-1)}{2})$	$\mathfrak{so}(n, \mathbb{C})_{\mathbb{R}}$	++	$n=4k+2 : 1$ otherwise: 2
$\mathfrak{so}(4t, 4(n-t))$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{so}(t, n-t)$	n and t are even: + otherwise : −	n is odd, $t=0 : 1$ n and t are even: 4 otherwise : 2
$\mathfrak{so}(4n)$	$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(t, n-t)$	n and t are even: + otherwise: −	n is odd: 1 n and t are even: 4 otherwise: 2
$\mathfrak{so}(n+2t, 3n-2t)$	$\mathfrak{sp}(2, \mathbb{C})_{\mathbb{R}} \oplus \mathfrak{so}(t, n-t)$	++	n is odd : 1 n is even: 2
$\mathfrak{u}^*(4n, \mathbb{H})$	$\mathfrak{sp}(2, \mathbb{C})_{\mathbb{R}} \oplus \mathfrak{u}^*(n, \mathbb{H})$	++	1
$\mathfrak{u}^*(4n, \mathbb{H})$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}^*(n, \mathbb{H})$	+	1
$\mathfrak{u}^*(4n, \mathbb{H})$	$\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{u}^*(n, \mathbb{H})$	+	1
$\mathfrak{sp}(4t, 4(n-t))$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{sp}(t, n-t)$	+	1

Table 3. (continued)

1		2	3
$\mathfrak{sp}(8n,\mathbb{R})$	$\mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{sp}(2n,\mathbb{R})$	$+$	1
$\mathfrak{sp}(4n)$	$\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sp}(t,n-t)$	$+$	1
$\mathfrak{sp}(8n,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sp}(2n,\mathbb{R})$	$+$	1
$\mathfrak{sp}(n+2t,3n-2t)$	$\mathfrak{sp}(2,\mathbb{C})_{\mathbb{R}}\oplus\mathfrak{sp}(t,n-t)$	$++$	1
$\mathfrak{sp}(8n,\mathbb{R})$	$\mathfrak{sp}(2,\mathbb{C})_{\mathbb{R}}\oplus\mathfrak{sp}(2n,\mathbb{R})$	$++$	1
$\mathfrak{sp}(4)$	$\mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(2)$	$++$	1
$\mathfrak{sp}(8,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(2)$	$++$	1
$\mathfrak{sp}(2,2)$	$\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{su}(2)$	$++$	1
$\mathfrak{sp}(8,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$	$++$	1
$\mathfrak{sp}(3,1)$	$\mathfrak{su}(2)\oplus\mathfrak{sp}(2,\mathbb{C})_{\mathbb{R}}$	$++$	1
$\mathfrak{sp}(8,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sp}(2,\mathbb{C})_{\mathbb{R}}$	$++$	1
$\mathfrak{so}(16)$	$4\mathfrak{su}(2)$	$+$	2
$\mathfrak{so}(8,8)$	$\mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})$	$+$	4
$\mathfrak{so}(8,8)$	$4\mathfrak{sl}(2,\mathbb{R})$	$+$	4
$\mathfrak{so}(12,4)$	$\mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{sp}(2,\mathbb{C})_{\mathbb{R}}$	$-$	2
$\mathfrak{so}(8,8)$	$\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sp}(2,\mathbb{C})_{\mathbb{R}}$	$-$	2
$\mathfrak{so}(10,6)$	$\mathfrak{sp}(2,\mathbb{C})_{\mathbb{R}}\oplus\mathfrak{sp}(2,\mathbb{C})_{\mathbb{R}}$	$++$	2
$\mathfrak{so}(2^{2r})$	$\bigoplus_{i=1}^{2r}\mathfrak{su}(2)$	$+$	2
$\mathfrak{so}(2^{2r-1},2^{2r-1})$	$\bigoplus_{i=1}^{2r}\mathfrak{sl}(2,\mathbb{R})$	$+$	4
$\mathfrak{so}(2^{2r-1}+2^{r-1},2^{2r-1}-2^{r-1})$	$\bigoplus_{i=1}^{r}\mathfrak{sp}(2,\mathbb{C})_{\mathbb{R}}$	$+$	1
$\mathfrak{sp}(2^{2r})$	$\bigoplus_{i=1}^{2r+1}\mathfrak{su}(2)$	$+$	1
$\mathfrak{sp}(2^{2r+1},\mathbb{R})$	$\bigoplus_{i=1}^{2r+1}\mathfrak{sl}(2,\mathbb{R})$	$+$	1
$\mathfrak{sp}(2^{2r}(n^{2r+1}+(n-2t)^{2r+1}),2^{2r}(n^{2r+1}-(n-2t)^{2r+1}))$	$\bigoplus_{i=1}^{2r+1}\mathfrak{sp}(t,n-t)$	$+$	1
$\mathfrak{sp}((2n)^{2r+1},\mathbb{R})$	$\bigoplus_{i=1}^{2r+1}\mathfrak{sp}(2n,\mathbb{R})$	$+$	1
$\mathfrak{so}(2^{2r-1}(n^{2r}+(n-2t)^{2r}),2^{2r-1}(n^{2r}-(n-2t)^{2r}))$	$\bigoplus_{i=1}^{2r}\mathfrak{sp}(t,n-t)$	$+$	$t=0:2$ $t\neq 0:4$
$\mathfrak{so}(2^{2r-1}n^{2r},2^{2r-1}n^{2r})$	$\bigoplus_{i=1}^{2r}\mathfrak{sp}(2n,\mathbb{R})$	$+$	4

Table 3. (continued)

1		2	3
$\mathfrak{so}(\frac{n^r+(n-2t)^r}{2}, \frac{n^r-(n-2t)^r}{2})$	$\overset{r}{\underset{i=1}{\oplus}} \mathfrak{so}(t, n-t)$	+	n is odd: 1 n is even, $t = 0 : 2$ $n = 4k+2$, $r = 3$, t is odd: 2 otherwise: 4
$\mathfrak{so}(2^{2r-1}n^{2r}, 2^{2r-1}n^{2r})$	$\overset{2r}{\underset{i=1}{\oplus}} \mathfrak{u}^*(n, \mathbb{H})$	+	4
$\mathfrak{u}^*(2^{2r}n^{2r+1}, \mathbb{H})$	$\overset{2r+1}{\underset{i=1}{\oplus}} \mathfrak{u}^*(n, \mathbb{H})$	+	1
$\mathfrak{u}^*(2^{2r}n^{2r+1}, \mathbb{H})$	$\overset{2r+1}{\underset{i=1}{\oplus}} \mathfrak{u}^*(n, \mathbb{H})$	+	1

Conditions on parameters in the real case are consistent with those in the complex case.

Table 3′. Irreducible case

Here we list all symmetric pairs corresponding to all almost primitive real forms of the complex primitive pair $D_6 \subset B_{247}$ (see Lemma 4, section 2.2).

1		2	3
$B_{247}(B_{247})$	$D_6(D_6)$	+	1
$B_{247}(D_{120}+B_{127})$	$D_6(D_5+\mathbb{C})$	+	2
$B_{247}(D_{128}+B_{119})$	$D_6(D_4+A_1+A_1)$	+	2
$B_{247}(D_{120}+B_{127})$	$D_6(A_3+A_3)$	+	2
$B_{247}(D_{165}+B_{82})$	$D_6(B_5)$	++	2
$B_{247}(D_{117}+B_{130})$	$D_6(B_4+A_1)$	++	2
$B_{247}(D_{125}+B_{122})$	$D_6(B_3+B_2)$	++	2

In tables 4–62 we list all real almost primitive pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$, where $\bar{\mathfrak{g}}$ is a simple exceptional Lie algebra and $\mathfrak{g}$ is a reductive subalgebra of $\bar{\mathfrak{g}}$. Such pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is uniquely defined by an involution $\bar{\theta}$ of the complex Lie algebra $\bar{\mathfrak{p}} = \bar{\mathfrak{g}}^{\mathbb{C}}$ such that $\bar{\theta}(\mathfrak{p}) = \mathfrak{p}$, where $\mathfrak{p} = \mathfrak{g}^{\mathbb{C}}$. Further for any complex almost primitive pair $(\bar{\mathfrak{p}}, \mathfrak{p})$ all its real forms are indicated in the following way:

first we indicate the involution $\bar{\theta}$ of $\bar{\mathfrak{p}}$;

then in parentheses for the algebras $\bar{\mathfrak{p}}$ and $\mathfrak{p}$ we indicate the subalgebras of fixed points for $\bar{\theta}$ and $\bar{\theta}|_{\mathfrak{p}}$ respectively;

and finally we indicate the type of the obtained real pair.

Before a table we describe the indexing of the base roots for $\bar{\mathfrak{p}}$ and $\mathfrak{p}$. If $\mathfrak{p}$ is regular, then it is shown how the Dynkin diagram of $\mathfrak{p}$ is embedded into that of $\bar{\mathfrak{p}}$.

We assume that a certain Chevalley system $(X_\alpha)_{\alpha \in \bar{R}}$ is fixed for $\bar{\mathfrak{p}}$. Here $\bar{R}$ is the set of all roots of $\bar{\mathfrak{p}}$. In addition, for any $\alpha \in \bar{R}$, $\mathfrak{sl}_2$-triple $(X_{-\alpha}, H_\alpha, X_\alpha)$ is defined.

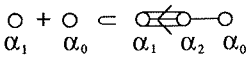
If $\mathfrak{p}$ is a subalgebra of maximal rank, we assume that its Chevalley system is imbedded into that of $\bar{\mathfrak{p}}$. Otherwise by $\bar{\alpha}$ and α we denote roots of $\bar{\mathfrak{p}}$ and $\mathfrak{p}$ respectively. In addition, by $\bar{\alpha}_i$ or α_i we denote the base roots of the root system of $\bar{\mathfrak{p}}$ or $\mathfrak{p}$ according to the indexing of the Dynkin diagram vertices. For the sake of simplicity we put

$$\bar{H}_i = H_{\bar{\alpha}_i}, \quad H_i = H_{\alpha_i},$$

$$\bar{X}_{\pm i_1 i_2 \dots i_k} = \left[X_{\pm\bar{\alpha}_{i_1}}, \left[\dots, X_{\pm\bar{\alpha}_{i_k}}\right]\dots\right]$$

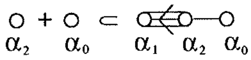
for all $i, i_1, \dots, i_k \in \mathbb{N}$ such that the corresponding vertices of the Dynkin diagram exist.

Table 4. $A_1 + A_1 \subset G_2$



1	2	2
$G_2(G_2) \supset A_1(A_1) + A_1(A_1)$ $\bar{\theta} = Id$	++	1
$G_2(A_1 + A_1) \supset A_1(A_1) + A_1(A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}H_0$	++	1
$G_2(A_1 + A_1) \supset A_1(A_1) + A_1(A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + H_0)$	++	1

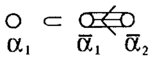
Table 5. $A_2 \subset G_2$



Let φ be an automorphism of the Lie algebra $\bar{\mathfrak{p}}$ such that $\varphi(X_2) = X_0\,, \varphi(X_0) = X_2$

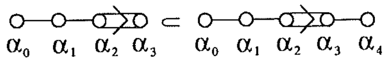
1	2	3
$G_2(G_2) \supset A_2(A_2)$ $\bar{\theta} = Id$	++	1
$G_2(A_1 + A_1) \supset A_2(A_1 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_2 + H_0)$	++	1
$G_2(E_2 + G_2) \supset A_2(A_1)$ $\bar{\theta} = \varphi$	++	1

Table 6. $A_1^{28} \subset G_2$



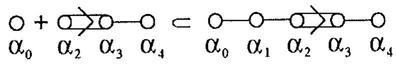
1	2	3
$G_2(G_2) \supset A_1(A_1)$ $\bar{\theta} = Id$	++	1
$G_2(A_1 + A_1) \supset A_1(\mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}H_1$	++	1

Table 7. $B_4 \subset F_4$



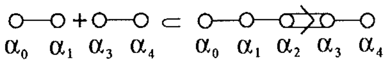
1	2	3
$F_4(F_4) \supset B_4(B_4)$ $\bar{\theta} = Id$	++	1
$F_4(B_4) \supset B_4(B_4)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(2H_0 + 2H_1 + 2H_2 + H_3)$	++	1
$F_4(C_3 + A_1) \supset B_4(B_2 + A_1 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + 2H_1 + 2H_2 + H_3)$	++	1
$F_4(B_4) \supset B_4(D_4)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + 2H_1 + 3H_2 + 2H_3)$	++	1

Table 8. $C_3 + A_1 \subset F_4$



1	2	3
$F_4(F_4) \supset A_1(A_1) + C_3(C_3)$ $\bar{\theta} = Id$	++	1
$F_4(A_1 + C_3) \supset C_3(C_3) + A_1(A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}H_0$	++	1
$F_4(B_4) \supset C_3(C_2 + A_1) + A_1(A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(2H_2 + 2H_3 + H_4)$	++	1
$F_4(C_3 + A_1) \supset C_3(C_2 + A_1) + A_1(A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + 2H_2 + 2H_3 + H_4)$	++	1
$F_4(C_3 + A_1) \supset C_3(A_2 + \mathbb{C}) + A_1(\mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_0 + \frac{3}{2}H_2 + H_3 + \frac{1}{2}H_4)$	++	1

Table 9. $A_2 + A_2 \subset F_4$



For any root α of the Lie algebra $\bar{\mathfrak{p}}$ we can construct the automorphism

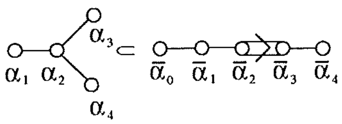
$$\theta_\alpha(t) = \exp \operatorname{ad} tX_\alpha \exp \operatorname{ad} t^{-1}X_{-\alpha} \exp \operatorname{ad} tX_\alpha,$$

where $t \in \mathbb{C}^*$. Put $\alpha = \alpha_1 + \alpha_2 + 2\alpha_3$ and $\beta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. Consider the automorphism φ defined by

$$\varphi = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_1 + \frac{3}{2}H_2 + \frac{1}{2}H_3)\theta_\alpha(1)\theta_\beta(-1).$$

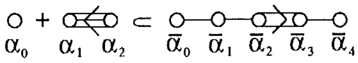
1	2	3
$F_4(F_4) \supset A_2(A_2) + A_2(A_2)$ $\bar{\theta} = Id$	++	1
$F_4(C_3 + A_1) \supset A_2(A_2) + A_2(A_1 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + H_1)$	++	1
$F_4(B_4) \supset A_2(A_2) + A_2(A_1 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_3 + H_4)$	++	1
$F_4(C_3 + A_1) \supset A_2(A_1 + \mathbb{C}) + A_2(A_1 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + H_1 + H_3 + H_4)$	++	1
$F_4(C_3 + A_1) \supset A_2(A_1) + A_2(A_1)$ $\bar{\theta} = \varphi$	++	1

Table 10. $D_4 \subset F_4$



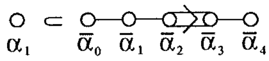
1	2	3
$F_4(F_4) \supset D_4(D_4)$ $\bar{\theta} = Id$	+	1
$F_4(C_3 + A_1) \supset A_4(A_1 + A_1 + A_1 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_2 + H_3 + H_4)$	+	1

Table 11. $A_1^8 + G_2^1 \subset F_4$



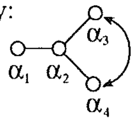
1	2	3
$F_4(F_4) \supset A_1(A_1) + G_2(G_2)$ $\bar{\theta} = Id$	++	1
$F_4(B_4) \supset A_1(\mathbb{C}) + G_2(G_2)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}H_0$	++	1
$F_4(C_3 + A_1) \supset G_2(A_1 + A_1) + A_1(A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(2H_1 + 3H_2)$	++	1
$F_4(C_3 + A_1) \supset A_1(\mathbb{C}) + G_2(A_1 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_0 + 2H_1 + 3H_2)$	++	1

Table 12. $A_1^{156} \subset F_4$



1	2	3
$A_1(A_1) \supset F_4(F_4)$ $\bar{\theta} = Id$	++	1
$A_1(\mathbb{C}) \supset F_4(A_1 + C_3)$ $\bar{\theta} = \exp \operatorname{ad} \frac{1}{2}\pi\sqrt{-1}H_1$	++	1

Let δ be the automorphism of the Lie algebra D_4 which acts on the Dynkin diagram in the following way:



There exists an automorphism φ of the Lie algebra D_4 which acts on the extended Dynkin diagram in the following way:

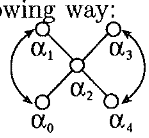
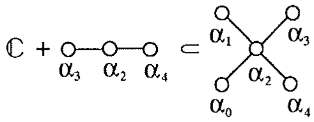
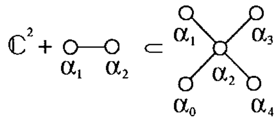


Table 13. $A_3 + \mathbb{C} \subset D_4$



1	2	3
$D_4(D_4) \supset A_3(A_3) + \mathbb{C}\,(\mathbb{C})$ $\bar{\theta} = Id$	++	3
$D_4(A_3 + \mathbb{C}) \supset A_3(A_3) + \mathbb{C}\,(\mathbb{C})$ $\bar{\theta} = \exp \mathrm{ad}\, \pi\sqrt{-1}(\frac{1}{2}H_1 - \frac{1}{2}H_0)$	++	1
$D_4(A_3 + \mathbb{C}) \supset A_3(A_2 + \mathbb{C}) + \mathbb{C}\,(\mathbb{C})$ $\bar{\theta} = \exp \mathrm{ad}\, \pi\sqrt{-1}(\frac{1}{2}H_3 - \frac{1}{2}H_0)$	++	1
$D_4(4A_1) \supset A_3(A_1 + A_1 + \mathbb{C}) + \mathbb{C}\,(\mathbb{C})$ $\bar{\theta} = \exp \mathrm{ad}\, \pi\sqrt{-1}\, H_0$	++	6
$D_4(A_3 + \mathbb{C}) \supset A_3(A_1 + A_1 + \mathbb{C}) + \mathbb{C}\,(\mathbb{C})$ $\bar{\theta} = \exp \mathrm{ad}\, \pi\sqrt{-1}(\frac{1}{2}H_0 + \frac{1}{2}H_1)$	++	1
$D_4(B_3) \supset A_3(B_2) + \mathbb{C}\,(\mathbb{C})$ $\bar{\theta} = \delta$	++	1
$D_4(B_2 + A_1) \supset A_3(B_2) + \mathbb{C}\,(\mathbb{C})$ $\bar{\theta} = \delta \circ \exp \mathrm{ad}\, \pi\sqrt{-1}(-\frac{1}{2}H_0 + \frac{1}{2}H_1)$	++	1
$D_4(B_2 + A_1) \supset A_3(A_1 + A_1) + \mathbb{C}\,(\mathbb{C})$ $\bar{\theta} = \delta \circ \exp \mathrm{ad}\, \pi\sqrt{-1}H_0$	++	1
$D_4(A_3 + \mathbb{C}) \supset A_3(B_2) + \mathbb{C}\,(0)$ $\bar{\theta} = \varphi$	+	1
$D_4(4A_1) \supset A_3(A_1 + A_1) + \mathbb{C}\,(0)$ $\bar{\theta} = \varphi \circ \exp \mathrm{ad}\, \pi\sqrt{-1}H_0$	+	3
$D_4(B_3) \supset A_3(A_3) + \mathbb{C}\,(0)$ $\bar{\theta} = \varphi \circ \delta$	+	1
$D_4(B_2 + A_1) \supset A_3(A_1 + A_1 + \mathbb{C}) + \mathbb{C}\,(0)$ $\bar{\theta} = \varphi \circ \delta \circ \exp \mathrm{ad}\, \pi\sqrt{-1}H_0$	+	1

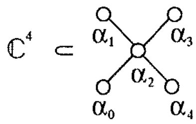
Table 14. $A_2 + \mathbb{C}^2 \subset D_4$



Let ψ be an automorphism of the Lie algebra $\bar{\mathfrak{p}}$ such that $\psi(\bar{X}_\alpha) = \bar{X}_{-\alpha}$ for all roots α.

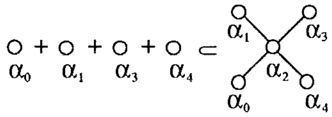
1	2	3
$D_4(D_4) \supset A_2(A_2) + \mathbb{C}^2(\mathbb{C}^2)$ $\bar{\theta} = Id$	$-$	1
$D_4(4A_1) \supset A_2(A_1 + \mathbb{C}) + \mathbb{C}^2(\mathbb{C}^2)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + H_2)$	$-$	4
$D_4(4A_1) \supset A_2(A_1) + \mathbb{C}^2(0)$ $\bar{\theta} = \psi$	$-$	1

Table 15. $\mathbb{C}^4 \subset D_4$



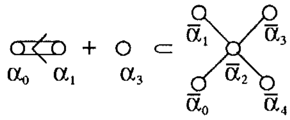
1	2	3
$D_4(D_4) \supset \mathbb{C}^4(\mathbb{C}^4)$ $\bar{\theta} = Id$	$+$	1
$D_4(4A_1) \supset \mathbb{C}^4(\mathbb{C}^4)$ $\bar{\theta} = \exp \operatorname{ad} 2\pi\sqrt{-1}(\frac{1}{2}H_1 + H_2 + \frac{1}{2}H_3 + \frac{1}{2}H_4)$	$+$	1

Table 16. $4A_1 \subset D_4$



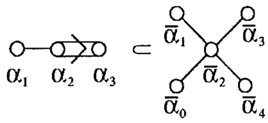
1	2	3
$D_4(D_4) \supset 4A_1(A_1)$ $\bar{\theta} = Id$	++	1
$D_4(4A_1) \supset 4A_1(A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}H_0$	++	1
$D_4(A_3 + \mathbb{C}) \supset 2A_1(A_1) + 2A_1(\mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_3 + \frac{1}{2}H_4)$	++	1
$D_4(4A_1) \supset 4A_1(\mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_0 + \frac{1}{2}H_1 + \frac{1}{2}H_3 + \frac{1}{2}H_4)$	++	1
$D_4(B_3) \supset 2A_1(A_1) + (A_1 + A_1)(A_1)$ $\bar{\theta} = \delta$	++	1
$D_4(B_2 + A_1) \supset 2A_1(A_1) + (A_1 + A_1)(A_1)$ $\bar{\theta} = \delta \circ \exp \operatorname{ad} \pi\sqrt{-1}H_0$	++	1
$D_4(B_2 + A_1) \supset A_1(\mathbb{C}) + A_1(\mathbb{C}) + (A_1 + A_1)(A_1)$ $\bar{\theta} = \delta \circ \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_0 + \frac{1}{2}H_1)$	++	1
$D_4(A_3 + \mathbb{C}) \supset (A_1 + A_1)(A_1) + (A_1 + A_1)(A_1)$ $\bar{\theta} = \varphi$	++	1
$D_4(4A_1) \supset (A_1 + A_1)(A_1) + (A_1 + A_1)(A_1)$ $\bar{\theta} = \varphi \circ \exp \operatorname{ad} \pi\sqrt{-1}H_0$	++	3

Table 17. $B_2^1 + A_1^2 \subset D_4$



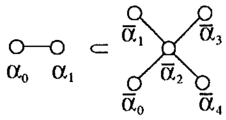
1	2	3
$D_4(D_4) \supset B_2(B_2) + A_1(A_1)$ $\bar{\theta} = Id$	++	3
$D_4(B_2 + A_1) \supset B_2(B_2) + A_1(A_1)$ $\bar{\theta} = \varphi \circ \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + H_1)$	++	1
$D_4(A_3 + \mathbb{C}) \supset B_2(A_1 + \mathbb{C}) + A_1(A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_0 + H_1)$	++	1
$D_4(B_2 + A_1) \supset B_2(A_1 + \mathbb{C}) + A_1(A_1)$ $\bar{\theta} = \varphi \circ \exp \operatorname{ad} \pi\sqrt{-1}(\frac{3}{2}H_0 + 2H_1)$	++	1
$D_4(4A_1) \supset B_2(A_1 + \mathbb{C}) + A_1(\mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_0 + H_1 + \frac{1}{2}H_3)$	++	6
$D_4(B_2 + A_1) \supset B_2(A_1 + \mathbb{C}) + A_1(\mathbb{C})$ $\bar{\theta} = \varphi \circ \exp \operatorname{ad} \pi\sqrt{-1}(\frac{3}{2}H_0 + 2H_1 + \frac{1}{2}H_3)$	++	1
$D_4(4A_1) \supset B_2(A_1 + A_1) + A_1(A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + H_1)$	++	6
$D_4(B_3) \supset B_2(A_1 + A_1) + A_1(A_1)$ $\bar{\theta} = \varphi \circ \exp \operatorname{ad} \pi\sqrt{-1}(2H_0 + 2H_1)$	++	1
$D_4(A_3 + \mathbb{C}) \supset B_2(A_1 + A_1) + A_1(\mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + H_1 + \frac{1}{2}H_3)$	++	1
$D_4(B_2 + A_1) \supset B_2(A_1 + A_1) + A_1(\mathbb{C})$ $\bar{\theta} = \varphi \circ \exp \operatorname{ad} \pi\sqrt{-1}(2H_0 + 2H_1 + \frac{1}{2}H_3)$	++	1
$D_4(A_3 + \mathbb{C}) \supset B_2(B_2) + A_1(\mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \frac{1}{2}\pi\sqrt{-1}H_3$	++	1
$D_4(B_3) \supset B_2(B_2) + A_1(\mathbb{C})$ $\bar{\theta} = \varphi \circ \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + H_1 + \frac{1}{2}H_3)$	++	1

Table 18. $B_3^1 \subset D_4$



1	2	3
$D_4(D_4) \supset B_3(B_3)$ $\bar{\theta} = Id$	++	3
$D_4(B_3) \supset B_3(B_3)$ $\bar{\theta} = \delta$	++	1
$D_4(A_3 + \mathbb{C}) \supset B_3(B_2 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + H_2 + \frac{1}{2}H_3)$	++	1
$D_4(B_2 + A_1) \supset B_3(B_2 + \mathbb{C})$ $\bar{\theta} = \delta \circ \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + H_2 + \frac{1}{2}H_3)$	++	1
$D_4(4A_1) \supset B_3(A_1 + A_1 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_2 + H_3)$	++	6
$D_4(B_2 + A_1) \supset B_3(A_1 + A_1 + A_1)$ $\bar{\theta} = \delta \circ \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_2 + H_3)$	++	1
$D_4(\mathbb{C} + A_3) \supset B_3(A_3)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_2 + \frac{3}{2}H_3)$	++	1
$D_4(B_3) \supset B_3(A_3)$ $\bar{\theta} = \delta \circ \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_2 + \frac{3}{2}H_3)$	++	1

Table 19. $A_2^3 \subset D_4$



Put $\varepsilon = \exp \frac{2\pi\sqrt{-1}}{3}$. Let ψ be an automorphism of the Lie algebra $\bar{\mathfrak{p}}$ such that

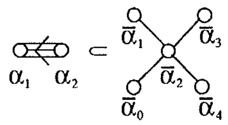
$$\psi(X_1) = [X_2, X_1],\ \psi(X_2) = -X_{-2},$$

$$\psi(X_3) = \varepsilon^2[X_2, X_4],\ \psi(X_4) = \varepsilon[X_2, X_3].$$

It is easy to see that ψ is an automorphism of second order.

1	2	3
$D_4(D_4) \supset A_2(A_2)$ $\bar{\theta} = Id$	++	1
$D_4(4A_1) \supset A_2(A_1 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_0)$	++	4
$D_4(B_2 + A_1) \supset A_2(A_1)$ $\bar{\theta} = \psi$	++	1

Table 20. $G_2^1 \subset D_4$



1	2	3
$D_4(D_4) \supset G_2(G_2)$ $\bar{\theta} = Id$	$-$	1
$D_4(4A_1) \supset G_2(A_1 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_2)$	$-$	1

Let φ be an automorphism of the Lie algebra E_6 which acts on the Dynkin diagram in the following way:

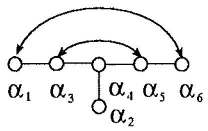
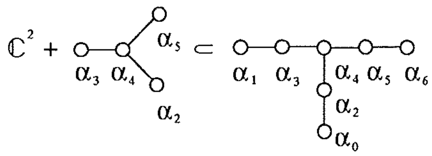


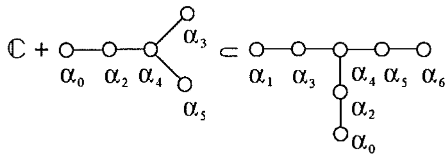
Table 21. $D_4 + \mathbb{C}^2 \subset E_6$



Let ψ be an automorphism of the Lie algebra $\bar{\mathfrak{p}}$ such that $\psi(\bar{X}_\alpha) = \bar{X}_{-\alpha}$ for all roots α and $f = \psi|_{[\mathfrak{p},\mathfrak{p}]}$. It is clear that $f = \exp \operatorname{ad} h$ for some $h \in [\mathfrak{p}, \mathfrak{p}]$ such that $f(h) = h$ and $(\exp \operatorname{ad} h)^2 = Id_{\bar{\mathfrak{g}}}$. Put $\xi = \psi \circ \exp \operatorname{ad} h$.

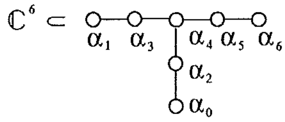
1	2	3
$E_6(E_6) \supset \mathbb{C}^2(\mathbb{C}^2) + D_4(D_4)$ $\bar{\theta} = Id$	$+$	1
$E_6(E_5 + A_1) \supset \mathbb{C}^2(\mathbb{C}^2) + D_4(4A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_4 + H_5 + H_2)$	$+$	1
$E_6(C_4) \supset \mathbb{C}^2(0) + D_4(4A_1)$ $\bar{\theta} = \psi$	$+$	1
$E_6(F_4) \supset \mathbb{C}^2(0) + D_4(D_4)$ $\bar{\theta} = \xi$	$+$	1

Table 22. $D_5 + \mathbb{C} \subset E_6$



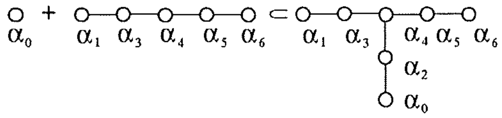
1	2	3
$E_6(E_6) \supset \mathbb{C}(\mathbb{C}) + D_5(D_5)$ $\bar{\theta} = Id$	++	1
$E_6(D_5 + \mathbb{C}) \supset \mathbb{C}(\mathbb{C}) + D_5(D_5)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(2H_1 + H_3 - H_5 - 2H_6)$	++	1
$E_6(D_5 + \mathbb{C}) \supset \mathbb{C}(\mathbb{C}) + D_5(D_4 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_2 + H_3 + 2H_4 + 2H_5 + 2H_6)$	++	1
$E_6(A_5 + A_1) \supset \mathbb{C}(\mathbb{C}) + D_5(A_3 + A_1 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + H_3 + H_4 + H_5 + H_6)$	++	1
$E_6(A_5 + A_1) \supset \mathbb{C}(\mathbb{C}) + D_5(A_4 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_6)$	++	1
$E_6(D_5 + \mathbb{C}) \supset \mathbb{C}(\mathbb{C}) + D_5(A_4 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(2H_1 + H_3 - H_5 - H_6)$	++	1
$E_6(F_4) \supset \mathbb{C}(0) + D_5(B_4)$ $\bar{\theta} = \varphi$	−	1
$E_6(C_4) \supset \mathbb{C}(0) + D_5(B_2 + B_2)$ $\bar{\theta} = \varphi \circ \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + H_3 + H_4 + H_5 + H_6)$	−	1

Table 23. $\mathbb{C}^6 \subset E_6$



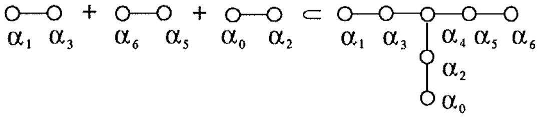
1	2	3
$E_6(E_6) \supset \mathbb{C}^6(\mathbb{C}^6)$ $\bar{\theta} = Id$	+	1
$E_6(F_4) \supset \mathbb{C}^6(0)$ $\bar{\theta} = \varphi$	+	1

Table 24. $A_5 + A_1 \subset E_6$



1	2	3
$E_6(E_6) \supset A_1(A_1) + A_5(A_5)$ $\bar\theta = Id$	++	1
$E_6(A_1 + A_5) \supset A_1(A_1) + A_5(A_5)$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}H_0$	++	1
$E_6(F_4) \supset A_1(A_1) + A_5(C_3)$ $\bar\theta = \varphi$	++	1
$E_6(C_4) \supset A_1(A_1) + A_5(C_3)$ $\bar\theta = \varphi \circ \exp \mathrm{ad}\, \pi\sqrt{-1}H_0$	++	1
$E_6(A_5 + A_1) \supset A_1(A_1) + A_5(A_1 + A_3 + \mathbb{C})$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}(H_1 + H_3 + H_4 + H_5 + H_6)$	++	1
$E_6(D_5 + \mathbb{C}) \supset A_1(A_1) + A_5(A_1 + A_3 + \mathbb{C})$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}(H_0 + H_1 + H_2 + H_3 + H_4 + H_5 + H_6)$	++	1
$E_6(D_5 + \mathbb{C}) \supset A_1(\mathbb{C}) + A_5(A_4 + \mathbb{C})$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}(\frac{1}{2}H_0 + \frac{5}{6}H_1 + \frac{2}{3}H_3 + \frac{1}{2}H_4 + \frac{1}{3}H_5 + \frac{1}{6}H_6)$	++	1
$E_6(A_5 + A_1) \supset A_1(\mathbb{C}) + A_5(A_2 + A_2 + \mathbb{C})$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}(\frac{1}{2}H_0 + \frac{1}{2}H_1 + H_3 + \frac{3}{2}H_4 + H_5 + \frac{1}{2}H_6)$	++	1
$E_6(C_4) \supset A_1(\mathbb{C}) + A_5(A_3)$ $\bar\theta = \varphi \circ \exp \mathrm{ad}\, \pi\sqrt{-1}(\frac{1}{2}H_0 + \frac{1}{2}H_1 + H_3 + \frac{3}{2}H_4 + H_5 + \frac{1}{2}H_6)$	++	1

Table 25. $A_2 + A_2 + A_2 \subset E_6$



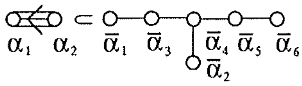
There exists an automorphism δ such that $\delta^2 = Id$ and

$$\delta(X_1) = X_5,\ \delta(X_2) = X_0,\ \delta(X_3) = X_6,\ \delta(X_4) = X_4,$$

$$\delta(X_5) = X_1,\ \delta(X_6) = X_3,\ \delta(X_0) = X_2.$$

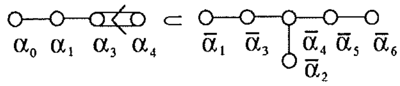
1	2	3
$E_6(E_6) \supset 3A_2(A_2)$ $\bar\theta = Id$	++	1
$E_6(A_5 + A_1) \supset 2A_2(A_2) + A_2(A_1 + \mathbb{C})$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}(H_0 + H_2)$	++	1
$E_6(D_5 + \mathbb{C}) \supset 2A_2(A_1 + \mathbb{C}) + A_2(A_2)$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}(H_1 + H_3 + H_6 + H_5)$	++	1
$E_6(A_5 + A_1) \supset 3A_2(A_1 + \mathbb{C})$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}(H_1 + H_3 + H_6 + H_5 + H_0 + H_2)$	++	1
$E_6(F_4) \supset (A_2 + A_2)(A_2) + A_2(A_2)$ $\bar\theta = \varphi$	++	1
$E_6(C_4) \supset (A_2 + A_2)(A_2) + A_2(A_1 + \mathbb{C})$ $\bar\theta = \varphi \circ \exp \mathrm{ad}\, \pi\sqrt{-1}(H_0)$	++	1
$E_6(A_5 + A_1) \supset (A_2 + A_2)(A_2) + A_2(A_1)$ $\bar\theta = \delta$	++	1
$E_6(C_4) \supset A_2(A_1) + A_2(A_1) + A_2(A_1)$ $\bar\theta = \varphi \circ \delta$	++	1

Table 26. $G_2^3 \subset E_6$



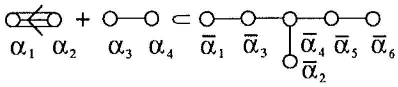
1	2	3
$E_6(E_6) \supset G_2(G_2)$ $\bar\theta = Id$	++	2
$E_6(A_1 + A_5) \supset G_2(A_1 + A_1)$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}(2H_1 + 3H_2)$	++	2

Table 27. $C_4^1 \subset E_6$



1	2	3
$E_6(E_6) \supset C_4(C_4)$ $\bar{\theta} = Id$	++	1
$E_6(C_4) \supset C_4(C_4)$ $\bar{\theta} = \varphi \circ \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + H_1 + H_3 + H_4)$	++	1
$E_6(A_5 + A_1) \supset C_4(A_3 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_0 + H_1 + \frac{3}{2}H_3 + 2H_4)$	++	1
$E_6(C_4) \supset C_4(A_3 + \mathbb{C})$ $\bar{\theta} = \varphi \circ \exp \operatorname{ad} \pi\sqrt{-1}(\frac{3}{2}H_0 + 2H_1 + \frac{5}{2}H_3 + 3H_4)$	++	1
$E_6(A_5 + A_1) \supset C_4(C_3 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + H_1 + H_3 + H_4)$	++	1
$E_6(F_4) \supset C_4(C_3 + A_1)$ $\bar{\theta} = \varphi \circ \exp \operatorname{ad} \pi\sqrt{-1}(2H_0 + 2H_1 + 2H_3 + 2H_4)$	++	1
$E_6(D_5 + \mathbb{C}) \supset C_4(C_2 + C_2)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + 2H_1 + 2H_3 + 2H_4)$	++	1
$E_6(C_4) \supset C_4(C_2 + C_2)$ $\bar{\theta} = \varphi \circ \exp \operatorname{ad} \pi\sqrt{-1}(2H_0 + 3H_1 + 3H_3 + 3H_4)$	++	1

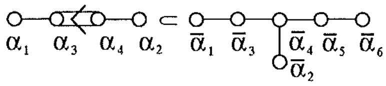
Table 28. $G_2^1 + A_2^{2''} \subset E_6$



Let δ be an automorphism of the Lie algebra $\bar{\mathfrak{p}}$ such that $\delta(\bar{X}_i) = \bar{X}_{-i}$, $1 \leqslant i \leqslant 6$.

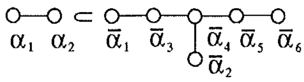
1	2	3
$E_6(E_6) \supset G_2(G_2) + A_2(A_2)$ $\bar{\theta} = Id$	++	1
$E_6(A_1 + A_5) \supset G_2(A_1 + A_1) + A_2(A_2)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(2H_1 + 3H_2)$	++	1
$E_6(D_5 + \mathbb{C}) \supset G_2(G_2) + A_2(A_1 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_3 + H_4)$	++	1
$E_6(A_1 + A_5) \supset G_2(A_1 + A_1) + A_2(A_1 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \frac{1}{2}\pi\sqrt{-1}(2H_1 + 3H_2 + H_3 + H_4)$	++	1
$E_6(F_4) \supset G_2(G_2) + A_2(A_1)$ $\bar{\theta} = \varphi$	++	1
$E_6(C_4) \supset G_2(A_1 + A_1) + A_2(A_1)$ $\bar{\theta} = \delta$	++	1

Table 29. $F_4^1 \subset E_6$



1	2	3
$E_6(E_6) \supset F_4(F_4)$ $\bar{\theta} = Id$	++	1
$E_6(F_4) \supset F_4(F_4)$ $\bar{\theta} = \varphi$	++	1
$E_6(D_5 + \mathbb{C}) \supset F_4(B_4)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(2H_1 + 3H_3 + 4H_4 + 2H_2)$	++	1
$E_6(F_4) \supset F_4(B_4)$ $\bar{\theta} = \varphi \circ \exp \operatorname{ad} \pi\sqrt{-1}(2H_1 + 3H_3 + 4H_4 + 2H_2)$	++	1
$E_6(A_5 + A_1) \supset F_4(C_3 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_3 + 3H_4 + 2H_2)$	++	1
$E_6(C_4) \supset F_4(C_3 + A_1)$ $\bar{\theta} = \varphi \circ \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_3 + 3H_4 + 2H_2)$	++	1

Table 30. $A_2^9 \subset E_6$



Let δ be an automorphism of the Lie algebra $\bar{\mathfrak{p}}$ such that $\delta^2 = Id$ and

$$\delta(\bar{X}_1) = \bar{X}_{-6},\ \delta(\bar{X}_2) = \bar{X}_{-2},\ \delta(\bar{X}_3) = \bar{X}_{-5},$$

$$\delta(\bar{X}_4) = \bar{X}_{-4},\ \delta(\bar{X}_5) = \bar{X}_{-3},\ \delta(\bar{X}_6) = \bar{X}_{-1}.$$

1	2	3
$E_6(E_6) \supset A_2(A_2)$ $\bar{\theta} = Id$	++	2
$E_6(A_1 + A_5) \supset A_2(A_1 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + H_2)$	++	2
$E_6(A_1 + A_5) \supset A_2(A_1)$ $\bar{\theta} = \delta$	++	2

There exists an automorphism φ of the Lie algebra E_7 which acts on the extended Dynkin diagram in the following way:

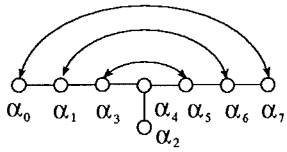
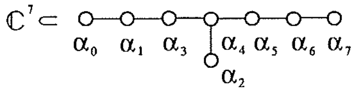
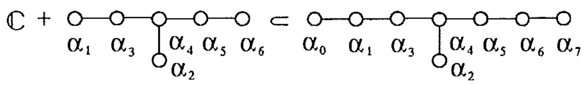


Table 31. $\mathbb{C}^7 \subset E_7$



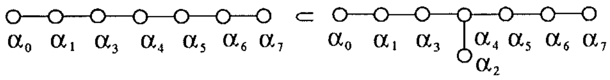
1	2	3
$\mathbb{C}^7(\mathbb{C}^7) \subset E_7(E_7)$ $\bar{\theta} = Id$	+	1
$\mathbb{C}^7(0) + E_6(E_6) \subset E_7(E_6 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(2H_1 + \frac{7}{2}H_2 + 4H_3 + 6H_4 + \frac{9}{2}H_5 + 3H_6 + \frac{3}{2}H_7)$	+	1

Table 32. $\mathbb{C} + E_6 \subset E_7$



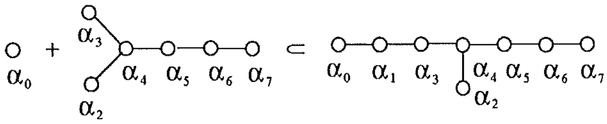
1	2	3
$\mathbb{C}(\mathbb{C}) + E_6(E_6) \subset E_7(E_7)$ $\bar{\theta} = Id$	++	1
$\mathbb{C}(\mathbb{C}) + E_6(E_6) \subset E_7(E_6 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + \frac{3}{2}H_2 + 2H_3 + 3H_4 + \frac{5}{2}H_5 + 2H_6 + \frac{3}{2}H_7)$	++	1
$\mathbb{C}(\mathbb{C}) + E_6(D_5 + \mathbb{C}) \subset E_7(D_6 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(2H_1 + 2H_2 + 3H_3 + 4H_4 + 3H_5 + 2H_6)$	++	1
$\mathbb{C}(\mathbb{C}) + E_6(D_5 + \mathbb{C}) \subset E_7(E_6 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(3H_1 + \frac{7}{2}H_2 + 5H_3 + 7H_4 + \frac{11}{2}H_5 + 4H_6 + \frac{3}{2}H_7)$	++	1
$\mathbb{C}(\mathbb{C}) + E_6(A_5 + A_1) \subset E_7(D_6 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_2 + 2H_3 + 3H_4 + 2H_5 + H_6)$	++	1
$\mathbb{C}(\mathbb{C}) + E_6(A_5 + A_1) \subset E_7(A_7)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(2H_1 + \frac{7}{2}H_2 + 4H_3 + 6H_4 + \frac{9}{2}H_5 + 3H_6 + \frac{3}{2}H_7)$	++	1
$\mathbb{C}(0) + E_6(F_4) \subset E_7(E_6 + \mathbb{C})$ $\bar{\theta} = \varphi$	+	1
$\mathbb{C}(0) + E_6(C_4) \subset E_7(A_7)$ $\bar{\theta} = \varphi \circ \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_2 + 2H_3 + 3H_4 + 2H_5 + H_6)$	+	1

Table 33. $A_7 \subset E_7$



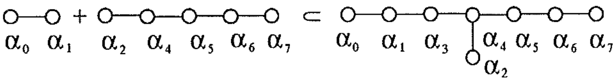
1	2	3
$A_7(A_7) \subset E_7(E_7)$ $\bar{\theta} = Id$	++	1
$A_7(A_7) \subset E_7(A_7)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{7}{4}H_0 + \frac{3}{2}H_1 + \frac{5}{4}H_3 + H_4 + \frac{3}{4}H_5 + \frac{1}{2}H_6 + \frac{1}{4}H_7)$	++	1
$A_7(C_4) \subset E_7(E_6 + \mathbb{C})$ $\bar{\theta} = \varphi$	++	1
$A_7(C_4) \subset E_7(A_7)$ $\bar{\theta} = \varphi \circ \exp \operatorname{ad} \pi\sqrt{-1}(\frac{7}{4}H_0 + \frac{3}{2}H_1 + \frac{5}{4}H_3 + H_4 + \frac{3}{4}H_5 + \frac{1}{2}H_6 + \frac{1}{4}H_7)$	++	1
$A_7(A_5 + A_1 + \mathbb{C}) \subset E_7(D_6 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + 2H_1 + 3H_3 + 3H_4 + 3H_5 + 2H_6 + H_7)$	++	1
$A_7(A_5 + A_1 + \mathbb{C}) \subset E_7(E_6 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{11}{4}H_0 + \frac{7}{2}H_1 + \frac{17}{4}H_3 + 4H_4 + \frac{15}{4}H_5 + \frac{5}{2}H_6 + \frac{5}{4}H_7)$	++	1
$A_7(A_3 + A_3 + \mathbb{C}) \subset E_7(D_6 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_0 + H_2 + \frac{3}{2}H_3 + 2H_4 + \frac{3}{2}H_5 + H_6 + \frac{1}{2}H_7)$	++	1
$A_7(A_3 + A_3 + \mathbb{C}) \subset E_7(A_7)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{9}{4}H_0 + \frac{5}{2}H_1 + \frac{11}{4}H_3 + 3H_4 + \frac{9}{4}H_5 + \frac{3}{2}H_6 + \frac{3}{4}H_7)$	++	1
$A_7(D_4) \subset E_7(A_7)$ $\bar{\theta} = \varphi \circ \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_0 + H_1 + \frac{3}{2}H_3 + 2H_4 + \frac{3}{2}H_5 + H_6 + \frac{1}{2}H_7)$	++	1

Table 34. $A_1 + D_6 \subset E_7$



1	2	3
$A_1(A_1) + D_6(D_6) \subset E_7(E_7)$ $\bar{\theta} = Id$	++	1
$A_1(A_1) + D_6(D_6) \subset E_7(D_6 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}H_0$	++	1
$A_1(A_1) + D_6(A_5 + \mathbb{C}) \subset E_7(E_6 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{3}{2}H_2 + H_3 + H_4 + \frac{3}{2}H_5 + H_6 + \frac{1}{2}H_7)$	++	1
$A_1(A_1) + D_6(A_5 + \mathbb{C}) \subset E_7(A_7)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + \frac{3}{2}H_2 + H_3 + H_4 + \frac{3}{2}H_5 + H_6 + \frac{1}{2}H_7)$	++	1
$A_1(A_1) + D_6(D_4 + A_1 + A_1) \subset E_7(D_6 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_2 + H_3 + 2H_4 + 2H_5 + 2H_6 + H_7)$	++	1
$A_1(\mathbb{C}) + D_6(D_5 + \mathbb{C}) \subset E_7(E_6 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_0 + \frac{1}{2}H_2 + \frac{1}{2}H_3 + H_4 + H_5 + H_6 + H_7)$	++	1
$A_1(\mathbb{C}) + D_6(A_5 + \mathbb{C}) \subset E_7(D_6 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_0 + H_2 + \frac{3}{2}H_3 + 2H_4 + \frac{3}{2}H_5 + H_6 + \frac{1}{2}H_7)$	++	1
$A_1(\mathbb{C}) + D_6(A_3 + A_3) \subset E_7(A_7)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_0 + \frac{3}{2}H_1 + \frac{3}{2}H_3 + 3H_4 + 3H_5 + 2H_6 + H_7)$	++	1

Table 35. $A_2 + A_5 \subset E_7$

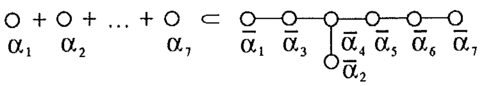


There exists an automorphism δ such that

$$\delta(X_1) = X_0,\ \delta(X_2) = X_7,\ \delta(X_3) = X_3,\ \delta(X_4) = X_6,$$
$$\delta(X_5) = X_5,\ \delta(X_6) = X_4,\ \delta(X_7) = X_2,\ \delta(X_0) = X_1.$$

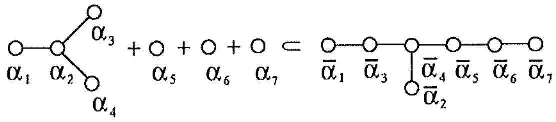
1	2	3
$A_2(A_2) + A_5(A_5) \subset E_7(E_7)$ $\bar{\theta} = Id$	++	1
$A_2(A_2) + A_5(A_4 + \mathbb{C}) \subset E_7(A_7)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{5}{2}H_2 + 2H_4 + \frac{3}{2}H_5 + H_6 + \frac{1}{2}H_7)$	++	1
$A_2(A_2) + A_5(A_3 + A_1 + \mathbb{C}) \subset E_7(D_6 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(2H_2 + 4H_4 + 3H_5 + 2H_6 + H_7)$	++	1
$A_2(A_2) + A_5(A_2 + A_2 + \mathbb{C}) \subset E_7(E_6 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_2 + H_4 + \frac{3}{4}H_5 + H_6 + \frac{1}{2}H_7)$	++	1
$A_2(A_1 + \mathbb{C}) + A_5(A_5) \subset E_7(D_6 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + H_1)$	++	1
$A_2(A_1 + \mathbb{C}) + A_5(A_4 + \mathbb{C}) \subset E_7(E_6 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + H_1 + \frac{5}{2}H_2 + 2H_4 + \frac{3}{2}H_5 + H_6 + \frac{1}{2}H_7)$	++	1
$A_2(A_1 + \mathbb{C}) + A_5(A_3 + A_1 + \mathbb{C}) \subset E_7(D_6 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + H_1 + 2H_2 + 4H_4 + 3H_5 + 2H_6 + H_7)$	++	1
$A_2(A_1 + \mathbb{C}) + A_5(A_2 + A_2 + \mathbb{C}) \subset E_7(A_7)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + H_1 + \frac{1}{2}H_2 + H_4 + \frac{3}{2}H_5 + H_6 + \frac{1}{2}H_7)$	++	1
$A_2(A_1) + A_5(C_3) \subset E_7(D_6 + A_1)$ $\bar{\theta} = \delta$	++	1
$A_2(A_1) + A_5(A_3) \subset E_7(A_7)$ $\bar{\theta} = \delta \circ \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_2 + H_4 + \frac{3}{2}H_5 + H_6 + \frac{1}{2}H_7)$	++	1

Table 36. $7A_1 \subset E_7$



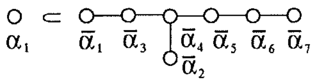
1	2	3
$7A_1(A_1) \subset E_7(E_7)$ $\bar{\theta} = Id$	+	1
$7A_1(\mathbb{C}) \subset E_7(A_7)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + \cdots + H_7)$	+	1

Table 37. $D_4 + A_1 + A_1 + A_1 \subset E_7$



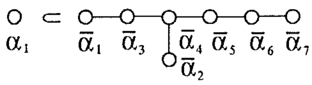
1	2	3
$D_4(D_4) + 3A_1(A_1) \subset E_7(E_7)$ $\bar{\theta} = Id$	+	1
$D_4(4A_1) + 3A_1(A_1) \subset E_7(D_6 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_2 + H_3 + H_4)$	+	1
$D_4(D_4) + 3A_1(\mathbb{C}) \subset E_7(E_6 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_5 + \frac{1}{2}H_6 + \frac{1}{2}H_7)$	+	1
$D_4(4A_1) + 3A_1(\mathbb{C}) \subset E_7(A_7)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_2 + H_3 + H_4 + \frac{1}{2}H_5 + \frac{1}{2}H_6 + \frac{1}{2}H_7)$	+	1

Table 38. $A_1^{399} \subset E_7$



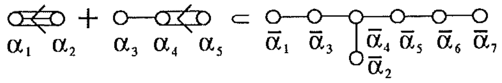
1	2	3
$A_1(A_1) \subset E_7(E_7)$ $\bar{\theta} = Id$	+	1
$A_1(\mathbb{C}) \subset E_7(A_7)$ $\bar{\theta} = \exp \operatorname{ad} \frac{1}{2}\pi\sqrt{-1}H_1$	+	1

Table 39. $A_1^{231} \subset E_7$



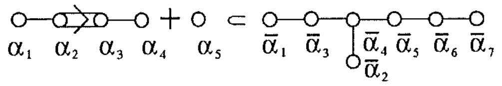
1	2	3
$A_1(A_1) \subset E_7(E_7)$ $\bar{\theta} = Id$	+	1
$A_1(\mathbb{C}) \subset E_7(A_7)$ $\bar{\theta} = \exp \operatorname{ad} \frac{1}{2}\pi\sqrt{-1}H_1$	+	1

Table 40. $G_2^1 + C_3^{1''} \subset E_7$



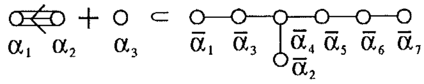
1	2	3
$G_2(G_2) + C_3(C_3) \subset E_7(E_7)$ $\bar\theta = Id$	++	1
$G_2(A_1 + A_1) + C_3(C_3) \subset E_7(D_6 + A_1)$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}(2H_1 + 3H_2)$	++	1
$G_2(G_2) + C_3(A_2 + \mathbb{C}) \subset E_7(E_6 + \mathbb{C})$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}(\frac{1}{2}H_3 + H_4 + \frac{3}{2}H_5)$	++	1
$G_2(G_2) + C_3(C_2 + A_1) \subset E_7(D_6 + A_1)$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}(H_3 + H_4 + H_5)$	++	1
$G_2(A_1 + A_1) + C_3(A_2 + \mathbb{C}) \subset E_7(A_7)$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}(2H_1 + 3H_2 + \frac{1}{2}H_3 + H_4 + \frac{3}{2}H_5)$	++	1
$G_2(A_1 + A_1) + C_3(C_2 + A_1) \subset E_7(D_6 + A_1)$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}(2H_1 + 3H_2 + H_3 + H_4 + H_5)$	++	1

Table 41. $F_4^1 + A_1^{3''} \subset E_7$



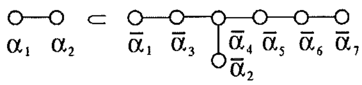
1	2	3
$E_7(E_7) \supset F_4(F_4) + A_1(A_1)$ $\bar\theta = Id$	++	1
$E_7(E_6 + \mathbb{C}) \supset F_4(F_4) + A_1(\mathbb{C})$ $\bar\theta = \exp \mathrm{ad}\, \frac{1}{2}\pi\sqrt{-1}H_5$	++	1
$E_7(D_6 + A_1) \supset F_4(C_3 + A_1) + A_1(A_1)$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}(2H_1 + 3H_2 + 2H_3 + H_4)$	++	1
$E_7(A_7) \supset F_4(C_3 + A_1) + A_1(\mathbb{C})$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}(2H_1 + 3H_2 + 2H_3 + H_4 + \frac{1}{2}H_5)$	++	1
$E_7(D_6 + A_1) \supset F_4(B_4) + A_1(A_1)$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}(2H_1 + 4H_2 + 3H_3 + 2H_4)$	++	1
$E_7(E_6 + \mathbb{C}) \supset F_4(B_4) + A_1(\mathbb{C})$ $\bar\theta = \exp \mathrm{ad}\, \pi\sqrt{-1}(2H_1 + 4H_2 + 3H_3 + 2H_4 + \frac{1}{2}H_5)$	++	1

Table 42. $G_2^2 + A_1^7 \subset E_7$



1	2	3
$E_7(E_7) \supset G_2(G_2) + A_1(A_1)$ $\bar\theta = Id$	++	1
$E_7(A_7) \supset G_2(G_2) + A_1(\mathbb{C})$ $\bar\theta = \exp \operatorname{ad} \frac{1}{2}\pi\sqrt{-1}H_3$	++	1
$E_7(D_6 + A_1) \supset G_2(A_1 + A_1) + A_1(A_1)$ $\bar\theta = \exp \operatorname{ad} \pi\sqrt{-1}(2H_1 + 3H_2)$	++	1
$E_7(A_7) \supset G_2(A_1 + A_1) + A_1(\mathbb{C})$ $\bar\theta = \exp \operatorname{ad} \pi\sqrt{-1}(2H_1 + 3H_2 + \frac{1}{2}H_3)$	++	1

Table 43. $A_2^{21} \subset E_7$

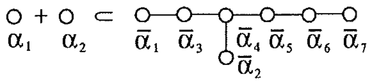


There exists an automorphism δ such that

$$\delta(\bar X_i) = \bar X_{-i} \ (1 \leqslant i \leqslant 7).$$

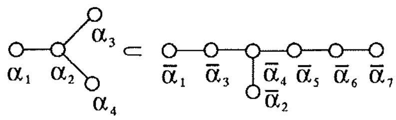
1	2	3
$E_7(E_7) \supset A_2(A_2)$ $\bar\theta = Id$	++	1
$E_7(D_6 + A_1) \supset A_2(A_1 + \mathbb{C})$ $\bar\theta = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + H_2)$	++	1
$E_7(A_7) \supset A_2(A_1)$ $\bar\theta = \delta$	++	2

Table 44. $A_1^{24} + A_1^{15} \subset E_7$



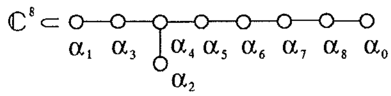
1	2	3
$E_7(E_7) \supset A_1(A_1) + A_1(A_1)$ $\bar{\theta} = Id$	++	1
$E_7(D_6 + A_1) \supset A_1(\mathbb{C}) + A_1(A_1)$ $\bar{\theta} = \exp \operatorname{ad} \frac{1}{2}\pi\sqrt{-1}H_1$	++	1
$E_7(A_7) \supset A_1(A_1) + A_1(\mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \frac{1}{2}\pi\sqrt{-1}H_2$	++	1
$E_7(A_7) \supset A_1(\mathbb{C}) + A_1(\mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \frac{1}{2}\pi\sqrt{-1}(H_1 + H_2)$	++	1

Table 45. $D_4^2 \subset E_7$



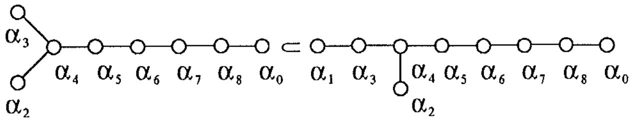
1	2	3
$E_7(E_7) \supset D_4(D_4)$ $\bar{\theta} = Id$	+	1
$E_7(D_6 + A_1) \supset D_4(A_1 + A_1 + A_1 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_2 + H_3 + H_4)$	+	1

Table 46. $\mathbb{C}^8 \subset E_8$



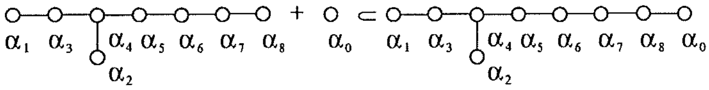
1	2	3
$E_8(E_8) \supset \mathbb{C}^8(\mathbb{C}^8)$ $\bar{\theta} = Id$	+	1
$E_8(D_8) \supset \mathbb{C}^8(0)$ $\bar{\theta} = \exp \operatorname{ad} 2\pi\sqrt{-1}(2H_1 + \frac{5}{2}H_2 + \frac{7}{2}H_3 + 5H_4 + 5H_5 + 3H_6 + 2H_7 + H_8)$	+	1

Table 47. $D_8 \subset E_8$



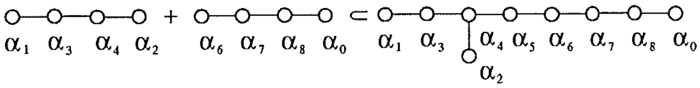
1	2	3
$E_8(E_8) \supset D_8(D_8)$ $\bar{\theta} = Id$	++	1
$E_8(D_8) \supset D_8(D_8)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_2 + H_3 + 2H_4 + 2H_5 + 2H_6 + 2H_7 + 2H_8 + 2H_0)$	++	1
$E_8(E_7 + A_1) \supset D_8(A_7 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{5}{2}H_2 + 3H_3 + 5H_4 + \frac{9}{2}H_5 + 4H_6 + \frac{7}{2}H_7 + 3H_8 + \frac{5}{2}H_0)$	++	1
$E_8(D_8) \supset D_8(A_7 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{5}{2}H_2 + 3H_3 + 5H_4 + \frac{9}{2}H_5 + 4H_6 + \frac{7}{2}H_7 + 3H_8 + \frac{5}{2}H_0)$	++	1
$E_8(E_7 + A_1) \supset D_8(D_6 + A_1 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_2 + H_3 + 2H_4 + 2H_5 + 2H_6 + 2H_7 + 2H_8 + H_0)$	++	1
$E_8(D_8) \supset D_8(D_4 + D_4)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(2H_2 + 2H_3 + 4H_4 + 4H_5 + 4H_6 + 3H_7 + 2H_8 + H_0)$	++	1

Table 48. $E_7 + A_1 \subset E_8$



1	2	3
$E_8(E_8) \supset E_7(E_7) + A_1(A_1)$ $\bar{\theta} = Id$	++	1
$E_8(E_7 + A_1) \supset E_7(E_7) + A_1(A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}H_0$	++	1
$E_8(E_7 + A_1) \supset E_7(D_6 + A_1) + A_1(A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(2H_1 + 2H_2 + 3H_3 + 4H_4 + 3H_5 + 2H_6 + H_7)$	++	1
$E_8(D_8) \supset E_7(D_6 + A_1) + A_1(A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_0 + 2H_1 + 2H_2 + 3H_3 + 4H_4 + 3H_5 + 2H_6 + H_7)$	++	1
$E_8(E_7 + A_1) \supset E_7(E_6 + \mathbb{C}) + A_1(\mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_0 + H_1 + \frac{3}{2}H_2 + 2H_3 + 3H_4 + \frac{5}{2}H_5 + 2H_6 + \frac{3}{2}H_7)$	++	1
$E_8(D_8) \supset E_7(A_7) + A_1(\mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(\frac{1}{2}H_0 + 2H_1 + \frac{7}{2}H_2 + 4H_3 + 6H_4 + \frac{9}{2}H_5 + 3H_6 + \frac{3}{2}H_7)$	++	1

Table 49. $A_4 + A_4 \subset E_8$



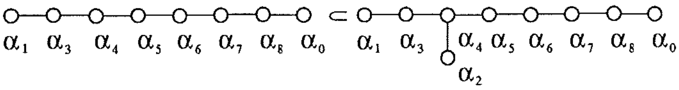
There exists an automorphism δ such that $\delta^2 = Id$ and

$$\delta(X_1) = X_2,\ \delta(X_2) = X_1,\ \delta(X_3) = X_4,\ \delta(X_4) = X_3,$$

$$\delta(X_6) = X_0,\ \delta(X_7) = X_8,\ \delta(X_8) = X_7,\ \delta(X_0) = X_6.$$

1	2	3
$E_8(E_8) \supset A_4(A_4) + A_4(A_4)$ $\bar{\theta} = Id$	++	1
$E_8(D_8) \supset A_4(A_4) + A_4(A_3 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(4H_6 + 3H_7 + 2H_8 + H_0)$	++	1
$E_8(E_7 + A_1) \supset A_4(A_4) + A_4(A_2 + A_1 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(3H_6 + 6H_7 + 4H_8 + 2H_0)$	++	1
$E_8(D_8) \supset A_4(A_3 + \mathbb{C}) + A_4(A_3 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(4H_1 + 3H_3 + 2H_4 + H_2 + 4H_6 + 3H_7 + 2H_8 + H_0)$	++	1
$E_8(E_7 + A_1) \supset A_4(A_3 + \mathbb{C}) + A_4(A_2 + A_1 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(4H_1 + 3H_3 + 2H_4 + H_2 + 3H_6 + 6H_7 + 4H_8 + 2H_0)$	++	1
$E_8(D_8) \supset A_4(A_2 + A_1 + \mathbb{C}) + A_4(A_2 + A_1 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(3H_1 + 6H_3 + 4H_4 + 2H_2 + 3H_6 + 6H_7 + 4H_8 + 2H_0)$	++	1
$E_8(D_8) \supset A_4(B_2) + A_4(B_2)$ $\bar{\theta} = \delta$	++	1

Table 50. $A_8 \subset E_8$



There exists an automorphism δ such that $\delta^2 = Id$ and

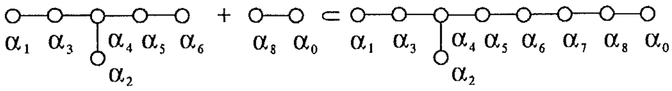
$$\delta(X_1) = X_0,\ \delta(X_3) = X_8,\ \delta(X_4) = X_7,$$

$$\delta(X_5) = X_6,\ \delta(X_6) = X_5,\ \delta(X_7) = X_4,$$

$$\delta(X_8) = X_3,\ \delta(X_0) = X_1.$$

1	2	3
$E_8(E_8) \supset A_8(A_8)$ $\bar{\theta} = Id$	++	1
$E_8(D_8) \supset A_8(A_7 + \mathbb{C})$ $\bar{\theta} = \exp \mathrm{ad}\, \pi\sqrt{-1}(8H_1 + 7H_3 + 6H_4 + 5H_5 + 4H_6 + 3H_7 + 2H_8 + H_0)$	++	1
$E_8(E_7 + A_1) \supset A_8(A_6 + A_1 + \mathbb{C})$ $\bar{\theta} = \exp \mathrm{ad}\, \pi\sqrt{-1}(7H_1 + 14H_3 + 12H_4 + 10H_5 + 8H_6 + 6H_7 + 4H_8 + 2H_0)$	++	1
$E_8(E_7 + A_1) \supset A_8(A_5 + A_2 + \mathbb{C})$ $\bar{\theta} = \exp \mathrm{ad}\, \pi\sqrt{-1}(6H_1 + 12H_3 + 18H_4 + 15H_5 + 12H_6 + 9H_7 + 6H_8 + 3H_0)$	++	1
$E_8(D_8) \supset A_8(A_4 + A_3 + \mathbb{C})$ $\bar{\theta} = \exp \mathrm{ad}\, \pi\sqrt{-1}(5H_1 + 10H_3 + 15H_4 + 20H_5 + 16H_6 + 12H_7 + 8H_8 + 4H_0)$	++	1
$E_8(D_8) \supset A_8(B_4)$ $\bar{\theta} = \delta$	++	1

Table 51. $E_6 + A_2 \subset E_8$



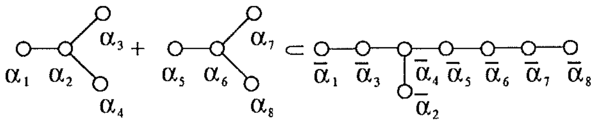
There exists an automorphism δ such that $\delta^2 = Id$ and

$$\delta(X_1) = X_6, \delta(X_2) = X_2, \ \delta(X_3) = X_5, \ \delta(X_4) = X_4,$$

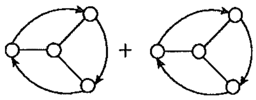
$$\delta(X_5) = X_3, \ \delta(X_6) = X_1, \ \delta(X_8) = X_0, \ \delta(X_0) = X_8.$$

1	2	3
$E_8(E_8) \supset E_6(E_6) + A_2(A_2)$ $\bar{\theta} = Id$	++	1
$E_8(D_8) \supset E_6(D_5 + \mathbb{C}) + A_2(A_2)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(2H_1 + 2H_2 + 3H_3 + 4H_4 + 3H_5 + 2H_6)$	++	1
$E_8(E_7 + A_1) \supset E_6(A_5 + A_1) + A_2(A_2)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_2 + 2H_3 + 3H_4 + 2H_5 + H_6)$	++	1
$E_8(E_7 + A_1) \supset E_6(E_6) + A_2(A_1 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_8 + H_0)$	++	1
$E_8(E_7 + A_1) \supset E_6(D_5 + \mathbb{C}) + A_2(A_1 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(2H_1 + 2H_2 + 3H_3 + 4H_4 + 3H_5 + 2H_6 + H_8 + H_0)$	++	1
$E_8(D_8) \supset E_6(A_5 + A_1) + A_2(A_1 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_2 + 2H_3 + 3H_4 + 2H_5 + H_6 + H_8 + H_0)$	++	1
$E_8(D_8) \supset E_6(A_5 + A_1) + A_2(A_1 + \mathbb{C})$ $\bar{\theta} = \delta \circ \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_2 + 2H_3 + 3H_4 + 2H_5 + H_6)$	++	1
$E_8(E_7 + A_1) \supset E_6(F_4) + A_2(A_1)$ $\bar{\theta} = \delta$	++	1

Table 52. $D_4 + D_4 \subset E_8$

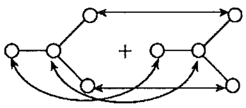


Consider the automorphism σ of the Lie algebra $\mathfrak{p}$ which acts on the Dynkin diagram in the following way:



And let ε be one of extensions of σ. Without loss of generality we can assume that $\varepsilon^3 = Id$. There exists an automorphism δ of the Lie algebra $\mathfrak{p}$ such that

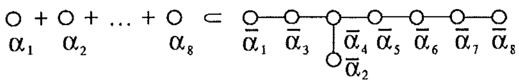
(1) $\delta(\mathfrak{p}) = \mathfrak{p}$;
(2) the restriction of δ to $\mathfrak{p}$ has the following diagram:



(3) $\varepsilon\delta = \delta\varepsilon$;
(4) δ acts identically on all weight vectors of $\mathfrak{p}$-module $\bar{\mathfrak{p}}$ corresponding to the weights invariant under the restriction $\delta|_{\mathfrak{p}}$.

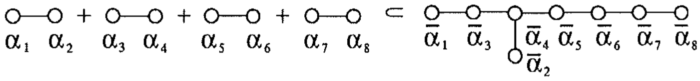
1	2	3
$E_8(E_8) \supset D_4(D_4) + D_4(D_4)$ $\bar{\theta} = Id$	+	1
$E_8(E_7 + A_1) \supset D_4(A_1 + A_1 + A_1 + A_1) + D_4(D_4)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_2 + H_3 + H_4)$	+	1
$E_8(D_8) \supset D_4(A_1 + A_1 + A_1 + A_1) + D_4(A_1 + A_1 + A_1 + A_1)$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + 2H_2 + H_3 + H_4 + H_5 + 2H_6 + H_7 + H_8)$	+	1
$E_8(E_7 + A_1) \supset (D_4 + D_4)D_4$ $\bar{\theta} = \delta$	+	1

Table 53. $8A_1 \subset E_8$



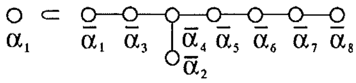
1	2	3
$E_8(E_8) \supset 8A_1(A_1)$ $\bar{\theta} = Id$	+	1
$E_8(D_8) \supset 8A_1(\mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}\frac{1}{2}(H_1 + \cdots + H_8)$	+	1

Table 54. $4A_2 \subset E_8$



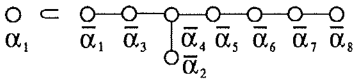
1	2	3
$E_8(E_8) \supset 4A_2(A_2)$ $\bar{\theta} = Id$	$+$	1
$E_8(D_8) \supset 2A_2(A_2) + 2A_2(A_1 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_5 + H_6 + H_7 + H_8)$	$+$	1
$E_8(D_8) \supset 4A_2(A_1 + \mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + \cdots + H_8)$	$+$	1

Table 55. $A_1^{1240} \subset E_8$



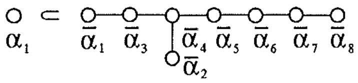
1	2	3
$E_8(E_8) \supset A_1(A_1)$ $\bar{\theta} = Id$	$++$	1
$E_8(D_8) \supset A_1(\mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \frac{1}{2}\pi\sqrt{-1}H_1$	$++$	1

Table 56. $A_1^{760} \subset E_8$



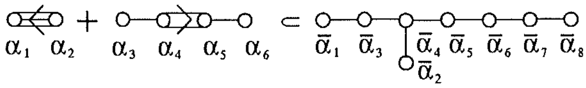
1	2	3
$E_8(E_8) \supset A_1(A_1)$ $\bar{\theta} = Id$	$++$	1
$E_8(D_8) \supset A_1(\mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \frac{1}{2}\pi\sqrt{-1}H_1$	$++$	1

Table 57. $A_1^{520} \subset E_8$



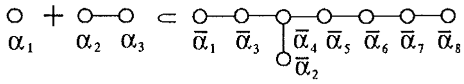
1	2	3
$E_8(E_8) \supset A_1(A_1)$ $\bar{\theta} = Id$	$++$	1
$E_8(D_8) \supset A_1(\mathbb{C})$ $\bar{\theta} = \exp \operatorname{ad} \frac{1}{2}\pi\sqrt{-1}H_1$	$++$	1

Table 58. $G_2^1 + F_4^1 \subset E_8$



1	2	3
$E_8(E_8) \supset G_2(G_2) + F_4(F_4)$ $\bar{\theta} = Id$	++	1
$E_8(E_7 + A_1) \supset G_2(G_2) + F_4(C_3 + A_1)$ $\bar{\theta} = \exp \mathrm{ad}\, \pi\sqrt{-1}(2H_3 + 3H_4 + 2H_5 + H_6)$	++	1
$E_8(D_8) \supset G_2(G_2) + F_4(B_4)$ $\bar{\theta} = \exp \mathrm{ad}\, \pi\sqrt{-1}(2H_3 + 4H_4 + 3H_5 + 2H_6)$	++	1
$E_8(E_7 + A_1) \supset G_2(A_1 + A_1) + F_4(F_4)$ $\bar{\theta} = \exp \mathrm{ad}\, \pi\sqrt{-1}(2H_1 + 3H_2)$	++	1
$E_8(D_8) \supset G_2(A_1 + A_1) + F_4(C_3 + A_1)$ $\bar{\theta} = \exp \mathrm{ad}\, \pi\sqrt{-1}(2H_1 + 3H_2 + 2H_3 + 3H_4 + 2H_5 + H_6)$	++	1
$E_8(E_7 + A_1) \supset G_2(A_1 + A_1) + F_4(B_4)$ $\bar{\theta} = \exp \mathrm{ad}\, \pi\sqrt{-1}(2H_1 + 3H_2 + 2H_3 + 4H_4 + 3H_5 + 2H_6)$	++	1

Table 59. $A_2^{6'} + A_1^{16} \subset E_8$

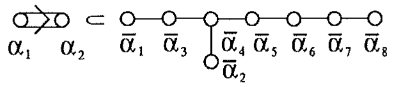


There exists an automorphism δ of the Lie algebra $\bar{\mathfrak{p}}$ such that $\delta^2 = Id$ and $\delta(\bar{X}_i) = \bar{X}_{-i}$. Let

$$\sigma = \exp \mathrm{ad}\, \pi\sqrt{-1}X_1 \circ \exp \mathrm{ad}\, \pi\sqrt{-1}X_{-1} \circ \exp \mathrm{ad}\, \pi\sqrt{-1}X_1.$$

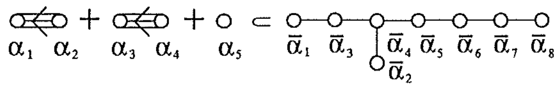
1	2	3
$E_8(E_8) \supset A_1(A_1) + A_2(A_2)$ $\bar{\theta} = Id$	++	1
$E_8(D_8) \supset A_1(\mathbb{C}) + A_2(A_2)$ $\bar{\theta} = \exp \mathrm{ad}\, \pi\sqrt{-1}\frac{1}{2}H_1$	++	1
$E_8(D_8) \supset A_1(A_1) + A_2(A_1 + \mathbb{C})$ $\bar{\theta} = \exp \mathrm{ad}\, \pi\sqrt{-1}\frac{1}{2}(H_2 + H_3)$	++	1
$E_8(D_8) \supset A_1(\mathbb{C}) + A_2(A_1 + \mathbb{C})$ $\bar{\theta} = \exp \mathrm{ad}\, \pi\sqrt{-1}\frac{1}{2}H_1$	++	1
$E_8(D_8) \supset A_1(\mathbb{C}) + A_2(A_1)$ $\bar{\theta} = \delta$	++	1
$E_8(E_7 + A_1) \supset A_1(A_1) + A_2(A_1)$ $\bar{\theta} = \sigma \circ \delta$	++	1

Table 60. $B_2^{12} \subset E_8$



1	2	3
$E_8(E_8) \supset B_2(B_2)$ $\bar\theta = Id$	++	1
$E_8(D_8) \supset B_2(A_1 + \mathbb{C})$ $\bar\theta = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + \frac{1}{2}H_2)$	++	1
$E_8(D_8) \supset B_2(A_1 + A_1)$ $\bar\theta = \exp \operatorname{ad} \pi\sqrt{-1}(H_1 + H_2)$	++	1

Table 61. $G_2^1 + G_2^1 + A_1^8 \subset E_8$



There exist an automorphism σ of the Lie algebra $\bar{\mathfrak{p}}$ such that $\sigma^2 = Id$ and

$$\sigma(\bar X_1) = \bar X_1,\ \sigma(\bar X_2) = -\bar X_{4256},\ \sigma(\bar X_3) = -\bar X_{4356},$$

$$\sigma(\bar X_4) = \bar X_7,\ \sigma(\bar X_5) = \bar X_8,\ \sigma(\bar X_6) = -\bar X_{-45678},$$

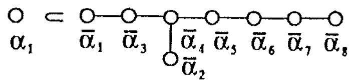
$$\sigma(\bar X_7) = \bar X_4,\ \sigma(\bar X_8) = \bar X_5,$$

and an automorphism φ of the Lie algebra $\bar{\mathfrak{p}}$ such that $\varphi^2 = Id$ and

$$\varphi(\bar X_1) = -\bar X_1, \quad \phi(\bar X_i) = \sigma(\bar X_i), \quad 2 \leqslant i \leqslant 8.$$

1	2	3
$E_8(E_8) \supset G_2(G_2) + G_2(G_2) + A_1(A_1)$ $\bar\theta = Id$	+	1
$E_8(D_8) \supset G_2(G_2) + G_2(G_2) + A_1(\mathbb{C})$ $\bar\theta = \exp \operatorname{ad} \pi\sqrt{-1}\frac{1}{2}H_5$	+	1
$E_8(D_8) \supset G_2(A_1 + A_1) + G_2(A_1 + A_1) + A_1(A_1)$ $\bar\theta = \exp \operatorname{ad} \pi\sqrt{-1}(2H_1 + 3H_2 + 2H_3 + 3H_4)$	+	1
$E_8(D_8) \supset G_2(A_1 + A_1) + G_2(A_1 + A_1) + A_1(\mathbb{C})$ $\bar\theta = \exp \operatorname{ad} \pi\sqrt{-1}(2H_1 + 3H_2 + 2H_3 + 3H_4 + \frac{1}{2}H_5)$	+	1
$E_8(E_7 + A_1) \supset (G_2 + G_2)(G_2) + A_1(A_1)$ $\bar\theta = \sigma$	++	1
$E_8(D_8) \supset G_2(A_1 + A_1) + G_2(A_1 + A_1) + A_1(\mathbb{C})$ $\bar\theta = \varphi$	++	1

Table 62. $A_1^{40} \subset E_8$



1	2	3
$E_8(E_8) \supset A_1(A_1)$ $\bar{\theta} = Id$	+	1
$E_8(D_8) \supset A_1(\mathbb{C})$ $\bar{\theta} = \exp \mathrm{ad}\, \frac{1}{2}\pi\sqrt{-1}H_1$	++	1

REFERENCES

1. S. Lie, *Theorie der transformations gruppen. 3*, Leipzig, 1893.
2. V.V. Morozov, *On nonsemisimple maximal subgroups of simple Lie groups*, Thesis, Kazan State University, 1943. (Russian)
3. M. Golubitsky, *Primitive actions and maximal subgroups of Lie groups*, J. Diff. Geom. **7** (1972), 175–191.
4. V.V. Morozov, *On primitive groups*, Mat. Sbornik **5** **(47)** (1939), 355–390.
5. F.I. Karpelevich, *On nonsemisimple maximal subalgebras of semisimple Lie algebras*, Doklady AN SSSR **76** (1951), 775–778. (Russian)
6. E.B. Dynkin, *Maximal subgroups of classical groups*, Amer. Math. Soc. Transl. (2) **6** (1957).
7. E.B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Transl. (2) **6** (1957).
8. A. Borel and J. de Siebenthal, *Les sous groupes fermèss de rang maximum des groupes de Lie compacts*, Comm. Math. Helv. **23** (1949), 200–221.
9. M. Golubitsky and B. Rothshild, *Primitive subalgebras of exceptional Lie algebras*, Pacific J. Math. **39, N2** (1971), 371–393.
10. I.V. Chekalov, *Primitive subalgebras of complex Lie algebras*, Doklady AN Byelorus. SSR **13, N9** (1979), 773–776. (Russian)
11. N.P. Mushits, *On maximal reducible subgroups of simple real Lie algebras*, Uchenye Zapiski Kazan. Univ. **114, kn.2** (1954), 195–204. (Russian)
12. M.S. Taufic, *Semisimple subalgebras of certain simple real Lie algebras*, Uspekhi Mat. Nauk **30, N4** (1975), 261–262. (Russian)
13. M.S. Taufic, *On maximal subalgebras in classical real Lie algebras*, Sbornik trudov (1979), 148–168, Yaroslavl. (Russian)
14. A.I. Maltsev, *On semisimple subgroups of Lie groups*, Amer. Math. Soc. Transl. (1) **9** (1962).
15. E. Cartan, *Les groupes projectifs qui ne laissent invariante ancune multiplicitè plane*, Bull. Soc. Math. de France, deuxième sèrie **41** (1913).
16. N. Ivahori, *On real irreducible representations of Lie algebras*, Nagoya Math. Journ. **14** (1959), 59–83.
17. F.I. Karpelevich, *Simple subalgebras of real Lie algebras*, Trudy Moskovskogo Mat. Ob. **4**, 3–112. (Russian)
18. A.S. Fedenko, *Symmetric spaces with noncompact fundamental groups*, Doklady AN SSSR **108** (1956), 1026–1029.
19. M. Berger, *Les èspaces simètriques non compacts*, Ann. Sci. Ecole. Norm. **74** (1957), 85–117.
20. A. Gray, *Riemannian manifolds with geodesic symmetries of order 3*, J. Diff. Geom. **7** (1972), 343–369.
21. B.P. Komrakov, *Real forms of singular subalgebras of exceptional Lie algebras*, Deposited at VINITI N6835-B87, 36 pages.
22. B.P. Komrakov and A.V. Churyumov, *The exceptional subalgebra A_1^{40} in E_8*, Deposited at VINITI N6836-B87 (1987). (Russian)
23. B.P. Komrakov, *Reductive subalgebras of reductive real Lie algebras*, Group theory methods in physics. Trudy Tretiego Sem. Yurmala **2** (1986), 205–212, "Nauka". (Russian)
24. B.P. Komrakov, *Reductive subalgebras of semisimple real Lie algebras*, Soviet Math. Dokl. **v.40** (1990), 329–333.
25. B.P. Komrakov, *Maximal subalgebras of semisimple real Lie algebras and the Sophus Lie problem*, Soviet Math. Dokl. **v.41** (1991), 269–273.
26. B.P. Komrakov, *Primitive actions and the Sophus Lie problem*, "Vysh. Shkola", Minsk, 1991. (Russian)