Rational Curves of low degree on a complete intersection Calabi-Yau threefold in $\mathbb{P}^3 \times \mathbb{P}^3$

by

Dag Einar Sommervoll
Rational Curves of low degree on a complete intersection Calabi-Yau threefold in $\mathbb{P}^3 \times \mathbb{P}^3$.

DAG EINAR SOMMERVOLL

In this note we compute the number of rational curves of degree one, two and three on Yau’s CICY (Complete Intersection Calabi-Yau) threefold in $\mathbb{P}_1^3 \times \mathbb{P}_2^3$. The number of lines and conics agree with the conjectured numbers worked out by Van Straten (private communication) using ”mirror techniques”. Yau’s manifold, which is a quotient of Yau’s CICY by a group acting without fixed points, gives the easiest known example of a manifold that corresponds to a three generational superstring model ([1] [6]). Note that there are infinitely many curves of every degree greater than 3 (this is true even for a generic choice of defining equations for the CICY) [5].

1. Preliminaries.

Yau’s CICY is defined by the following:

$$X = Z(\sum x_i^3, \sum x_iy_i, \sum y_i^3) \subseteq \mathbb{P}_1^3 \times \mathbb{P}_2^3$$

It is defined by three polynomials of bidegrees (3,0), (1,1) and (0,3). When we speak of a generic Yau’s CICY, we will mean a generic choice of polynomials of these bidegrees.

By a rational curve we will mean a nonsingular rational curve.

We introduce the following notation: $F_1$ (resp. $F_2$) is the cubic surface in $\mathbb{P}_1^3$ (resp. $\mathbb{P}_2^3$) defined by the polynomial of degree (3,0), (resp. (0,3)). The ”incidence variety” $G$ is defined by a polynomial of bidegree (1,1). In other words:

$$X = F_1 \times F_2 \cap G$$

On this biprojective space we have a natural notion of degree of a rational curve, by defining it to be the degree of it’s image in $\mathbb{P}^{15}$ by the Segre embedding.

Every rational curve of degree $d$ has a bidegree $(i,j)$ with $i+j = d$. The Hilbert scheme $\text{Hilb}^{dm+1}_X$ therefore has a natural partition in open-closed disjoint subschemes $\text{Hilb}^{(i,j)m+1}_X$ with $i + j = d$.

We are primarily interested in curves of degree 1, 2 and 3, and have the following partitions:

Typeset by \LaTeX
\[ \text{Hilb}_X^{m+1} \cong \text{Hilb}_X^{(1,0)m+1} \cup \text{Hilb}_X^{(0,1)m+1} \]
\[ \text{Hilb}_X^{2m+1} \cong \text{Hilb}_X^{(2,0)m+1} \cup \text{Hilb}_X^{(1,1)m+1} \cup \text{Hilb}_X^{(0,2)m+1} \]
\[ \text{Hilb}_X^{3m+1} \cong \text{Hilb}_X^{(3,0)m+1} \cup \text{Hilb}_X^{(2,1)m+1} \cup \text{Hilb}_X^{(1,2)m+1} \cup \text{Hilb}_X^{(0,3)m+1} \]

One immediate observation is that the curves we want to study, always lie in the inverse image of a line in one factor. In the next section we will investigate the geometry of \( G \) over a line in one factor. In the last section we enumerate the curves of degree 2 and 3 on Yau’s CICY. The Appendix contains a description of conics on the cubic \( F_i \) in \( \mathbb{P}_i^3 \).


Let \( G = Z(\sum \alpha_{ij}x_iy_j) \in \mathbb{P}_1^3 \times \mathbb{P}_2^3 \).

**Definition 2.1.**

Let \( L \) be a line in \( \mathbb{P}_2^3 \) (resp. \( \mathbb{P}_3^3 \)). Define \( V(L) \) in \( \mathbb{P}_1^3 \) (resp. \( \mathbb{P}_2^3 \)) to be the unique linear subspace of maximal dimension such that \( V \times L \) (resp. \( L \times V \)) is contained in \( G \).

The following lemma assures that the definition above makes sense.

**Lemma 2.2.**

Let \( G = Z(\sum \alpha_{ij}x_iy_j) \in \mathbb{P}_1^3 \times \mathbb{P}_2^3 \) and \( L \) a line in \( \mathbb{P}_2^3 \) (resp. \( \mathbb{P}_1^3 \)). For any given \( L \), \( V(L) \) has dimension at least one, and it is unique.

**Proof.**

It is enough to prove the assertion in the case where the line \( L \) is in \( \mathbb{P}_2^3 \), the other case follows by symmetry.

Choose coordinates s.t. \( L \) is defined by \((y_0, y_1, 0, 0)\). \( G \) becomes with respect to these new coordinates \( Z(\sum \alpha_{ij}^1x_iy_j) \). Let \( \tilde{G} = G|_{\mathbb{P}^3 \times L} \), then \( \tilde{G} \) is defined by the following equation:

\[ (\sum \alpha_{i0}^1x_i)y_0 + (\sum \alpha_{i1}^1x_i)y_1 = 0 \]

\[ Z(\sum \alpha_{i0}^1x_i, \sum \alpha_{i1}^1x_i) \] is obviously both maximal and unique, we get

\[ V(L) = Z(\sum \alpha_{i0}^1x_i, \sum \alpha_{i1}^1x_i) \]

\( \blacklozenge \)

**Remark 2.3.** In fact we proved more, every point \( a \in \mathbb{P}_2^3 \) with the property that \( a \times L \subset G \), is contained in \( V(L) \).

**Remark 2.4.** \( \dim V = 1, 2, 3 \) all occur.

The general case is clearly \( \dim V = 1 \). The definition of \( V \) depends on \( L \) as well as on \( G \). We are primarily interested in the case when we are in the general situation for all \( L \subset \mathbb{P}_i^3 \) \( i = 1, 2 \).

We make the following definition:
Definition 2.5.

G is nice if \( \dim V(L) = 1 \) for all \( L \subseteq \mathbb{P}^3_i, \ i = 1, 2 \).

When \( G \) is nice, we have a map:

\[
l : \text{Grass}(\mathbb{P}^3) \longrightarrow \text{Grass}(\mathbb{P}^3)_i
\]

This is defined by sending \( L \) to \( V(L) \). This map is obviously bijective since \( l(l(L)) = L \) by definition.

We have a criterion for when \( G \) is nice:

Lemma 2.6.

Let \( G = Z(\sum \alpha_{ij} x_i y_j) \).

If the matrix \( [\alpha_{ij}] \) is of maximal rank, then \( G \) is nice.

Proof.

Introduce the following notation for \( \sum \alpha_{ij} x_i y_j, \; x A y^t \), where \( x = (x_0, \ldots, x_3) \) (likewise for \( y \)) and

\[
A = \begin{pmatrix}
\alpha_{00} & \cdots & \alpha_{03} \\
\cdots & \cdots & \cdots \\
\alpha_{30} & \cdots & \alpha_{33}
\end{pmatrix}
\]

For a generic choice of the \( \alpha_{ij} \)'s \( A \) is invertible. We have to prove that for every line \( L \) in \( \mathbb{P}^3_i \), \( V(L) \) is of minimal dimension. (\( V(L) \) is defined to be the maximal linear subspace of \( \mathbb{P}^3_i \) such that \( V(L) \times L \subseteq G \).) Consider first the special case where \( L = (y_0, y_1, 0, 0) \). This gives the following \( V(L) \):

\[
Z(\sum \alpha_{i1} x_i, \sum \alpha_{i0} x_i) \times L \subseteq \mathbb{P}^3_1 \times \mathbb{P}^3_0
\]

Assume that \( V(L) \) is not of minimal dimension, i.e. \( \dim Z(\sum \alpha_{i1} x_i, \sum \alpha_{i0} x_i) \geq 2 \), then there are two different possibilities:

1. \( \sum \alpha_{i1} x_i \equiv 0 \) or \( \sum \alpha_{i0} x_i \equiv 0 \). In this case \( \alpha_{i0} \) or \( \alpha_{i1} \) is zero for all \( i \), which contradicts that \( A \) is of maximal rank.

2. \( Z(\sum \alpha_{i1} x_i) = Z(\sum \alpha_{i0} x_i) \). This implies that \( \sum \alpha_{i1} x_i = \lambda \sum \alpha_{i0} x_i \), giving \( \alpha_{i1} = \lambda \alpha_{i0} \). In other words the first two rows are proportional, which contradicts that \( A \) is of maximal rank.

The final step is reducing the general situation to the special case considered above. This is done in the following way: Choose a general \( L \) in \( \mathbb{P}^3_0 \). It is possible to change the coordinates on the second factor, such that \( L \) is parametrised by \( (y_0', y_1', 0, 0) \). Call this coordinate change matrix \( P \) (i.e., \( (y_1', \ldots, y_3')^t = P \cdot (y_1, \ldots, y_3)^t \)). We make the following change of coordinates on the first factor:

\[
x'^t = (A^{-1})^t P^t A^t x^t
\]

This gives \( G = Z(\sum \alpha_{ij} x'_i y'_j) \) with respect to the new coordinates, since

\[
\sum \alpha_{ij} x_i y_j = x A y^t = (x' A P A^{-1} (A P^{-1} y'^t) = x' A y' = \sum \alpha_{ij} x'_i y'_j.
\]

The result now follows from the special case considered above.

\[\mathfrak{M}\]

When \( G \) is nice, we write \( l(L) \) for \( V(L) \) to signify that it is a line.
Proposition 2.7.

Let $G$ be nice, and let $\tilde{G} = G_{|p_3 \times L}$.

Then $\tilde{G}$ is isomorphic to the blowing-up of $\mathbb{P}^3_1$ in $l(L)$.

Proof.

We can without loss of generality assume that $\tilde{G}$ is defined by

$$Z(x_1y_2 - x_2y_1) \subseteq \mathbb{P}^3(x_0, \ldots, x_3) \times \mathbb{P}^1(y_1, y_2)$$

(by change of coordinates).

In this situation $l(L)$ is defined by $x_1 = x_2 = 0$. It is enough to check the statement locally, take for instance $x_0 = 1$. Then we have

$$Z(x_1y_2 - x_2y_1) \subseteq \mathbb{A}^3 \times \mathbb{P}^1.$$

This is in fact the blowing-up of $\mathbb{A}^3$ with center $Z(x_1, x_2)$ ([3] II.7.12.1).

We have the following important corollary:

Corollary 2.8.

Let $G$ be nice, and let $\tilde{G} = G_{|H \times L}$, where $H$ is a hyperplane and $L$ is a line. Denote the blow-up map $\tilde{G} \to \mathbb{P}^3_3$ by $\pi$.

Then $\tilde{G}$ is isomorphic to $\pi^{-1}(H)$. In the case $l(L) \not\subseteq H$ then $\tilde{G}$ is isomorphic to $H$ blown up in the point $H \cap l(L)$.

3. Curves of degree 1, 2 and 3 on Yau’s CICY.

In view of section 1, we have to study rational curves of bidegree $(1,0),(1,1),(2,0),(2,1)$ and $(3,0)$.

Let $C$ be a rational curve in $\mathbb{P}^3_1 \times \mathbb{P}^3_2$. Let $f$ be a parametrisation:

$$\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^3_1 \times \mathbb{P}^3_2 \xrightarrow{\pi} \mathbb{P}^3_3$$

Definition 3.1.

A rational curve in $\mathbb{P}^3_1 \times \mathbb{P}^3_2$ is of type $(\tilde{m}, \tilde{n})$, if the image of the first (resp. second) projection is of degree $m$ ( resp. $n$).

Recall that a generic Yau CICY was defined to be a complete intersection:

$$X = F_1 \times F_2 \cap G$$

where $F_1$ (resp. $F_2$) is defined by a polynomial of bidegree $(3,0)$ (resp. $(0,3)$), and the incidence variety $G$ is defined by a polynomial of bidegree $(1,1)$.

We start out by discussing the curves of bidegree $(1,0)$ and $(1,1)$. 
Proposition 3.2. Let $G$ be nice, and let $L \subseteq \mathbb{P}^3_1$ and $L' \subseteq \mathbb{P}^3_2$ be a pair of lines, such that $l(L) \neq L'$. Then

1) If $L' \cap l(L) = \emptyset$, then $D = L' \times L \cap G$ is the irreducible rational curve of bidegree $(1,1)$.

2) If $L' \cap l(L) \neq \emptyset$, then $D = L' \times L \cap G$ is the union of a rational curve of bidegree $(1,0)$ and a rational curve of bidegree $(0,1)$.

Proof.

In either case, choose a plane $H$ such that $l(L) \nsubseteq H$ and $L \subseteq H$. By Corollary 2.8 $\tilde{G}$ is isomorphic to the blow up of $H$ in the point $p = l(L) \cap H$. Let $C$ denote the strict transform of $L$ and $\pi$ the blow up map. Then $\pi_1|_C : C \longrightarrow L'$ is an isomorphism. The degree on the first factor is 1 in either case.

Case 1.

$D = C$. Note that $(a, b) \in D$ is equivalent to $a \in Z(\sum \alpha_i b_j x_i) \cap L$, giving that the degree on the second factor must be 1.

$C$ is a rational curve of bidegree $(1,1)$.

Case 2.

$D = \pi^{-1}(L) = C \cup E$, where $E$ denotes the exceptional fiber of the blow up of $H$. The degree on the second factor has to be zero since $\pi_2 : D \longrightarrow L$ has generically one point in the fiber, and $\pi_2 : E \longrightarrow L$ is obviously an isomorphism. Thus $C$ is a $(1,0)$ curve and $E$ is a $(0,1)$ curve.

Proposition 3.3.

The number of $(1,1)$ curves on Yau’s CICY is 567.

Proof.

In view of Proposition 3.2 we have to find the number of irreducible intersections $L' \times L \cap G$ where $L' \subseteq F_1$ and $L \subseteq F_2$. We have 729 pairs of lines to consider. We start out by finding the reducible pairs which give reducible intersections. In other words we want to determine the number of pairs such that $l(L) \cap L'$ is a point.

Choose the line $L$ to be $F_{00}^{1}$ (in the notation the Appendix):

\begin{align*}
y_0 + y_1 &= 0 \\
y_2 + y_3 &= 0
\end{align*}

This gives $l(L)$:

\begin{align*}
x_0 - x_1 &= 0 \\
x_2 - x_3 &= 0
\end{align*}

We look at planes in the first factor that contain three lines. From the Appendix we know that every line is contained in such a plane, and that any line intersect the two other
lines in two distinct points. (From this follows directly that \( l(L) \) can at most intersect two of these lines.) The planes are defined by:

\[
\alpha^{i_0}x_0 + \alpha^{i_1}x_1 + \alpha^{i_2}x_2 + \alpha^{i_3}x_3 = 0
\]

with \( i_j \in \{0, 1, 2\}, \quad j = 0, 1, 2, 3. \)

Denote such a plane by \( H. \)

The intersection point \( p \) between \( l(L) \) and \( H \) is of the form \((1, 1, -\alpha^l, -\alpha^l)\) with \( l \in \{0, 1, 2\}. \)

The plane \( H \) contains the following lines:

\[
\begin{align*}
L_1 : & \quad \alpha^{i_0}x_0 + \alpha^{i_1}x_1 = \alpha^{i_2}x_2 + \alpha^{i_3}x_3 = 0 \\
L_2 : & \quad \alpha^{i_0}x_0 + \alpha^{i_2}x_2 = \alpha^{i_1}x_1 + \alpha^{i_3}x_3 = 0 \\
L_3 : & \quad \alpha^{i_0}x_0 + \alpha^{i_3}x_3 = \alpha^{i_1}x_1 + \alpha^{i_2}x_2 = 0
\end{align*}
\]

Clearly \( p \) cannot lie on \( L_1 \). If \( p \) is on \( L_2 \) then:

\[
\begin{align*}
\alpha^{i_0} - \alpha^{i_2 + l} &= 0 \\
\alpha^{i_1} - \alpha^{i_3 + l} &= 0
\end{align*}
\]

(To be contained in the plane gives the condition \( \alpha^l(\alpha^{i_0} + \alpha^{i_1}) = \alpha^{i_2} + \alpha^{i_3}. \))

We get the following three planes:

\[
\alpha^{i_0}x_0 + \alpha^{i_1}x_1 + \alpha^{i_0 - l}x_2 + \alpha^{i_1 - l}x_3 = 0
\]

It is easy to check that demanding \( p \) to lie on \( L_3 \) gives the same three planes and the same intersection point as before.

The total number of planes \( H \) giving intersections between a \( l(L) \), \( L \subseteq F_2 \) and a line \( L' \) contained in both \( F_1 \) and \( H \) is \( 27 \cdot 3 = 81 \). Since every such plane contains two lines that are in \( F_1 \), and intersects \( l(L) \), the number of irreducible intersections \( L' \times L \cap G \) is equal to \( 27 \cdot 27 - 2 \cdot 81 = 567 \).

\[\clubsuit\]

A Corollary of the proof is that there are at least 81 \((1, 0)\) curves. These are in fact the only ones.

**Proposition 3.4.** There are 81 \((1, 0)\) curves on Yau’s CICY.

**Proof.**

By symmetry it is enough to consider how many points \( b \in F_2 \) there are with the property \( F_1^{00} \times b \subseteq G \). Parametrise \( F_1^{00} \times b \) by \((t, -t, u, -u) \times (b_0, b_1, b_2, b_3)\) Imposing the conditions \( \sum x_i b_i = 0 \) and \( \sum b_i^3 = 0 \), gives that the solutions for \( b \) are \((1, 1, -\alpha^l, -\alpha^l)\). Thus we get 81 curves of degree \((1, 0)\).

\[\clubsuit\]
Corollary 3.5. There are 162 curves of degree 1 on Yau’s CICY.

\[
\begin{align*}
\bullet & \\
\text{The number of curves of bidegree } (0,1), (1,0) \text{ and } (1,1) \text{ is therefore } 567 + 162 = 729. \\
\text{This number has been calculated previously, using different techniques ([2],[4]).} \\
\text{Now consider a conic } C_1 \text{ of degree 2 contained in a plane } H \in \mathbb{P}_1^3. \text{ Furthermore, fix a} \\
\text{line } L \text{ in } \mathbb{P}_1^3. \text{ Define } D \text{ by the following:} \\
D := C_1 \times L \cap G \subseteq H \times L \cap G = \tilde{G} \\
\end{align*}
\]

By Corollary 2.8 \( \tilde{G} \) is isomorphic to \( \pi^{-1}(H) \). \( D \) is a one-dimensional subscheme, since it is isomorphic to \( \pi^{-1}(C_1) \).

The structure of \( D \) depends on \( L \) and \( C_1 \). There are three different cases:
1. \( \lg(l(L) \cap C_1) = 2 \)
2. \( \lg(l(L) \cap C_1) = 1 \)
3. \( \lg(l(L) \cap C_1) = 0 \)

1. In this case \( l(L) \) is contained in \( H \). If \( l(L) \) is not tangent to \( C_1 \), then \( l(L) \) and \( C_1 \) intersect in two distinct points. Then \( D \cong C \cup E_1 \cup E_2 \). Here \( C \) denotes the strict transform of \( C_1 \) and the \( E_i \) are the exceptional fibers above the two intersection points \( l(L) \cap C_1 \). \( C \) is a rational curve. We want to determine its bidegree.

\[
\pi_C : C \longrightarrow C_1 \text{ is an isomorphism, giving that the degree on the first factor is 2.} \\
\text{Consider} \\
\pi_2|_D : D \longrightarrow L
\]

I claim that \( \pi_2|_D \) has generically two points in the fiber over a point in \( L \). Let \((a,b)\) be a point in \( D \). Then \( a \) is a point on \( C_1 \), and since \((a,b)\) is on \( G \), we have the relation \( \sum \alpha_{ij}a_ib_j = 0 \). Hence, \( a \in C_1 \cap Z(\sum \alpha_{ij}b_jx_i) \), which gives \((a,b) \in D \) if and only if \( a \in C_1 \cap Z(\sum \alpha_{ij}b_jx_i) \). Bezout’s theorem then gives that for almost all \( b \in L \) there are two points in the fiber of \( \pi_2|_D \), since not all the \( Z(\sum \alpha_{ij}b_jx_i) \) contains a tangent of \( C_1 \).

If \( l(L) \) is a tangent of \( C_1 \), then all the \( Z(\sum \alpha_{ij}b_jx_i) \) contains this tangent, and the projection has generically one point in the fiber. Moreover, in this case \( D = C \cup E_1 \) as sets.

Since the maps \( \pi|_{E_i} : E_i \longrightarrow L \) are isomorphisms, \( \pi|_C : C \longrightarrow L \) is not dominant. The degree of \( C \) on the second factor is then 0. The curve \( C \) has bidegree \((2,0)\).

In the remaining cases \( l(L) \) is not contained in \( H \) and \( H \times L \cap G \) is isomorphic to the blow up of \( H \) in the point \( p = l(L) \cap H \). Let \( C \) be the strict transform of \( C_1 \). The map \( \pi_2|_D : D \longrightarrow L \) is 2:1 (by the same argument as in 1). Since \( \pi_1|_C : C \longrightarrow C_1 \) is an isomorphism, the degree on the first factor of the rational curve \( C \) is 2.

2. In this case the curve \( C_1 \) intersects \( l(L) \), implying that \( D = C \cup E \), where \( C \) is the strict transform of \( C_1 \) and \( E \) is the exceptional divisor of the blow up.

Since \( \pi_2|_D : D \longrightarrow L \) is 2:1, and \( \pi_2|_E : E \longrightarrow L \) is an isomorphism, \( \pi_2|_C : C \longrightarrow L \) is an isomorphism. The degree of \( C \) on the second factor is 1. The curve \( C \) has bidegree \((2,1)\).
3. Since \( l(L) \cap C_1 = \emptyset \), \( D \) is isomorphic to the strict transform of \( C_1 \). In this case
\[ D = C. \]
\( \pi_2| C : C \longrightarrow L \) is 2:1, which implies that the degree on the second factor is 2. The curve \( C \) has bidegree \((2,2)\).

In short, we have established the following.

**Proposition 3.6.**

Let \( C_1 \) be a conic in \( \mathbb{P}^3 \) and \( L \) be a line in \( \mathbb{P}^3 \), such that \( \lg(l(L) \cap C_1) = j \). This gives rise to a rational curve in \( \mathbb{P}^3 \times \mathbb{P}^3 \) of bidegree \((2,2-j)\).

\[ \bullet \]

**Remark.** Note that all three cases occur, since there are examples where \( l : \text{Grass}(\mathbb{P}^3) \longrightarrow \text{Grass}(\mathbb{P}^3) \) is a bijective map of sets (Proposition 2.6).

Now with the explicit criterions for curves of low degree at hand, we want to study Yau’s CICY through these.

**Theorem 3.7.**

There are 81 curves of bidegree \((2,0)\) on Yau’s CICY

**Proof.**

A rational curve \( C \) of bidegree \((2,0)\) is of the form \( C_1 \times p \), where \( C_1 \) is a curve of degree 2 and \( p \) is a point. Let \( p \)'s coordinates be \( b \). Let \( H \) be the plane \( Z(\sum b_i x_i) \). By the definition of \( C \), \( C_1 \subseteq H \). Moreover, we have determined all the families of planes that give rise to curves of degree 2 on \( F_1 \). Take one such family, we can without loss of generality assume that it is of the form given in Appendix:

\[ H(A) = Ax_0 + \alpha^2 x_1 + x_2 + Ax_3 = 0 \]

So in order to have a \((2,0)\) curve, we must demand \( H = H(A) \), giving \( b = (A, \alpha^2, 1, A) \). \( p \in F_2 \) gives \( \sum b_i^3 = 0 \) or \( A^3 + 1 = 0 \). There are three solutions \( A = -1, -\alpha, -\alpha^2 \). All the curves \( C_1(A) \) corresponding to these values of \( A \), are irreducible. So for each of the 27 families we get 3 curves of bidegree \((2,0)\), giving the total number 81.

**Theorem 3.8.**

There are 918 curves of bidegree \((2,1)\) on Yau’s CICY.

**Proof.**

Consider one of the 27 families of curves, we can without loss of generality assume that it is of the form \( C_1(A) \subseteq H(A) \) given in the Appendix.

There are no curves of bidegree \((2,1)\) and of type \((1,1)\). Assume for contradiction that there is one, and denote it by \( C \). Let \( L_i = \pi_i(C) \). Choose a plane \( H \) in \( \mathbb{P}^3 \) such that \( L_1 \subseteq H \). Since \( H \times L_2 \cap G \) is the blow up of \( H \) in \( l(L_2) \) and \( L_1 \neq l(L_2) \), \( D = L_1 \times L_2 \cap G \) is of dimension one. Assume that \( L_1 \cap l(L_2) = \emptyset \), then \( L_1 \simeq D \). This implies that \( C \)'s degree on the first factor is at worst 1.

Assume that \( L_1 \cap L_2 \neq \emptyset \), then \( D \) consists of a curve \( \tilde{C} \) the strict transform of \( L_1 \), and an exceptional divisor \( E \). \( C \) has to be contained in \( \tilde{C} \), since it is dominant on the first factor. This implies that \( \tilde{C} = \tilde{C} \). This implies that \( C \)'s degree on the first factor is 1.
Curves of degree \((2, 1)\) have to be of type \((\bar{2}, \bar{1})\). So our candidates have to be contained in \(H(A) \times L\), where \(L\) is one of the 27 lines on \(F_2\). As noted earlier there are three groups of lines. We study each of these groups separately.

**GROUP 1.**

Here \(L\) is defined by \(y_0 + \alpha^i y_1 = y_2 + \alpha^j y_3 = 0\), giving \(l(L) = Z(\alpha^i x_0 - x_1, \alpha^j x_2 - x_3)\).

A point on \(l(L) \cap F_1\) is given by \((1, \alpha^i, -\alpha^l, -\alpha^{j+l})\) \(l \in \{0, 1, 2\}\). Demanding the point \(p\) to lie in \(H(A)\), gives the following condition for \(A\):

\[
A(1 - \alpha^{j+l}) = \alpha^{l+2} - \alpha^i
\]

The condition on \(A\) for a given choice of \((l, i, j)\), is given in Table 1 in the Appendix.

Eight of the choices of \((l, i, j)\) correspond to planes containing three lines. These do not give rise to bidegree \((2, 1)\) curves. There are three choices that do not give any condition on \(A\). In these cases \(p\) is on the line \(G_1\) (notation of the Appendix) contained in all the planes \(H(A)\). We have to check if there exist values of \(A\), such that \(p\) is contained in \(C_1(A)\) also. These values are tabulated in the third column, in table 1. We see that these planes do not contain conics. \(A = -1, -\alpha, -\alpha^2\) correspond to planes that give bidegree \((2, 0)\) curves (see the proof of Proposition 3.7).

The number of \((2, 1)\) curves arising from group 1 is \(27 - 14 = 13\)

**GROUP 2.**

This is essentially the same calculation as in group 1. Here \(L\) is defined by \(y_0 + \alpha^i y_2 = y_1 + \alpha^j y_3 = 0\), giving \(l(L) = Z(\alpha^i x_0 - x_2, \alpha^j x_1 - x_3)\).

The point \(p\) is here given by \((1, -\alpha^l, \alpha^i, -\alpha^{l+j})\), and the condition on \(A\) is:

\[
A(1 - \alpha^{l+j}) = \alpha^{l+2} - \alpha^i
\]

The different cases are tabulated in the Appendix. The number of \((2, 1)\) curves arising from group 2 is \(27 - 14 = 13\)

**GROUP 3.**

Here \(L\) is defined by \(y_0 + \alpha^i y_3 = y_1 + \alpha^j y_2 = 0\), giving \(l(L) = Z(\alpha^i x_0 - x_3, \alpha^j x_1 - x_2)\).

The point \(p\) is here given by \((1, -\alpha^l, \alpha^{l+j}, -\alpha^i)\), and the condition on \(A\) is:

\[
A(1 + \alpha^j) = \alpha^{2+l} + \alpha^{l+j}
\]

The different cases are tabulated in the Appendix. The number of \((2, 1)\) arising from group 3 is \(27 - 19 = 8\).

In other words there are \(13 + 13 + 8 = 34\) bidegree \((2, 1)\) curves for each 1-dimensional family of conics in \(F_1\). The total number of bidegree \((2, 1)\) curves is \(27 \cdot 34\).
Proposition 3.9.

Let \( L \) be a line in \( \mathbb{P}_2^3 \), and let \( C_1 \) be a rational curve of degree \( m \) in \( \mathbb{P}_1^3 \). Furthermore let \( G = Z(\sum \alpha_{ij} x_i y_j) \subseteq \mathbb{P}_1^3 \times \mathbb{P}_2^3 \) be nice, and denote \( G|_{\mathbb{P}_1^3 \times L} \) by \( \bar{G} \). Let \( C \) be the unique component of \( V = C_1 \times L \cap \bar{G} \) such that \( \pi_1(C) = C_1 \), where \( \pi_1 \) is the projection map on the first factor. Let \( i = \lg(C_1 \cap l(L)) \).

Then \( C \) is a rational curve of bidegree \( (m, m - i) \).

Proof.

\( \bar{G} \) is isomorphic to the blow up of \( \mathbb{P}_1^3 \) with center \( l(L) \), so \( C \) is by definition the strict transform of \( C_1 \). Moreover \( V = C \cup E_1 \cup \cdots \cup E_i \), where \( E_i \) are the exceptional fibers corresponding to the intersection points \( p_1 \ldots p_i \) in \( C_1 \cap l(L) \). \( C \) is rational ( [7] V.3.7). The bidegree is determined the same way as in Proposition 3.6. (The degree on the second factor drops by one for each intersection point counted with multiplicity.)

\( \blacklozenge \)

Proposition 3.10.

There are no curves of bidegree \( (3, 0) \) on \( G \).

Proof.

Any nonsingular rational curve \( C_1 \) of degree 3 in \( \mathbb{P}^3 \) is a twisted cubic. In order to give rise to a \( (3, 0) \) curve on \( X \), \( \lg(l(L) \cap C_1) = 3 \) by the preceding proposition. This is impossible. All curves of bidegree \( (3, 0) \) would arise in this way.

\( \blacklozenge \)

Corollary 3.11.

There are no curves of bidegree \( (3, 0) \) on Yau's CICY.

Proof.

\( \blacklozenge \)

Corollary 3.11.

There are 729 curves of degree 2 on Yau's CICY.

Proof.

Since

\[
\text{Hilb}_X^{2m+1} \cong \text{Hilb}_X^{(2,0)m+1} \cup \text{Hilb}_X^{(1,1)m+1} \cup \text{Hilb}_X^{(0,2)m+1}
\]

we get immediately \( 81 + 567 + 81 = 729 \).

\( \blacklozenge \)

Remark 3.12.

The number of curves predicted by Duco van Straten is 891. Some of these 891 curves may be degenerations of rational curves. If we count \( L_1 \times L_2 \cap G \) where \( L_1 \) (resp. \( L_2 \)) is a line on \( F_1 \) (resp. \( F_2 \)) as a bidegree \( (1, 1) \) curve, we get 729 bidegree \( (1, 1) \) curves.

This gives \( 81 + 729 + 81 = 891 \) curves of degree 2.
Corollary 3.13.

There are 1836 curves of degree 3 on Yau’s CICY.

Proof.

Since

$$\text{Hilb}_X^{3m+1} \cong \text{Hilb}_X^{(3,0)m+1} \cup \text{Hilb}_X^{(2,1)m+1} \cup \text{Hilb}_X^{(1,2)m+1} \cup \text{Hilb}_X^{(0,3)m+1},$$

we get immediately $0 + 918 + 918 + 0 = 1836$

\[\heartsuit\]

Theorem 3.14. There are 4374 rational curves of degree 3 on a generic Yau’s CICY.

Proof.

Let $X = F_1 \times F_2 \cap G$ where $F_i$ are cubics and $G$ is the incidence variety. $G$ is nice for a generic choice of defining equations by Proposition 2.7.

Denote the lines on $F_1$ by $L_i^k$ where $k \in \{1, \ldots, 27\}$.

In the generic situation $l(L_i^k) \cap L_i^{k'} = \emptyset$ for all $k, k' \in \{1, \ldots, 27\}$. Since there are only finitely many planes $H$ in $\mathbb{P}_1$ such that $H \cap F_1$ is the union of three lines, the $l(L_i^k)$ are in the general situation not contained in any of these planes. This gives that for each 1-dimensional family of conics and each $l(L_i^k)$ we get three planes such that $l(L_i^k)$ intersect a conic. By Proposition 3.6 each of these cases gives rise to one bidegree $(2,1)$ curve and every bidegree $(2,1)$ curve has to arise this way. This gives the total number of bidegree $(2,1)$ curves: $3 \cdot 27 \cdot 27 = 2187$. By symmetry the number of $(1,2)$ is equal to the number of $(2,1)$ curves. Since $G$ is nice there are no $(3,0)$ or $(0,3)$ curves by Proposition 3.10.

The number of degree 3 curves is $0 + 2187 + 2187 + 0 = 4374$.

\[\heartsuit\]

Remark 3.15.

The number of degree 3 curves conjectured by Duco van Straten is 4410. The discrepancy may be due to nodal cubics.
Appendix: The Fermat Cubic in Projective Threespace.

**Lines on the Fermat Cubic**

The lines on the Fermat cubic are easily determined, they are naturally divided into three groups:

\[
F_1^{ij} : \quad x_0 + \alpha^i x_1 = x_2 + \alpha^j x_3 = 0 \\
F_2^{ij} : \quad x_0 + \alpha^i x_2 = x_1 + \alpha^j x_3 = 0 \\
F_3^{ij} : \quad x_0 + \alpha^i x_3 = x_1 + \alpha^j x_2 = 0
\]

with \(\alpha^3 = 1\), \(\alpha \neq 1\) and \(i, j\) integers.

It is trivial to check that the planes defined by:

\[
\alpha^{i_0} x_0 + \alpha^{i_1} x_1 + \alpha^{i_2} x_2 + \alpha^{i_3} x_3 = 0
\]

with \(i_j \in \{0, 1, 2\}, \quad j = 0, 1, 2, 3\), contain three lines, one from each group of lines. These lines are explicitly given by,

\[
\alpha^{i_0} x_0 + \alpha^{i_1} x_1 = \alpha^{i_2} x_2 + \alpha^{i_3} x_3 = 0 \\
\alpha^{i_0} x_0 + \alpha^{i_2} x_2 = \alpha^{i_1} x_1 + \alpha^{i_3} x_3 = 0 \\
\alpha^{i_0} x_0 + \alpha^{i_3} x_3 = \alpha^{i_1} x_1 + \alpha^{i_2} x_2 = 0
\]

**Conics on the Fermat Cubic**

By a conic we shall mean a nonsingular irreducible curve of degree 2. We want to determine the conics on the Fermat cubic. Any conic is contained in a plane. On the other hand a plane cuts out a one dimensional scheme of degree 3 by Bezout’s theorem. A plane containing only one of the the 27 lines on the cubic, necessarily contains a conic. Obviously, the curves obtained in this way, are the only conics. Here we want to determine these curves explicitly.

Choose the pair of skew lines:

\[
G_1 : \quad x_0 + x_3 = x_1 + \alpha x_2 = 0 \quad (F_3^{01}) \\
E_1 : \quad x_0 + x_1 = x_2 + x_3 = 0 \quad (F_1^{00})
\]

Choose a point on \(E_1\), \((\gamma_0, -\gamma_0, \gamma_1, -\gamma_1)\). This point and \(G_1\) defines a plane:

\[
H(\gamma_0, \gamma_1) = Z(\alpha^2 \gamma_0 - \gamma_1)x_0 + \alpha^2(\gamma_0 - \gamma_1)x_1 + (\gamma_0 - \gamma_1)x_2 + (\alpha^2 \gamma_0 - \gamma_1)x_3 = 0
\]

We write:

\[
(1) \quad Ax_0 + \alpha^2 x_1 + x_2 + Ax_3 = 0
\]
where

\[ A = \frac{\alpha^2 \gamma_0 - \gamma_1}{\gamma_0 - \gamma_1} \]

(Note: Here \( \gamma_0 = \gamma_1 \) is allowed and corresponds to the plane \( x_0 + x_3 = 0 \).)

On \( H \) we use the induced coordinates \( x_0, x_1, x_3 \). The line \( G_1 \) is defined by \( x_0 + x_3 = 0 \) in \( \mathbb{P}^2(x_0, x_1, x_3) \cong H \). It is straightforward to compute the conic defining \( C_1 \) in these coordinates.

\[ (2) \quad C_1 = Z((1 - A^3)x_0^2 - 3\alpha A x_1^2 + (1 - A^3)x_3^2 - 3\alpha^2 A^2 x_0 x_1 - (1 + 2A^3)x_0 x_3 - 3\alpha^2 A^2 x_1 x_3) \]

This curve is generically irreducible, but it degenerates for \( A = 1, \alpha, \alpha^2 \). It is easy to check that in these cases the plane contains three lines, one from each group of lines.

We have constructed a 1-dimensional quasiprojective family of conics with a natural parametrisation through \((\gamma_0, \gamma_1)\) on the line \( E_1 \). In retrospect we see that any line in \( \mathbb{P}^3 \), skew to \( G_1 \), would do.

Note that the automorphisms of type \( x_i = \alpha^k x_j \) on \( \mathbb{P}^3 \), maps the Fermat cubic to itself. This gives us a mapping of the set of 27 lines to itself. It is clear that given any ordered pair of lines, there exists an automorphism \( \sigma \) of the described type, taking the first one to the second one. Take a line \( F_{ij}^k \), and let \( \sigma \) be an automorphism that maps \( G_1 \) to \( F_{ij}^k \). The family of conics \( \hat{C}_1(A, x) \in H(A, x) \) \((x = (x_0, \cdots, x_3))\) derived from taking planes \( H \) through \( F_{ij}^k \), is given by:

\[ \hat{C}_1(A, \sigma x) = H(A, \sigma x) \]

For later reference we state the following:

On the Fermat cubic there are 27 1-dimensional families of conics. Each family are of the following form, up to change of coordinates of the type \( x_i = \alpha^k x_j \) on \( \mathbb{P}^3 \).

\[ C_1(A) = H(A) \cap \]

\[ Z((1 - A^3)x_0^2 - 3\alpha A x_1^2 + (1 - A^3)x_3^2 - 3\alpha^2 A^2 x_0 x_1 - (1 + 2A^3)x_0 x_3 - 3\alpha^2 A^2 x_1 x_3) \]

where

\[ H(A) = Z(Ax_0 + \alpha^2 x_1 + x_2 + Ax_3) \]
Tables for the proof of Proposition 3.8.

In the first column are the different choices of \( \{i, j, k\} \). The second column gives the values of \( A \) such that the point \( p \) defined by \( \{i, j, k\} \) is contained in \( H(A) \). In the cases where \( p \) is contained in all the planes, then the values of \( A \) such that \( p \) is contained in more than one component of \( H(A) \cap F_1 \) are tabulated in the column designated \( OC \) (Other Component). Finally, if \( H(A) \) is a union of three lines then "red" is printed in the last column. A \((2,0)\) printed in the last column indicates that this case gives a curve of bidegree \((2,0)\).

Table 1

<table>
<thead>
<tr>
<th>((l, i, j))</th>
<th>(A)</th>
<th>(OC)</th>
<th>Red</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0,0,0))</td>
<td>(\infty)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>((0,0,1))</td>
<td>(-\alpha^2)</td>
<td></td>
<td>((2,0))</td>
</tr>
<tr>
<td>((0,0,2))</td>
<td>(1)</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>((0,1,0))</td>
<td>all</td>
<td>1</td>
<td>red</td>
</tr>
<tr>
<td>((0,1,1))</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>((0,1,2))</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>((0,2,0))</td>
<td>(\infty)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>((0,2,1))</td>
<td>1</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>((0,2,2))</td>
<td>(-\alpha)</td>
<td></td>
<td>((2,0))</td>
</tr>
<tr>
<td>((1,0,0))</td>
<td>(\alpha)</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>((1,0,1))</td>
<td>(-\alpha^2)</td>
<td></td>
<td>((2,0))</td>
</tr>
<tr>
<td>((1,0,2))</td>
<td>(\infty)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>((1,1,0))</td>
<td>(-1)</td>
<td></td>
<td>((2,0))</td>
</tr>
<tr>
<td>((1,1,1))</td>
<td>(\alpha)</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>((1,1,2))</td>
<td>(\infty)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>((1,2,0))</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>((1,2,1))</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>((1,2,2))</td>
<td>all</td>
<td>(\alpha)</td>
<td>red</td>
</tr>
<tr>
<td>((2,0,0))</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>((2,0,1))</td>
<td>all</td>
<td>(\alpha^2)</td>
<td>red</td>
</tr>
<tr>
<td>((2,0,2))</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>((2,1,0))</td>
<td>(-1)</td>
<td></td>
<td>((2,0))</td>
</tr>
<tr>
<td>((2,1,1))</td>
<td>(\infty)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>((2,1,2))</td>
<td>(\alpha^2)</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>((2,2,0))</td>
<td>(\alpha^2)</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>((2,2,1))</td>
<td>(\infty)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>((2,2,2))</td>
<td>(-\alpha)</td>
<td></td>
<td>((2,0))</td>
</tr>
</tbody>
</table>
Table 2

<table>
<thead>
<tr>
<th>$(l, i, j)$</th>
<th>$A$</th>
<th>OC</th>
<th>Red</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0, 0)</td>
<td>$\infty$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, 0, 1)</td>
<td>$\alpha^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, 0, 2)</td>
<td>$-1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, 1, 0)</td>
<td>$\infty$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, 1, 1)</td>
<td>$-\alpha$</td>
<td></td>
<td>(2, 0)</td>
</tr>
<tr>
<td>(0, 1, 2)</td>
<td>$\alpha^2$</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(0, 2, 0)</td>
<td>all</td>
<td>$\alpha^2$</td>
<td>red</td>
</tr>
<tr>
<td>(0, 2, 1)</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, 2, 2)</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1, 0, 0)</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1, 0, 1)</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1, 0, 2)</td>
<td>all</td>
<td>1</td>
<td>red</td>
</tr>
<tr>
<td>(1, 1, 0)</td>
<td>1</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(1, 1, 1)</td>
<td>$-\alpha$</td>
<td></td>
<td>(2, 0)</td>
</tr>
<tr>
<td>(1, 1, 2)</td>
<td>$\infty$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1, 2, 0)</td>
<td>$-\alpha^2$</td>
<td></td>
<td>(2, 0)</td>
</tr>
<tr>
<td>(1, 2, 1)</td>
<td>1</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(1, 2, 2)</td>
<td>$\infty$</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(2, 0, 0)</td>
<td>$\alpha$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2, 0, 1)</td>
<td>$\infty$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2, 0, 2)</td>
<td>$-1$</td>
<td></td>
<td>(2, 0)</td>
</tr>
<tr>
<td>(2, 1, 0)</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2, 1, 1)</td>
<td>all</td>
<td>$\alpha$</td>
<td>red</td>
</tr>
<tr>
<td>(2, 1, 2)</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2, 2, 0)</td>
<td>$-\alpha^2$</td>
<td></td>
<td>(2, 0)</td>
</tr>
<tr>
<td>(2, 2, 1)</td>
<td>$\infty$</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(2, 2, 2)</td>
<td>$\alpha$</td>
<td></td>
<td>red</td>
</tr>
</tbody>
</table>
Table 3

<table>
<thead>
<tr>
<th>((i,j,l))</th>
<th>(A)</th>
<th>OC</th>
<th>Red</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0, 0)</td>
<td>(-\frac{a}{2})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, 0, 1)</td>
<td>(-\frac{a}{2})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, 0, 2)</td>
<td>(-\frac{1}{2})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, 1, 0)</td>
<td>(-\frac{a}{2})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, 1, 1)</td>
<td>(-\frac{a}{2})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, 1, 2)</td>
<td>(-\frac{a}{2})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0, 2, 0)</td>
<td>(a^2)</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(0, 2, 1)</td>
<td>1</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(0, 2, 2)</td>
<td>(a)</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(1, 0, 0)</td>
<td>(a^2)</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(1, 0, 1)</td>
<td>1</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(1, 0, 2)</td>
<td>(a)</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(1, 1, 0)</td>
<td>(a)</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(1, 1, 1)</td>
<td>(a^2)</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(1, 1, 2)</td>
<td>1</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(1, 2, 0)</td>
<td>(-2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1, 2, 1)</td>
<td>(-2a)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1, 2, 2)</td>
<td>(-2a^2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2, 0, 0)</td>
<td>1</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(2, 0, 1)</td>
<td>(a)</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(2, 0, 2)</td>
<td>(a^2)</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(2, 1, 0)</td>
<td>(a^2)</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(2, 1, 1)</td>
<td>1</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(2, 1, 2)</td>
<td>(a)</td>
<td></td>
<td>red</td>
</tr>
<tr>
<td>(2, 2, 0)</td>
<td>(-2a)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2, 2, 1)</td>
<td>(-2a^2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(2, 2, 2)</td>
<td>(-2)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
REFERENCES


5. D.E. Sommervoll, Rational curves of higher degree on a complete intersection Calabi-Yau threefold in $\mathbb{P}^3 \times \mathbb{P}^2$, Preprint (to appear).