MAXIMUM PRINCIPLES FOR
A CLASS OF CONSERVATION LAW

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ABSTRACT. We prove maximum principles for a class of conservation laws, \( u_t + f(u)_x = 0 \), and the corresponding regularized parabolic system, \( u_t + f(u)_x = \epsilon u_{xx} \). The class of conservation laws is determined by requiring the flux function \( f \) to be constant along certain coordinate directions in state space. The class includes models of multi-phase flow in porous media, polymer flooding, chemical chromatography as well as gas dynamics. The maximum principle is first derived for the Cauchy problem for the parabolic equation and then for the Riemann problem of the hyperbolic equation. Finally, we conclude that the maximum principle also holds for approximate solutions to the hyperbolic equation generated by the Lax-Friedrichs, the Godunov and the Glimm schemes. Hence the the maximum principle also holds for weak solutions of the Cauchy problem for the hyperbolic equation, when these are limits of approximate solutions generated by such schemes.

0. Introduction. Maximum principles are fundamental in the theory of partial differential equations, and they constitute important elements in existence and uniqueness proofs. Furthermore, in physical applications they often reflect basic physical properties.

Well-known is the example of the heat equation for which one can easily prove that the solution never exceeds the maximum value of the initial data. For the Dirichlet problem the solution is bounded by the maximum on the boundary. The notion of invariant regions is closely connected to that of maximum principles. If \( u|_{t=0} \leq a \) implies \( u(x, t) \leq a \), then the region \( \{ u \leq a \} \) is invariant. More generally, for evolution equations we say that a region \( R \) is invariant if \( u|_{t=0} \in R \) implies \( u(\cdot, t) \in R \). For examples of maximum principles and invariant regions for reaction-diffusion equations, we refer to Smoller [13] (and references therein).

We will in this paper address the question of maximum principles for conservation laws. Our principal motivation comes from the theory of three-phase flow in porous media.

Let \( u = u(x, t), v = v(x, t), w = w(x, t) \) denote saturations (i.e., relative volume fractions) of each of the three phases (e.g., oil, gas and water) in a porous medium. By definition

\[
(0.1) \quad u + v + w = 1
\]

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for all $x \in \mathbb{R}^3, t > 0$. Assuming conservation of masses for each of the phases we find that [10]

\begin{equation}
U_t + \nabla \cdot F(U) = 0
\end{equation}

where $U = (u, v)$ and $w = 1 - u - v$. It is of importance to know whether the solution $U = U(x, t)$ will remain inside the phase triangle $R = \{(u, v) \in \mathbb{R}^2 | 0 \leq u, v, u + v \leq 1\}$ when the initial data satisfies $U(x, 0) \in R$, i.e., whether $R$ is an invariant region for (0.2). If one can show that $u(x, 0) \geq 0$ implies $u(x, t) \geq 0$, and similarly for $v$ and $w$, the invariance of $R$ will follow. Furthermore, it is of importance to show that also approximate solutions of (0.2) are inside $R$ if the approximate initial data are. In fact, a common problem with simulation of three-phase flow, is that one frequently encounters non-physical values (e.g., negative saturations) as a result of the simulation. A theoretical analysis of invariance properties of numerical schemes is therefore called for.

In this paper we will address these questions in a setting that includes the above example of three-phase flow in one-dimension.

Consider a system of conservation laws in one space dimension, viz.

\begin{equation}
u_t + f(u)_x = 0\end{equation}

where $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$, $f = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function. Here we will consider flux functions $f$ which have the property that there is an index $i$ and a constant $u^*_i$ such that

\begin{equation}
\frac{\partial f_i}{\partial u_j}(u_1, \ldots, u_{i-1}, u^*_i, u_{i+1}, \ldots, u_n) = 0, \quad j = 1, \ldots, n, \quad j \neq i
\end{equation}

In particular we will not assume the system to be strictly hyperbolic; hyperbolic will suffice. For systems satisfying (0.4) we study the Cauchy problem with Riemann initial data, i.e.

\begin{equation}
u(x, 0) = u_0(x) = \begin{cases} u^\ell & \text{for } x < 0 \\ u^r & \text{for } x \geq 0 \end{cases}
\end{equation}

where $u^\ell, u^r$ are constant vectors. Riemann initial data are vital in existence proofs for strictly hyperbolic systems, see [4], [12], as well as for systems of the above type, see [14], [3].

Furthermore, we study the corresponding parabolic system

\begin{equation}
u_t + f(u)_x = \epsilon u_{xx}.
\end{equation}

and show that for flux functions satisfying condition (0.4), solutions to this parabolic system satisfy certain invariance properties also. In addition, we consider invariance of numerical schemes approximating the solutions to conservation laws (0.3), and deduce some invariance properties of the solutions generated by these from the corresponding invariance for the solutions of Riemann problems.
Invariant regions for conservation laws have been studied in depth in the paper by Hoff [5]. He obtains necessary and sufficient conditions for a region to be invariant for a system of strictly hyperbolic conservation laws. Furthermore he establishes invariance for certain finite difference schemes, e.g. Lax–Friedrichs’ scheme and Godunov’s scheme. His analysis is based on Lax’ theorem for the solution of the Riemann problem with small initial data for systems of conservation laws that are strictly hyperbolic and genuinely nonlinear. The system that models three-phase flow will not be of that type [7], and assuming property (0.4) of the system we will be able to give rather straight-forward proofs of our assertions.

1. A class of conservation laws. In this section we will give some examples of systems of conservation laws where the flux function satisfies (0.4). For ease of reference, we will denote this class of systems by $\mathcal{M}$. It is important to note that this condition does not imply hyperbolicity, in fact some of the systems we study are known to have elliptic regions, cf. [7] and the reference given therein.

In the examples presented below, we show that the systems are of type $\mathcal{M}$, and we will return to these examples in section 5, where we state the invariance properties of the systems based on the results derived in section 3.

The systems of type $\mathcal{M}$ includes the following important examples:

Example 1. Multi-phase flow in porous media. Consider a one-dimensional porous medium, and let $u_i = u_i(x,t)$ denote the saturation, i.e., the relative volume fraction, of phase $i$. Assuming mass conservation of each phase we have, neglecting capillary pressure, that

\begin{equation}
   u_t + f(u)_x = 0
\end{equation}

with $u = (u_1, \ldots, u_n)$, and $f = (f_1, \ldots, f_n)$. The flux function $f$ is determined from Darcy’s law and incompressibility, and one can deduce that [10]

\begin{equation}
   f_i(u) = \frac{\mu_i k_i(u)}{\sum_{j=1}^n \mu_j k_j(u)}, \quad i = 1, \ldots, n
\end{equation}

where $\mu_i$ and $k_i(u)$ denote the mobility and relative permeability of phase $i$, respectively. If phase $i$ is absent, the flux function of that phase is zero, thus

\begin{equation}
   f_i(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n) = 0
\end{equation}

and hence the system is of type $\mathcal{M}$. The above model (1.6) applies in the absence of gravitational effects or when the geology is assumed to be homogeneous. If one includes gravitation and geological variation one obtains a model of the following type [3]

\begin{equation}
   f_i(u) = \frac{\mu_i k_i(u)}{\sum_{j=1}^n \mu_j k_j(u)} \left(1 - g(x) \sum_{j \neq i} k_j(u) \right).
\end{equation}

Here $g$ is a factor describing the geology and the gravitational effects which may vary from point to point in the reservoir. One can then extend the system by adding a “conservation law” for $g$, namely $g_t = 0$, see [3]. Also this extended system is of type $\mathcal{M}$.  

Example 2. Polymer flooding. Consider the displacement of oil by water, now containing dissolved polymer, in a one-dimensional porous medium. This process can be modelled by the following system of conservation laws,

\begin{align}
    s_t + f(s,c)_x &= 0, \\
    (sc + a(c))_t + (cf(s,c))_x &= 0.
\end{align}

 Cf. [8] for a discussion of this model. Here \( s \) denotes the saturation variable, \( c \) denotes the concentration of polymer in the aqueous phase, \( a = a(c) \) models adsorption of polymer on the rock, and \( f \) is usually referred to as the fractional flow function. The functions \( f \) and \( a \) satisfies the following requirements

\begin{align}
    f(0,c) &= 0, \quad f(1,c) = 1, \quad a(c) \geq 0, \quad a(0) = 0, \quad a'(c) \geq 0
\end{align}

for all \( c \geq 0 \). From (1.6) it follows directly that \( f_s(0,c) = f_c(1,c) = 0 \) for all \( c \geq 0 \), and hence, the polymer system (1.5) is of type \( M \). Moreover, introducing the conserved quantity \( b = sc + a(c) \) we find that \( \frac{\partial}{\partial s} (c(s,b)f(s,c(s,b))) \rvert_{s=0} = 0 \), again confirming that (1.5) is of type \( M \). We remark that there is a bijection from \((s,c)\) to \((s,b)\). It should also be noted that [8] presented a detailed analysis of the solution to the Riemann problem for (1.5).

A parabolic version of (1.5) was studied in [15] and it was shown that if the initial data was in the state space \( S = \{(s,c)|0 \leq s \leq 1, 0 \leq c\} \), then the solution of the Cauchy problem will remain in \( S \) for all finite time.

Example 3. Chemical chromatography. The following system of conservation laws arises in chemical chromatography,

\begin{align}
    \frac{\partial}{\partial t} (\eta_i + c_i) + \frac{\partial c_i}{\partial x} &= 0 \quad i = 1, \ldots, n,
\end{align}

cf. [1], [11]. The system models one phase flow at a constant volumetric flow rate containing \( n \) dissolved chemical components. The concentrations of the components are given by \( c_i \), whereas \( \eta_i \) represents the adsorption of chemicals.

The generalized Langmuir adsorption relations asserts that the functions \( \{c_i\} \) and \( \{\eta_i\} \) are related as follows,

\begin{align}
    \eta_i = \frac{K_i c_i}{1 + \sum_{j=1}^{n} K_j c_j},
\end{align}

where \( \{K_j\} \) denote positive constants satisfying \( 0 < K_1 < K_2 \cdots < K_n \). Following Hoff [5], we define the conserved quantities

\begin{align}
    u_i = \eta_i + c_i = c_i (1 + \frac{K_i}{1 + \sum_j K_j c_j}),
\end{align}

and observe that \( u \) is locally an invertible function of \( c \). Thus, we may rewrite the system (1.7) in a form more convenient for our analysis,

\begin{align}
    \frac{\partial u}{\partial t} + \frac{\partial c(u)}{\partial x} &= 0.
\end{align}
Next we show that this system is of type $\mathcal{M}$. Let $i$ be a fixed index, and let $k \neq i$. From the $i$–th equation of (1.9), we get

$$u_i(1 + \sum_j K_j c_j(u)) = c_i(u)(K_i + 1 + \sum_j K_j c_j(u)).$$

By differentiating this equation with respect to $u_k$, and setting $u_i = 0$, we obtain

$$\frac{\partial c_i}{\partial u_k} = 0.$$

Thus the system (1.12) is of type $\mathcal{M}$, with $u_i^* = 0$.

**Example 4. Gas dynamics.** The continuity equation of the Euler equations of gas dynamics reads

$$\rho_t + (\rho v)_x = 0$$

where $\rho$ and $v$ denote the density and velocity of the gas respectively. We see that also this flux function is of the above type.

**Example 5. “Factored systems”** More generally, consider a system of conservation laws of the type

$$u_t + f(u)_x = 0$$

with $f = (f_1, \ldots, f_n)$, where there exists an $i$ such that

$$f_i(u) = \phi(u_i)\Phi(u_1, \ldots, u_n)$$

where $\phi$ vanishes for $u_i = u_i^*$ and $\phi$ is bounded. Then condition $\mathcal{M}$ is ensured.

**2. The parabolic problem.** The main purpose of the present paper is to study maximum principles for the class of hyperbolic conservation laws described in the previous section. But it is well known that a weak solution of the system (1.1) is not unique, and one usually appeals to the associated parabolic system

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u^\epsilon_{xx}$$

$$u^\epsilon(x, 0) = u_0(x)$$

by defining the entropy solution of (0.3) to be the limit, as $\epsilon \to 0^+$, of the $\{u^\epsilon\}$–family generated by the solution operator of (2.1). It is therefore convenient to motivate the maximum principles for the system (0.3) by deriving similar properties for the corresponding parabolic system (2.1). For more general results concerning maximum principles for parabolic systems, we refer to Smoller [13] and Chueh, Conley and Smoller [2].

In order to derive a maximum principle for the system (2.1), we introduce the following auxiliary system

$$v_t + f(v)_x = \epsilon v_{xx} - \delta^2$$

$$v(x, 0) = u_0(x).$$
Here $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function satisfying condition $\mathcal{M}$. Furthermore, we assume that the initial condition satisfies

$$u_i(x, 0) \leq u_i^* \quad \forall x \in \mathbb{R},$$

and that, for $\delta^2$ sufficiently small, the system (2.2) has a unique and smooth solution for $t \leq T$. We will show that the condition $\mathcal{M}$ assures that (2.3) is satisfied for all $t \leq T$.

Assume that $v_i(x, t) < u_i^*$ for all $x \in \mathbb{R}$ and $t < \bar{t} \leq T$, and that $v_i(\bar{x}, \bar{t}) = u_i^*$. By putting $p = (\bar{x}, \bar{t})$, the $i$-th equation of (2.2) reads

$$\frac{\partial v_i(p)}{\partial t} = \epsilon \frac{\partial^2 v_i(p)}{\partial x^2} - \sum_{j=1}^{n} \frac{\partial f_i(v(p))}{\partial v_j} \frac{\partial v_j(p)}{\partial x} - \delta^2,$$

where $v(p) = (v_1(p), \ldots, u_i^*, \ldots, v_n(p))$. Using that $f$ is of class $\mathcal{M}$, we get

$$\frac{\partial v_i(p)}{\partial t} = \epsilon \frac{\partial^2 v_i(p)}{\partial x^2} - \frac{\partial f_i(v(p))}{\partial v_i} \frac{\partial v_i(p)}{\partial x} - \delta^2.$$

Since, by assumption, $v$ takes on its maximum value at $x = \bar{x}$, we obtain

$$\frac{\partial v_i(p)}{\partial t} \leq -\delta^2 < 0,$$

which shows that

$$v_i(x, t) \leq u_i^*$$

for all $t \leq T$.

Since the solution, $u$, of (2.1) is the limit in $L^\infty$ of the $\{v^\epsilon\}$-family generated by (2.2), as $\delta^2 \to 0$, we conclude that $u$ inherits the property derived for $v^\epsilon$. The convergence of the $\{v^\epsilon\}$-family is proved in [2], and it is interesting to note that the convergence is independent of the diffusion parameter $\epsilon$.

A similar argument applies when $u_i(x, 0) \geq u_i^*$, and we can summarize our maximum principle for the parabolic system as follows:

**Theorem 2.1.** Consider the parabolic system (2.1), and assume that it is of type $\mathcal{M}$. Furthermore, we assume that (2.1) has a unique smooth solution for $t \leq T$. Then the $i$-th component of the solution has the following properties:

$$u_i(x, 0) \leq u_i^* \Rightarrow u_i(x, t) \leq u_i^* \quad \forall (x, t) \in \mathbb{R} \times [0, T]$$

$$u_i(x, 0) \geq u_i^* \Rightarrow u_i(x, t) \geq u_i^* \quad \forall (x, t) \in \mathbb{R} \times [0, T].$$

We note that the argument given above is valid for any $\epsilon > 0$, hence, for those solutions of the hyperbolic problem (1.1) which are generated by the "vanishing viscosity method", a similar maximum principle is valid, and again we remark that more general results concerning invariant regions for parabolic systems can be found in [2] and [13].
3. The hyperbolic equation. We will in this section consider the hyperbolic conservation law

\[ u_t + f(u)_x = 0 \]

where \( f = (f_1, \ldots, f_n) \) satisfies condition \( M \) of section 1 with Riemann initial data

\[ u(x, 0) = u_0(x) = \begin{cases} u^L & \text{for } x < 0 \\ u^R & \text{for } x \geq 0 \end{cases} \]

We assume throughout this section that the Riemann problem (3.1), (3.2), has a unique solution. Our purpose is to derive some qualitative properties of such a solution. Recall that condition \( M \) says that there exists an \( i \) and a \( u_i^j \) such that

\[ \frac{\partial f_i}{\partial u_j}(u_1, \ldots, u_{i-1}, u_i^j, u_{i+1}, \ldots, u_n) = 0, \quad j = 1, \ldots, n, \quad j \neq i \]

Assuming hyperbolicity, i.e., real eigenvalues, the speed of propagation is finite. Based on this assumption, we consider \( u = u(x, t) \) in a finite rectangle \( R = [-X, X] \times [0, T] \), where \( X \) is chosen such that

\[ u(-X, t) = u^L, \quad u(X, t) = u^R \]

for all \( t \leq T \). Let the average of \( u \) at \( t = T \) be defined by

\[ \bar{u}(u^L, u^R) = \frac{1}{2X} \int_{-X}^{X} u(x, T) \, dx. \]

Conservation of each component \( u_j \) of \( u \) implies that

\[ \int_{-X}^{X} u(x, t) \, dx = \int_{-X}^{X} u(x, 0) \, dx + \int_{0}^{T} [f(u(-X, t)) - f(u(X, t))] \, dt \]

\[ = X u^L + X u^R + T (f(u^L) - f(u^R)), \]

and therefore

\[ \bar{u}(u^L, u^R) = \frac{1}{2} (u^L + u^R) - \frac{\mu}{2} (f(u^L) - f(u^R)), \]

where \( \mu = T/X \). If we differentiate the \( j \)-th component of (3.6) with respect to \( u_j^L \) we get

\[ \frac{\partial \bar{u}_j}{\partial u_j^L} = \frac{1}{2} + \frac{\mu}{2} \frac{\partial f_j}{\partial u_j^L} = \frac{1}{2} (1 - \mu \| \frac{\partial f_j}{\partial u_j^L} \|_\infty) > 0 \]

for \( \mu \) sufficiently small. If we differentiate \( \bar{u}_j \) with respect to \( u_j^R \) we get similarly

\[ \frac{\partial \bar{u}_j}{\partial u_j^R} = \frac{1}{2} - \frac{\mu}{2} \frac{\partial f_j}{\partial u_j^R} = \frac{1}{2} (1 - \mu \| \frac{\partial f_j}{\partial u_j^R} \|_\infty) > 0. \]

We now have that

\[ \frac{\partial \bar{u}_j}{\partial u_j^L} > 0 \quad \text{and} \quad \frac{\partial \bar{u}_j}{\partial u_j^R} > 0 \quad \text{for } j = 1, \ldots, n. \]

By using the monotonicity property of the average of the solution, we will prove that the value \( u_i^j \) acts as a barrier in the solution.
Lemma 3.1. Assume that \( f \) satisfies condition \( M \). If both \( u^l_i \leq u^*_i \) and \( u^r_i \leq u^*_i \) then also \( \bar{u}_i \leq u^*_i \). Similarly, if \( u^l_i \geq u^*_i \) and \( u^r_i \geq u^*_i \) then \( \bar{u}_i \geq u^*_i \).

Proof. We show the first of these statements, the proof of the second is identical. Let

\[
(3.10) \quad v^l = (u^l_1, \ldots, u^l_i, \ldots, u^l_n) \quad \text{and} \quad v^r = (u^r_1, \ldots, u^r_i, \ldots, u^r_n).
\]

Since \( u^l_i \leq u^*_i \) and \( u^r_i \leq u^*_i \) (3.9) implies that

\[
(3.11) \quad \bar{u}_i(u^l_i, u^r_i) \leq \bar{u}_i(v^l_i, v^r_i) = u^*_i
\]

by (3.6) and (3.9). □

We shall now show that we can divide the phase space into two regions \( A = \{u|u_i \leq u^*_i\} \) and \( B = \{u|u_i \geq u^*_i\} \) each of which are invariant for the Riemann problem in the sense that if the initial data is in \( A \) or \( B \), then so is the solution for each \( T \) and for almost all \( x \).

Theorem 3.1. Assume that \( f \) satisfies condition \( M \), and let \( u^l, u^r \in A(B) \), then also \( u(x,T) \) is in \( A(B) \) for almost all \( x \).

Proof. Let \( u^l, u^r \) be in \( A \) and assume that there exist \( x_1 \) and \( x_2 \) such that \( x_1 < x_2 \) and that \( u(x,T) \notin A \) for \( x \in [x_1, x_2] \). Define \( u^1 = u(x_1,T) \) and \( u^2 = u(x_2,T) \). Since \( u(x,T) \) is the solution of a Riemann problem the function

\[
(3.12) \quad v(x,T) = \begin{cases} 
    u^1 & \text{for } x < x_1 \\
    u(x,T) & \text{for } x_1 \leq x \leq x_2 \\
    u^2 & \text{for } x > x_2 
\end{cases}
\]

is the solution for \( t = T \) of the Riemann problem

\[
(3.13) \quad v(x,0) = \begin{cases} 
    u^1 & \text{for } x < 0 \\
    u^2 & \text{for } x \geq 0 
\end{cases}
\]

In particular,

\[
(3.14) \quad u_i(x,T) = \begin{cases} 
    u^1_i & \text{for } x < x_1 \\
    u_i & \text{for } x_1 \leq x \leq x_2 \\
    u^2_i & \text{for } x > x_2 
\end{cases}
\]

with \( u_i(x) > u^*_i \) for \( x \in [x_1, x_2] \). Therefore \( \bar{v}_i > u^*_i \) and consequently \( \bar{v} \notin A \). Since \( u^1 \) and \( u^2 \) both are in \( A \) this contradicts Lemma 3.1. □

4. Consequences for numerical methods. The invariance property has important implications for approximate solutions generated by various schemes as well as for the exact solution.
Let \( u^n_m \) denote the approximation of the solution \( u \) at \( x = m\Delta x, t = n\Delta t \), i.e. \( u^n_m \approx u(m\Delta x, n\Delta t) \) for \( \Delta x, \Delta t > 0 \). The Lax-Friedrichs scheme can be written

\[
 u^{n+1}_m = \frac{1}{2\Delta x} \int_{(m-1)\Delta x}^{(m+1)\Delta x} v(x, (n+1)\Delta t) \, dx
\]

where \( v(x, t), t \geq n\Delta t, \) is the solution of the Riemann problem with initial data at \( t = n\Delta t \) given by

\[
v(x, n\Delta t) = \begin{cases} u^n_{m-1} & \text{for } x < m\Delta x \\ u^n_m & \text{for } x \geq m\Delta x \end{cases}
\]

\( \Delta t \) and \( \Delta x \) are chosen so that the proper CFL condition is satisfied. For computations, (4.1) is usually written in the more practical form

\[
u^{n+1}_m = \frac{1}{2} \left( u^n_{m+1} + u^n_{m-1} \right) - \frac{\Delta t}{2\Delta x} \left( f(u^n_{m+1}) - f(u^n_{m-1}) \right).
\]

Thus we see that invariance for the Riemann problem implies that those approximate solutions of the Cauchy problem which are generated by the Lax-Friedrichs method have the same invariance properties as those of the solution of the Riemann problem.

The Godunov scheme is written

\[
 u^{n+1}_m = \frac{1}{\Delta x} \int_{(m-1/2)\Delta x}^{(m+1/2)\Delta x} v(x, (n+1)\Delta t) \, dx,
\]
in this case \( v \) is the solution of the multiple Riemann problem with initial data at \( t = n\Delta t \) given by

\[
v(x, n\Delta t) = \begin{cases} u^n_{m-1} & \text{for } x \leq (m - \frac{1}{2}) \Delta x \\ u^n_m & \text{for } (m - \frac{1}{2}) \Delta x < x \leq (m + \frac{1}{2}) \Delta x \\ u^n_{m+1} & \text{for } (m + \frac{1}{2}) \Delta x < x. \end{cases}
\]

Therefore also this scheme will generate approximations which are invariant on the same regions as the Riemann problem.

Also the Glimm scheme is written using the solution of the multiple Riemann problem (4.6):

\[
u^{n+1}_m = v \left( m\Delta x + \frac{\theta^n}{2}, (n+1)\Delta t \right),
\]

here \( \{\theta^n\} \) is an equidistributed sequence taking values in \([-1, 1]\). We see that solutions generated by the Glimm scheme also are invariant on the same regions as the solutions of the Riemann problem. In general, we see that any numerical method which takes its data from solutions of Riemann problems will have the same invariance properties as the solution of the Riemann problems. In particular, this applies to approximations generated by the front tracking method as defined in [12].
5. **Examples revisited.** In this final section, we reconsider the examples from section 1. Based on the results derived in section 3, we will state invariant regions for the solution of the systems. By *invariant* we will in the following mean that a region in phase space is invariant for:

I. the solution of Riemann problems associated the system (1.1),
II. numerical solutions of the system (1.1) generated by the Glimm scheme, the Lax-Friedrichs scheme or the Godunov scheme provided the proper CFL-condition is satisfied,
III. the solution of parabolic regularizations of the system (2.1), i.e. for the system 
\[ u_t + f(u)_x = \epsilon u_{xx}, \quad \epsilon > 0. \]

**Corollary 5.1.** Consider multi-phase flow in a porous medium as defined in Example 1. Then the region

\[
R = \{ (u_1, \ldots, u_n) | u_i \in [0, 1], \sum_{i=1}^{n} u_i \leq 1 \}
\]

is invariant.

**Corollary 5.2.** Consider polymer flooding in a porous medium as defined in Example 2. Then the region

\[
R = \{ (s, c) | s \in [0, 1], c \geq 0 \}
\]

is invariant.

**Corollary 5.3.** Consider chemical chromatography as defined in Example 3. Then the region

\[
R = \{ c_i \geq 0 \quad i = 1, \ldots, n \}
\]

is invariant.

We have that similar invariance properties hold for the systems defined in examples 4 and 5.

An important application of maximum principles can be found in the the derivation of existence and uniqueness results. One powerful result in this direction was proved by Nishida and Smoller [9], see also [6] and [5]. They considered parabolic systems of the form (2.1) with \( \epsilon > 0 \), and proved that if the approximate solutions of such systems generated by the Lax-Friedrichs scheme are uniformly bounded, then a unique solution of the Cauchy problem exists. By appealing to this result, we see that the existence and uniqueness of a solution of the system modelling multiphase flow in porous media (example 1 and Corollary 5.1) follows directly.

**References**


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