C*-dynamical systems for which the tensor product formula for entropy fails

by

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1. Introduction.

In the present paper we study C*-dynamical systems which are highly non-asymptotically abelian. More specifically we consider a unital C*-algebra $A$ with an automorphism $\alpha$ such that there is a self-adjoint subset $S$ of $A$ which together with the identity spans a dense subset of $A$, and with the property that the anticommutators $[\alpha^n(w), w^*]_+$ converge to 0 for some properly chosen sequence of $n$'s depending on $w$ for $w \in S$. It turns out that then there exists a unique $\alpha$-invariant state $\phi$, $\phi$ restricted to $S$ is zero, and the entropy of $\alpha$ with respect to $\phi$ in the sense of [ST] in zero. Hence if $A$ is nuclear the entropy in the sense of [CNT] vanishes, thus we have another example of a highly nonabelian C*-dynamical system with vanishing entropy.

Examples of systems as above can be found among the C*-algebras introduced by Powers [P], see also [Pr], in the study of binary shifts of the hyperfinite $II_1$-factor. The set $S$ will consist of finite products of self-adjoint unitary operators $\{s_i\}_{i \in \mathbb{Z}}$ with the property that $s_i s_j = \pm s_j s_i$, and $\alpha$ is the shift $\alpha(s_i) = s_{i+1}$. If $w_1, w_2 \in S$ then $w_1 w_2 = \pm w_2 w_1$, hence the C*-subalgebra $C$ of $A \otimes A$ generated by $w \otimes w, w \in S$, is abelian. As pointed out in [AN] $\alpha \otimes \alpha$ restricted to $C$ is the baker's transform so has entropy log 2. It follows that $h_{\phi \otimes \phi}(\alpha \otimes \alpha) \geq h_{\phi \otimes \phi}(C(\alpha \otimes \alpha)) = \log 2 > 0 = h_{\phi}(\alpha) + h_{\phi}(\alpha)$, hence the tensor product formula 
$h_{\phi \otimes \phi}(\alpha \otimes \beta) = h_{\phi}(\alpha) + h_{\phi}(\beta)$, see [SV], is false in general.

2. General results.

Let $A$ be a unital C*-algebra, $\alpha \in \text{Aut } A$, and $\phi$ an $\alpha$-invariant state. We shall show that if the C*-dynamical system $(A, \alpha)$ is highly nonasymptotically abelian then the entropy $H_{\phi}(\alpha)$ in the sense of [ST] in zero. For our purposes it is unnecessary to repeat the definition, only the following. Let $B$ be an abelian C*-algebra with an automorphism $\beta, \mu$ a $\beta$-invariant state on $B$, and $\lambda$ an $\alpha \otimes \beta$-invariant state on $A \otimes B$ such that $\lambda(\alpha \otimes 1) = \phi(a), \lambda(1 \otimes b) = \mu(b)$ for $a \in A, b \in B$. If $\lambda = \phi \otimes \mu$ then by [ST, 2.1] the “mutual information” $\varepsilon_{\lambda}(A, B) = 0$, hence the “conditional entropy” $H_{\lambda}(B|A) = H_{\mu}(B)$, so by [ST, 3.1] the entropy $h(B, \lambda) = 0$. If $\lambda = \phi \otimes \mu$ is the only $\alpha \otimes \beta$-invariant state as above then by [ST, Lemma 3.2] the entropy $H_{\phi}(\alpha) = 0$.

It was also shown in [ST, Prop. 4.1] that if $A$ is nuclear then $H_{\phi}(\alpha) = h_{\phi}(\alpha)$, where $h_{\phi}(\alpha)$ is the entropy of $\alpha$ with respect to $\phi$ in the sense of [CNT].
Recall that a subset $S \subset A$ is said to be self-adjoint if $a \in S$ implies $a^* \in S$, and total if its linear span is norm dense in $A$. We shall use the notation $[a, b]_+$ for the anticommutator $[a, b]_+ = ab + ba$, $a, b \in A$.

**Theorem 2.1.** Let $A$ be a unital C*-algebra and $\alpha \in \text{Aut}A$. Suppose $S$ is a self-adjoint subset of $A$ such that $S \cup \{1\}$ is total in $A$ and for which the following condition holds:

$$(*) \quad \forall w \in S, \forall \varepsilon > 0, \forall N \in \mathbb{N} \text{ there exist } n_1, \ldots, n_N \in \mathbb{N}$$

such that if $i \neq j$ then

$$\|[[\alpha^{n_i}(w^*), \alpha^{n_j}(w)]_+]_+ \| < \varepsilon, \quad i, j = 1, \ldots, N.$$ 

Then there exists a unique $\alpha$-invariant state $\phi$. $\phi$ satisfies $\phi|S = 0$, and the entropy $H_\phi(\alpha) = 0$.

The theorem generalizes [NT] and is an easy consequence of the following lemma, where we use the notation $\|x\|_{2,\phi} = \phi(x^*x)^{\frac{1}{2}}$ if $x \in A$ and $\phi$ is a state.

**Lemma 2.2.** Let $A$ be a unital C*-algebra, $\alpha \in \text{Aut}A$, and $\phi$ an $\alpha$-invariant state. Suppose $S$ is a self-adjoint subset of $A$ for which $S \cup \{1\}$ is total in $A$ and such that the following condition holds:

$$(***) \quad \forall w \in S, \forall \varepsilon > 0, \forall N \in \mathbb{N} \text{ there exist } n_1, \ldots, n_N \in \mathbb{N}$$

such that if $i \neq j$ then

$$\|[[\alpha^{n_i}(w^*), \alpha^{n_j}(w)]_+]_+ \|_{2,\phi} < \varepsilon, \quad i, j = 1, \ldots, N.$$ 

Suppose $B$ is an abelian C*-algebra, $\beta \in \text{Aut}B$, and $\mu$ a $\beta$-invariant state on $B$. Let $\lambda$ be an $\alpha \otimes \beta$ invariant state on $A \otimes B$ such that $\lambda(a \otimes 1) = \phi(a), \lambda(1 \otimes b) = \mu(b), a \in A, b \in B$. Then $\lambda|S \otimes B = 0$ and $\lambda = \phi \otimes \mu$.

**Proof.** Note that if $a \in A, b \in B$ then by the Cauchy-Schwarz inequality

$$|\lambda(a \otimes b)| = |\lambda((1 \otimes b)(a \otimes 1))| \leq \lambda((1 \otimes b)(1 \otimes b)^*)^{\frac{1}{2}} \lambda((a \otimes 1)^*(a \otimes 1))^{\frac{1}{2}}$$

$$= \mu(bb^*)^{\frac{1}{2}} \phi(a^*a)^{\frac{1}{2}} = \|a\|_{2,\phi} \|b\|_{2,\mu},$$

Furthermore, again since $B$ is abelian

$$[a^* \otimes b^*, a \otimes b]_+ = [a^*, a]_+ \otimes b^*b.$$ 

Let now $w \in S, b \in B$, where we for simplicity assume $\|b\| \leq 1$. Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ so large that

$$\frac{1}{N} \lambda([w^*, w]_+ \otimes b^*b) < \varepsilon.$$ 

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Using the inequality $|\rho(x)|^2 \leq \frac{1}{2}\rho([x^*, x]_+)$ for a state $\rho$, we have for $n_1, \cdots, n_N$ as given by (**)

$$
|\lambda(w \otimes b)|^2 = \left| \frac{1}{N} \lambda \left( \sum_{i=1}^{N} \alpha^{n_i}(w) \otimes \beta^{n_i}(b) \right) \right|^2 \\
\leq \frac{1}{2N^2} \lambda \left( \sum_{i} \alpha^{n_i}(w^*) \otimes \beta^{n_i}(b^*) \sum_{j} \alpha^{n_j}(w) \otimes \beta^{n_j}(b) \right) \\
= \frac{1}{2N^2} \sum_{i,j} \lambda \left( \alpha^{n_i}(w^*) \otimes \alpha^{n_j}(w) \otimes \beta^{n_i}(b^*) \beta^{n_j}(b) \right) \\
+ \frac{1}{2N^2} \sum_{i \neq j} \lambda \left( \alpha^{n_i}(w^*) \otimes \alpha^{n_j}(w) \otimes \beta^{n_i}(b^*) \beta^{n_j}(b) \right) \\
\leq \frac{1}{2N} \lambda([w^*, w]_+ \otimes b^* b) + \frac{1}{2N^2} N(N-1) \|\alpha^{n_i}(w^*), \alpha^{n_j}(w)\|_2^2 \phi \|eta^{n_i}(b^*) \beta^{n_j}(b)\|_2^2 \\
< \frac{\epsilon}{2} + \frac{1}{2}\epsilon \|b\|\phi(b^* b)^{\frac{1}{2}} \\
< \epsilon.
$$

Since $\epsilon$ is arbitrary, $\lambda(w \otimes b) = 0$, hence $\lambda|S \otimes B = 0$. In particular $\phi(w) = \lambda(w \otimes 1) = 0$. Thus if $c \in C$ then

$$
\lambda((c1 + w) \otimes b) = c\lambda(1 \otimes b) = c\mu(b) = \phi(c1 + w)\mu(b) \\
= \phi \otimes \mu((c1 + w) \otimes b).
$$

Since $(S \cup \{1\}) \otimes B$ is total in $A \otimes B$, $\lambda = \phi \otimes \mu$. QED.

**Proof of Theorem 2.1.** Since the group $\{\alpha^n : n \in \mathbb{Z}\}$ is amenable there exists an $\alpha$-invariant state $\phi$ on $A$. For all $x \in A \|x\|_2 \leq \|x\|$, so that condition (**) of Lemma 2.2 follows from (*). If we apply Lemma 2.2 to the case $B = C$ we conclude that $\phi|S = 0$, so $\phi$ is unique by the assumption that $S \cup \{1\}$ is total in $A$. The conclusion of Lemma 2.2 holds for all triples $(B, \beta, \mu)$ and all $\lambda$, hence from the discussion preceding the statement of the theorem, $H_\phi(\alpha) = 0$. QED.

**Corollary 2.3.** If in Theorem 2.1 $A$ is nuclear then $h_\phi(\alpha) = 0$.

**Proof.** As remarked before $h_\phi(\alpha) = H_\phi(\alpha)$ if $A$ is nuclear.

**Remark 2.4.** If we as in Lemma 2.2 assume the existence of the invariant state satisfying (**), then as in the proof of Theorem 2.1 we obtain $H_\phi(\alpha) = 0$. 

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Remark 2.5. If we in Theorem 2.1 assume \( S \) has the property that \( z, w \in S \) implies \( zw \in S \cup C_1 \), and \( zw \in C_1 \) implies \( zw = wz \), then \( \phi \) is a trace. Indeed, if \( a, b \in C \) then
\[
\phi((a1 + w)(b1 + z)) = ab + \phi(wz) = \begin{cases} ab & \text{if } wz \in S \\ ab + wz = ba + zw & \text{if } wz \in C_1 \end{cases}
\]
hence the assertion follows from the totality of \( S \cup \{1\} \).

Remark 2.6. If there exists a tracial state \( \tau \) on \( A \) (e.g. if \( A \) is a unital AF-algebra) then the unique invariant state is tracial. Indeed, we can take an invariant mean over \( \tau \circ \alpha^n \) to obtain the invariant state.

3. A number theoretic lemma.

In order to find sequences \((n_i)\) as in Theorem 2.1 we need a result on the existence of certain sequences in \( N \). We use the notation
\[
[k, m] = \{k, k + 1, \ldots, m\} \text{ when } k \leq m \text{ in } \mathbb{Z}
\]

Lemma 3.1. For each \( j \in N \) there exists a sequence \((x_s^j)_{s \in N}\) in \( N \) such that \( x_1^j = j, j + (j + 2) \) \( < x_s^j - x_{s-1}^j \), and if
\[
W_j = \{x_s^j - x_t^j + i : i \in [-j, j], 1 \leq t < s, s, t \in N\}
\]
then the sets \( W_j \) are pairwise disjoint.

We shall need an estimate for the growth of the cardinality of unions of sets of the form \( W_j \cap [1, m] \) as above.

Lemma 3.2. If \( W_1, \ldots, W_k \) are constructed as in Lemma 3.1 then
\[
\text{card}(\bigcup_{j=1}^k W_j \cap [1, m]) \leq ((k + 1) \log m)^2, \quad m \in N.
\]

Proof. Fix \( m \) and let \( 1 \leq j \leq k \). Let \( s \) be chosen as the minimal natural number such that \( x_{s+1}^j - x_s^j - j > m \) for all \( 1 \leq r \leq s \). Then
\[
\text{card}(W_j \cap [1, m]) \leq \sum_{r=2}^s \text{card}\{x_r^j - x_t^j + i : i \in [-j, j], 1 \leq t < r\}
\]
\[
= (2j + 1) \sum_{r=2}^s (r - 1) < (j + 1)s^2.
\]
By choice of $s$

$$x^j_s - x^j_{s-1} - j \leq m$$

by minimality of this element among the numbers $x^i_s - x^i_{t} + i$, $i \in [-j, j]$. By assumption then,

$$(j + 2)^s \leq x^j_s - x^j_{s-1} - j \leq m,$$

hence

$$s \leq \frac{\log m}{\log(j + 2)} < \log m,$$

using that $\log(j + 2) \geq \log 3 > 1$. Thus

$$\text{card}(W_j \cap [1, m]) < (j + 1)(\log m)^2.$$

It follows that

$$\text{card}\left(\bigcup_{j=1}^{k} W_j \cap [1, m]\right) \leq \sum_{j=1}^{k} \text{card}(W_j \cap [1, m])$$

$$< (\log m)^2 \sum_{j=1}^{k} (j + 1)$$

$$< ((k + 1) \log m)^2.$$

QED.

**Proof of Lemma 3.1.** We shall construct the sets $W_j$ by induction on $j$. If $j = 1$ put $x^1_1 = 1$ and choose $x^1_s \in \mathbb{N}$ such that $3^s + 1 < x^1_s - x^1_{s-1}$, and put

$$W_1 = \{x^1_s - x^1_t + i : i \in [-1, 1], \ 1 \leq t < s, s, t \in \mathbb{N}\}$$

Suppose the sequences $(x^j_s)_{s \in \mathbb{N}}$, $j = 1, \ldots, p-1$, are chosen such that $x^j_s - x^j_{s-1} > (j+2)^s + j$ and such that the sets $W_1, \ldots, W_{p-1}$ are pairwise disjoint.

Put $x^p_2 = p$. We first seek $x^p_2$ such that

(i) $[x^p_2 - 2p, x^p_2] \cap \bigcup_{j=1}^{p-1} W_j = \emptyset$

(ii) $x^p_2 > x^p_1 + (p + 2)^2 + p = p^2 + 6p + 4$.

By Lemma 3.2 $\text{card}\left(\bigcup_{j=1}^{p-1} W_j \cap [1, m]\right) \leq (p \log m)^2$ for all $m$, hence for $m$ sufficiently large

$$(2p + 1)\text{card}\left(\bigcup_{j=1}^{p-1} W_j \cap [1, m]\right) \leq m - (p^2 + 6p + 4)$$
Thus there exists $x_2^p \in \mathbb{N}$, $x_2^p \leq m$ satisfying (ii) such that

$$[x_2^p - 2p, x_2^p] \cap \bigcup_{j=1}^{p-1} W_j = \emptyset,$$

from which (i) follows.

Let $r \geq 2$ and suppose $x_1^p, \ldots, x_r^p$ are constructed such that $x_s^p - x_{s-1}^p > (p + 2)^s + p$ and such that the sets

$$\{x_s^p - x_t^p + i : i \in [-p, p], 1 \leq t < s\}, \quad 2 \leq s \leq r$$

are disjoint from $\bigcup_{j=1}^{p-1} W_j$.

Let $m_0 = x_r^p + (p + 2)^{r+1} + p + 1$. By Lemma 3.2

(1) $$\text{card}([m_0, m] \setminus \bigcup_{j=1}^{p-1} W_j) \geq m - m_0 - (p \log m)^2.$$  

Choose by Lemma 3.2 $m$ so large that

$$\sum_{j=1}^{p-1} (x_r^p + p + 1) \text{card}(W_j \cap [1, m]) \leq m - m_0 - (p \log m)^2.$$ 

Then by (1) we can find $x_{r+1}^p \in [m_0, m]$ such that

(2) $$\{x_{r+1}^p - n : n \in [0, x_r^p + p]\} \cap \bigcup_{j=1}^{p-1} W_j = \emptyset.$$ 

If $1 \leq t \leq r$ we find for $i \in [-p, p]$ 

$$x_{r+1}^p \geq x_{r+1}^p - x_t^p + i \geq x_{r+1}^p - x_r^p - p = x_{r+1}^p - (x_r^p + p).$$

Thus by (2)

$$\{x_{r+1}^p - x_t^p + i : i \in [-p, p], 1 \leq t \leq r\} \cap \bigcup_{j=1}^{p-1} W_j = \emptyset.$$ 

This completes the induction, since by choice of $m_0$, $x_{r+1}^p - x_r^p > (p + 2)^s + p$. QED.

In [P] Powers introduced a class of C*-algebras obtained from binary shifts of the hyperfinite II_1-factor, see also [Pr]. The definition is as follows. Let \( X \subset \mathbb{N} \) be a subset considered as a subset of \( \mathbb{Z} \), and let \( g \) be its characteristic function. Put

\[
\sigma(n) = (-1)^{\varphi(n)}.
\]

Changing Powers’ definition slightly we let \((s_i)_{i \in \mathbb{Z}}\) be a sequence of self-adjoint unitary operators satisfying the commutation relations

\[
s_is_j = \sigma(|i - j|)s_js_i.
\]

We denote by

\[
I = \{i_1 < \cdots < i_r\}
\]

the ordered subset \(\{i_k : k \in [1, r], i_1 < i_2 < \cdots < i_r\}\), and we denote by

\[
w_I = s_{i_1}s_{i_2}\cdots s_{i_r}, \quad w_\emptyset = 1.
\]

If \( J = \{j_1 < \cdots < j_s\} \) an easy calculation yields

\[
w_Iw_J = \prod_{k,l} \sigma(|i_k - j_l|)w_tw_I
\]

Put

\[
S = \{w_I : I = \{i_1 < \cdots < i_r\}, \quad I \neq \emptyset\}.
\]

Let \( A(X) \) denote the C*-algebra generated by the set of \( s_i, \quad i \in \mathbb{Z} \). Then \( S \cup \{1\} \) is total in \( A(X) \). We define \( \alpha \in \text{Aut} A(X) \) to be the shift \( \alpha(s_i) = s_{i+1} \).

**Theorem 4.1.** With the above notation there exists \( X \subset \mathbb{N} \) such that the C*-dynamical system \((A(X), \alpha)\) satisfies the assumptions of Theorem 2.1.

**Proof.** For each \( j \in \mathbb{N} \) let \((x_j^s)_{s \in \mathbb{N}}\) be the sequence found in Lemma 3.1. Put

\[
U_j = \{x_s^t - x_s^j + j : 1 \leq t < s, \quad s, t \in \mathbb{N}\},
\]

and put

\[
X = \bigcup_{j=1}^{\infty} U_j.
\]

Let \( I = \{i_1 < \cdots < i_r\} \) and \( N \in \mathbb{N} \). We shall find \( n_1 < n_2 < \cdots < n_N \in \mathbb{N} \) such that

\[
[\alpha^{n_s}(w_I), \alpha^{n_t}(w_I)]_+ = 0 \text{ if } s \neq t.
\]

Note that this is sufficient since \( w_I^* = \pm w_I \). Since (3) holds if we replace \( n_s \) and \( n_t \) by \( n_s + n \) and \( n_t + n \) for any \( n \in \mathbb{Z} \), we may assume \( 1 \leq i_1 < \cdots < i_r \). Since also \([a, b]_+ = [b, a]_+\) for
all \( a, b \), it suffices to show (3) for \( n_s > n_t \). Put \( j = i_r - i_1, n_s = x^j_s, n_t = x^j_t \). Then with \( W_j \) as in Lemma 3.1 we have

\[
\begin{align*}
  n_s - n_t + i_t - i_m & \in W_j \setminus U_j \text{ if } i_t - i_m < j, \\
  n_s - n_t + i_r - i_1 & \in U_j.
\end{align*}
\]

By Lemma 3.1 the sets \( W_k \) are pairwise disjoint, so the only contribution to \( \sigma \) applied to the numbers \( n_s - n_t + i_t - i_m \) comes from \( U_j \subset W_j \). Thus we have

\[
\prod_{l,m} \sigma(|n_s - n_t + i_t - i_m|) = \prod_{l,m} \sigma(n_s - n_t + i_t - i_m) = \sigma(n_s - n_t + i_r - i_1) = -1.
\]

Thus (3) holds whenever \( n_s = x^j_s, n_t = x^j_t \). This completes the proof. QED.

By Remark 2.5 or by [P] the unique invariant state \( \phi \) found in Theorem 2.1 is a trace. Also by [P] \( A(X) \) is an AF-algebra, hence is nuclear, so the entropies \( H_\phi(\alpha) \) and \( h_\phi(\alpha) \) coincide. We shall now prove that with \( X \) and \( \alpha \) as in Theorem 4.1 the tensor product formula for entropy fails for \( \alpha \otimes \alpha \) and \( \phi \otimes \phi \).

**Theorem 4.2.** Let \( A(X) \) and \( \alpha \) be as in Theorem 4.1, and let \( \phi \) be the unique \( \alpha \)-invariant trace. Then the tensor product formula fails for \( \alpha \otimes \alpha \) and \( \phi \otimes \phi \). More specifically we have

\[
h_{\phi \otimes \phi}(\alpha \otimes \alpha) \geq \log 2, \ h_\phi(\alpha) = 0.
\]

**Proof.** Let \( A_0 \) denote the \( C^* \)-subalgebra of \( A(X) \otimes A(X) \) generated by operators of the form \( w_I \otimes w_I \). Since \( w_I w_J = \pm w_J w_I, A_0 \) is abelian. Since \( A_0 \) is generated by the self-adjoint unitaries \( s_i \otimes s_i \), and \( \alpha \otimes \alpha \) is the shift, and the invariant state \( \phi \otimes \phi \) vanishes on each \( s_i \otimes s_i \), the dynamical system \( (A_0, \alpha \otimes \alpha, \phi \otimes \phi) \) is isomorphic to the two shift, or equivalently to the baker’s transform, see [AN], and has entropy \( \log 2 \). Thus

\[
h_{\phi \otimes \phi}(\alpha \otimes \alpha) \geq h_{\phi \otimes \phi}(\alpha \otimes \alpha | A_0) = \log 2.
\]

By Theorem 2.1 \( h_\phi(\alpha) = 0 \), so that

\[
h_{\phi \otimes \phi}(\alpha \otimes \alpha) > h_\phi(\alpha) + h_\phi(\alpha),
\]

proving the theorem. QED.

By [P, Theorem 3.9] the trace \( \phi \) in the above theorem is a factor state. Since in the GNS-representation of \( \phi \) the entropy of [CNT] equals that of [CS] we have

**Corollary 4.3.** There exists an automorphism \( \alpha \) of the hyperfinite \( II_1 \)-factor such that the tensor product formula fails for \( \alpha \otimes \alpha \).
Remark 4.4. In a recent paper [V] Voiculescu has introduced some alternative definitions of entropy in AF and hyperfinite von Neumann algebras based on approximation of given operators by operators in finite dimensional C*-subalgebras. For all these entropies Voiculescu showed the inequality

\[ h'_{\tau \otimes \sigma} (\alpha \otimes \beta) \leq h'_t (\alpha) + h'_\sigma (\beta), \]

hence by Theorem 4.2 and Corollary 4.3 his entropies are in general different from the entropies of [CS] and [CNT] considered in the present paper.

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References


