Stochastic partial differential equations
A mathematical connection between
macrocosmos and microcosmos

by

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STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS
A MATHEMATICAL CONNECTION BETWEEN MACROCOSMOS AND MICROCOSMOS

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§1. Introduction. What is a stochastic differential equation?

This paper gives a brief, non-technical introduction to some of the theory and methods of stochastic partial differential equations. No attempt has been made to be complete in any sense. The exposition is mainly based on recent joint works with a group of people, including H. Gjessing, H. Holden, T. Lindstrøm, N.H. Risebro, J. Ubøe and T.S. Zhang. These people are, however, not responsible for any errors produced in this article. Nor should they be blamed for the partly philosophical comments and interpretations I have added in this survey.

The purpose here is just to give a flavour of the theory and convince the reader of its usefulness and of the mathematical challenge that it represents. The reader is referred to the relevant publications listed in the back for proofs and further details.

The theory of *stochastic differential equations* (SDE) stems from, and is motivated by, the attempts to describe mathematically the stochastic dynamic phenomena that occur in physics, biology, economics etc. The term "stochastic phenomena" includes phenomena which may not really be stochastic in nature, but appear stochastic or random to us because of our lack of information. (What's the difference anyway? Is the outcome of a throw of a die really random or just appearing to be random because of our lack of information about the movement of the hand throwing it?)

When we describe a situation in physics or economics by means of a differential equation we are always making simplifying assumptions, for example about the coefficients of the equation. Coefficients which are assumed to be constant, may not really be so. They may be subject to fluctuations which to us appear random, or they may be constant but impossible to measure exactly. How will this randomness affect the solution of the differential equation? Questions like these lead to delicate questions about how to model various types of "noise" mathematically in such a way that it can be adopted as a proper (and rigorous) part of a differential equation.

Roughly speaking, a *stochastic differential equation* is a differential equation where some of the coefficients are subject to some properly defined versions of "noise".

In order to illustrate this, let us consider an example from population growth:

EXAMPLE 1.1

Let X_t denote the size of a population at time t. Suppose K is the carrying capacity of the environment for this population. Then a simple classical model for the growth of X_t is the differential equation

$$(1.1) \qquad \frac{dX_t}{dt} = r(K - X_t), \quad \text{where } r \text{ is a constant.}$$

The solution of this equation is

$$(1.2) \qquad X_t = K + (X_0 - K)e^{-rt}.$$

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Now suppose that r is not a constant but subject to random fluctuations due to unpredictable changes in the environment. Then we could try to put

(1.3) $$ r = r_t = a + b \cdot W_t, \quad (a, b \text{ constants}) $$

where $\{W_t(\omega); t \geq 0; \omega \in \Omega\}$, is some stochastic process modelling "noise". (The probability law P of this process is defined on a σ-algebra $\mathcal{F}$ of subsets of the given "set of outcomes" Ω). This gives - formally - the stochastic differential equation

(1.4) $$ \frac{dX_t}{dt} = a(K - X_t) + b(K - X_t) \cdot W_t $$

In some cases it may be justified to assume that the noise W_t is *white*, in the sense that W_t has the following properties:

(i) If $t_1 \neq t_2$ then the random variables $W_{t_1}(\cdot)$ and $W_{t_2}(\cdot)$ are independent.

(ii) $\{W_t\}_{t \geq 0}$ is a *stationary* process, i.e. the law of

$$ \{W_{t_1 + h}, \ldots, W_{t_n + h}\} $$

is independent of $h > 0$ for all $t_1, \ldots, t_n$ and all n.

(iii) $E[W_t] = 0$ and $E[W_t^2] = 1$ for all t, where E denotes expectation (i.e. integration) with respect to the probability measure P.

It turns out that no (measurable) process W_t satisfying (i), (ii) and (iii) exists! To overcome this difficulty, the following approaches are natural:

Alternative 1. Weaken the requirement (i) to allow dependence between W_{t_1} and W_{t_2} if t_1 and t_2 are close.

Alternative 2. Interpret equation (1.4) in a weak sense, i.e. as an *integral* equation instead of a differential equation:

(1.5) $$ X_t = X_0 + \int_0^t a(K - X_s)ds +'' \int_0^t b(K - X_s)W_s ds'' ; \quad t \geq 0 $$

where the integral in quotation marks needs to be explained.

It turns out that there is not a big difference between these two approaches. We will return to Alternative 1 in §5-6. Let us first look at Alternative 2:

We now recall that *Brownian motion* $\{B_t(\omega)\}_{t \geq 0}$ is a stochastic process with independent, stationary *increments* of mean zero (in fact, it is the only t-continuous process with these properties). This indicates that the t-derivative of B_t, $\frac{dB_t}{dt}$, could have been a good model for W_t - if it existed! Unfortunately, the paths $t \to B_t(\omega)$ of Brownian motion has infinite

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variation for almost all ω. Nevertheless, we could try to replace "$W_s ds$" by "dB_s" in (1.5) and then try to make sense out of integrals of the form

$$(1.6) \qquad \int_0^t f(s, \omega) dB_s(\omega)$$

for a reasonably large class of integrands $f(s, \omega)$. For deterministic functions $f(s, \omega) = f(s)$ such integrals were defined by N. Wiener [Wie] in 1923 and the general construction for a class of random integrands $f(s, \omega)$ was carried out by K. Ito [I1] in 1944. Such integrals are now called *Ito integrals*. The most notable necessary requirement that such integrands must satisfy is that $f(t, \cdot)$ is measurable w.r.t. the σ-algebra $\mathcal{F}_t$ generated by $\{B_s(\cdot); s \leq t\}$. Such processes $f(t, \omega)$ are called $\mathcal{F}_t$-*adapted*. See e.g. [Ø] for more details.

In terms of Ito integrals (1.5) gets the form

$$(1.7) \qquad X_t = X_0 + a \int_0^t (K - X_s) ds + b \int_0^t (K - X_s) dB_s ; \quad t \geq 0$$

This gives a precise mathematical interpretation of the equation (1.4), but what is its solution? Using the *Ito formula* (see e.g. [Ø]) one can prove that if X_0 is independent of the σ-algebra $\mathcal{F}_\infty$ generated by $\{B_t(\cdot); t \geq 0\}$ then

$$(1.8) \qquad X_t = X_t(\omega) = K - (K - X_0) \exp(-(a + \frac{1}{2}b^2)t - bB_t(\omega)) ; \quad t \geq 0$$

Note that this solution coincides with the solution (1.2) if $b = 0$ i.e. if the noise is zero.

Since we know a lot about the probabilistic behaviour of $B_t(\cdot)$, we can from (1.8) easily deduce interesting properties about X_t. For example, $X_t \to K$ as $t \to \infty$, a.s. (a.s.= "almost surely", i.e. with probability one) and if X_0 is a constant (deterministic) then

$$(1.9) \qquad E[X_t] = K - (K - X_0)e^{-at},$$

i.e. the expected value of X_t coincides with the solution (1.2) of the no-noise equation.

We will return to the solution of this equation in §6.

§2. Stochastic partial differential equations: The need for a more general framework.

Encouraged by the success in SDE let us now consider a stochastic *partial* differential equation (SPDE):

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EXAMPLE 2.1 (Membrane in a sand storm)

If a membrane is exposed to a (vertical) force $F(t,x,y)$ at time t and at the point with horizontal coordinates (x,y), then the vertical coordinate $z = u(t,x,y)$ of the membrane will satisfy the wave equation

$$(2.1) \qquad \frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = F(t,x,y)$$

Now suppose the force is coming from the bombardement of infinitesimal sand particles. Then a natural mathematical model for $F(t,x,y)$ would be a *3-parameter white noise process* $W_{t,x,y}(\omega)$. Analogous to the 1-parameter case discussed in §1 we can relate this noise to *3-parameter Brownian motion* (or Brownian sheet) $B_{t,x,y}(\omega)$ as follows:

$$(2.2) \qquad W_{t,x,y}(\omega) = \frac{\partial^3}{\partial t \partial x \partial y} B_{t,x,y}(\omega) \quad \text{(see §3)}$$

With this force (2.1) becomes

$$(2.3) \qquad \frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial^3}{\partial t \partial x \partial y} B_{t,x,y}$$

Again this does not make sense as it stands, but it is natural to try the weak, integral interpretation:

$$(2.4) \qquad \int\limits_0^t \int\limits_{\mathbf{R}^2} u \frac{\partial^2 \phi}{\partial s^2} ds\,dx\,dy - \int\limits_0^t \int\limits_{\mathbf{R}^2} u\Delta\phi \, ds\,dx\,dy = \int\limits_0^t \int\limits_{\mathbf{R}^2} \phi \, dB_{t,x,y},$$

for all test functions $\phi(t,x,y) \in C_0^\infty(\mathbf{R}^3)$ (the smooth functions with compact support in $\mathbf{R}^3$). Here the right hand side is a 3-parameter Ito integral, which can be defined in a similar way as in the 1-parameter case. We have put $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta$, the Laplace operator.

So far, so good. But now the surprise is that (2.4) has no 3-parameter stochastic process solution $u_{t,x,y}(\omega)$! This was proved by J. Walsh [Wa]. However, Walsh also showed that (2.4) has a solution u in a more general setting, namely as a *distribution valued* stochastic process $u_\phi(\omega)$.

The reader's first reaction to such a result might be disappointment or disbelief: The physicall membrane must have a position at time t over (x,y), no matter what the mathematics says! But is that really the case? When we measure the position of the membrane we are really taking averages of microscopic quantities over small periods of time and space and the actual macroscopic measurement really depends on what "scale" or microscope we use. Mathematically, the process of taking averages corresponds to applying a test function $\phi(t,x,y)$ to the distribution $u(\omega)$. This is exactly how we interpret the Walsh solution.

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The singular white noise force $W_{t,x,y}$ is really itself a distribution valued process (see §3), so from this point of view it is not so surprising that the solution u is also. We could also say that solving equation (2.3) really corresponds to trying to find what the *macroscopic* value u is if the membrane is exposed to the singular force W working on the *microscopic* level.

This mathematical connection between micro- and macro-cosmos will be illustrated again in connection with fluid flow in porous media (§4). But first we need to develop some mathematical machinery. Two basic questions need to be clarified:

Question 1. What is the *right* mathematical formulation of a "noisy" partial differential equation?

By "right" we mean both that it is mathematically rigorous and that it (or rather its solution) actually gives a realistic model of the situation we are studying.

Question 2. Having found the right mathematical formulation, how do we proceed to solve the corresponding stochastic partial differential equation? By "solve" we here mean proving the existence and uniqueness of the solution, finding some of its probabilistic properties and - if possible - obtaining an explicit solution formula like (1.8) in Example 1.1.

§3. White noise, chaos expansion, Wick products and Skorohod integrals

The previous examples illustrate the need for a rigorous mathematical framework for concepts like white noise and distribution valued processes. In this section we give a brief summary of such a framework. More details can be found, for example, in [HP] or [HKPS].

It was Hida's original idea [H] that the basic object to consider is not the classical Brownian motion, but rather the more troublesome concept of white noise. In §2 white noise was introduced as the (non-existing) derivative of Brownian motion. It is however, surprisingly simple to construct white noise directly - and rigorously - as a distribution valued process:

The white noise probability space.

Fix a natural number d (the parameter dimension) and let $\mathcal{S} = \mathcal{S}(\mathbf{R}^d)$ denote the Schwartz space of rapidly decreasing smooth functions on $\mathbf{R}^d$. The dual of $\mathcal{S}$ is the space $\mathcal{S}' = \mathcal{S}'(\mathbf{R}^d)$ of *tempered distributions*. According to the Bochner-Minlos theorem [GV] there exists a probability measure μ on $\mathcal{S}'$ with the property that

$$(3.1) \qquad \int\limits_{\mathcal{S}'} e^{i<\omega,\phi>} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2} ; \quad \phi \in \mathcal{S}$$

where $<\omega,\phi> = \omega(\phi)$ is the result of applying $\phi \in \mathcal{S}$ to the distribution $\omega \in \mathcal{S}'$ and $\|\phi\| = (\int_{\mathbf{R}^d} |\phi(x)|^2 dx)^{1/2}$ is the classical $L^2(\mathbf{R}^d)$-norm of ϕ.
It is not hard to show that (3.1) implies that

$$(3.2) \qquad \int_{\mathcal{S}'} f(<\omega,\phi>)d\mu(\omega) = (2\pi\|\phi\|^2)^{-d/2} \int_{\mathbf{R}^d} f(x)e^{-\frac{x^2}{2\|\phi\|^2}}dx\,; \quad \phi \in \mathcal{S}$$

for all f such that the integral on the right hand side converges. (It suffices to show (3.2) for functions f which are the inverse Fourier transform of their Fourier transforms $\hat{f}$ and for such functions (3.2) follows from (3.1) and the Fubini theorem).

In particular, if we apply this to $f(x) = x^2$ we get

$$(3.3) \qquad \int_{\mathcal{S}'} |<\omega,\phi>|^2 d\mu(\omega) = \|\phi\|^2\,; \quad \phi \in \mathcal{S}$$

Using this we can extend the definition of $<\omega,\phi>$ from $\phi \in \mathcal{S}$ to $\psi \in L^2(\mathbf{R}^d)$ as follows:

$$(3.4) \qquad <\omega,\psi> = \lim_{n\to\infty} <\omega,\phi_n> \qquad (\text{limit in } L^2(\mu))$$

where $\phi_n \in \mathcal{S}$ and $\phi_n \to \psi$ in $L^2(\mathbf{R}^d)$. (It follows from (3.3) that the limit exists in $L^2(\mu)$ and that it is independent of the actual choice of approximating sequence $\{\phi_n\} \subset \mathcal{S}$.)

In particular, for all $t_1,\dots,t_d \geq 0$ we can choose

$$(3.5) \qquad \psi(x) = \chi_{[0,t_1]\times\cdots\times[0,t_d]}(x)$$

(i.e. $\psi(x) = 1$ if $x \in [0,t_1] \times \cdots \times [0,t_d]$ and 0 otherwise) and define

$$(3.6) \qquad \tilde{B}_{t_1,\dots,t_d}(\omega) = <\omega, \chi_{[0,t_1]\times\cdots\times[0,t_d]}>$$

Then $\tilde{B}$ is a d-parameter Gaussian process with stationary, independent increments of mean zero. Moreover by (3.3) the covariance is (from now on $E = E_\mu$ means expectation with respect to the measure μ):

$$(3.7.) \qquad E[\tilde{B}_{t_1,\dots,t_d} \cdot \tilde{B}_{s_1,\dots,s_d}] = \int_{\mathcal{S}'} <\omega, \chi_{[0,t_1]\times\cdots\times[0,t_d]}> \cdot <\omega, \chi_{[0,s_1]\times\cdots\times[0,s_d]}> d\mu(\omega)$$

$$= \int_{\mathbf{R}^d} \chi_{[0,t_1]\times\cdots\times[0,t_d]} \cdot \chi_{[0,s_1]\times\cdots\times[0,s_d]}dx = \prod_{k=1}^d (s_k \wedge t_k).$$

One can prove that there exists a t-continuous stochastic process $B_{t_1,\dots,t_d}$ which is a *version* of $\tilde{B}_{t_1,\dots,t_d}$, in the sense that

$$(3.8) \qquad \mu(\{\omega; B_t(\omega) = \tilde{B}_t(\omega)\}) = 1 \quad \text{for all } t = (t_1,\dots,t_d).$$

In view of the properties of $\tilde{B}_t$ stated above, it is natural to call B_t the *d-parameter Brownian motion* (or *Brownian sheet*).

DEFINITION 3.1. The *white noise process* is the map $W : \mathcal{S} \times \mathcal{S}' \to \mathbf{R}$ defined by

$$(3.9) \qquad W_\phi(\omega) = \, <\omega, \phi> \quad \text{for } \phi \in \mathcal{S}, \ \omega \in \mathcal{S}'$$

By (3.4), (3.6) and the isometry (3.3) we see that if $\phi \in \mathcal{S}$ with supp $\phi \subset (\mathbf{R}^+)^d$ and $\{\psi_n\}$ are step functions converging to ϕ in $L^2(\mathbf{R}^d)$ then

$$(3.10) \qquad W_\phi(\omega) = \lim_n <\omega, \psi_n> = \lim \int_{\mathbf{R}^d} \psi_n(x) dB_x = \int_{\mathbf{R}^d} \phi(x) dB_x$$

where the limit is in $L^2(\mu)$ and the last term is the *d-parameter Ito integral* of ϕ. For general $\phi \in L^2(\mathbf{R}^d)$ formula (3.10) is used to *define* the Ito integral of ϕ over $\mathbf{R}^d$. The identity (3.10) may be regarded as a precise way of saying that $W.(\omega)$ is the distributional derivative of $B.(\omega)$:

$$(3.11) \qquad W = \frac{\partial}{\partial x_1 \ldots \partial x_d} B_{x_1,\ldots,x_d}$$

as claimed earlier.

The Wiener-Ito chaos expansion.

Let h_n be the *n'th order Hermite polynomial* defined by

$$(3.12) \qquad h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} (e^{-\frac{x^2}{2}}); \quad n = 0, 1, 2 \ldots$$

Thus

$$h_0(x) = 1, \ h_1(x) = x, \ h_2(x) = x^2 - 1, \ h_3(x) = x^3 - 3x,$$
$$h_4(x) = x^4 - 6x^2 + 3, \ h_5(x) = x^5 - 10x^3 + 15x, \ldots$$

For $n = 1, 2, \ldots$ let ξ_n be *the Hermite function of order* n defined by

$$(3.13) \qquad \xi_n(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{n-1}(\sqrt{2}x); \quad x \in \mathbf{R}$$

Then $\{\xi_n\}_{n=1}^\infty$ forms an orthonormal basis for $L^2(\mathbf{R})$. Therefore the family of tensor products

$$(3.14) \qquad e_\alpha := e_{\alpha_1,\ldots,\alpha_d} := \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_d},$$

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where α denotes the multi-index $(\alpha_1, \ldots, \alpha_d)$, forms an orthonormal basis for $L^2(\mathbf{R}^d)$. With a slight abuse of notation let $e_1, e_2, \ldots$ denote a fixed ordering of the family $\{e_\alpha\}_\alpha$ from now on. Put

$$(3.15) \qquad \theta_j = \theta_j(\omega) = \int_{\mathbf{R}^d} e_j(x) dB_x(\omega)$$

and define, for each multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$,

$$(3.16) \qquad H_\alpha(\omega) = \prod_{j=1}^m h_{\alpha_j}(\theta_j)$$

The Wiener-Ito chaos theorem states that the family $\{H_\alpha\}_\alpha$ forms an orthogonal basis for $L^2(\mu)$. This gives that any $X \in L^2(\mu)$ has the (unique) expansion

$$(3.17) \qquad X(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega),$$

the sum being taken over all multi-indices of non-negative integers. Moreover, we have the isometry

$$(3.18) \qquad \|X\|_{L^2(\mu)}^2 = \sum_\alpha \alpha! c_\alpha^2,$$

where $\alpha! = \alpha_1! , \alpha_2! \ldots \alpha_m!$ if $\alpha = (\alpha_1, \ldots, \alpha_m)$.

EXAMPLE 3.2. For each $\phi \in L^2(\mathbf{R}^d)$ we have $X := W_\phi \in L^2(\mu)$. The expansion of W_ϕ is

$$(3.19) \qquad W_\phi = \int \phi dB = \sum_j \int (\phi, e_j) e_j dB = \sum_j (\phi, e_j) h_1(\theta_j) = \sum_j (\phi, e_j) H_{(j)},$$

where $(j) = (0, 0, \ldots, 0, 1)$ with 1 on the j'th place and $(\cdot, \cdot)$ denotes inner product in $L^2(\mathbf{R}^d)$.

The Wick product

DEFINITION 3.3. It $X = \sum_\alpha a_\alpha H_\alpha$ and $Y = \sum_\beta b_\beta H_\beta$ and two functions in $L^2(\mu)$ we define their *Wick product* $X \diamond Y$ as follows

$$(3.20) \qquad X \diamond Y = \sum_{\alpha, \beta} a_\alpha b_\beta H_{\alpha+\beta} = \sum_\gamma (\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta) H_\gamma \quad \text{(when convergent)}$$

For general $X, Y \in L^2(\mu)$ this sum may or may not converge in L^p for some $p \geq 1$.

REMARKS

1) Note that if one of the two factors is constant (does not depend on ω) the the Wick product coincides with the ordinary product.

2) The Wick product is not local: It is not enough to know the value of $X(\omega_0)$ and $Y(\omega_0)$ in order to know the value of $(X \diamond Y)(\omega_0)$. In fact, not even the knowledge of $X(\omega)$, $Y(\omega)$ for ω in some neighborhood of ω_0 is sufficient in general. ([GHLØUZ]).

EXAMPLE 3.4.
If $X(\omega) = W_\phi(\omega) = \int_{\mathbf{R}^d} \phi(x)dB_x(\omega)$ and $Y(\omega) = W_\psi(\omega) = \int_{\mathbf{R}^d} \psi(x)dB_x(\omega)$ with $\phi, \psi \in L^2(\mathbf{R}^d)$ then it can be proved that

$$(3.21) \qquad (X \diamond Y)(\omega) = \int_{\mathbf{R}^d \times \mathbf{R}^d} (\phi \hat{\otimes} \psi)(x,y)dB^{\otimes 2}_{x,y},$$

where $\hat{\otimes}$ denotes *symmetrized tensor product* (i.e. $(\phi \hat{\otimes} \psi)(x,y) = \frac{1}{2}[\phi(x)\psi(y) + \phi(y)\psi(x)]$) and the right hand side of (3.20) is *the double Ito integral* (see e.g. [GHLØUZ] for details).

EXAMPLE 3.5. If $X(\omega) = B_t(\omega)$ for a fixed t, then (for example by (3.21))

$$B_t \diamond B_t := B^{\diamond 2}_t = B^2_t - t$$

and

$$B^{\diamond 3}_t := B_t \diamond (B^{\diamond 2}_t) = B^3_t - 3tB_t$$

(see e.g. [GHLØUZ] for extensions).

EXAMPLE 3.6. It is easy to see that $\diamond$ is an associative binary operation (when defined) so we can define, for $X \in L^2(\mu)$ and n a natural number, the *Wick power*

$$(3.22) \qquad X^{\diamond n} = X \diamond X \diamond \cdots \diamond X \quad (n \text{ times})$$

without specifying parenthesis on the right hand side (assuming the Wick products exist). In particular, for $X = W_\phi$ we can define the *Wick exponential* of W_ϕ, $\text{Exp}\,W_\phi$, by

$$(3.23) \qquad \text{Exp}\,W_\phi = \sum_{n=0}^\infty \frac{1}{n!}W^{\diamond n}_\phi, \quad \phi \in L^2(\mathbf{R}^d).$$

One can in fact show that (see e.g. [GHLØUZ])

$$(3.24) \qquad \text{Exp}\,W_\phi = \exp(W_\phi - \frac{1}{2}\|\phi\|^2)$$

In particular, this shows that $\text{Exp}\,W_\phi$ is *positive*, in the sense that

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$$\operatorname{Exp} W_\phi(\omega) > 0 \quad \text{for all } \phi \in L^2(\mathbf{R}^d), \ \omega \in \mathcal{S}'.$$

This property makes it a natural model for certain "positive noises" occurring for example in fluid flow in porous media. See §6.

The Wick product (or slightly different version of it) was originally introduced by G.C. Wick [Wi] in 1950 in connection with quantum field theory, where it corresponds to a kind of renormalization. In a stochastic analysis context the Wick product was first used in 1965 by T. Hida and N. Ikeda [HI]. It is remarkable that the Wick product concept should prove natural in two so different contexts. The full reason for - and all implications of - this connection is not yet fully understood (at least not by me).

In order to explain why the Wick product is natural in stochastic analysis we need to define the following concept:

The Skorohod integral

Let us for a moment assume that $d = 1$ and consider a stochastic process $\{X_t\}_{t \geq 0} \subset L^2(\mu)$. Such a process can be represented by the expansion

$$(3.25) \qquad X_t(\omega) = \sum_\alpha c_\alpha(t) H_\alpha(\omega).$$

Define *the Skorohod integral of X_t*, $\int_{\mathbf{R}} X_t \delta B_t$, by

$$(3.26) \qquad \int_{\mathbf{R}} X_t(\omega) \delta B_t(\omega) := \sum_{\alpha,j} \left(\int_{\mathbf{R}} c_\alpha(t) e_j(t) dt \right) H_{\alpha+(j)}(\omega) = \sum_{\alpha,j} (c_\alpha, e_j) H_{\alpha+(j)}(\omega)$$

(when convergent), where as before $(j) = (0, 0, \ldots, 0, 1)$ with a 1 on the j'th place and $(\cdot, \cdot)$ denotes the usual $L^2(\mathbf{R})$ inner product.

Note that if X_t is deterministic, i.e. $X_t = c_0(t)$, then

$$
(3.27) \qquad
\begin{aligned}
\left(\int_{\mathbf{R}} X_t(\omega) \delta B_t(\omega) \right. &= \sum_j (c_0, e_j) H_{(j)}(\omega) = \sum_j (c_0, e_j) \int_{\mathbf{R}} e_j dB \\
&= \int_{\mathbf{R}} c_0(t) dB_t = \int_{\mathbf{R}} X_t(\omega) dB_t.
\end{aligned}
$$

Thus the Skorohod integral coincides with the Ito integral in this case. In fact, the Skorohod integral can be shown to coincide with the Ito integral if the integrand is adapted [NZ].

For a general $\phi \in L^2(\mathbf{R})$, $t \in \mathbf{R}$ let $\phi_t(\cdot)$ denote the *t-shift of ϕ*, i.e.

$$(3.28) \qquad \phi_t(s) = \phi(s - t)$$

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The connection between Wick products, white noise and Skorohod integrals can now be formulated as follows:

$$(3.29) \qquad \int_{\mathbf{R}} (\phi * X)_t \delta B_t = \int_{\mathbf{R}} X_t \diamond W_{\phi_t} dt; \quad \phi \in \mathcal{S}(\mathbf{R})$$

where $*$ denotes convolution with respect to Lebesgue measure on $\mathbf{R}$, i.e.

$$(3.30) \qquad (\phi * X)_t(\omega) = \int_{\mathbf{R}} \phi(s+t) X_s(\omega) ds$$

(See [LØU 2], [AP], [ØZ]).

If we let $\phi \to \delta$ (the Dirac measure at 0) as measures on $\mathbf{R}$ one can prove that W_{ϕ_t} converges in a weak sense (in the space $(\mathcal{S})^*$ of Hida distributions) to an object which we - by an abuse of notation - denote by W_t (the "pointwise" version of white noise). The Wick product can be extended to $(\mathcal{S})^*$, so taking the limit in (3.39) we get

$$(3.31) \qquad \int_{\mathbf{R}} X_t \delta B_t = \int_{\mathbf{R}} X_t \diamond W_t dt$$

In particular, if X_t is adapted this says that *Ito integration is equivalent to Wick multiplication by white noise followed by Lebesque integration*. Here is the key to the fundamental importance of the Wick product in Ito stochastic calculus.

§4. The Hermite transform and its inverse

The Wiener - Ito expansion allows us to associate to any given $X \in L^2(\mu)$ a complex valued function $\mathcal{H}(X)$ of infinitely many complex variables $z_1, z_2, \ldots$:

DEFINITION 4.1. Let $X \in L^2(\mu)$ have the expansion

$$(4.1) \qquad X(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega)$$

Then the *Hermite transform* of X, denoted by $\mathcal{H}(X)$ or $\tilde{X}$, is the function defined on the space $\mathbf{C}_0^{\mathbf{N}}$ of all finite sequences of complex numbers $z_1, \ldots, z_n$ by

$$(4.2) \qquad \mathcal{H}(X)(z_1, z_2, \ldots) = \tilde{X}(z_1, z_2, \ldots) = \sum_\alpha c_\alpha z^\alpha$$

where we again use the multi-index notation: If $\alpha = (\alpha_1, \ldots, \alpha_m)$, $z = (z_1, z_2, \ldots)$ then

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_m^{\alpha_m}.$$

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EXAMPLE 4.2. The Hermite transform of white noise is, by (3.19),

$$(4.3) \qquad \mathcal{H}(W)(z) = \tilde{W}_\phi(z) = \sum_j (\phi, e_j) z_j ; \qquad (z = z_1, z_2, \ldots)$$

A crucial feature of the Hermite transform is that it changes Wick products into ordinary complex products:

LEMMA 4.3 If X, Y, $X \diamond Y \in L^2(\mu)$ then

$$\mathcal{H}(X \diamond Y)(z) = \mathcal{H}(X)(z) \cdot \mathcal{H}(Y)(z)$$

Moreover, it is possible to recover X from $\tilde{X}$ by performing an integration with respect to an infinite product of Gaussian measures:

Let $d\lambda(y) = d\lambda(y_1, y_2 \ldots)$ be the probability measure on $\mathbf{R}^N = \mathbf{R} \times \mathbf{R} \times \cdots$ defined by

$$(4.4) \qquad \int\limits_{\mathbf{R}^N} f(y_1, \ldots y_n) d\lambda(y) = (2\pi)^{-n/2} \int\limits_{\mathbf{R}^n} f(y) e^{-\frac{1}{2}|y|^2} dy$$

if f is a bounded function depending only on the first n coordinates $(y_1, \ldots, y_n)$ of y. Then we have

LEMMA 4.5. Let $X \in L^2(\mu)$. Then

$$(4.5) \qquad X(\omega) = \int\limits_{\mathbf{R}^N} \tilde{X}(\theta_1 + iy_1, \theta_2 + iy_2, \ldots) d\lambda(y) \qquad (i = \sqrt{-1})$$

where $\theta_j = \int e_j dB$ as before and the integral is interpreted as a limit of the integrals of the truncations of $\tilde{X}$. See [HLØUZ 1] for details.

EXAMPLE 4.6. Suppose $\tilde{X}(z) = z_1 = x_1 + iy_1$. Then

$$X(\omega) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbf{R}} (\theta_1 + iy_1) e^{-\frac{1}{2}y_1^2} dy_1 = \theta_1 = \int e_1 dB,$$

so $x = W_{e_1}$.

§5. A scheme for solving stochastic partial differential equations.

Example 2.1 illustrates that for a SPDE we cannot in general expect to find a solution which is an ordinary stochastic process. The reason is that white noise, albeit it has the ideal probabilistic properties, is too singular to produce solutions of this kind, even when the equation is interpreted in a weak, integrated sense. In view of this, and in view of equations (3.29) and (3.31), it is natural to adopt *Alternative 1* in §1 rather than Alternative 2 in the general setting.

Thus we replace the singular, pointwise white noise W_t by the "smeared out" version W_ϕ, where $\phi \in \mathcal{S}$ is fixed (at least for a while). We may regard ϕ as the "window" or the microcope we use to measure the noise. Then we consider translates of this window, $\phi_x(\cdot)$ and use $W_{\phi_x}(\cdot)$ as our approximate white noise in the equation. By integration by parts we see that distributional derivative with respect to ϕ is the same as ordinary derivative with respect to the shift x. Thus, with W_x replaced by W_{ϕ_x} we may regard the corresponding SPDE as a PDE in x for each ω, except that we use Wick products instead of ω-pointwise products (see below). In special cases the solution $X = X(x, \phi, \omega)$ of the SPDE may have limit as $\phi \to \delta$, but in general not. So in general we regard the equation as solved if we have found $X(\phi, x, \omega)$ for each $\phi \in \mathcal{S}$.

As we will illustrate in the next section this works well and gives us the same result as classical methods in the linear case. In the non-linear case, however, the question arises what kind of products one should use: Wick products or ordinary (pointwise) products. See [LØU2] for a more detailed discussion about this. As we have pointed out already, the Wick product is natural in the context of Ito integration. If we start with such a Wick SPDE, then in view of §4 the canonical solution procedure in the following:

1) Apply the Hermite transform to convert the original equation into a deterministic PDE with complex parameters $z_1, z_2, \ldots$
2) Solve this equation (if possible).
3) Apply the inverse Hermite transform to the solution in 2) to obtain a solution of the original equation.

§6. Applications

To illustrate the method outlined above let us use it to solve the SDE (1.7) of Example 1.1:

$$(6.1) \qquad X_t = X_0 + a \int_0^t (K - X_s)ds + b \int_0^t (K - X_s)dB_s$$

Adopting the point of view of §5 we choose $\phi \in \mathcal{S}(\mathbf{R})$ and consider the approximate equation in $X_t = X(\phi, t, \omega)$:

$$X_t = X_0 + \int_0^t (K - X_s) \diamond (a + bW_{\phi_s})ds$$

Taking Hermite transforms we get

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t (K - \tilde{X}_s)(a + b\tilde{W}_{\phi_s})ds$$

or

$$(6.2) \qquad \frac{d\tilde{X}_t}{dt} = (K - \tilde{X}_t)(a + b\tilde{W}_{\phi_t}).$$

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Recall that $\tilde{W}_{\phi_t} = \tilde{W}_{\phi_t}(z_1, z_2, \ldots) = \sum_k (\phi_t, e_k) z_k$ where $z_k \in \mathbf{C}$ so (6.2) is a differential equation with respect to t with complex parameters $z_1, z_2, \ldots$. It is easily verified that the solution of (6.2) is

$$(6.3) \qquad \tilde{X}_t = K - (K - \tilde{X}_0) \cdot \exp(-\int_0^t (a + b\tilde{W}_{\phi_s}) ds)$$

To find X_t we can apply the inverse Hermite transform. However, it is easier to note that by (4.4) we have

$$(6.3) \qquad \mathcal{H}^{-1}(\exp(-b\int_0^t \tilde{W}_{\phi_s} ds)) = \mathrm{Exp}(-b\int_0^t W_{\phi_s} ds)$$

which gives, using (4.4) again,

$$(6.4) \qquad X_t = K - (K - X_0) \diamond \mathrm{Exp}(-at - b\int_0^t (\int_{\mathbf{R}} \phi_s(u) dB_u) ds)$$

Now

$$\int_0^t (\int_{\mathbf{R}} \phi_s(u) dB_u) ds = \int_{\mathbf{R}} (\int_0^t \phi(u - s) ds) dB_u \to \int_{\mathbf{R}} X_{[0,t]}(u) dB_u = B_t$$

as $\phi \to \delta$. Therefore (6.4) gives us

$$(6.5) \qquad \lim_{\phi \to \delta} X(\phi, t, \omega) = K - (K - X_0) \diamond \mathrm{Exp}(-at - bB_t(\omega))$$

This is the same as the solution (1.8) we gave earlier, because of the identity (3.24).

Note that not only did we get the solution quickly by this method, but we also got a more general result than in (1.8): Our method here did not assume that X_0 is independent of the σ-algebra $\mathcal{F}$ generated by $\{B_t(\cdot); t \geq 0\}$ and for general X_0 the solution is expressed by the Wick product in (6.4). This Wick product reduces to the ordinary product if X_0 is $\mathcal{F}$-independent.

The ability to handle such *non-adaptive* stochastic differential equations is an additional useful feature of this method. Non-adaptive SDE's occur for example in problems regarding economic investments under uncertainty. See [ØZ], where (adapted and non-adapted) stochastic Volterra equations are studied.

We proceed by giving some examples of SPDE's which can be handled by the method outlined above.

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EXAMPLE 6.1. (Fluid flow in porous media).
In a porous rock the permeability k will often vary rapidly from point to point and it can be hard to measure. Accordingly, in the equation for the fluid pressure $p(x)$ at the point x for one phase flow,

$$(6.6) \qquad \begin{cases} \mathrm{div}(k(x)\nabla p(x)) = -f(x) \ ; & x \in D \subset \mathbf{R}^d \\ p(x) = 0 \ ; & x \in \partial D \end{cases}$$

(where f is the given source rate) it is natural to represent the quantity $k(x)$ by some *positive noise* $K(x,\omega); \omega \in \mathcal{S}'$. This makes the pressure stochastic, too: $p(x) = p(x,\omega)$. Representing $K(x,\omega)$ by $\mathrm{Exp}(W_{\phi_x}(\cdot))$ (see (3.23)) we arrive at the SPDE

$$(6.7) \qquad \begin{cases} \mathrm{div}(\mathrm{Exp}(W_{\phi_x}(\cdot)) \diamond \nabla p(x, \cdot)) = -f(x) \ ; & x \in D \\ p(x, \cdot) = 0 \ ; & x \in D \end{cases}$$

The question is now: How will the *microscopic* feature $k(x)$ of the medium affect the *macroscopic* properties of the flow? This equation is studied in [LØU 3].

EXAMPLE 6.2. (The stochastic Schrödinger equation)
A model for the Schrödinger equation with a random, positive potential is

$$(6.8) \qquad \begin{cases} \Delta u(x, \cdot) + V(x, \cdot) \diamond u(x, \cdot) = -f(x) \ ; & x \in D \subset \mathbf{R}^d \\ u(x, \cdot) = 0 \ ; & x \in \delta D, \end{cases}$$

where $\Delta = \sum\limits_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$ is the Laplacian and $V(x, \cdot) = \epsilon\mathrm{Exp}(W_{\phi_x}(\cdot))$ is the positive potential, $\epsilon > 0$.

In [HLØUZ 1] it is proved that for ϵ small enough the unique solution u of (6.6) is given by

$$(6.9) \qquad u(x, \cdot) = \hat{E}^x[\int\limits_0^{\tau_D} \mathrm{Exp}[\epsilon \int\limits_0^t \mathrm{Exp}(W_{\phi_{b_s}})ds]f(b_t)dt],$$

where $(b_t, \hat{P}^x)$ is a classical, 1-parameter Brownian motion in $\mathbf{R}^d$, $\hat{E}^x$ denotes expectation with respect to $\hat{P}^x$ and

$$\tau_D = \inf\{t > 0; b_t \notin D\}$$

is the first exit time from D of b_t.

EXAMPLE 6.3. (The transport equation in a turbulent medium)
If we model the turbulent motion of the medium by some d-dimensional noise $\vec{v}(t,x,\omega)$ an equation modelling the transport of a substance in this medium is

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$$(6.10) \qquad \frac{\partial}{\partial t} u(t, x, \cdot) + \vec{v}(t, x, \cdot) \diamond \nabla u(t, x, \cdot) = \frac{1}{2} \Delta u(t, x, \cdot)$$

where the gradient ∇ and the Laplacian Δ work on the x-variable, $x = (x_1, \ldots, x_d)$.

Here $\vec{v}$ could be modelled as d-dimensional, $(d+1)$-parameter white noise

$$(6.11) \qquad \vec{v} = (W_1(t, x, \cdot), \ldots, W_d(t, x, \cdot))$$

where the W_i's are independent ([G]). This equation has been studied in [CP].

EXAMPLE 6.4. (SPDE's arising in non-linear filtering).
The Wong-Zakai equation for the unnormalized conditional density ρ_t of the filtered estimate has the form

$$(6.12) \qquad d\rho_t(x; \omega) = A^* \rho_t(x, \omega) dt + \rho_t(x, \omega) h^T(x) dB_t$$

where A^* is a semi-elliptic second order partial differential operator acting on the space variable $x \in \mathbf{R}^d$ (A^* is the adjoint of the generator of an Ito diffusion) and $h : \mathbf{R}^d \to \mathbf{R}^m$ is given.

A general existence and uniqueness theorem for SPDEs of this type has been given by Pardoux [Pa]. Using Wick calculus one can in fact obtain an explicit solution [B].

EXAMPLE 6.5. (The Burgers equation with a noisy force)
The Burgers equation (in dimension 1) is the non-linear partial differential equation

$$(6.13) \qquad \frac{\partial u}{\partial t} + \lambda u \cdot \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$$

where λ, ν are constants. ($R = \frac{1}{\nu}$ is the Reynolds number.) It was orginally introduced as a model for turbulence but has later found many other applications as well.

With a noisy force the Burgers equation gets the form

$$(6.13) \qquad \frac{\partial u}{\partial t}(t, x, \cdot) + \lambda u(t, x, \cdot) \diamond \frac{\partial u}{\partial x}(t, x, \cdot) = \nu \frac{\partial^2 u}{\partial x^2}(t, x, \cdot) + M(t, x, \cdot),$$

where M is a (t, x)-parameter noise. By a certain Wick-substitution this equation can be transformed into a *linear* stochastic heat equation of the form

$$(6.14) \qquad \frac{\partial Y}{\partial t} = \nu \Delta Y + \frac{\lambda}{2\nu} Y \diamond N,$$

where N is another noise. This equation can be solved by Hermite transforms as outlined in §5. See [HLØUZ 2].

The Burgers equation is a special case of the conservation law

$$(6.15) \qquad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(f(u)) = F$$

where f and F are given functions.

This equation appears for example in fluid flow in porous media, where $u(t,x)$ is the saturation of the fluid, f is the flux function and F is the source. Various stochastic versions of this equation appear naturally in applications. See [HR], [HLR] and the references there.

§7. Concluding remarks. Towards a random fractal calculus?

SPDE is a mathematical machinery developed to handle PDEs where some of the coefficients are subject to random fluctuations or noise (or modelled as noise because of lack of information). The point is that even though there is noise in the coefficients one can still say something about the probabilistic properties of the solution. Moreover, it is of interest to see explicitly how the noise in the coefficients affects the solution. Often the noise comes from (the basically unknown) *microscopic* properties of the medium (e.g. permeability) or the surroundings and one seeks the corresponding macroscopic properties of the solution (e.g. the fluid flow). The stochastic analysis that is used to handle these questions turns out to involve in a natural way the *Wick product*, a concept which has been developed earlier in connection with renormalization in quantum physics [S]. This confirms that there is a deep relation between stochastic analysis and quantum physics.

Another interesting aspect of this theory is that it seems to be able to handle analytically some classes of *random fractals*. This means that random fractals can be more than just a tool to describe certain phenomena; they can be adopted as rigorous parts of a stochastic differential equation: We can do calculus with it. More precisely, as soon as the random fractal can be represented as some kind of noise or white noise functional as explained in §3, the whole stochastic calculus machinery applies. For example, the Wick exponential of white noise appears to be a good model not just for permeability, but also for the multifractals appearing in connection with turbulence or in connection with oil distribution [M].

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