IGUSA'S EXACT SEQUENCE AND THE BÖKSTEDT TRACE.

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1. Introduction.

Let $\mathcal{P}(M)$ be the space of smooth pseudoisotopies of a differentiable manifold M. In [I1], Igusa constructed an exact sequence

$$(1.1) \quad K_3(\mathbb{Z}[\pi_1 M]) \xrightarrow{\chi} Wh_1^+(\pi_1 M; \mathbb{Z}/2 \oplus \pi_2 M) \to \pi_0 \mathcal{P}(M) \xrightarrow{\lambda} Wh_2(\pi_1 M) \to 0$$

in order to analyze the kernel of Hatcher and Wagoner's obstruction λ. He also gave examples to show that χ is nonzero in general, thus disproving Hatcher's claim that $\pi_0\mathcal{P}(M)$ is a direct sum of $Wh_2(\pi_1)$ and $Wh_1^+(\pi_1 M; \mathbb{Z}/2 \oplus \pi_2 M)$.

However, for pseudoisotopy, the important question is how much of the Wh_1^+-term actually survives to $\pi_0\mathcal{P}(M)$ — in other words: what is the cokernel of χ?

It is clear that in general the cokernel can be very large. A simple but striking example is when $\pi_1 M = \mathbb{Z}$. Then $K_3(\mathbb{Z}[\pi_1 M]) \approx \mathbb{Z}/48 \oplus \mathbb{Z}/2$, and $Wh_1^+(\mathbb{Z}; \mathbb{Z}/2)$ is infinitely generated. Therefore $\pi_0\mathcal{P}(M)$ will also be infinitely generated!

But it is hard to say anything in the general case. In particular, it would be interesting to know the answer to the following:

Question 1.2. Is $\pi_0\mathcal{P}(M)$ nontrivial if $\pi_1 M$ is nontrivial?

(Note that $Wh_1^+(\pi_1 M; \mathbb{Z}/2 \oplus \pi_2 M) \neq 0$ if $\pi_1 M \neq 0$, so this would follow immediately if χ were trivial.)

Igusa constructed the sequence (1.1) as an extension of Hatcher and Wagoner's work ([HW], especially part II, correcting the mistakes). Today, the natural approach to pseudoisotopy is through Waldhausen's A-theory and Igusa's stabiity theorem. Then computing $\pi_0\mathcal{P}(M)$ is essentially the same as computing $\pi_2 A(M)$, and (1.1) is the analogue of the fibration sequence one gets by comparing $A(M)$ to $K(\mathbb{Z}[\pi_1 M])$.

In section 2 we set up Igusa's sequence from this point of view. In fact, it turns out that we almost for free get an extension of (1.1) one step to the left as follows:

$$(1.2) \quad \pi_1\mathcal{P}(M) \to Wh_3(\pi_1 M) \xrightarrow{\chi}$$
$$\to Wh_1^+(\pi_1 M; \mathbb{Z}/2 \oplus \pi_2 M) \to \pi_0\mathcal{P}(M) \to Wh_2(\pi_1 M) \to 0\,.$$

($Wh_3(\pi)$ is a certain quotient of $K_3(\mathbb{Z}[\pi])$, see definition 2.6.) Hence the kernel of χ tells us something about $\pi_1\mathcal{P}(M)$. (1.2) was also constructed by Igusa, but only when $\pi_2 M = 0$, and by completely different methods (see [I3]).

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For the proof we need Theorem 2.4, which says that the "monomial" homomorphism $\pi_2^S(B\pi_+) \to K_2(\mathbb{Z}[\pi])$ is (split) injective. This result is certainly known to other people, but since it appears hard to find in the literature, a proof is given in section 3.

The map $Wh_1^+(\pi_1 M; \mathbb{Z}/2 \oplus \pi_2 M) \to \pi_0\mathcal{P}(M)$ comes essentially from the inclusion of '1×1-matrices' into $\widehat{GL}(Q(M))$. It turns out that a large part of its image can be detected using Bökstedt's trace map $A(M) \to THH(M) \simeq \Omega^\infty S^\infty(\Lambda M_+)$. This is the theme of the rest of the paper. The basic constructions and proofs are given in section 4, and the application to $\pi_0\mathcal{P}(M)$ is in section 5. It follows, for example, that Question 1.2 has an affirmative answer whenever $\pi_1(M)$ is finitely generated abelian and has elements of either infinite or even order (Theorem 5.3).

I would like to thank Marcel Bøkstedt for many discussions about the material in section 4, during our very enjoyable stay at the Mittag–Leffler institute during the spring of 1994.

2. Waldhausen's results and Igusa's sequence.

If Y is a topological space, we let Y_+ be Y with an extra basepoint added. Set $\widetilde{Q}(Y) = \Omega^\infty S^\infty(Y)$ and $Q(Y) = \widetilde{Q}(Y_+)$. If X is a connected space, the definition of Waldhausen's algebraic K-theory of X that we shall use is

$$A(X) = \Omega B(\coprod_n B\widehat{GL}_n(Q(GX)))$$

('group completion with respect to direct sum of matrices'), where GX is a topological group model for the loop space ΩX ([W1]).

One of the main results of the theory is that

$$A(X) \simeq Wh^{\mathrm{DIFF}}(X) \times Q(X), \tag{2.1}$$

where $Wh^{\mathrm{DIFF}}(X)$ is a space, functorial in X, such that if X is a manifold, then $\pi_{k+2}Wh^{\mathrm{DIFF}}(X) \approx \pi_k(\mathcal{P}(X \times I^l))$ for l large. The map from $Q(X)$ to $A(X)$ is given by inclusion of the permutation matrices with entries in $GX \subset Q(GX)$ into $\widehat{GL}(Q(X))$, and then using the Barratt-Priddy-Quillen-Segal theorem:

$$Q(X) \simeq \Omega B(\coprod B(\Sigma_q \wr GX) \to \Omega B(\coprod_n B\widehat{GL}_n(Q(GX))) = A(X)$$

From now on, we set $\pi = \pi_1 X$. Let $K(R) = \Omega B(\coprod_n BGL_n(R))$ be the (free module) K-theory space of a ring R, and denote by $\mathcal{F}$ be the homotopy fiber of the "linearization" map $A(X) \to K(\mathbb{Z}[\pi])$. Then there is a composed mapping $Q(B\pi) \to A(B\pi) \to K(\mathbb{Z}[\pi])$, and we have a map of fibrations up to homotopy

$$\begin{array}{ccccc} \Omega\widetilde{Q}(B\pi/X) & \longrightarrow & Q(X) & \longrightarrow & Q(B\pi) \\ \big\downarrow{\scriptstyle \phi_{rel}} & & \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \phi} \\ \mathcal{F} & \longrightarrow & A(X) & \longrightarrow & K(\mathbb{Z}[\pi]), \end{array} \tag{2.2}$$

where $B\pi/X$ is the homotopy cofibre of $X \to B\pi$.

Recall that $Wh_2(\pi) = \operatorname{coker}(\pi_2^S(B\pi_+) \to K_2(\mathbb{Z}[\pi])$, and define (for the moment) $W_3(\pi) = \operatorname{coker}(\pi_3^S(B\pi_+) \to K_3(\mathbb{Z}[\pi]))$. Then we have the following commutative diagram, with the two leftmost columns and all rows exact:

$$\begin{tikzcd}
0 \arrow[r] & \pi_3^S(X_+) \arrow[r] \arrow[d] & \pi_3 A(X) \arrow[r] \arrow[d] & \pi_3 Wh^{\mathrm{DIFF}}(X) \arrow[r] \arrow[d] & 0 \\
 & \pi_3^S(B\pi_+) \arrow[r] \arrow[d] & K_3(\mathbb{Z}[\pi]) \arrow[r] \arrow[d, "\delta"] & W_3(\pi) \arrow[r] \arrow[d, "\delta"] & 0 \\
 & \pi_3^S(B\pi/X) \arrow[r, "\phi_{rel}"] \arrow[d] & \pi_2\mathcal{F} \arrow[r] \arrow[d, "\alpha"] & \operatorname{coker}(\phi_{rel}) \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & \pi_2^S(X_+) \arrow[r, "\phi"] \arrow[d, "\gamma"'] & \pi_2 A(X) \arrow[r] \arrow[d, "\beta"] & \pi_2 Wh^{\mathrm{DIFF}}(X) \arrow[r] \arrow[d, "\lambda"] & 0 \\
 & \pi_2^S(B\pi_+) \arrow[r, "\phi_\pi"] \arrow[d] & K_2(\mathbb{Z}[\pi]) \arrow[r] \arrow[d] & Wh_2(\pi) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

(The surjectivity of β follows from [W1], Prop. 1.1.) We shall refer to this as the "main diagram".

The group $\pi_2\mathcal{F}$ may be determined by [W1, Prop. 1.2.]. Let $Q_1(GX) \subset Q(GX)$ denote the component of the map $S^n = S^n \wedge 1_+ \subset S^n \wedge (GX_+)$. $Q_1(GX)$ is a submonoid of $\widehat{GL}_1(Q(GX))$, and we obtain an induced map $BQ_1(GX) \to A(X)$ which clearly lifts to a map $BQ_1(GX) \to \mathcal{F}$. A simple calculation with the Atiyah-Hirzebruch spectral sequence gives

$$\pi_2 BQ_1(GX) \approx \pi_1^S(GX_+) \approx H_1(GX; \pi_0^S) \oplus H_0(GX; \pi_1^S) \approx (\pi_2 X)[\pi] \oplus (\mathbb{Z}/2)[\pi].$$

This has an obvious action of π induced by conjugation on itself and the standard action on $\pi_2 X$.

Waldhausen's result can then be formulated as

$$\begin{aligned}\pi_2\mathcal{F} &\approx \pi_2 BQ_1(GX)/(xz - zx) \quad \text{where } x \in \pi_1^S(GX_+),\ z \in \mathbb{Z}[\pi] \\ &\approx H_0(\pi; (\pi_2 X \oplus \mathbb{Z}/2)[\pi]).\end{aligned}$$

Lemma 2.3. $\pi_3^S(B\pi/X) \approx H_0(\pi; \pi_2 X)$, *and* ϕ_{rel} *is the inclusion of the summand* $H_0(\pi; (\pi_2 X[1]))$.

Proof. $B\pi/X$ is 2-connected, hence we have isomorphisms

$$\pi_3^S(B\pi/X) \approx \widetilde{H}_3(B\pi/X) \approx H_3(B\pi, X),$$

and by the Hurewicz theorem the last group is isomorphic to the quotient of $\pi_3(B\pi, X)$ by the action of $\pi_1 X = \pi$, i.e. $H_0(\pi; \pi_3(B\pi, X))$. But $\pi_3(B\pi, X) \approx \pi_2 X$.

To prove the statement about ϕ_{rel} we consider the diagram

$$\begin{tikzcd} G_0X \arrow[r, "\subset"] \arrow[d, "\subset"] & Q_1(GX) \arrow[d, "\subset"] \\ GX \arrow[r, "\subset"] \arrow[d] & \widehat{GL}_1(Q(GX)) \arrow[d] \\ \pi \arrow[r, "\subset"] & GL_1(\mathbb{Z}[\pi]) \end{tikzcd}$$

where G_0X is the component of the trivial loop; i.e. $G_0X \simeq \Omega\widetilde{X}$. Taking classifying spaces, this maps into diagram (2.2). But then we see that both $\pi_2\Omega Q(B\pi/X)$ and the '1'–component of $\pi_2\mathcal{F}$ are identified with the same quotient of $\pi_2 BG_0X = \pi_2\widetilde{X} \approx \pi_2 X$. $\square$

The following theorem ought to be well known, but seems hard to find in the literature. Since the result has independent interest, a proof is given in section 3.

Theorem 2.4. *$\phi_\pi : \pi_2^S(B\pi_+) \to K_2(\mathbb{Z}[\pi])$ is split injective.*

By standard diagram chasing (or, consider the main diagram as a short exact sequence of complexes and take homology) we then get

Corollary 2.5. *All three columns in the main diagram are exact, and*

$$\operatorname{coker}(\phi_{rel}) \approx H_0(\pi; (\pi_2 X \oplus \mathbb{Z}/2)[\pi])/H_0(\pi; \pi_2 X)$$

However, we can do slightly better than this by observing that since the main diagram is functorial, the diagram for $X =$ one point sits inside as a direct summand. In $\pi_2\mathcal{F}$ this is $H_0(1; \mathbb{Z}/2[1]) \approx \mathbb{Z}/2$, and $\delta : K_3(\mathbb{Z}) \to \mathbb{Z}/2$ is surjective (See e.g. [W3], Cor. 3.7. + remark). Hence it induces an isomorphism $\delta : W_3(1) \to \mathbb{Z}/2$.

Definition 2.6.

(1) $Wh_3(\pi) = K_3(\mathbb{Z}[\pi])/(K_3(\mathbb{Z}) + \operatorname{im} \pi_3^S(B\pi_+))$
(2) $Wh_1^+(\pi_1 X; \mathbb{Z}/2 \oplus \pi_2 X) = H_0(\pi; (\pi_2 X \oplus \mathbb{Z}/2)[\pi])/H_0(\pi; (\pi_2 X \oplus \mathbb{Z}/2)[1])$

Corollary 2.7 (Igusa's exact sequence). *There is a functorial exact sequence*

$$\pi_3 Wh^{\mathrm{DIFF}}(X) \to Wh_3(\pi_1 X) \xrightarrow{\chi} \\ \to Wh_1^+(\pi_1 X; \mathbb{Z}/2 \oplus \pi_2 X) \to \pi_2 Wh^{\mathrm{DIFF}}(X) \to Wh_2(\pi_1 X) \to 0 .$$

3. Proof of theorem 2.4.

Let $\widetilde{K}_*(\mathbb{Z}[\pi]) = \ker(K_*(\mathbb{Z}[\pi]) \to K_*(\mathbb{Z}))$ be reduced K-theory. Since $\pi_2^S \approx K_2(\mathbb{Z})$, it suffices to prove that $\pi_2^S(B\pi) \to \widetilde{K}_2(\mathbb{Z}[\pi])$ is a split injection.

We first compute $\pi_2^S(B\pi)$.

Lemma 3.1. *For connected X there is a natural, split exact sequence*

$$0 \to H_1(X;\mathbb{Z}/2) \to \pi_2^S(X) \to H_2(X) \to 0 \tag{3.1.1}$$

Proof. Consider the Atiyah–Hirzebruch spectral sequence for $\pi_*^S(X)$. The part of the E^2-term involved in $\pi_2^S(X)$ is

$$\begin{array}{llll}
\widetilde{H}_0(X;\pi_2^S) = 0 & & & \\
\widetilde{H}_0(X;\pi_1^S) = 0 & \widetilde{H}_1(X;\pi_1^S) & & \\
\widetilde{H}_0(X;\pi_0^S) = 0 & \widetilde{H}_1(X;\pi_0^S) & \widetilde{H}_2(X;\pi_0^S) & \widetilde{H}_3(X;\pi_0^S)
\end{array}$$

The only possible nonzero differential here is

$$d_2 = d_2^X : H_3(X) = \widetilde{H}_3(X;\pi_0^S) \to \widetilde{H}_1(X;\pi_1^S) = H_1(X;\mathbb{Z}/2),$$

which sits in an exact sequence

$$\pi_3^S(X) \xrightarrow{\eta} H_3(X) \xrightarrow{d_2} H_1(X;\mathbb{Z}/2) \to \pi_2^S(X) \to H_2(X) \to 0. \tag{3.1.2}$$

First observe that η is surjective. This follows from the fact that $\pi_3^S(X_+) \approx \Omega_3^{fr}(X)$, since H_3 is representable by orientable manifolds, and all orientable 3-manifolds are parallelizable. Hence d_2 must be trivial.

We therefore have the exact sequence in (3.1.1), and it remains to prove that it splits.

For this, we use naturality of the sequence. Recall that $H_1(X;\mathbb{Z}/2) \approx H_1(\pi;\mathbb{Z}/2) \approx \pi/[\pi,\pi] \otimes \mathbb{Z}/2$, which we shall denote by $\pi/2$. Observe also that for $\pi = \mathbb{Z}/2$ the exact sequence reduces to an isomorphism $H_1(\mathbb{Z}/2;\mathbb{Z}/2) \approx \pi_2^S(B(\mathbb{Z}/2))$.

By naturality we then get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1(X;\mathbb{Z}/2) & \longrightarrow & \pi_2^S(X) & \longrightarrow & H_2(X) & \longrightarrow & 0 \\
 & & \approx\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_1(\pi;\mathbb{Z}/2) & \longrightarrow & \pi_2^S(B\pi) & \longrightarrow & H_2(\pi) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Pi\phi_*} & & \downarrow{\scriptstyle \Pi B\phi_*} & & \downarrow & & \\
0 & \longrightarrow & \Pi_\phi H_1(\mathbb{Z}/2;\mathbb{Z}/2) & \xrightarrow{\approx} & \Pi_\phi \pi_2^S(B(\mathbb{Z}/2)) & \longrightarrow & 0 & &
\end{array}$$

where the product is over all homomorphisms $\phi : \pi \to \mathbb{Z}/2$. Each such ϕ factors through $\phi' : \pi/2 \to \mathbb{Z}/2$, which may be identified with ϕ_* via the natural isomorphisms $H_1(\pi;\mathbb{Z}/2) \approx \pi/2$ and $H_1(\mathbb{Z};\mathbb{Z}/2) \approx \mathbb{Z}/2$. Hence ϕ_* is surjective when ϕ is nontrivial. It then follows that the bottom isomorphism identifies the images of $\Pi\phi_*$ and $\Pi B\phi_*$. Since $\Pi\phi_*$ is injective, we obtain a canonical splitting. □

It is now easy to extend to the unreduced theory and arbitrary X, and we record for later use:

Theorem 3.2. *We have a natural isomorphism*

$$\begin{aligned}\pi_2^S(X_+) &\approx H_2(X;\pi_0^S)\oplus H_1(X;\pi_1^S)\oplus H_0(X;\pi_2^S)\\ &\approx H_2(X;\mathbb{Z})\oplus H_1(X;\mathbb{Z}/2)\oplus H_0(X;\mathbb{Z}/2).\end{aligned}$$

Terminology. Observe that the map $\pi_2^S(X)\to H_2(X)$ in (3.1.1) can be thought of as evaluation on a fundamental class. For convenience we shall generally use the name *Hurewicz homomorphism* for such homomorphisms. The main examples are maps like $\pi_k(X)\to\pi_k^S(X_+)\to H_k(X)$.

Next we want to compare the maps in (3.1.1) with K-theory. First we study the composition $\phi\circ\iota: H_1(\pi;\mathbb{Z}/2)\approx\pi/2\xrightarrow{\iota}\pi_2^S(B\pi)\to\widetilde{K}_2(\mathbb{Z}[\pi])$:

Lemma 3.3. *$\phi\circ\iota$ injects onto a direct summand.*

Proof. Again we exploit the functoriality. Just as in the proof of the splitting in Lemma 3.1, let $\phi:\pi\to\mathbb{Z}/2$ be a homomorphism, and consider the commutative diagram

$$\begin{array}{ccccc}\pi/2 & \longrightarrow & \pi_2^S(B\pi) & \longrightarrow & \widetilde{K}_2(\mathbb{Z}[\pi])\\ \downarrow{\scriptstyle \phi'} & & \downarrow & & \downarrow\\ (\mathbb{Z}/2) & \xrightarrow{\approx} & \pi_2^S(B(\mathbb{Z}/2)) & \longrightarrow & \widetilde{K}_2(\mathbb{Z}[\mathbb{Z}/2]).\end{array}$$

$\pi_2^S(B(\mathbb{Z}/2))\to\widetilde{K}_2(\mathbb{Z}[\mathbb{Z}/2])$ is an isomorphism by [D], so taking product over all ϕ and arguing as in lemma 3.1, we again get a (natural) splitting. □

To complete the proof of theorem 2.4, we use the Dennis trace map. Recall that this is a homomorphism $D: K_n(R)\to HH_n(R,R)$ for every n, where HH_* is Hochschild homology. If R is a group ring $\mathbb{Z}[\pi]$, we have a homomorphism $HH_n(\mathbb{Z}[\pi],\mathbb{Z}[\pi])\to HH_n(\mathbb{Z}[\pi],\mathbb{Z})\approx H_n(B\pi)$, induced by the augmentation. (Note that this is not the usual map — in particular it does not commute with the cyclic action. However, we also have $HH_n(\mathbb{Z}[\pi],\mathbb{Z}[\pi])\approx H_n(\Lambda B\pi)$, where $\Lambda B\pi$ is the free loop space on $B\pi$, and the augmentation corresponds to the map to $H_n(B\pi)$ induced by evaluation in a basepoint in S^1. We could, in fact, also have used the usual map in the following, but I find it more convenient to use the augmentation.)

Lemma 3.4. *The composition*

$$\pi_n^S(B\pi_+)\xrightarrow{\phi_\pi}K_n(\mathbb{Z}[\pi])\xrightarrow{D}HH_n(\mathbb{Z}[\pi],\mathbb{Z}[\pi])\to H_n(B\pi)$$

is the Hurewicz homomorphism.

Remark. In [W2], appendix, Waldhausen states a similar result as the K-theory analogy of the splitting of the map $Q(X)\to A(X)$. He uses a different map, but the equivalence between the two approaches follows from [DM]. The following proof is much more direct and explicit.

Proof of Lemma 3.4. If Y is a simplicial space, let $\mathbb{Z}Y$ be the simplicial abelian group generated by Y. Then $\mathbb{Z}Y\simeq\Omega B(\coprod Y^q/\Sigma_q)$ and $Q(B\pi)\simeq\Omega B(\coprod B(\Sigma_q\wr\pi))$, and the Hurewicz homomorphism is determined by the natural maps

$$\psi: B(\Sigma_q\wr\pi)\approx E\Sigma_q\times_{\Sigma_q}(B\pi)^q\to(B\pi)^q/\Sigma_q$$

In fact, ψ extends to a homomorphism $\mathbb{Z}\psi : \mathbb{Z}B(\Sigma_q \wr \pi) \to \mathbb{Z}B\pi$ such that the Hurewicz homomorphism $\pi_n^S(B\pi_+) \to H_n(B\pi)$ for q large factors as

$$\pi_n^S(B\pi_+) \approx \pi_n(Q(B\pi)) \to H_n(Q(B\pi)) \approx H_n(B(\Sigma_q \wr \pi)) \xrightarrow{\mathbb{Z}\psi_*} H_n(B\pi).$$

We write the elements in $\Sigma_q \wr \pi$ as $\sigma \cdot g$, where $\sigma \in \Sigma_q$ and $g = (g_1, \dots, g_q) \in \pi^q$. Then $g \cdot \sigma = \sigma \cdot \sigma^{-1}(g) = \sigma \cdot (g_{\sigma(1)}, \dots g_{\sigma(q)})$. From the explicit identification of $B(\Sigma_q \wr \pi)$ with $E\Sigma_q \times_{\Sigma_q} (B\pi)^q$ in [M], p.278, we then see that

$$\psi(\sigma^1 \cdot g^1, \dots, \sigma^k \cdot g^k) = \sum_i (g^1_{\sigma^2 \dots \sigma^k(i)}, \dots, g^{k-1}_{\sigma^k(i)}, g^k_i).$$

ϕ_π is induced by the homomorphism $\Sigma_q \wr \pi \to BGL_q(\mathbb{Z}[\pi])$ taking $\sigma \cdot g$ to the permutation matrix σg with

$$(\sigma g)_{i,j} = \begin{cases} g_j & \text{if } i = \sigma(j) \\ 0 & \text{if } i \neq \sigma(j) \end{cases} \tag{3.4.1}$$

D is given by the Hurewicz homomorphism $K_n(\mathbb{Z}[\pi]) \to H_n(GL(\mathbb{Z}[\pi])$ followed by the homomorphism given on generators for the chain complex for $H_n(GL_q(\mathbb{Z}[\pi])$ by

$$(G^1, \dots, G^n) \to \sum_{i_0, \dots, i_n} (G^1 G^2 \cdots G^n)^{-1}_{i_0, i_1} \otimes G^1_{i_1, i_2} \otimes \cdots \otimes G^n_{i_n, i_0} \in \mathbb{Z}[\pi]^{\otimes^{n+1}}$$

in the Hochschild complex $C_*(\mathbb{Z}[\pi], \mathbb{Z}[\pi])$.

The last map $HH_n(\mathbb{Z}[\pi], \mathbb{Z}[\pi]) \to HH_n(\mathbb{Z}[\pi], \mathbb{Z}) \approx H_n(B\pi)$ can be thought of as being induced by the projection $\pi^{n+1} \to \pi^n$ onto the last n factors.

The commutative diagram

$$\begin{array}{ccc} \pi_n(B(\Sigma_q \wr \pi)^+) & \longrightarrow & \pi_n(BGL_q(\mathbb{Z}[\pi])^+) \\ \downarrow & & \downarrow \\ H_n(\Sigma_q \wr \pi) & \longrightarrow & H_n(GL_q(\mathbb{Z}[\pi])) \end{array}$$

now shows that what we need to do is to identify the composition

$$H_n(\Sigma_q \wr \pi) \to H_n(GL_q(\mathbb{Z}[\pi])) \to H_n(B\pi)$$

with the homomorphism $\mathbb{Z}\psi_*$ defined above.

The composition $D \circ \phi_\pi$ is given on k-simplexes by

$$(\sigma^1 \cdot g^1, \dots, \sigma^k \cdot g^k) \mapsto \sum (\sigma^1 g^1 \cdots \sigma^k g^k)^{-1}_{i_0, i_1} \otimes (\sigma^1 g^1)_{i_1, i_2} \otimes \cdots \otimes (\sigma^k g^k)_{i_k, i_0}$$

The conditions (3.4.1) now imply that these terms vanish unless $i_k = \sigma(i_0)$ and $i_j = \sigma(i_{j+1})$ for $1 \leq j \leq k-1$. Therefore the result can be written

$$\sum_i (\sigma^1 g^1 \cdots \sigma^k g^k)^{-1}_{i, \sigma^1 \dots \sigma^k(i)} \otimes g^1_{\sigma^2 \dots \sigma^k(i)} \otimes \cdots \otimes g^{k-1}_{\sigma^k(i)} \otimes g^k_i.$$

To see what these terms are mapped to in $C_k(\mathbb{Z}[\pi],\mathbb{Z})$, we have to analyze $(\sigma^1 g^1\cdots\sigma^k g^k)^{-1}_{i,\sigma^1\cdots\sigma^k(i)}$. It is clear that $\Pi=\sigma^1 g^1\cdots\sigma^k g^k$ is a permutation that can be written as $\sigma^1\sigma^2\cdots\sigma^k h=\sigma h$, with $h\in\pi^n$. But then $\Pi^{-1}_{i,\sigma(i)}=(\sigma h^{-1})_{\sigma(i),i}=h_i^{-1}\in\pi$. Hence the image in $C_k(\mathbb{Z}[\pi],\mathbb{Z})$ is $\sum_i g^1_{\sigma^2\cdots\sigma^k(i)}\otimes\cdots\otimes g^{k-1}_{\sigma^k(i)}\otimes g^k_i$.

But this is precisely the formula for the Hurewicz homomorphism above. □

Proof of Theorem 2.4. Let ρ be a left inverse of $\phi\circ\iota$ as provided by Lemma 3.3. then $h:\pi_2^S(B\pi)\to H_2(\pi)$ has a splitting (right inverse) f which is well defined by the formula $f(y)=\hat{y}-\iota\rho\phi(\hat{y})$, where $h(\hat{y})=y$.

Let D' be the homomorphism $K_2(\mathbb{Z}[\pi])\to H_2(\pi)$ from Lemma 3.4. Then we can define a left inverse g of ϕ by

$$g(z)=\iota\rho(z)+fD'(z).$$

If $z=\phi(x)$, we can chose $\widehat{D'(z)}=x$ by Lemma 3.4. Therefore

$$g(\phi(x))=\iota\rho\phi(x)+x-\iota\rho\phi(x)=x.\quad \square$$

Remark. Lemma 3.4 can obviously be used to construct nontrivial elements of $K_n(\mathbb{Z}[\pi])$. For example, the surjectivity of $\pi_3^S(B\pi_+)\to H_3(\pi)$ implies:

Corollary 3.5. *The homomorphism $D':K_3(\mathbb{Z}[\pi])\to H_3(\pi)$ is surjective.* □

Take for example $\pi=\mathbb{Z}/n$. Then it follows that there is a surjective homomorphism

$$K_3(\mathbb{Z}[\mathbb{Z}/n])\to\mathbb{Z}/n.$$

(The analogous theorem for K_2 is also true, but (2.4) is stronger).

A similar application of Lemma 3.4 to higher K-theory is to a result of Dennis, saying that if π is a group with vanishing homology below dimension n, then $D':K_n(\mathbb{Z}[\pi])\to H_n(B\pi)$ is surjective ([I1]). (For $n=3$, this is a special case of Corollary 3.5.) Dennis' result follows because $\pi_n^S(B\pi_+)\to H_n(B\pi)$ in that case will be surjective by the Atiyah-Hirzebruch spectral sequence.

4. Bøkstedt's trace on $\pi_2\mathcal{F}$.

From the formulation of Waldhausen's calculation of $\pi_2\mathcal{F}$ section 2, it follows that we can think of the homomorphism $\pi_2\mathcal{F}\to\pi_2 A(X)$ as induced by the inclusion

$$BQ_1(GX)\to B\widehat{GL}_1(Q(GX))\to A(X).$$

Using the group completion model for $A(X)$, it is natural to think of this as lying in the 1-component $A_1(X)$.

Recall now that Bökstedt has defined a generalization of Hochschild homology and the Dennis trace which in the case of $A(X)$ gives an infinite loop map

$$\tau:A(X)\to Q(\Lambda X),$$

where ΛX is the free loop space $\mathrm{Map}(\mathrm{S}^1, X)$ of X. (See [B], [BHM].) In this section we shall compute the composition

$$\pi_2 BQ_1(GX) \to \pi_2 A_1(X) \to \pi_2 Q_1(\Lambda X).$$

($Q_1(\Lambda X)$ is the component of $\mathrm{S}^n = \mathrm{S}^n \wedge (*_+) \subset \mathrm{S}^n \wedge (\Lambda X_+)$, where the basepoint in ΛX is the trivial loop at the basepoint of X.)

From now on we adjust the notation slightly, and write $G = GX$, $X \simeq BG$. Then Bökstedt's trace is induced by composition of maps of the homotopy type of

$$B\widehat{GL}_n(QG) \xrightarrow{\iota} \Lambda B\widehat{GL}_n(QG) \xrightarrow{\tau_n'} Q(\Lambda BG)$$

where ι as before is the inclusion of the constant loops. τ_n' is a more complicated construction, involving Morita equivalence (for $n > 1$) and replacing (stably) compositions with smash products. Taking restriction to $BQ_1G \subset B\widehat{GL}_1(QG)$), we see that the maps we need to compute are

$$BQ_1G \xrightarrow{\iota} \Lambda BQ_1G \xrightarrow{\tau_1'} Q_1(\Lambda BG).$$

Hence Morita equivalence does not enter, and it turns out that the crucial property of τ_1' for our computation is that it is equivariant with respect to the natural S^1–actions. Thus we have a diagram

$$\begin{array}{ccc}
\mathrm{S}^1 \times Q_1G & = & \mathrm{S}^1 \times Q_1G \\
\downarrow & & \downarrow \\
\mathrm{S}^1 \times \Lambda BQ_1G & \xrightarrow{\mathrm{S}^1 \times \tau_1'} & \mathrm{S}^1 \times Q_1(\Lambda BG) \\
\downarrow & & \downarrow \\
\Lambda BQ_1G & \xrightarrow{\tau_1'} & Q_1(\Lambda BG)
\end{array}$$

(The top maps are induced by the natural inclusions $Q_1G \subset \Omega BQ_1G \subset \Lambda BQ_1G$ and $G \subset \Omega BG \subset \Lambda BG$.) In fact, both actions leave the basepoints fixed, so we have

$$\begin{array}{ccc}
\mathrm{S}^1_+ \wedge Q_1G & = & \mathrm{S}^1_+ \wedge Q_1G \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \bar{\mu}} \\
\Lambda BQ_1G & \xrightarrow{\tau_1'} & Q_1(\Lambda BG)
\end{array} \qquad (4.1)$$

We now calculate the homomorphisms induced by μ on π_2. Observe that for any monoid Y we have a diagram like (4.1) with Q_1G replaced by Y.

Lemma 4.2. *Let Y be a connected monoid, and let $I[\pi_1 Y]$ be the augmentation ideal in $\mathbb{Z}[\pi_1 Y]$, with its additive structure. Then there are isomorphisms*

$$\pi_2(S^1_+ \wedge Y) \approx \pi_2 Y \oplus I[\pi_1 Y] \tag{4.2.1}$$

$$\pi_n(\Lambda BY) \approx \pi_n Y \oplus \pi_{n-1} Y \quad \text{for } n \geq 1 \tag{4.2.2}$$

and μ induces the obvious homomorphism for $n = 2$ (note that $\pi_1 Y$ is abelian).

Proof. Since $\mathrm{S}^1_+ \wedge Y = \mathrm{S}^1 \times Y/\mathrm{S}^1 \times * \simeq \mathrm{S}^1 \times Y \cup D^2 \times *$, it is not hard to see that the universal covering $\widetilde{\mathrm{S}^1_+ \wedge Y}$ is weakly equivalent to $\mathrm{S}^1 \times \widetilde{Y} \cup D^2 \times p^{-1}(*)$, where $p : \widetilde{Y} \to Y$ is the universal covering space of Y. (4.2.1) then follows from the Hurewicz isomorphism $\pi_2(\mathrm{S}^1_+ \wedge Y) \approx H_2(\widetilde{\mathrm{S}^1_+ \wedge Y})$.

The inclusion of the summand $\pi_2 Y$ is induced by the inclusion $Y \subset \mathrm{S}^1_+ \wedge Y$, and $I[\pi_1 Y] \to \pi_2(\mathrm{S}^1_+ \wedge Y)$ is the restriction of the (additive) homomorphism γ from $\mathbb{Z}[\pi_1 Y]$ taking the free generator $g : \mathrm{S}^1 \to Y$ to the composition

$$g' : \mathrm{S}^2 \to \mathrm{S}^1_+ \wedge \mathrm{S}^1 \xrightarrow{1 \wedge g} \mathrm{S}^1_+ \wedge Y .$$

(Note that $\mathrm{S}^1_+ \wedge \mathrm{S}^1$ can be identified with S^2 with two points identified.)

(4.2.2) is standard, and comes from homotopy sequence of the fibration $\Omega BY \to \Lambda BY \xrightarrow{\rho} BY$ where ρ is evaluation in $1 \in \mathrm{S}^1$ and the section ι defined above (taking $x \in BY$ to the constant path in x). Here is an explicit description of the projection $\pi_n(\Lambda BY) \to \pi_n Y$:

Let $f : \mathrm{S}^n \to \Lambda BY$ represent an element in $\pi_n(\Lambda BY)$. f has an adjoint map $\mathrm{S}^1_+ \wedge \mathrm{S}^n \to BY$, which may be composed with $\mathrm{S}^{n+1} \to \mathrm{S}^1_+ \wedge \mathrm{S}^n$ to produce an element in $\pi_{n+1} BY \approx \pi_n Y$.

Let $\nu : \Sigma Y \to BY$ be the adjoint of $Y \simeq \Omega BY$ (or inclusion of '1-skeleton'). In order to calculate μ_* we use the diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "\simeq"] & \mathrm{S}^1_+ \wedge Y \arrow[r] \arrow[d, "\mu"] & \Sigma Y \arrow[d, "\nu"] \\
\Omega BY \arrow[r] & \Lambda BY \arrow[r, "\rho"] & BY
\end{tikzcd}$$

which is easily seen to be commutative. It follows that $\rho \circ \mu \circ g'$ is equal to the composition

$$\mathrm{S}^2 \xrightarrow{g'} \mathrm{S}^1_+ \wedge Y \to \Sigma Y \xrightarrow{\nu} BY ,$$

which corresponds precisely to g via the isomorphism $\pi_2 BY \approx \pi_1 Y$.

It remains to prove that the $\pi_2 Y$-component of $\mu_*[g]$ is trivial for every $[g] \in \pi_1 Y$. Think of g as a map $\mathrm{S}^1 \to \Omega BY$. By the description above, the component of $\mu_*[g]$ in $\pi_2 Y \approx \pi_3 BY$ is given by the composition

$$\mathrm{S}^3 \to \mathrm{S}^1_+ \wedge \mathrm{S}^2 \to \mathrm{S}^1_+ \wedge \mathrm{S}^1_+ \wedge \mathrm{S}^1 \xrightarrow{1 \wedge g} \mathrm{S}^1_+ \wedge \mathrm{S}^1_+ \wedge \Omega BY \xrightarrow{(s,t,\omega) \mapsto \omega(st)} BY.$$

This is easily seen to factor through the composition

$$\mathrm{S}^3 \to \mathrm{S}^1_+ \wedge \mathrm{S}^2 \to \mathrm{S}^1_+ \wedge \mathrm{S}^1_+ \wedge \mathrm{S}^1 \to \mathrm{S}^1_+ \wedge \mathrm{S}^1, \tag{4.2.3}$$

where the last map is the suspension of the product map $\mathrm{S}^1 \times \mathrm{S}^1_+ \to \mathrm{S}^1_+$. But this composition is null homotopic; the product map is equivalent to projection to one factor, and then (4.2.3) is the inclusion of one wedge summand and projection to another in $\mathrm{S}^1_+ \wedge \mathrm{S}^1_+ \wedge \mathrm{S}^1 \simeq \mathrm{S}^3 \vee \mathrm{S}^2 \vee \mathrm{S}^1_+ \wedge \mathrm{S}^1$. □

Let now $\alpha \in \pi_2 BQ_1G$ be represented by $\bar{\alpha} \in \pi_1 Q_1 G$. By lemma 4.2 we may write $\iota_*(\alpha) = \mu_*\gamma(\beta - 1)$, where $\beta \in \pi_1 Q_1 G$, and hence $\tau_*(\alpha) = \bar{\mu}_*\gamma(\beta - 1) = \bar{\mu}_*\gamma(\beta)$. We therefore need to calculate the composition

$$\mathbb{Z}[\pi_1 Q_1 G] \xrightarrow{\gamma} \pi_2(\mathrm{S}^1_+ \wedge Q_1 G) \xrightarrow{\bar{\mu}_*} \pi_2(Q_1(\Lambda BQ_1 G)).$$

Note that $\bar{\mu}$ is the composition

$$\mathrm{S}^1_+ \wedge Q_1 G \to Q_1(\mathrm{S}^1_+ \wedge G) \xrightarrow{Q_1(\mu)} Q_1(\Lambda BG),$$

where the first map is defined by $(t, f : \mathrm{S}^n \to \mathrm{S}^n \wedge (G_+)) \mapsto (f : \mathrm{S}^n \to \mathrm{S}^n \wedge (t \times G)_+)$. (This makes sense if the basepoint in $Q_1 G$ is the 'identity', represented by maps $\mathrm{S}^n = \mathrm{S}^n \wedge (1_+)$, where 1 is the basepoint (unit) in G.) Therefore $\bar{\mu}_*\gamma = Q(\mu)_*\gamma'$, where γ' is the map $\mathbb{Z}\pi_1 Q_1 G \to \pi_2 \mathrm{S}^1_+ \wedge Q_1 G \to \pi_2 Q_1(\mathrm{S}^1_+ \wedge G$.

For any space, $\pi_i Q_1 Y \approx \pi_i^S(Y_+)$. For $i = 2$ this was computed in Proposition 3.2, and it is also easy to see that $\pi_1^S(Y_+) \approx H_1(Y; \pi_0^S) \oplus H_0(Y; \pi_1^S)$.

Our computation of τ_* can now be formulated in the following theorem:

Theorem 4.3. *$\tau_* : \pi_2 BQ_1 G \to \pi_2 Q_1(\Lambda BG)$ is the vertical composition in the following diagram:*

$$\begin{tikzcd}
\pi_2 BQ_1 G \arrow[d, "\approx"] & \\
\pi_1 Q_1 G \arrow[r, "\approx"] \arrow[d, "\gamma'"] & H_1(G;\mathbb{Z}) \oplus H_0(G;\mathbb{Z}/2) \arrow[d, "\gamma'"] \\
\pi_2 Q_1(S^1_+ \wedge G) \arrow[r, "\approx"] \arrow[d, "Q\mu_*"] & H_2(S^1_+ \wedge G;\mathbb{Z}) \oplus H_1(S^1_+ \wedge G;\mathbb{Z}/2) \oplus H_0(S^1_+ \wedge G;\mathbb{Z}/2) \arrow[d, "\mu_*"] \\
\pi_2 Q_1(\Lambda BG) \arrow[r, "\approx"] & H_2(\Lambda BG;\mathbb{Z}) \oplus H_1(\Lambda BG;\mathbb{Z}/2) \oplus H_0(\Lambda BG;\mathbb{Z}/2)
\end{tikzcd}$$

γ' is given by cross product with the fundamental glass of S^1.

Proof. The only statement that needs proof, is the identification of γ'. The simplest way to see this, is to use the isomorphisms $\pi_i Q_1 Y \approx \Omega_i^{fr}(Y)$. Then γ' corresponds to the map $\Omega_1^{fr}(G) \to \Omega_2^{fr}(\mathrm{S}^1_+ \wedge G)$ given by $(f : M \to G) \mapsto (\mathrm{S}^1 \times f : \mathrm{S}^1 \times M \to \mathrm{S}^1_+ \wedge M)$. But this corresponds precisely to the cross product in homology. □

Remark. It follows that γ' is a homomorphism, even though it involves the lifting $(g \in \pi_1 Q_1 G) \mapsto (1 \cdot g \in \mathbb{Z}[\pi_1 Q_1 G])$, which is certainly *not* a homomorphism!

Here are some explicit computations. First we consider the case where G is a discrete group. Let $<G>$ be the set of representatives of the conjugacy classes of G, and let $C(g)$ be the centralizer of the group element g. Then there is a well known homotopy equivalence

$$\Lambda BG \simeq \coprod_{g \in <G>} BC(g)\,. \tag{5.1}$$

Hence $\pi_2 BQ_1 G \approx H_0(G;\mathbb{Z}/2) \approx \mathbb{Z}/2[G]$ (additively), and according to Theorem 4.3, τ_* maps into $H_1(\Lambda BG;\mathbb{Z}/2) \approx \oplus_{g \in <G>} C(g)/2$.

Lemma 4.4. *The resulting $\tau_* : \mathbb{Z}/2[G] \to \oplus_{g \in <G>} C(g)/2$ maps the generator $1 \cdot h$ to $[g] \in C(g)/2$, if h is in the conjugacy class of $g \in <G>$.*

Proof. Choose representatives $\bar{g} \in \Omega BG \subset \Lambda BG$ for all $g \in <G>$, and let $\Lambda_g BG \subset \Lambda BG$ be the component of $\bar{g}$ (with $\bar{g}$ as basepoint). Then $\rho : \Lambda_g BG \to BG$ is equivalent to $BC(g) \to BG$, inducing the inclusion on π_1.

The homomorphism γ' in Theorem 4.3 takes $h \in G \subset \Omega BG$ to the homology class of the map $h' : \mathrm{S}^1 \to \Lambda BG$ defined by $h'(s)(t) = h(st)$. If h is conjugate to $\bar{g}$, then $[h'] = [\bar{g}]$ in $H_1(\Lambda BG;\mathbb{Z}/2)$. The result now follows from the diagram

$$\begin{array}{ccc} \pi_1(\Lambda_g BG) & \xrightarrow{\approx} & C(g) \\ \downarrow & & \downarrow \\ H_1(\Lambda_g BG;\mathbb{Z}/2) & \xrightarrow{\approx} & C(g)/2 \end{array}$$

□

Remark. Since τ_* factors through $\pi_2 \mathcal{F}$, we already knew that it has to be constant on conjugacy classes in G.

Examples. If G is abelian, we get

$$\tau_* : \mathbb{Z}/2[G] \to \oplus_{g \in G} G/2\,.$$

This is nontrivial if G has elements of even order or is infinite of finite rank. We note two special cases:

(1) $G = \mathbb{Z}/2$ (or more generally, a direct sum of $\mathbb{Z}/2$'s). Then τ_* is 'almost injective'; the kernel is the summand corresponding to the identity element.

(2) $G = \mathbb{Z}$. Then $C(n) = G = \mathbb{Z}$ for all $n \in \mathbb{Z}$, so the map is $n \mapsto n \mod 2$. Hence $\tau_*(n) \neq 0$ if and only if n is odd. Thus θ_* in this case only detects about "half" of $\pi_2 \mathcal{F}$, since we know that $\ker \alpha = \operatorname{im} \chi$ is finite.

For a general G, τ_* is nontrivial if and only if there is at least one $g \neq 1$ such that $C(g)/2$ is nontrivial. For finite groups of even order, a simple transfer argument shows that this holds e.g. if G has a normal, cyclic 2–Sylow subgroup. Other simple examples include symmetric groups, fundamentalgroups of 2–manifolds, free products of nontrivial groups etc. On the other hand, for finite groups of odd order, we detect nothing.

Now let $G = \Omega X$ for a connected space X. Then we have $\pi_0 G \approx \pi_1 X$, and $\pi_1(G,g) = \pi_1\Omega_g X \approx \pi_2 X$, where $\Omega_g X$ denotes the component of $g \in \pi_1 X$. Hence

$$H_1(G;\mathbb{Z}) = \oplus_{\pi_1 X} H_1(\Omega_g X) \approx \oplus_{\pi_1 X}\pi_2 X\,.$$

ΛX has one connected component $\Lambda_g X$ for each conjugacy class $<g>$ in $\pi_1 X$, and τ_* maps $H_1(\Omega_g X)$ to $H_2(\Lambda_g X)$.

Instead of computing $H_2(\Lambda_g X)$ in general, we offer the following observation, which is sufficient for many applications. The evaluation map $\rho : \Lambda X \to X$ splits into a sum of components $\rho_g : \Lambda_g X \to X$, and we may consider the composition

$$\oplus_{\pi_1 X}\pi_2 X \approx H_1(G;\mathbb{Z}) \xrightarrow{\tau_*} H_2(\Lambda X : \mathbb{Z}) \xrightarrow{\rho_*} \oplus_{<\pi_1 X>} H_2(X;\mathbb{Z})\,.$$

Then it is straightforward to check that the following is true:

Lemma 4.5. *This composition is the Hurewicz homomorphism on each summand.*

Hence elements mapped nontrivially by the Hurewicz homomorphism h_2 give rise to elements mapped nontriviallly by $\alpha : \pi_2\mathcal{F} \to \pi_2 A(X)$. Note that the kernel of the $h_2(X)$ is determined precisely by the first k-invariant $k_1(X)$ of X:

Let $\pi_2' X = H_0(\pi_1 X;\pi_2 X)$, i.e. $\pi_2 X$ divided by the action of $\pi_1 X$. Then the Hurewicz homomorphism factors through $h' : \pi_2' X \to H_2(X)$, and there is an exact sequence

$$H_3(X) \xrightarrow{p_*} H_3(B\pi_1 X) \xrightarrow{k_1(X)_*} \pi_2' X \xrightarrow{\overline{h}_2} H_2(X) \xrightarrow{p_*} H_2(B\pi_1 X) \longrightarrow 0$$

(From the Atiyah-Hirzebruch spectral sequence for the fibration $\widetilde{X} \to X \xrightarrow{p} B\pi_1 X$.) This should be compared to Igusa's counterexamples [I1], which depended on the non-vanishing of the first k-invariant in an essential way.

5. Applications to $\pi_0\mathcal{P}(M)$.

Before we can apply these results to detect elements in $\pi_0\mathcal{P}(M)$, we need

Lemma 5.1. *The composition $Q(M) \xrightarrow{\phi} A(M) \xrightarrow{\tau} Q(\Lambda M)$ is $Q(\iota)$, where $\iota : M \subset \Lambda M$ is the 'trivial loop' embedding.*

Proof. We know that τ and ϕ are infinite loop maps. Hence $\tau\circ\phi$ is determined by its restriction to $M \subset Q(M)$. But in the group completion model for $Q(M)$, this subset corresponds to 1×1-matrices with entries in GM. The result now follows from the description of τ in 4 above. □

Corollary 5.2. *The Bökstedt trace induces a map* $\tau : Wh^{\mathrm{DIFF}}(M) \to \widetilde{Q}(\Lambda M/M)$, *and the composed homomorphism*

$$Wh_1^+(\pi_1 M; \mathbb{Z}/2 \oplus \pi_2 M) \to \pi_2 Wh^{\mathrm{DIFF}}(M) \to \pi_2^S(\Lambda M/M)$$

can be computed by Theorem 4.3 (with $M = BG$).

In fact, we do not loose any information passing to $\pi_2^S(\Lambda M/M)$, since we already divided out everything that mapped to $\pi_2^S(M_+)$. Hence, for example, the calculations in the previous section immediately give results about the homomorphism $Wh_1^+(\pi_1 M; \mathbb{Z}/2 \oplus \pi_2 M) \to \pi_0 \mathcal{P}(M)$.

In particular, we obtain the following partial answers to Question 1.2:

Theorem 5.3.

(1) *If $\pi_1 M$ has an element $g \neq 1$ such that $H_1(C(g); \mathbb{Z}/2) = C(g)/2$ is nontrivial, then $\pi_0 \mathcal{P}(M)$ is nontrivial. (For examples, see section 4.)*

(2) *If $\pi_1 M \neq (1)$ and the Hurewicz homomorphism in degree two is nontrivial, then $\pi_0 \mathcal{P}(M)$ is nontrivial.*

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