# Limit Cycles and Complex Geometry

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**Abstract**

We study the relation between dynamical systems and linear connections in domains of the complex plane **C**. Any dynamical system generates a unique connection without torsion such that the operation of parallel displacement is compatible with the complex structure and preserves the direction field of the system. The holonomy group consists of homothetic transformations. The operation of parallel displacement along each limit cycle does not depend on the choice of any particular complex structure and is given by the eigenvalue of the Poincaré map. We give an example of the dynamical system whose connection is flat and limit cycles are structurally stable.

## 1 Introduction

Let $\gamma$ be a limit cycle of the dynamical system

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y).$$

Then the eigenvalue $\lambda$ of the Poincaré map, *i.e.* the first return map, is given by

$$\lambda = \exp \int_\gamma (P_x + Q_y) dt.$$

Our main goal is to give a differential geometric interpretation of this formula. In other words, we introduce the operator of parallel displacement

$$H_\gamma : T_{z_0}(M) \to T_{z_1}(M)$$

along each path

$$\gamma : [0, 1] \to M, \qquad \gamma(0) = z_0, \quad \gamma(1) = z_1,$$

in the domain

$$M = R^2 \setminus \{\text{stationary points of the dynamical system}\},$$

such that

- $H_\gamma$ is compatible with the complex structure $z = x + iy$, *i.e.* $H_\gamma$ is given by a matrix of the form
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$
with respect to the tangent frame $\partial/\partial x,\ \partial/\partial y$;
- $H_\gamma$ is a linear connection without torsion on the tangent bundle $T(M)$;
- $H_\gamma$ preserves the field of directions $Qdx - Pdy = 0$.

In Section 3, we prove that for each differential equation

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$$

there exists a unique such connection $H_\gamma$. The holonomy group of $H_\gamma$ consists of homothetic transformations. In Section 4, we show that for each limit cycle $\gamma$ the holonomy transformation $H_\gamma$ is given by the multiplicator $\lambda$, *i.e.*

$$H_\gamma = \lambda I.$$

Thus, we conclude that the operation of parallel displacement along limit cycles does not depend on the choice of any particular complex structure.

Finally, we study differential equations with a flat connection. Such equations are generated by meromorphic differential 1-forms. The poles correspond to the stationary points. The residues contain some information about limit cycles. A spectacular example is

$$(1 + i\nu)\frac{dz}{z}.$$

This meromorphic differential generates the dynamical system

$$\frac{dx}{dt} = x\cos\phi - y\sin\phi, \qquad \frac{dy}{dt} = x\sin\phi + y\cos\phi,$$

where

$$\phi = \nu \log\sqrt{x^2 + y^2}.$$

This system has a unique stationary point of index 1 and has an infinite set of limit cycles

$$x^2 + y^2 = \exp\frac{\pi(2k+1)}{\nu}, \qquad k = 0, \pm 1, \pm 2, \ldots,$$

with the multiplicator

$$\lambda = e^{2\pi\nu\sigma}, \qquad \sigma = (-1)^{k+1}.$$

We must emphasize that the idea to study differential equations within the framework of the theory of connections dates back to Elie Cartan [1]. On the other hand, we would like to mention a sporadic example of P. Chirokoff [2] and P. Hendlé [3], which illustrates the relation between differential equations, complex structures and linear connections. We discuss this example in Section 2.

## 2 The Chirokoff equation

In 1917, P. Chirokoff [2] raised the following question. *Trouver l'intégrale de l'équation différentielle suivante:*

$$\frac{dy}{dx} = \tan(xy).$$

In 1918, P. Hendlé [3] proved that the equation can be integrated by quadrature, with solution

$$\int_0^y e^{\frac{1}{2}(t^2-x^2)} \cos(xt)\, dt = const\,.$$

The integration procedure of Hendlé looks like a somewhat enigmatical trick. An interesting feature of that trick is the use of complex numbers. Keeping in mind this feature only, we will solve the Shirokoff equation following the integration strategy of Sophus Lie.

An arbitrary differential equation can be wtitten as

$$\frac{dy}{dx} = \tan(\alpha), \qquad \alpha = \alpha(x,y),$$

or as a Pfaff equation defined by the 1-form

$$\Omega^\alpha = \sin\alpha\, dx - \cos\alpha\, dy.$$

According to Lie's integration strategy, we ought to find a symmetry, *i.e.* a vector field

$$V = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

such that

$$d\,\frac{\Omega^\alpha}{\Omega^\alpha(V)} = 0.$$

Then the solution of the equation is given by

$$\int \frac{\Omega^\alpha}{\Omega^\alpha(V)} = const\,.$$

Now, the problem is to find the symmetry $V$. For this aim, we will identify $V$ with the complex valued function

$$w = \xi + i\eta$$

and assume that $w$ holomorphically depends on the complex variable

$$z = x + iy,$$

that is

$$\frac{\partial w}{\partial \bar{z}} = 0.$$

A direct computation shows that the function $w(z)$ is a symmetry, if and only if $\eta_x = \alpha_x \xi + \alpha_y \eta$, which is equivalent to the condition

$$Im\left(\frac{dw}{dz} - \Gamma^\alpha w\right) = 0,$$

where the coefficient $\Gamma^\alpha$ is given by

$$\Gamma^\alpha = 2i\frac{\partial \alpha}{\partial z}.$$

In particular, if $\Gamma^\alpha$ depends holomorphically on $z$, then we can construct the field of "parallel" vectors $w(z)$ such that

$$w(z_1) = \lambda(z_0, z_1)\, w(z_0), \qquad \lambda(z_0, z_1) = \exp\int_{z_0}^{z_1} \Gamma^\alpha dz.$$

For the Chirokoff equation, we have

$$\Gamma^{xy} = z, \qquad w(z) = w(0)\exp\frac{z^2}{2}.$$

Setting $w(0) = i$, we get the Hendlé solution.

# 3 Real Connections

Consider the differential 1-forms

$$\vartheta = \frac{dz}{w}, \qquad \omega = \frac{dw}{w} - \Gamma dz,$$

where $w = \xi + i\eta$ is the complex coordinate of the vector tangent to $M \subseteq R^2$ at the point with the complex coordinate $z = x + iy$, and $\Gamma$ is a complex valued function of $(x, y) \in M$. Since $M$ is of complex dimension one, the *complex tangent vector*

$$w\,\frac{\partial}{\partial z}$$

can be identified with the *complex frame* of $M$. Therefore, $\vartheta$ and $\omega$ can be viewed as 1-forms on the bundle of all complex frames over $M$, which we denote by $F_c(M)$. This bundle is a G-structure. Its structure group is the multiplicative group $C \setminus \{0\}$. The 1-form $\vartheta$ is nothing else but the *canonical differential form* on the G-structure $F_c(M)$. Roughly speaking, the value $\vartheta_u(v)$ is the coordinate of the complex tangent vector $v$ with respect to the complex frame $u$. The 1-form $\omega$ defines a linear connection on the G-structure $F_c(M)$. The *parallel displacement* along each path $\gamma$ is given by

$$w\,\frac{\partial}{\partial z}\ \text{ at }\ z_0 \quad \xrightarrow{H_\gamma} \quad \lambda(\gamma)\,w\,\frac{\partial}{\partial z}\ \text{ at }\ z_1,$$

where

$$\lambda(\gamma) = a(\gamma) + i\,b(\gamma) = \exp\int_\gamma \Gamma dz.$$

It is not hard to see that the 1-forms $\vartheta$ and $\omega$ satisfy the structure equation of Elie Cartan

$$d\vartheta + \omega\wedge\vartheta = 0,$$

which means that the connection $\omega$ has vanishing *torsion*. The *curvature* of $\omega$ is given by the exterior derivative

$$d\omega = K\,\vartheta\wedge\overline{\vartheta},$$

where

$$K = w\,\overline{w}\,\frac{\partial\Gamma}{\partial\overline{z}}.$$

We say that the connection $\omega$ is *flat*, if the curvature $K$ vanishes.

Suppose $\Gamma = \Gamma^\alpha$ is the function associated to the differential equation $\Omega^\alpha = 0$ (see Section 2). Then we have

$$\Gamma^\alpha dz = i\,d\alpha - *\,d\alpha,$$

where $*$ is the Hodge operator:

$$*\,(p\,dx + q\,dy) = -q\,dx + p\,dy.$$

Thus, if $\gamma$ is a closed path, then the imaginary part of the integral

$$\oint_\gamma \Gamma^\alpha dz = i\oint_\gamma d\alpha + \oint_\gamma(-*\,d\alpha)$$

is an integer multiple of $2\pi$, and the holonomy transformation $H_\gamma$ is given by the real number

$$\lambda(\gamma) = \exp\oint_\gamma(-*\,d\alpha).$$

This is why the differential 1-form

$$\omega^\alpha = \frac{dw}{w} - \Gamma^\alpha dz$$

will be called the *real connection* associated with the differential equation $\Omega^\alpha = 0$.

**Proposition.** *There is an intrinsic correspondence between differential equations and real connections.*

Actually, the change of variables

$$z \mapsto \varphi(z), \qquad w \mapsto \varphi'(z)w$$

takes back the forms

$$\Omega^\alpha \qquad \textit{and} \qquad \omega^\alpha$$

to

$$|\varphi'|\Omega^{\alpha-\beta} \qquad \textit{and} \qquad \omega^{\alpha-\beta}$$

respectively, where

$$\beta = \arg \varphi'.$$

This proves that the correspondence between differential equations and real connections is independent of any particular coordinate system. Moreover, this correspondence is analogous to the well known relation between Riemannian metrics and Levi-Civita connections. Let us follow up this analogy.

1. Any Riemannian metric is parallel with respect to the associated Levi-Civita connection. This is a unique linear connection, which has vanishig torsion and preserves the Riemannian metric.

   *Any differential equation is parallel with respect to the associated real connection. This is a unique linear connection on $F_c(M)$, which has vanishig torsion and preserves the differential equation.*

   *Proof.* An arbitrary linear connection on $F_c(M)$ is given by

   $$\omega = \frac{dw}{w} - \Gamma dz - \Pi d\overline{z}.$$

   The structure equation

   $$d\vartheta + \omega \wedge \vartheta = T\, d\vartheta \wedge d\overline{\vartheta}, \qquad T = w\overline{w}\,\Pi,$$

   shows that $\omega$ has vanishing torsion, if and only if $\Pi = 0$. In this case, $\omega$ preserves the equation $\Omega^\alpha = 0$, if and only if $\Gamma = \Gamma^\alpha$.

2. The Levi-Civita connection is flat if and only if the Riemannian metric is locally Euclidean.

   *The real connection is flat if and only if the differential equation is locally holomorphically equivalent to the differential equation of straight lines.*

   *Proof.* The curvature $K$ of $\omega^\alpha$ vanishes, if and only if

$$\frac{\partial^2 \alpha}{\partial z\, \partial \bar{z}} = 0.$$

   This means that the angle $\alpha$ is a harmonic function and thereby is the argument of a local holomorphic map $z \mapsto \varphi(z)$, which takes back the equation $\Omega^\alpha = 0$ to $\Omega^0 = 0$.

3. The holonomy group of the Levi-Civita connection consists of rotations.

   *The holonomy group of the real connection consists of homothetic transformations.*

4. Christoffel's symbols of the Levi-Civita connection can be expressed in terms of the Riemannian metric.

   *Christoffel's symbols of the real connection can be expressed in terms of the differential equation as follows*

$$\begin{pmatrix} \Gamma^1_{11}dx + \Gamma^1_{12}dy & \Gamma^1_{21}dx + \Gamma^1_{22}dy \\ \Gamma^2_{11}dx + \Gamma^2_{12}dy & \Gamma^2_{21}dx + \Gamma^2_{22}dy \end{pmatrix} = \begin{pmatrix} *d\alpha & d\alpha \\ -d\alpha & *d\alpha \end{pmatrix}.$$

# 4 Limit Cycles

Each dynamical system on the plane $\mathbf{R}^2$

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y),$$

defines on the open set

$$M = R^2 \setminus \{(x,y) \mid P(x,y) = Q(x,y) = 0\}$$

the field of directions

$$\Omega^\alpha \equiv \frac{Q\, dx - P\, dy}{\sqrt{P^2+Q^2}} = 0,$$

where the angle $\alpha$ is defined locally by the cocycle

$$d\alpha = \frac{P\, dQ - Q\, dP}{P^2+Q^2}.$$

Suppose $M$ is supplied with the *complex structure* $z = x + iy$. Then, as we have seen in Section 3, the differential equation $\Omega^\alpha = 0$ generates on $F_c(M)$ a real connection $\omega^\alpha$. The holonomy transformations of $\omega^\alpha$ define on the loop space a real valued functional

$$\gamma \mapsto \lambda(\gamma) = \exp \oint_\gamma \Gamma^\alpha dz = \exp \oint_\gamma (- * d\alpha).$$

**Theorem.** *If $\gamma$ is a limit cycle of the dynamical system, then $\lambda(\gamma)$ is the eigenvalue of the Poincaré map.*

*Proof.* Applying the Hodge star to the cocycle $d\alpha$, we find

$$- * d\alpha = \frac{(PQ_y - QP_y)dx + (QP_x - PQ_x)dy}{P^2 + Q^2}.$$

Using the time $t$ as a parameter on the limit cycle $\gamma$, we compute the restriction

$$(- * d\alpha)|_\gamma = (P_x + Q_y)dt + d \log \frac{1}{\sqrt{P^2 + Q^2}}.$$

Therefore, we have

$$\lambda(\gamma) = \exp \int_\gamma (P_x + Q_y)dt.$$

The proof is complete.

**Corollary.** *If $\gamma$ is a limit cycle of the dynamical system, then the number $\lambda(\gamma)$ does not depend on the choice of any particular complex structure.*

**Remark.** This Corollary is analagous to the Gauss–Bonnet theorem, which states that the integral of the curvature over a compact surface does not depend on the choice of any particular Riemannian metric, because this integral is the Eulerian characteristic of the surface multiplied by $2\pi$.

Now, suppose the real connection is flat. Then, for any closed path $\gamma$, homotopic in $M$ to a given limit cycle, $\lambda(\gamma)$ is also equal to the eigenvalue of the Poincaré map. Moreover, in the flat case the functional $\lambda(\gamma)$ defines a homomorphism

$$\lambda \colon \pi_1(M) \to R^+$$

of the fundamental group $\pi_1(M)$ to the group of positive real numbers $R^+$. Consider the induced homomorphism

$$h \colon \pi_1(M) \to C$$

such that

$$h(\gamma) = \log \lambda(\gamma) + i\,2\pi n = \oint_\gamma \Gamma^\alpha dz,$$

where $n$ is an interger equal to the index of the stationary point surrounded by the generator $\gamma \in \pi_1(M)$, the orientation of $\gamma$ is supposed to be positive. In the theory of Riemann surfaces, the homomorphism $h$ is called the *period homomorphism*. The problem is to restore the dynamical system with a prescribed period homomorphism.

**Example.** Suppose we want the dynamical system to have one stationary point and a srtucturally stable limit cycle. Then we have

$$M = C^*, \qquad \pi_1(C^*) = Z, \qquad h(1) = \chi + i\,2\pi, \quad \chi \neq 0.$$

In other words, the index of the stationary point should be equal to one, the number

$$\lambda(\pm 1) = e^{\pm\chi}$$

is supposed to be the eigenvalue of the Poincaré map. Clearly, the described homomorphism $h$ can be given by the meromorphic differential

$$\Gamma^\alpha dz = \left(1 - i\frac{\chi}{2\pi}\right)\frac{dz}{z},$$

where

$$\alpha = \arg z + \phi, \qquad \phi = \frac{\chi}{2\pi}\log\frac{1}{|z|} + const\,.$$

The corresponding dynamical system is

$$\frac{dz}{dt} = \rho e^{i\phi} z, \qquad \rho = \rho(x,y) > 0.$$

In this system the dynamics is oscillatory, that is every trajectory is asymptotic to a structurally stable periodic orbit

$$\phi = \frac{\pi}{2} + \pi k, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

Finally we note that in the same way one can construct more complicated systems with oscillatory dynamics. Let $M = C \setminus \{z_1, \ldots, z_s\}$. Then each period homomorphism $h$ can be given by

$$\Gamma^\alpha dz = \sum_{j=1}^{s} \frac{h_j}{2\pi i}\,\frac{dz}{z - z_j} + df(z),$$

where $f(z)$ is a polynomial, $h_j = \chi_j + i\,2\pi n_j$,

$$\alpha = \sum_{j=1}^{s} n_j \arg(z - z_j) + \phi,$$

$$\phi = \sum_{j=1}^{s} \frac{\chi_j}{2\pi} \log \frac{1}{|z - z_j|} + Im\, f(z).$$

The corresponding dynamical system is

$$\frac{dz}{dt} = \rho e^{i\phi}(z - z_1)^{n_1} \cdots (z - z_r)^{n_r} (\overline{z} - \overline{z}_{r+1})^{-n_{r+1}} \cdots (\overline{z} - \overline{z}_s)^{-n_s}.$$

We assume here that

$$n_1 > 0, \ldots, n_r > 0 \qquad and \qquad n_{r+1} < 0, \ldots, n_s < 0.$$

If $n_1 = 1$ and $\chi_1 \neq 0$, then, in a small neighbourhood of $z_1$, the dynamical system takes the form

$$\frac{d}{dt}(z - z_1) = \rho_1 e^{i(\phi_1 + \varepsilon_1)}(z - z_1),$$

where

$$\phi_1 = \frac{\chi_1}{2\pi} \log \frac{1}{|z - z_1|} + c_1,$$

and $\varepsilon_1 \to 0$ as $|z - z_1| \to 0$. This means that the stationary point $z_1$ is surrounded by a structurally stable periodic orbit with the multiplicator

$$\lambda = e^{\pm \chi_1}.$$

# References


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3. P. Hendlé, *Intermédiaire Math.* **XXV** (1918) 45–46.