Existence and uniqueness theorems for some stochastic parabolic partial differential equations

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Abstract

In this paper we will prove existence and uniqueness theorems for the stochastic differential equations

$$\frac{\partial u}{\partial t} = \begin{cases} \Delta u + \vec{W}_{\phi_{(t,x)}} \diamond \nabla u + g(t,x) \\ \operatorname{Exp}\{W_{\phi_{(t,x)}}\} \diamond \Delta u + g(t,x) \\ \nabla \cdot \{\operatorname{Exp}\{\mathcal{W}^{s}_{\phi_{(t,x)}}\} \diamond \nabla u\} + g(t,x) \end{cases} \qquad (t,x) \in [0,T] \times \mathbb{R}^n$$

with initial conditions $u(0,x) = f(x)\ \forall x \in \mathbb{R}^n$ where $\vec{W}_{\phi_{(t,x)}}$ is a white noise vector, $\operatorname{Exp}\{W_{\phi_{(t,x)}}\}$ is positive white noise, $\operatorname{Exp}\{\mathcal{W}^{s}_{\phi_{(t,x)}}\}$ is a positive white noise matrix and f, g are real functions. We will show that these equations have solutions in the space $(\mathcal{S})^{-1}$ of generalized white noise distributions in a strong differentiation sense.

Keywords: Generalized white noise distributions, Wick product, Hermite Transform.

§1 Introduction

We will in this paper apply white noise analysis to obtain existence and uniqueness theorems for some stochastic partial differential equations. Within the white noise analysis, there are several choices of possible solution spaces, but we will only work in the space of generalized white noise distributions, known as the Kondratiev distribution space. This space is used in [VÅGE] to obtain Hilbert space methods for solving several classes of stochastic partial differential equations, including time-independent versions of the equations we are going to solve. Other interesting stochastic partial differential equations, mostly which are possible to solve explicitly, are

- The transport equation ([GjHØUZ]).
- The pressure equation for fluid flow ([HLØUZ3]).

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}^n$, and

$$\mathcal{N}^* := (\prod_{i=1}^{m} \mathcal{S}(\mathbb{R}^n))^* \approx \prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n)$$

where $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions.

Let $\mathcal{B} := \mathcal{B}(\mathcal{N}^*)$ denote the Borel σ-algebra on $\mathcal{N}^*$ equipped with the weak star topology and set

$$\mathcal{H} := \bigoplus_{i=1}^{m} \mathcal{L}^2(\mathbb{R}^n)$$

where $\oplus$ denotes orthogonal sum.

Since $\mathcal{N}$ is a countably Hilbert nuclear space (cf. eg.[Gj]) we get, using Minlos' theorem, a unique probability measure ν on $(\mathcal{N}^*, \mathcal{B})$ such that

$$\int_{\mathcal{N}^*} e^{i\langle\omega,\phi\rangle}\, d\nu(\omega) = e^{-\frac{1}{2}\|\phi\|^2_{\mathcal{H}}} \quad \forall \phi \in \mathcal{N}$$

where $\|\phi\|^2_{\mathcal{H}} = \sum_{i=1}^{m} \|\phi_i\|^2_{\mathcal{L}^2(\mathbb{R}^n)}$.

Note that if $m = 1$ then ν is usually denoted by μ.

THEOREM 2.1 [Gj] We have the following

1. $\otimes_{i=1}^{m} \mathcal{B}(\mathcal{S}'(\mathbb{R}^n)) = \mathcal{B}(\prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n))$
2. $\nu = \times_{i=1}^{m} \mu$

DEFINITION 2.2 [Gj] The triple

$$(\prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n), \mathcal{B}, \nu)$$

is called the (m-dimensional) (n-parameter) **white noise probability space.**

For $k = 0, 1, 2, \ldots$ and $x \in \mathbb{R}$ let

$$h_k(x) := (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k}(e^{-\frac{x^2}{2}})$$

be the Hermite polynomials and

$$\xi_k(x) := \pi^{-\frac{1}{4}}((k-1)!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{k-1}(\sqrt{2}x) \ ; \ k \geq 1$$

the Hermite functions.

It is well known that the family $\{\tilde{e}_\alpha\} \subset \mathcal{S}(\mathbb{R}^n)$ of tensor products

$$\tilde{e}_\alpha := \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_n}$$

1. $\{W^{(i)}(\phi, \cdot)\}_{i=1}^{m}$ is a family of independent normal random variables.
2. $W^{(i)}(\phi, \cdot) \in \mathcal{L}^2(\nu)$ for $1 \leq i \leq m$.

DEFINITION 2.5 [HLØUZ3] Let $0 \leq \rho \leq 1$.

- Let $(\mathcal{S}_n^m)^\rho$, the space of **generalized white noise test functions,** consist of all

$$f = \sum_\alpha c_\alpha H_\alpha \in \mathcal{L}^2(\nu)$$

such that

$$\|f\|_{\rho,k}^2 := \sum_\alpha c_\alpha^2 (\alpha!)^{1+\rho} (2\mathbb{N})^{\alpha k} < \infty \quad \forall k \in \mathbb{N}$$

- Let $(\mathcal{S}_n^m)^{-\rho}$, the space of **generalized white noise distributions,** consist of all formal expansions

$$F = \sum_\alpha b_\alpha H_\alpha$$

such that

$$\sum_\alpha b_\alpha^2 (\alpha!)^{1-\rho} (2\mathbb{N})^{-\alpha q} < \infty \text{ for some } q \in \mathbb{N}$$

where

$$(2\mathbb{N})^\alpha := \prod_{i=1}^{k} (2^n \beta_1^{(i)} \cdots \beta_n^{(i)})^{\alpha_i} \text{ if } \alpha = (\alpha_1, \ldots, \alpha_k).$$

We know that $(\mathcal{S}_n^m)^{-\rho}$ is the dual of $(\mathcal{S}_n^m)^\rho$ (when the later space has the topology given by the seminorms $\|\cdot\|_{\rho,k}$) and if $F = \sum b_\alpha H_\alpha \in (\mathcal{S}_n^m)^{-\rho}$ and $f = \sum c_\alpha H_\alpha \in (\mathcal{S}_n^m)^\rho$ then

$$\langle F, f \rangle = \sum_\alpha b_\alpha c_\alpha \alpha!.$$

It is obvious that we have the inclusions

$$(\mathcal{S}_n^m)^1 \subset (\mathcal{S}_n^m)^\rho \subset (\mathcal{S}_n^m)^{-\rho} \subset (\mathcal{S}_n^m)^{-1} \quad \rho \in [0, 1]$$

and in the remaining of this paper we will consider the larger space $(\mathcal{S}_n^m)^{-1}$.

DEFINITION 2.6 [HLØUZ3] The Wick product of two elements in $(\mathcal{S}_n^m)^{-1}$ given by

$$F = \sum_\alpha a_\alpha H_\alpha \ , \quad G = \sum_\beta b_\beta H_\beta$$

is defined by

$$F \diamond G = \sum_\gamma c_\gamma H_\gamma$$

where

$$c_\gamma = \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta$$

EXAMPLE 2.14 Define the x-shift of ϕ, denoted by ϕ_x, by $\phi_x(y) := \phi(y-x)$.Then

$$\text{Exp}\{W^{(i)}_{\phi_x}\} \in (\mathcal{S}^m_n)^{-1} \quad 1 \le i \le m, \forall x \in \mathbb{R}^n$$

which is an immediate consequence of proposition 2.4 and lemma 2.11.

EXAMPLE 2.15 Let a symmetric $u \times u$-matrix $\mathcal{W}^s(\phi,\cdot)$ be given by

$$(\mathcal{W}^s)_{ij}(\phi,\cdot) := W^{(\hat{\sigma}(i,j))}(\phi,\cdot) \in (\mathcal{S}_n^{\frac{u(u+1)}{2}})^{-1}$$

where

$$\hat{\sigma}(i,j) := \begin{cases} \sigma(j-u+i(u-\frac{i-1}{2})) & \text{when } i \le j \\ \hat{\sigma}(j,i) & \text{when } i > j \end{cases}$$

and σ is an arbitrary element in the permutation group of $\frac{u(u+1)}{2}$ elements. We are now able, using lemma 2.11, to construct the white noise exponential matrix, with components in $(\mathcal{S}_n^{\frac{u(u+1)}{2}})^{-1}$, as the matrix

$$\text{Exp}\{\mathcal{W}^s\} := \sum_{k=0}^{\infty} \frac{1}{k!}(\mathcal{W}^s)^{\diamond k}$$

where the Wick-exponents are in ordinary matrix multiplication sense.

§3 Some parabolic partial differential equations

The following notation will be used in this and the following sections:

- $C^{\alpha,t}_b([0,T]\times\mathbb{R}^n)$ are the bounded functions on $[0,T]\times\mathbb{R}^n$ which are Hölder continuous with exponent α in $x \in \mathbb{R}^n$, uniformly in $t \in [0,T]$.
- $C^{2+\alpha}_b(\mathbb{R}^n)$ are the functions on $\mathbb{R}^n$ with bounded partial derivatives up to 2'nd order which are Hölder continuous with exponent α.

DEFINITION 3.1 We say that a function $\mathbf{B}_q(\delta) \cap \mathbb{R}^N_0 \ni \lambda \mapsto \tilde{u}(\lambda)$ is in $C^\omega_u(\mathbf{B}_q(\delta)\cap\mathbb{R}^N_0)$ if the restriction of $\mathcal{H}u$ on the set $\mathbf{B}_q(\delta)\cap\mathbb{R}^n_0$ is real analytic $\forall n \in \mathbb{N}$ and $\exists(M>0, r>0)$ (independent of n) such that $|[\partial^\beta \tilde{u}](0)| \le M|\beta|! r^{-|\beta|}$ $\forall \beta \in \mathbb{Z}^n_0$ $\forall n \in \mathbb{N}$.

LEMMA 3.2 Suppose there exists $q \in \mathbb{N}$ and $\delta > 0$ such that $\lambda \mapsto \tilde{v}(\lambda)$ is in $C^\omega_u(\mathbf{B}_q(\delta)\cap\mathbb{R}^N_0)$. Then there exists $\hat{q} \in \mathbb{N}$, $\hat{\delta} > 0$ and an analytic function $\tilde{v}_c \in A_b(\mathbf{B}_{\hat{q}}(\hat{\delta}))$ such that $\tilde{v}(\lambda) = \tilde{v}_c(\lambda)$ whenever $\lambda \in \mathbf{B}_q(\delta)\cap\mathbf{B}_{\hat{q}}(\hat{\delta})\cap\mathbb{R}^N_0$.

PROOF:

It follows from [Fritz, page 65] that there exists $M > 0$ and $r > 0$ such that

$$|[\partial^\beta \tilde{v}](0)| \le M|\beta|! r^{-|\beta|} \quad \forall \beta \in \mathbb{Z}^l_0 \quad \forall l \in \mathbb{N}$$

where e_k is the k'th unit vector in $\mathbb{R}^n$, exists together with the other derivatives. This is, because of lemma 2.12, equivalent with the existence of $\delta > 0, q \in \mathbb{N}$ such that

$$\lim_{\epsilon \to 0} \frac{\mathcal{H}u(t, x + \epsilon e_k, z) - \mathcal{H}u(t, x, z)}{\epsilon} = \frac{\partial}{\partial x_k} \mathcal{H}u(t, x, z)$$

pointwise, uniformly bounded, whenever $z \in \mathbf{B}_q(\delta)$.

PROOF:

We will find $q \in \mathbb{N}, \delta > 0$ and a function $[0, T] \times \mathbb{R}^n \ni (t, x) \mapsto \tilde{v}(t, x) \in A_b(\mathbf{B}_q(\delta))$, the space of all bounded analytic functions on $\mathbf{B}_q(\delta)$, which solves the equation

$$\frac{\partial \tilde{v}}{\partial t} = \Delta \tilde{v} + \tilde{\vec{W}}_{\phi_{(t,x)}} \cdot \nabla \tilde{v} + \Delta f + \tilde{\vec{W}}_{\phi_{(t,x)}} \cdot \nabla f + g(t, x) \quad (t, x) \in (0, T] \times \mathbb{R}^n \tag{3}$$

$$\tilde{v}(0, x) = 0 \qquad x \in \mathbb{R}^n \tag{4}$$

when $z \in \mathbf{B}_q(\delta)$. Our solution will then be $u(t, x) := \mathcal{H}^{-1}\tilde{v}(t, x, z) + f(x)$. Since we will use existence and uniqueness results based on real-valued functions, we will solve equation (3) for all $\lambda \in \mathbf{B}_q(\delta) \cap \mathbb{R}_0^N$ (for suitable $q \in \mathbb{N}, \delta > 0$) and then show that we may apply lemma 3.2 below. The proof consists of several lemmas:

LEMMA 3.5 Let $q \in \mathbb{N}$ and $\delta > 0$ be arbitrary. Then

1. $|\tilde{W}^{(i)}_{\phi_{(t,x)}}(\lambda)|^2 \leq \delta^2 \|\phi\|^2$ whenever $\lambda \in \mathbf{B}_q(\delta) \cap \mathbb{R}_0^N$.
2. $\frac{\partial}{\partial \lambda_l} \tilde{W}^{(i)}_{\phi_{(t,x)}} = (\phi^{(i)}_{(t,x)}, e^{(i)}_l)$ if $\lambda = (\lambda_1, \cdots, \lambda_k)$ with $l \leq k$.

PROOF:

We have

$$W^{(i)}_{\phi_{(t,x)}}(\omega) = \sum_{k=0}^{\infty} (\phi^{(i)}_{(t,x)}, e^{(i)}_k) \langle \omega_i, e^{(i)}_k \rangle$$

which gives the second result and

$$\begin{aligned} |\tilde{W}^{(i)}_{\phi_{(t,x)}}(\lambda)|^2 &= |\sum_{k=0}^{\infty} (\phi^{(i)}_{(t,x)}, e^{(i)}_k) \lambda_k|^2 \leq \sum_{k=0}^{\infty} (\phi^{(i)}_{(t,x)}, e^{(i)}_k)^2 \sum_{k=0}^{\infty} |\lambda_k|^2 \\ &\leq \|\phi\|^2 \sum_{\alpha \neq 0} |\lambda^\alpha| (2\mathbf{N})^{\alpha q} \leq \delta^2 \|\phi\|^2 \end{aligned}$$

which gives the first result. ■

LEMMA 3.6 Let $q \in \mathbb{N}, \delta > 0$ and $0 < \alpha \leq 1$ be given. Then $\exists A > 0$, independent of $(t, x) \in [0, T] \times \mathbb{R}^n, \lambda \in \mathbf{B}_q(\delta) \cap \mathbb{R}_0^N$ and $\alpha \in (0, 1]$, such that for all $t \in [0, T], x, x^0 \in \mathbb{R}^n$ and $1 \leq i \leq n$

$$|\tilde{W}^{(i)}_{\phi_{(t,x)}}(\lambda) - \tilde{W}^{(i)}_{\phi_{(t,x^0)}}(\lambda)| \leq A|x - x^0|^\alpha$$

It is not hard, by using induction and the results of [Fri, page 14–16], to show that

$$|(LZ)^{\lambda}_{\nu}(x,t;\xi,\tau)| \le \frac{C_0 C_1^{\nu}}{\Gamma(\nu\alpha)}(t-\tau)^{\nu\alpha-\frac{n}{2}-1} e^{-\frac{\lambda^*|x-\xi|^2}{4(t-\tau)}} \tag{7}$$

and that each $(LZ)^{\lambda}_{\nu}$ is C^1 in the parameter with

$$|\frac{\partial}{\partial\lambda_i}(LZ)^{\lambda}_{\nu}(x,t;\xi,\tau)| \le \frac{\nu C_0 C_1^{\nu}}{\Gamma(\nu\alpha)}(t-\tau)^{\nu\alpha-\frac{n}{2}-1} e^{-\frac{\lambda^*|x-\xi|^2}{4(t-\tau)}} \quad (\forall i \in \mathbb{N}) \tag{8}$$

where C_0, C_1 and λ^* are constants independent of λ, based on estimates of Z (which is independent of λ) and upper bounds of $\{\tilde{W}^{(i)}_{\phi(t,x)}\}_{i=1}^{n}$. It is now clear that the series (6) and the series

$$\sum_{\nu=1}^{\infty} \frac{\partial}{\partial\lambda_i}(LZ)^{\lambda}_{\nu}(x,t;\xi,\tau) \quad (\forall i \in \mathbb{N}) \tag{9}$$

converges uniformly in λ for each fixed $(t,x;\xi,\tau)$ which proves the λ-differentiability of Φ^{λ}. It follows from [Fri, page 14–16] that

$$|\frac{\partial}{\partial\lambda_i}\Phi^{\lambda}(x,t;\xi,\tau)| \le C_2(t-\tau)^{(-n-2+\alpha)/2} e^{-\frac{\lambda^*|x-\xi|^2}{4(t-\tau)}} \quad (\forall i \in \mathbb{N})$$

where C_2 is a constant, which proves the λ-differentiability of Γ^{λ}. Finally, we are able to show that

$$|\frac{\partial}{\partial\lambda_i}\Gamma(x,t;\xi,\tau)| \le C_3(t-\tau)^{-n/2} e^{-\frac{\lambda^*|x-\xi|^2}{4(t-\tau)}} \quad (\forall i \in \mathbb{N})$$

where C_3 is a constant, which proves the λ-differentiability of $\tilde{v}$. By using uniqueness ([Fri, theorem 16, page 29]), it is easy to show that

$$\frac{\partial}{\partial\lambda_i}\tilde{v}(t,x,\lambda) = -\int_0^t\int_{\mathbb{R}^n} \Gamma^{\lambda}(x,t;\xi,\tau)\left(\sum_{k=1}^{n}\frac{\partial}{\partial\lambda_i}\tilde{W}^{(k)}_{\phi(t,x)}\frac{\partial}{\partial x_k}\tilde{v}\right) d\xi\, d\tau \quad (\forall i \in \mathbb{N})$$

which is clearly λ-differentiable. The result now follows by an induction argument. ■

LEMMA 3.8 $\exists K_1 > 0$ such that the solution $\tilde{v}$ of (3) satisfies an inequality of the type

$$\max\{\sup_{t,x,\lambda}|\tilde{v}(t,x,\lambda)|, \sup_{t,x,\lambda,i}|\frac{\partial}{\partial x_i}\tilde{v}(t,x,\lambda)|\} \le K_1 \sup_{t,x,\lambda}|\Delta f(t,x) + \tilde{\tilde{W}}_{\phi(t,x)} \cdot \nabla f(t,x) + g(t,x)| \tag{10}$$

where K_1 is independent of (t,x) and $\lambda \in \mathbf{B}_q(\delta)$.

PROOF:

This follows immediately from the estimates (see [Fri, page 28])

$$|\Gamma^{\lambda}(x,t;\xi,\tau)| \le \hat{K}_1(t-\tau)^{-\frac{n}{2}} e^{-\frac{\lambda^*|x-\xi|^2}{4(t-\tau)}}$$

and

$$|\frac{\partial}{\partial x_i}\Gamma^{\lambda}(x,t;\xi,\tau)| \le \hat{K}_2(t-\tau)^{-\frac{n+1}{2}} e^{-\frac{\lambda^*|x-\xi|^2}{4(t-\tau)}}$$

where the constants $\hat{K}_1$ and $\hat{K}_2$ are independent of λ, based on the estimates of Z (independent of λ) and estimates of Φ (independent of λ as in the proof of lemma 3.7). ■

LEMMA 3.10 The strong derivatives in equation (1) exist.

PROOF:

Let $\delta > 0$ and $q \in \mathbb{N}$ be given from lemma 3.2 and put

$$\tilde{v}_k^{\epsilon}(t,x,z) := \frac{\tilde{v}(t,x+\epsilon e_k,z) - \tilde{v}(t,x,z)}{\epsilon}$$

where e_k is the k'th unit vector in $\mathbb{R}^n$, then

$$\tilde{v}_k^{\epsilon}(t,x,z) = \sum_{\alpha} \frac{1}{\alpha!}\left(\frac{\partial^{\alpha}\tilde{v}(t,x+\epsilon e_k,0) - \partial^{\alpha}\tilde{v}(t,x,0)}{\epsilon}\right)z^{\alpha}.$$

Now, by using the mean-value theorem and an obvious modification of lemma 3.9, we obtain the inequality

$$\left|\frac{\partial^{\alpha}\tilde{v}(t,x+\epsilon e_k,0) - \partial^{\alpha}\tilde{v}(t,x,0)}{\epsilon}\right| \leq K_4|\alpha|!(nK_4\|\phi\|^2)^{|\alpha|} \quad \forall \epsilon > 0$$

so by using the dominated convergence theorem

$$\lim_{\epsilon \to 0} \tilde{v}_k^{\epsilon}(t,x,z) = \frac{\partial \tilde{v}}{\partial x_k}(t,x,z)$$

which is, because of remark 3.4, what we wanted to prove. The other derivatives are proven to exist in a similar manner. ■

Theorem 3.3 now finally follows by combining lemma 3.7 and lemma 3.9 with lemma 3.2. ■

EXAMPLE 3.11 (Singular white noise) The m-dimensional singular white noise is defined as

$$W_{t,x}(\omega) := (W_{t,x}^{(1)}(\omega), \cdots, W_{t,x}^{(m)}(\omega))$$

where

$$W_{t,x}^{(i)}(\omega) := \sum_{k=1}^{\infty} e_k^{(i)}(t,x)H_{\epsilon(k)}(\omega) \quad (1 \leq i \leq m)$$

and

$$\epsilon(k) := (\overbrace{0,\cdots,0}^{k-1},1) \quad \text{is a multi-index.}$$

We now from [HP, 21.3.3] that $\|e_k\|_{\infty} = O(n^{-\frac{1}{12}})$ so there exists a constant C such that $\|e_k\|_{\infty} \leq C \ \forall k \in \mathbb{N}$. From this, using definition 2.5, it is easy to show that $W_{t,x}^{(i)} \in (\mathcal{S}_{n+1}^m)^{-1}$ $(1 \leq i \leq m)$. Let now $\psi \in \mathcal{S}(\mathbb{R}^{n+1})$ with $\int \psi \, dt\, dx = 1$ and $\|\psi\|_1 \leq K$ be given. Define $\psi^k(t,x) := k^{n+1}\psi((k+1)(t,x))$, then $\psi^k(t,x) \to \delta_0$ in $\mathcal{S}'(\mathbb{R}^{n+1})$ by [RAUCH, propositon 2.2.3]. We have

$$\tilde{W}_{\psi_{(t,x)}^k}^{(i)}(z) = \sum_{l=1}^{\infty}((\psi_{(t,x)}^l)^{(i)}, e_k^{(i)})z_l \to \sum_{l=1}^{\infty}(e_l)_{(t,x)}z_l = \tilde{W}_{t,x}(z) \quad (1 \leq i \leq m)$$

and

$$\begin{aligned} |\tilde{W}_{\psi_{(t,x)}^k}^{(i)}(z)| &\leq \sum_{l=1}^{\infty} |((\psi_{(t,x)}^k)^{(i)}, e_l^{(i)})||z_l| \\ &\leq KC\delta \sum_{\alpha}(2N)^{-\frac{\alpha q}{2}} \end{aligned} \tag{17}$$

We will only prove the first inequality, the second follows similarly. It is enough, by the triangle inequality, to find constants A_1 and A_2, independent of $(t,x) \in [0,T] \times \mathbb{R}^n$ and $\lambda \in \mathbf{B}_q(\delta) \cap \mathbb{R}_0^N$ such that we have the inequalities

$$|\exp\{\tilde{W}_{\Phi_{(t,x)}}\} - \exp\{\tilde{W}_{\Phi_{(t,x^0)}}\}| \leq A_1|x - x^0|^\alpha \quad \forall t \in [0,T] \tag{20}$$

and

$$|\exp\{\tilde{W}_{\Phi_{(t,x^0)}}\} - \exp\{\tilde{W}_{\Phi_{(t^0,x^0)}}\}| \leq A_2|t - t^0|^{\frac{\alpha}{2}} \quad \forall x^0 \in \mathbb{R}^n, \tag{21}$$

but this is immediate from an obvious modification of the proof of lemma 3.6. ■

LEMMA 3.14 The Cauchy equation given by (18) has an unique solution $\tilde{v}(t,x,\lambda)$ for each $\lambda \in \mathbf{B}_q(\delta) \cap \mathbb{R}_0^N$. Moreover, the function $\lambda \mapsto \tilde{v}(t,x,\lambda)$ is C^∞ in every parameter λ_k for each fixed $(t,x) \in [0,T] \times \mathbb{R}^n$.

PROOF:

Equation (18) has an unique solution given by

$$\tilde{v}(t,x,\lambda) = -\int_0^t \int_{\mathbb{R}^n} \Gamma^\lambda(x,t;\xi,\tau)(\exp\{\tilde{W}_{\Phi_{(\tau,\xi)}}\} \cdot \Delta f(\tau,\xi) + g(\tau,\xi))\, d\xi\, d\tau$$

where Γ^λ is the fundamental solution given by (5), but with $Z(x,t;\xi,\tau) := Z^\lambda(x,t;\xi,\tau)$ where

$$Z^\lambda(x,t;\xi,\tau) := (2\sqrt{\pi})^{-n} \exp\{-\frac{1}{2}\tilde{W}_{\Phi_{(t,x)}}\}(t-\tau)^{-\frac{n}{2}} \exp\{-\frac{1}{4}(t-\tau)^{-1} \exp\{-\tilde{W}_{\Phi_{(\tau,\xi)}}\}|x-\xi|^2\}$$

and $\Phi^\lambda(x,t;\xi,\tau)$ is as in (6) but with

$$LZ(x,t;\xi,\tau) := \sum_{i=1}^n [e^{\tilde{W}_{\Phi(t,x)}} - e^{\tilde{W}_{\Phi(\tau,\xi)}}] \frac{\partial^2}{\partial x_i^2} Z^\lambda(x,t;\xi,\tau).$$

Each $(LZ)_\nu^\lambda$ satisfies, with possible different constants C_0 and C, the bound given in (7). It follows that the series (6) converges (uniformly) in λ for each fixed $(t,x;\xi,\tau)$. Now,

$$\frac{\partial}{\partial \lambda_h} \exp\{-\frac{1}{4}(t-\tau)^{-1} \exp\{-\tilde{W}_{\Phi_{(\tau,\xi)}}\}|x-\xi|^2\}$$
$$\leq C_1 \frac{|x-\xi|^2}{t-\tau} \exp\{-\frac{1}{4}(t-\tau)^{-1} \exp\{-\tilde{W}_{\Phi_{(\tau,\xi)}}\}|x-\xi|^2\}$$

where C_1 is a constant, so by the trick given in [Fri, 3.2 page 7], we obtain that there exist a constant such that both $|LZ|$ and $|\frac{\partial}{\partial \lambda_h} LZ|$ satisfies the same bound as in (8) (with $\nu = 0$). We may now use induction to obtain similar bounds on $\frac{\partial}{\partial \lambda_i}(LZ)_\nu^\lambda$ as in (8) and from this it follows that the series given in (9) converges uniformly in λ which proves the λ-differentiability of $\Phi^\lambda(x,t;\xi,\tau)$ and by an boundedness argument also the λ-differentiability of $\tilde{v}(t,x,\lambda)$. The final result now follows by an induction argument. ■

By further differentiation we obtain the formula

$$|\frac{\partial^p \tilde{v}}{\partial \lambda_1^p}| \le \sum_{k=1}^{p} \binom{p}{k} \|\phi\|^k K_3 e^{\|\phi\|\delta} |\frac{\partial^{(p-k)}\tilde{v}}{\partial \lambda_1^{(p-k)}}| + \|\phi\|^p e^{\|\phi\|\delta} K_3$$

which, by an easy induction argument in connection with [LSU], gives us the estimate

$$|\frac{\partial^p \tilde{v}}{\partial \lambda_1^p}| \le K_3 e^{\|\phi\|\delta}((2 + K_3 e^{\|\phi\|\delta})\|\phi\|)^p p!$$

and by an symmetry argument we obtain the wanted inequality

$$|\partial^\alpha \tilde{v}| \le K_3 e^{\|\phi\|\delta}((2 + K_3 e^{\|\phi\|\delta})\|\phi\|)^{|\alpha|}|\alpha|!$$

■

Theorem 3.12 now finally follows by combining lemma 3.14 and lemma 3.16 with lemma 3.2. ■

§3.3 SPDE #3

THEOREM 3.17 Let $T > 0$ be given and suppose we are given functions $g \in C_b^{\alpha,t}([0,T] \times \mathbb{R}^n)$ and $f \in C_b^{2+\alpha}(\mathbb{R}^n)$. Then the stochastic Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla \cdot \{\mathrm{Exp}\{\mathcal{W}^s_{\phi(t,x)}\} \diamond \nabla u\} + g(t,x) \quad && (t,x) \in (0,T] \times \mathbb{R}^n \\ u(0,x) &= f(x) && x \in \mathbb{R}^n \end{aligned}$$

has an unique solution $[0,T] \times \mathbb{R}^n \ni (t,x) \mapsto u(t,x) \in (\mathcal{S}_{n+1}^{\frac{n(n+1)}{2}})^{-1}$.

PROOF: We will find $q \in \mathbb{N}, \delta > 0$ and a function $[0,T] \times \mathbb{R}^n \ni (t,x) \mapsto \tilde{v}(t,x) \in A_b(\mathbf{B}_q(\delta))$ which solves the equation

$$\frac{\partial \tilde{v}}{\partial t} = \nabla \cdot \{e^{\tilde{\mathcal{W}}^s_{\phi(t,x)}} \cdot \nabla \tilde{v}\} + \nabla \cdot \{e^{\tilde{\mathcal{W}}_{\phi(t,x)}} \cdot \nabla f\} + g(t,x) \quad (t,x) \in (0,T] \times \mathbb{R}^n \qquad (23)$$

$$u(0,x) = 0 \qquad x \in \mathbb{R}^n \qquad (24)$$

when $z \in \mathbf{B}_q(\delta)$. Our solution will then be $u(t,x) := \mathcal{H}^{-1}\tilde{v}(t,x) + f(x)$. The proof consists of several lemmas:

LEMMA 3.18 $\exists K_1 > 0$ such that the solution $\tilde{v}$ of (23) satisfies an inequality of the type

$$\max\{\sup_{t,x,\lambda} |\tilde{v}(t,x,\lambda)|, \sup_{t,x,\lambda,i} |\frac{\partial}{\partial x_i}\tilde{v}(t,x,\lambda)|, \sup_{t,x,\lambda,i,j} |\frac{\partial^2}{\partial x_i \partial x_j}\tilde{v}(t,x,\lambda)|\}$$

$$\le K_1 \left(\sup_{t,x,\lambda} |\nabla \cdot \{e^{\tilde{\mathcal{W}}^s_{\phi(t,x)}} \cdot \nabla f\} + g| \right) \qquad (25)$$

where K_1 is independent of $(t,x) \in [0,T] \times \mathbb{R}^n$ and $\lambda \in \mathbf{B}_q(\delta) \cap \mathbb{R}_0^{\mathbb{N}}$.

PROOF:

As in lemma 3.15. ■

by lemma 3.18. Let

$$K_3 := \begin{cases} K_2 & \text{if } n^2A_1 \geq 1 \\ K_2 n^{-2} A_1^{-1} & \text{if } n^2 A_1 < 1. \end{cases}$$

By applying the differential operator $\frac{\partial}{\partial \lambda_1}$ on equation (23) we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial \tilde{v}}{\partial \lambda_1}\right) = \nabla \cdot \{e^{\tilde{\mathcal{W}}^s_{\phi(t,x)}} \cdot \nabla\left(\frac{\partial \tilde{v}}{\partial \lambda_1}\right)\} + \nabla \cdot \{\frac{\partial}{\partial \lambda_1} e^{\tilde{\mathcal{W}}^s_{\phi(t,x)}} \cdot \nabla \tilde{v}\} + \nabla \cdot \{\frac{\partial}{\partial \lambda_1} e^{\tilde{\mathcal{W}}^s_{\phi(t,x)}} \cdot \nabla f\}$$

and from lemma 3.18 and lemma 3.20 we obtain the estimate

$$\left|\frac{\partial \tilde{v}}{\partial \lambda_1}\right| \leq K_3(n^2A_1A_2 + 2n^2A_1A_2)K_3 + 3K_3n^2A_1A_2.$$

By further differentiation we obtain the formula

$$\left|\frac{\partial^p \tilde{v}}{\partial \lambda_1^p}\right| \leq \sum_{k=1}^{p} \binom{p}{k} K_3(2+k)n^2A_1A_2^k \left|\frac{\partial^{(p-k)}\tilde{v}}{\partial \lambda_1^{(p-k)}}\right| + K_3n^2A_1A_2^p(2+p)$$

which, by an easy induction argument, gives us the estimate

$$\left|\frac{\partial^p \tilde{v}}{\partial \lambda_1^p}\right| \leq n^2K_3A_1((A_1n^2K_3 + 6)A_2)^p p!$$

and by an symmetry argument we obtain the wanted inequality

$$|\partial^\alpha \tilde{v}| \leq n^2K_3A_1((A_1n^2K_3+6)A_2)^{|\alpha|}|\alpha|!$$

■

Theorem 3.17 now follows by combining a modified version of lemma 3.14 with lemma 3.21 and lemma 3.2. ■

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