States and shifts on infinite free products of $C^*$-algebras

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Dedicated to Richard V. Kadison on occasion of his 70th birthday

Abstract. We study quasi equivalence of states of free products of $C^*$-algebras together with free shifts, compute their entropy and show a strong form of unique ergodicity.

1 Introduction

Let for each element $i$ in index set $I$, $A_i$ be a unital $C^*$-algebra and $\phi_i$ a state on $A_i$. Let $(A, \phi) = (A_i, \phi_i)_{i \in I}$ be the corresponding free product $C^*$-algebra as defined in [6, 1.5]. In the present paper we shall study states on $A$, and if $I = \mathbb{Z}$ and all the pairs $(A_i, \phi_i)$ are equal, the shift automorphism on $A$ arising from the shift $i \to i + 1$. Our results will, except for those in the last section, extend those in [5] for the $\text{II}_1$-factor $L(F_\infty)$ defined by the left regular representation of the free group $F_\infty$ in infinite number of generators. Our main result is for general infinite products and shows the existence of a universal function $r : (0, 1] \to \mathbb{N}$ such that whenever $(A, \phi)$ is as above and $\omega$ is a state whose GNS-representation is quasi contained in that of $\phi$, then there is for each $\varepsilon > 0$ a subset $J \subset I$ of cardinality $\text{card } J \leq r(\varepsilon)$, such that $\| (\phi - \omega)|_{A_i} \| < \varepsilon$ for all $i \notin J$.

In the two last sections we assume $I = \mathbb{Z}$ and all the $(A_i, \phi_i)$ are equal and let $\alpha$ denote the free shift of $A$ which arises as mentioned above from the shift on $\mathbb{Z}$. Analogously to the free shift on $L(F_\infty)$ we use the above result to show that the entropy in the sense of Connes, Narnhofer and Thirring [1], called CNT-entropy in the sequel, of $\alpha$ with respect to the invariant state $\phi$ is zero. Then in the last section we show that $\alpha$ satisfies a very strong unique

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ergodicity property. Namely, if \((B, \beta, \mu)\) is a unital \(C^*\)-dynamical system and \(\lambda\) is an \(\alpha \otimes \beta\)-invariant state on \(A \otimes B\) such that \(\lambda(1 \otimes b) = \mu(b)\) for \(b \in B\), then \(\lambda = \phi \otimes \mu\). An immediate corollary of this is that the entropy of Sauvageot and Thouvenot of \(\alpha\) with respect to \(\phi\) is also zero.

We remark that it is not necessary for the above to restrict attention to the free shift. Our arguments work for an arbitrary infinite index set \(I\) and an automorphism arising from a bijection \(\sigma\) of \(I\) such that for all finite subsets \(J \subset I\) there exists \(p \in \mathbb{N}\) such that the sets \(\sigma^m(J), n \in \mathbb{N}\), are all disjoint.

We refer the reader to the book [6] of Voiculescu, Dykema and Nica for the theory of free products of \(C^*\)-algebras.

2 States on free products

Let \(I\) be an index set, and for each \(i \in I\) let \(A_i\) be a unital \(C^*\)-algebra and \(\phi_i\) a state on \(A_i\). Following [6, 1.5.1] we shall define the free product \((A, \phi) = (A_i, \phi_i)_{i \in I}\) with its canonical cyclic representation \(\pi\).

Let \((\pi_i, \mathcal{H}_i, \xi_i)\) be the GNS-representation of \(\phi_i, i \in I\). Let \(\mathcal{H}_i^c = \mathcal{H}_i \otimes \mathbb{C}\xi_i\), and \((\mathcal{H}, \xi) = \sum_{i \in I} (\mathcal{H}_i, \xi_i)\). Put

\[
\mathcal{H}(\iota) = \mathcal{C}\xi \oplus \bigoplus_{n \geq 1} \bigoplus_{\sum_{i_1, \ldots, i_n} \neq \emptyset} \mathcal{H}_{i_1}^c \otimes \cdots \otimes \mathcal{H}_{i_n}^c.
\]

We have unitary operators \(V_i : \mathcal{H}_i \otimes \mathcal{H}(\iota) \to \mathcal{H}\) defined by

\[
\xi_i \otimes \xi \to \xi
\]
\[
\mathcal{H}_i^c \otimes \xi \to \mathcal{H}_i^c \text{ by } \eta \otimes \xi \to \eta
\]
\[
\xi_i \otimes (\mathcal{H}_{i_1}^c \otimes \cdots \otimes \mathcal{H}_{i_n}^c) \to \mathcal{H}_{i_1}^c \otimes \cdots \otimes \mathcal{H}_{i_n}^c \text{ by } \xi_i \otimes \eta \to \eta, i_1 \neq i
\]
\[
\mathcal{H}_i^c \otimes (\mathcal{H}_{i_1}^c \otimes \cdots \otimes \mathcal{H}_{i_n}^c) \to \mathcal{H}_i^c \otimes \mathcal{H}_{i_1}^c \otimes \cdots \otimes \mathcal{H}_{i_n}^c \text{ by } \psi \otimes \eta \to \psi \otimes \eta, i_1 \neq i
\]

The representation \(\lambda_i : A_i \to B(\mathcal{H})\) is defined by

\[
\lambda_i(a) = V_i(\pi_i(a) \otimes 1_{\mathcal{H}(\iota)})V_i^*, \ a \in A_i.
\]

The free product representation \(\pi = \star\pi : \star A_i \to B(\mathcal{H})\) is the \(*\)-homomorphism of the free product \(C^*\)-algebra \((\star A_i, \star \lambda_i) \to B(\mathcal{H})\), using the universal property of the free product. When we write \((A, \phi) = (A_i, \phi_i)_{i \in I}\) we shall mean \(\star A_i\) in the representation \(\pi\) i.e. we shall mean \(\pi(\star A_i) \subset B(\mathcal{H})\).

We can now state the main result of this section, which is a direct generalization of [5, Lem. 2.4]. Since \(C^*\)-algebras isomorphic to the scalars are
redundant in the definition of free products, we shall in order to avoid complications assume the $C^*$-algebras in the product to have linear dimension at least 2. Hence we shall exclude homomorphisms in the theorem. We denote by $|J|$ the cardinality of a set $J$.

**Theorem 1.** For each $\varepsilon \in (0, 1]$ let $r(\varepsilon) = \lfloor 100\varepsilon^{-2} \rfloor + 1$. Then the following holds. Let $(A, \phi) = (A_\varepsilon, \phi_\varepsilon)_{\varepsilon \in I}$ be a free product of unital $C^*$-algebras $A_\varepsilon$ with states $\phi_\varepsilon$ which are not homomorphisms. Suppose $\omega$ is a state of $A$ of the form $\omega = \omega' \circ \pi$ with $\omega'$ a normal state on $\pi(A)''$. Then for each $\varepsilon \in (0, 1]$ there exists a subset $J = J(\omega, \varepsilon) \subset I$ with $|J| \leq r(\varepsilon)$ such that

$$
\|(\phi - \omega)|_{A_\varepsilon}\| < \varepsilon \quad \forall \varepsilon \not\in J, \varepsilon \in I.
$$

**Proof.** We first assume the state $\omega'$ is a vector-state $\omega_\eta$. We use the convention that whenever we write $J \subset I$ we mean a finite ordered subset of $I$ of the form

$$
J = \{\iota_1, \iota_2, \cdots, \iota_n\}, \quad \iota_1 \neq \iota_2 \neq \cdots \neq \iota_n.
$$

Here $n = |J|$. For each $\iota \in I$ we let

$$
I(\iota) = \{J \subset I : \iota_1 = \iota\}.
$$

Since each vector $\xi_\iota$ is cyclic for $\pi_\iota(A_\iota)$ we may (by approximation) assume

$$
\eta = \lambda \xi + \sum_{\iota \in I} \sum_{J \in I(\iota)} \eta_J,
$$

where

$$
\eta_J = \eta_{\iota_1} \otimes \eta_{\iota_2} \otimes \cdots \otimes \eta_{\iota_n}, \quad J = \{\iota_1, \cdots, \iota_n\},
$$

and

$$
\eta_{\iota_k} = \pi_{\iota_k}(a_{\iota_k}^J)\xi_{\iota_k}, \quad k = 1, \cdots, n, \quad a_{\iota_k}^J \in A_{\iota_k}.
$$

Furthermore, $\eta_{\iota_k} \in \mathcal{H}_{\iota_k}^o$, so that $\phi_{\iota_k}(a_{\iota_k}^J) = 0$, and we have that all vectors $\eta_J$ are mutually orthogonal since they are the orthogonal projections of $\eta$ on the Hilbert spaces $\mathcal{H}_{\iota_1}^o \otimes \cdots \otimes \mathcal{H}_{\iota_n}^o$, hence in particular, if $J = K$ then the corresponding vectors $\eta_J$ and $\eta_K$ coincide.

From now on we fix an index $\iota_0 \in I$, and let $a \in A_{\iota_0}$ be self-adjoint with $\phi(a) = \phi_{\iota_0}(a) = 0$. Denote by $\lambda_0 = \lambda_{\iota_0}$, $\pi_0 = \pi_{\iota_0}$, $\xi_0 = \xi_{\iota_0}$, $V_0 = V_{\iota_0}$. Computing we find
\[
\omega(a) = |\lambda|^2 \phi(a) + 2Re \lambda \sum \sum_{i \in I, J \in I(i)} (\lambda_0(a) \xi, \eta_J) \\
+ \sum \sum_{i \in I(i)} (\lambda_0(a) \eta_J, \eta_J) + \sum \sum_{i, p \in I, J \in I(i), K \in I(p)} (\lambda_0(a) \eta_J, \eta_K)
\]
\[
= \sum \sum_{i \in I(i)} (\lambda_0(a) \eta_J, \eta_J) + 2Re \lambda \sum \sum_{i \in I(i)} (\pi_0(a) \xi, V_0^* \eta_J) \\
+ \sum \sum_{i, p \in I, J \in I(i), K \in I(p)} (\pi_0(a) \xi, V_0^* \eta_J, V_0^* \eta_K).
\]

We shall compute the scalar products case by case. We use the notation when \( |J| > 1, J = \{i_1, \cdots, i_n\} \).

\[\eta_J^1 = \eta_{n_2} \otimes \eta_{n_3} \otimes \cdots \otimes \eta_n.\]

(1) \( J \in I(i_0) \). Then

\[
(\pi_0(a) \xi, V_0^* \eta_J) = \begin{cases} 
(\pi_0(a) \xi_0 \otimes \xi, \pi_0(a_{i_0}) \xi_0 \otimes \eta_J^1) & \text{if } |J| > 1 \\
\phi(a_{i_0}^* a) & \text{if } |J| = 1
\end{cases}
\]

\[
= \begin{cases} 
0 & \text{if } |J| > 1 \\
\phi(a_{i_0}^* a) & \text{if } |J| = 1
\end{cases},
\]

where we denote by \( a_0 \) the element \( a_{i_0}^J \in A_{i_0} \) when \( J = \{i_0\} \). We next consider

\[X = (\pi_0(a) \xi, V_0^* \eta_J, V_0^* \eta_K) = (\lambda_0(a) \eta_J, \eta_K).\]

(2) \( J = K = \{i_0\} \). Then as in (1)

\[X = \phi(a_{i_0}^* aa_0).\]

(3) \( J = \{i_0\}, K \in I(i_0), |K| > 1 \). Then

\[X = (\pi_0(a) \xi, \eta_K) = (\pi_0(a) \pi_0(a_0) \xi_0, \pi_0(a_{i_0}^K) \xi_0) = 0\]

(4) \( J, K \in I(i_0), |J| > 1, K = \{i_0\} \). Then as in (3) \( X = 0 \).

(5) \( J, K \in I(i_0), |J| > 1, |K| > 1 \). Then \( V_0^* \eta_J = \eta_J, V_0^* \eta_K = \eta_K \). Thus

\[X = (\pi_0(a) \pi_0(a_{i_0}^J) \xi_0, \pi_0(a_{i_0}^K) \xi_0)(\eta_J^1, \eta_K^1)
\]

\[= \phi(a_{i_0}^{K*} aa_{i_0}^J)(\eta_J, \eta_K)
\]

\[= \begin{cases} 
0 & \text{if } J \neq K \\
\phi(a_{i_0}^{J*} aa_{i_0}^J)||\eta_J||^2 & \text{if } J = K.
\end{cases}
\]
(6) \( J = \{\iota_0\}, K \not\in I(\iota_0). \) Then
\[
X = (\pi_0(a) \otimes 1_{\eta(\iota_0)} \otimes \xi, \xi_0 \otimes \eta_K) = 0.
\]

(7) \( J \not\in I(\iota_0), K = \{\iota_0\}. \) Then similarly \( X = 0. \)

(8) \( J \in I(\iota_0), |J| > 1, K \not\in I(\iota_0). \) Then
\[
X = (\pi_0(a) \otimes 1)\eta_J, \xi_0 \otimes \eta_K
\]
\[
= \phi(aa^*_{\iota_0})(\eta^*_J, \eta_K).
\]

(9) \( J \not\in I(\iota_0), K \in I(\iota_0), |K| > 1. \) Then as in (8)
\[
X = \phi(a_{\iota_0}^{K*}a)(\eta_J, \eta_K^*).
\]

(10) \( J, K \not\in I(\iota_0). \) Then since \( \phi(a) = 0, \)
\[
X = (\pi_0(a) \otimes 1(\xi_0 \otimes \eta_J), \xi_0 \otimes \eta_K)
\]
\[
= \phi(a)(\eta_J, \eta_K)
\]
\[
= 0.
\]

Summing up (1)–(10) we obtain
\[
\omega(a) = \phi(a_{\iota_0}^*aa_0) + \sum_{J \in I(\iota_0), |J| > 1} \phi(a_{\iota_0}^{J*}aa_{\iota_0})||\eta_J^*||^2
\]
\[
+ 2Re\lambda \phi(a_{\iota_0}^*a) + \sum_{J \in I(\iota_0), |J| > 1} \sum_{i \neq \iota_0} \sum_{K \in I(i)} \phi(aa_{\iota_0})\eta_J, \eta_K
\]
\[
+ \sum_{K \in I(\iota_0), |K| > 1} \sum_{J \in I(i)} \phi(a_{\iota_0}^{K*}a)(\eta_J, \eta_K^*).
\]

If \( J = \{\iota_1, \ldots, \iota_n\} \) put \( J_1 = \{\iota_2, \ldots, \iota_n\}. \) Then \( (\eta_J, \eta_K) = 0 \) unless \( K = J_1. \)

We therefore have for \( ||a|| \leq 1, \)
\[
|\omega(a)| \leq 2|\lambda||\eta(\iota_0)|| + ||\eta_{\iota_0}||^2 + \sum_{J \in I(\iota_0), |J| > 1} ||\eta_J||^2
\]
\[
+ \sum_{J \in I(\iota_0), |J| > 1} \phi(a_{\iota_0}^{J*}a)||\eta_J^*||||\eta_{J_1}|| + \sum_{J \in I(\iota_0), |J| > 1} \phi(a_{\iota_0}^{J*}a)||\eta_{J_1}||||\eta_J||
\]
\[
\leq 2|\lambda||\eta(\iota_0)|| + \sum_{J \in I(\iota_0), |J| > 1} ||\eta_J||^2 + 2 \sum_{J \in I(\iota_0), |J| > 1} ||\eta_J||||\eta_{J_1}||,
\]
where the last term arises from the fact that
\[
|\phi(a_{\iota_0}^*a)(\eta_J^*, \eta_K)| \leq ||\eta_{\iota_0}||||\eta_J^*||||\eta_K|| = ||\eta_J||||\eta_K||.
Since \( \sum_{J \notin I(t_0)} \|\eta_J\|^2 \leq 1 \) we thus have from the Cauchy-Schwarz inequality and the fact that the sum \( \|\eta_{J_1}\|^2 \) is the same as over \( K \)'s with \( K \notin I(t_0) \),

\[
|\omega(\alpha)| \leq 2|\lambda|\|\eta_{\{t_0\}}\| + \sum_{J \in I(t_0)} \|\eta_J\|^2 + 2 \left( \sum_{\substack{J \in I(t_0) \atop |J| > 1}} \|\eta_J\|^2 \right)^{1/2} \left( \sum_{\substack{J \in I(t_0) \atop |J| > 1}} \|\eta_J\|^2 \right)^{1/2} \\
\leq 2|\lambda|\|\eta_{\{t_0\}}\|^{1/2} + 3 \left( \sum_{J \in I(t_0)} \|\eta_J\|^2 \right)^{1/2} \\
\leq 5 \left( \sum_{J \in I(t_0)} \|\eta_J\|^2 \right)^{1/2}.
\]

(11)

Note that we have

\[
1 = \|\eta\|^2 = |\lambda|^2 + \sum_{i \in I} \sum_{J \in I(i)} \|\eta_J\|^2.
\]

(12)

If \( C_i \geq 0 \ \forall i \in I \) and \( \sum_{i \in I} C_i \leq 1 \), then given \( \delta > 0 \) and \( r(\delta) = [100\delta^{-2}] + 1 \) then there exists a subset \( J \) of \( I \) with \( |J| \leq r(\delta) \) such that

\[
C_i < \frac{\delta^2}{100}, \quad i \notin J.
\]

Thus by (12) there is a subset \( J(\omega, \varepsilon) \) of \( I \) with \( |J(\omega, \varepsilon)| \leq r(\varepsilon) \) such that

\[
\sum_{J \in I(i)} \|\eta_J\|^2 < \frac{\varepsilon^2}{100} \quad \text{for} \quad i \notin J(\omega, \varepsilon)
\]

In particular by (11)

\[
|\omega(\alpha)| < \varepsilon/2 \quad \text{for} \quad \alpha_0 \notin J(\omega, \varepsilon).
\]

We have thus shown the essence of the theorem in the case \( \omega = \omega' \circ \pi \) with \( \omega' \) a vector state. For the general case let \( \xi_i \in \mathcal{H}, \|\xi_i\| = 1, \ i \in \mathbb{N} \). Let \( \alpha_i \geq 0, \sum_{i=1}^{\infty} \alpha_i = 1, \ i \in \mathbb{N} \). Put \( \omega = \sum_{i=1}^{\infty} \alpha_i \omega_{\xi_i} \circ \pi \), i.e. \( \omega' = \sum_{i=1}^{\infty} \alpha_i \omega_{\xi_i} \). Then

\[
1 = \|\omega(1)\| = \sum_{i=1}^{\infty} \alpha_i \|\xi_i\|^2.
\]

Write as before

\[
\xi_i = \lambda_i \xi + \sum_{i \in I} \sum_{J \in I(i)} \eta_J.
\]

Thus

\[
1 = \|\xi_i\|^2 = |\lambda_i|^2 + \sum_{i \in I} \sum_{J \in I(i)} \|\eta_J\|^2,
\]

6
\[ 1 = \sum_i \alpha_i |\lambda_i|^2 + \sum_{i \in I} \sum_i \sum_{J_i \in I(i)} \|\eta_{J_i}\|^2. \]

As above given \( \varepsilon > 0 \) there is \( J = J(\omega, \varepsilon) \subset I \) with \( |J| \leq r(\varepsilon) \) such that

\[ \sum_i \sum_{J_i \in I(i)} \alpha_i \|\eta_{J_i}\|^2 < \frac{\varepsilon^2}{100}, \quad i \notin J. \]

Therefore, if \( a \in A_{\omega_0}, \; \mu_0 \notin J, \; \phi(a) = 0 \), we have by (11)

\[ |\omega(a)| \leq \sum_i \alpha_i |\omega_{\xi_i}(\pi(a))| \]

\[ \leq 5|a| \sum_i \alpha_i \left( \sum_{J_i \in I(\mu_0)} \|\eta_{J_i}\|^2 \right)^{\frac{1}{2}} \]

\[ = 5|a| \sum_i \alpha_i^{\frac{1}{2}} \left( \sum_{J_i \in I(\mu_0)} \alpha_i \|\eta_{J_i}\|^2 \right)^{\frac{1}{2}} \]

\[ \leq 5|a| \left( \sum_i \alpha_i \right)^{\frac{1}{2}} \left( \sum_{J_i \in I(\mu_0)} \sum_i \alpha_i \|\eta_{J_i}\|^2 \right)^{\frac{1}{2}} \]

\[ = 5|a| \left( \sum_{J_i \in I(\mu_0)} \sum_i \alpha_i \|\eta_{J_i}\|^2 \right)^{\frac{1}{2}} \]

\[ < 5|a| \frac{\varepsilon}{10} \]

\[ = \frac{\varepsilon}{2} |a|. \]

Finally, if \( a \in A_{\omega_0}, \; a = \phi(a)1 + a_0 \) with \( \phi(a_0) = 0, \; a_0 \in A_{\omega_0} \). Then \( |a_0| \leq \|a\| + |\phi(a)| \leq 2|a| \). Thus if \( \mu_0 \notin J \)

\[ |(\phi - \omega)(a)| = |\omega(a_0)| \leq \frac{\varepsilon}{2} |a_0| \|a\| \leq \varepsilon \|a\|. \]

Thus \( |(\phi - \omega)|_{A_{\omega_0}} \| < \varepsilon \) if \( \mu_0 \notin J(\omega, \varepsilon) = J \).

QED.

Following [1] if \( B \) is a C*-algebra and \( \mu \) a state of \( B \) we denote by \( \|x\|_\mu = \mu(x^\ast x)^{1/2} \). Then \( \|x\|_\mu \leq \|x\| \). If \( A \) is another C*-algebra and \( \rho : A \to B \) a linear map we put

\[ \|\rho\|_\mu = \sup_{\|x\| \leq 1} \|\rho(x)\|_\mu. \]

**Corollary 1.** Let \((A, \phi), r\) be as in Theorem 1. Let \( B \) be a finite dimensional Abelian C*-algebra generated by its minimal projections \( p_1, \ldots, p_n \). Suppose \( P : A \to B \) is a positive unital linear map, so that \( P(x) = \sum_{i=1}^n \psi_i(x)p_i \)

\[ 7 \]
with \( \psi_i \) a state of \( A \). Suppose \( \mu \) is a state of \( B \) such that \( \mu \circ P = \phi \). Then given \( \varepsilon > 0 \) there exists \( J = J(P, \varepsilon) \subset I \) with \( |J| \leq r(\varepsilon) \) such that

\[
\| P(x) - \phi(x)1 \|_\mu < \varepsilon \| x \|, \quad x \in A_i, \ i \notin J.
\]

Proof. We have

\[
\phi(x) = \mu \circ P(x) = \sum_{i=1}^{n} \psi_i(x)\mu(p_i).
\]

Thus if \( \mu(p_i) \neq 0, \psi_i \leq \mu(p_i)^{-1}\phi \), so that \( \psi_i = \omega_{\xi_i} \circ \pi \) with \( \xi_i \) a unit vector in \( H \). Therefore, if \( a \in A_{i_0} \), is self-adjoint and \( \phi(a) = 0 \), it follows by (11) that

\[
|\psi_i(a)|^2 \leq 25 \sum_{J_i \in I(\iota_0)} \| \eta_{J_i} \|^2,
\]

where the notation is as in the proof of Theorem 1 and \( \xi_i = \lambda_i\xi + \sum_{J_i \in I(\iota_0)} \eta_{J_i} \).

Given \( \varepsilon > 0 \) choose \( J = J(P, \varepsilon) \subset I \) with \( |J| \leq r(\varepsilon) \) such that

\[
\sum_{i} \left( \sum_{J_i \in I(\iota_0)} \| \eta_{J_i} \|^2 \right) \mu(p_i) < \frac{\varepsilon^2}{100} \quad \text{for} \quad \iota_0 \notin J.
\]

Thus we have for \( \iota_0 \notin J \), since the cases \( \mu(p_i) = 0 \) don’t matter,

\[
\sum_{i} \psi_i(a)^2 \mu(p_i) \leq 25\|a\|^2 \sum_{i} \left( \sum_{J_i \in I(\iota_0)} \| \eta_{J_i} \|^2 \right) \mu(p_i)
\]

\[
< 25\|a\|^2 \cdot \frac{\varepsilon^2}{100} = \|a\|^2(\varepsilon/2)^2.
\]

Since \( \phi(a) = 0 \) and \( a \) in self-adjoint,

\[
\| P(a) - \phi(a)1 \|_\mu = \sum_{i=1}^{n} \psi_i(a)^2 \mu(p_i) < \|a\|^2 \left( \frac{\varepsilon}{2} \right)^2.
\]

For general self-adjoint \( a \in A_{i_0}, \ a = \phi(a) + a_0, \ \phi(a_0) = 0 \). Hence if \( \iota_0 \notin J \),

\[
\| P(a) - \phi(a)1 \|_\mu = \mu(\| \phi(a) + P(a_0) - \phi(a) \|^2)^{1/2}
\]

\[
= \mu((P(a_0))^2)^{1/2}
\]

\[
< \|a_0\|\varepsilon/2
\]

\[
\leq 2\|a\|\varepsilon/2
\]

\[
= \varepsilon\|a\|.
\]

The conclusion follows. QED

3 Entropy of free shifts

In [5] it was shown that the entropy of the free shift on \( L(F_\infty) \) is zero. We shall now generalize this result to arbitrary free shifts.
Theorem 2. Let $A_0$ be a unital $C^*$-algebra and $\phi_0$ a state of $A_0$. Let $A_i = A_0$, $\phi_i = \phi_0$, $i \in \mathbb{Z}$, and let $(A, \phi) = (\star A_i, \star \phi_i)_{i \in \mathbb{Z}}$. Let $\alpha$ be the free shift on $A$, i.e. $\alpha$ is the automorphism of $A$ arising from the shift $n \to n + 1$ on $\mathbb{Z}$. Then the CNT-entropy of $\alpha$ with respect to $\phi$, $h_\phi(\alpha) = 0$.

Proof. If $\phi_0$ is a homomorphism each $\phi_i$ can be identified with its GNS-representation, hence the GNS-representation of $\phi$ is one dimensional, so $\phi$ is a homomorphism, thus $h_\phi(\alpha) = 0$. We therefore assume each $\phi_i$ is not a homomorphism.

Let $C$ be a finite dimensional $C^*$-algebra and $\gamma : C \to A$ a unital completely positive map. Let $\varepsilon > 0$ and $r = r(\varepsilon/2)$ as in Theorem 1. Choose $k \in \mathbb{N}$ so large that

$$k^{-1}rS(\phi \circ \gamma) < \varepsilon.$$

Let $B$ be a finite dimensional Abelian $C^*$-algebra generated by its minimal projections $p_1, \cdots, p_n$. Suppose $P : A \to B$ is a positive unital map and $\mu$ a state of $B$ such that $\mu \circ P = \phi$. For each $i \in \{1, \cdots, n\}$ there is a state $\psi_i$ of $A$ such that

$$P(x) = \sum_{i=1}^n \psi_i(x)p_i.$$

Then

$$\phi = \mu \circ P = \sum_{i=1}^n \mu(p_i)\psi_i,$$

is $\phi$ written as a convex combination of states. In the notation of [1] we have

$$\varepsilon_\mu(P) = \sum_{i=1}^n \mu(p_i)S(\phi|\phi_i)$$

$$s_\mu(P) = S(\mu) - \varepsilon_\mu(P).$$

Since $A = \star A_i$ and $C$ is finite dimensional there is $n_0 \in \mathbb{N}$ such that if $x \in C$ then there is $a \in A_0 = \bigstar_{i=-n_0}^{n_0} A_i$ such that $\|\gamma(x) - a\| < \varepsilon/4\|x\|$, and $\|a\| \leq \|x\|$. Let $p = 2n_0 + 1$ and $\tilde{A}_m = \alpha^{mp}(A_0)$, $\tilde{\phi}_0 = \bigstar_{i=-n_0}^{n_0} \phi_i$, $\tilde{\phi}_m = \tilde{\phi}_0 \circ \alpha^{mp}$. By distributivity [6, 2.5.5] and the uniqueness of the GNS-representation of $\phi$, we can write $(A, \phi)$ as the free product

$$(A, \phi) = (\star \tilde{A}_m, \star \tilde{\phi}_m)_{m \in \mathbb{Z}}.$$

Furthermore, $\alpha^p$ acts as a free shift on $A$.

For each $j \in \{0, \cdots, k - 1\}$ let $B_j$ be a $C^*$-subalgebra of $B$, and let $E_j : B \to B_j$ be the unique $\mu$-invariant conditional expectation of $B$ onto
Then \(E_j\) satisfies the Cauchy-Schwarz inequality \(E_j(x^*x) \geq E_j(x)^*E_j(x)\). If \(\Phi\) is a self-adjoint linear map of a \(C^*\)-algebra \(D\) into \(B\), we have for \(y \in D\),

\[
\|E_j \circ \Phi(y)\|_\mu^2 = \mu(E_j(\Phi(y^*))E_j(\Phi(y))) \\
\leq \mu(E_j(\Phi(y)^*)\Phi(y))) \\
= \mu(\Phi(y)^*\Phi(y)) \\
= \|\Phi(y)\|_\mu^2.
\]

Now put

\[P_j = E_j \circ P \circ \alpha^{jp} \circ \gamma : C \to B_j.\]

Choose the set \(J \subset \mathbb{N}\) with \(|J| \leq r(\varepsilon/2)\) corresponding to \(P\) as in Corollary 1. Let \(J_0 = J \cap \{0, 1, \cdots, k\}\). Then

\[
\|P(x) - \phi(x)1\|_\mu < \varepsilon/2\|x\|, \ x \in (\tilde{A}_j), j \notin J_0.
\]

For \(x \in C\), \(\|x\| \leq 1\) choose \(y \in \tilde{A}_j\) such that \(\|y\| \leq 1\), and

\[
\|y - \alpha^{jp} \circ \gamma(x)\| < \varepsilon/4.
\]

Then we have by the above estimates applied to \(\Phi_j = (P - \phi) \circ \alpha^{jp} \circ \gamma\),

\[
\|P_j(x) - \phi \circ \gamma(x)\|_\mu = \|E_j \circ P \circ \alpha^{jp} \circ \gamma(x) - E_j \circ \phi \circ \alpha^{jp} \circ \gamma(x)\|_\mu \\
\leq \|\mu(P - \phi) \circ \alpha^{jp} \circ \gamma(x)\|_\mu \\
\leq \|\mu(P - \phi)(y)\|_\mu + \|\mu(P - \phi)(y - \alpha^{jp}\gamma(x))\|_\mu \\
< \varepsilon/2\|y\| + \varepsilon/2\|x\| \\
< \varepsilon,
\]

for \(j \notin J_0\), since \(\|P - \phi\|_\mu \leq 2\).

If we in the notation of [1, VI.2] let \(\rho = P_j, \rho' = \phi \circ \gamma\) then we have for \(j \notin J_0\)

\[(13) \quad |s_\mu(P_j) - s_\mu(\phi \circ \gamma)| \leq \delta(n, d, \varepsilon),\]

where \(d\) is the linear dimension of \(C\) and \(\lim_{\varepsilon \to 0} \delta(n, d, \varepsilon) = 0\). By definition the entropy of the Abelian model \((B, E_j, P, \mu)\) for \((A, \phi, \gamma, \gamma \circ \alpha^p, \cdots, \gamma \circ \alpha^{p(k-1)})\) is

\[S(\mu| \bigvee_{0}^{k-1} B_j) - \sum_{j=0}^{k-1} s_\mu(P_j).\]
By subadditivity of $S$ and [1, III.3] we therefore have that the entropy of the Abelian model is smaller than

$$
S(\mu | \bigvee_{j \in J_0} B_j) + S(\mu | \bigvee_{j \notin J_0} B_j) - \sum_{j \in J_0} s_\mu(P_j) - \sum_{j \notin J_0} s_\mu(P_j) \\
= (S(\mu | \bigvee_{j \in J_0} B_j) - \sum_{j \in J_0} s_\mu(P_j)) + (S(\mu | \bigvee_{j \notin J_0} B_j) - \sum_{j \notin J_0} s_\mu(P_j)) \\
\leq \sum_{j \in J_0} S(\phi \circ \gamma^p) + \text{ Entropy of Abelian model } (B, E_j, P, \mu; j \notin J_0).
$$

As in the proof [1, VI. 3] it follows by (13) that the entropy of the Abelian model $(B, E_j, P, \mu; j \notin J_0)$ differs from that defined by $\phi \circ \gamma$ by at most $(k - |J_0|) \varepsilon' \leq k \varepsilon'$, where $\varepsilon' > 0$ is a number which converges to zero with $\varepsilon$. It follows that the entropy of the Abelian model $(B, E_j, P, \mu)$ differs from that of $(B, \phi)$ by less than

$$r(\varepsilon/2)S(\phi \circ \gamma) + k\varepsilon'. $$

We therefore have

$$|H_{\phi}(\gamma, \alpha^p \circ \gamma, \ldots, \alpha^{p(k-1)} \circ \gamma) - H_{\phi}(\phi \circ \gamma)| < r(\varepsilon/2)S(\phi \circ \gamma) + k\varepsilon'. $$

By [1, III.3] $H_{\phi}(\phi \circ \gamma) \leq S(\phi \circ \gamma)$. Therefore, with our original choice of $k$ as satisfying $k^{-1}r(\varepsilon/2)S(\phi \circ \gamma) < \varepsilon$, we find

$$\frac{1}{k}H_{\phi}(\gamma, \alpha^p \circ \gamma, \ldots, \alpha^{p(k-1)} \circ \gamma) \leq \frac{1}{k}S(\phi \circ \gamma) + \frac{1}{k}r(\varepsilon/2)S(\phi \circ \gamma) + \varepsilon' \leq \varepsilon + \varepsilon + \varepsilon',
$$

which can be made arbitrarily small. As in [5, Lem. 3.4] this means that $\frac{1}{m}H_{\phi}(\gamma, \alpha \circ \gamma, \ldots, \alpha^{m-1} \circ \gamma)$ can be made arbitrarily small, hence $H_{\phi, \gamma}(\alpha) = 0$ for all $\gamma$, i.e. $h_{\phi}(\alpha) = 0$. QED

4 Unique ergodicity of free shifts

In ergodic theory an automorphism of $C(X)$, $X$ compact Hausdorff, is said to be uniquely ergodic if there exists a unique invariant probability measure on $X$, or equivalently a unique invariant state. For free shifts we shall prove a much stronger property. Our proof is a modification of an argument of Powers [3].

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Theorem 3. Let $A_0$ be a unital $C^*$-algebra with a state $\phi_0$. Let $A_i = A_0$, $\phi_i = \phi_0, i \in \mathbb{Z}$, and put $(A, \phi) = (\ast A_i, \ast \phi_i)_{i \in \mathbb{Z}}$. Let $\alpha$ be the free shift and suppose $(B, \beta, \mu)$ is a $\mathcal{C}^*$-dynamical system (i.e. $B$ is a $\mathcal{C}^*$-algebra, $\beta$ an automorphism and $\mu$ a $\beta$-invariant state). Suppose $\lambda$ is an $\alpha \otimes \beta$-invariant state on $A \otimes B$ such that $\lambda(1 \otimes b) = \mu(b)$ for $b \in B$. Then $\lambda = \phi \otimes \mu$.

Proof. For each $i \in \mathbb{Z}$ let

$A_i = \{ a \in A_i : \phi_i(a) = 0 \}$
$\bar{A} = \text{span}\{ \prod_{k=1}^{n} a_{i_k} : a_{i_k} \in A_{i_k}, i_k \neq i_{k+1} \}$.

Then $\mathcal{C}1 + \bar{A}$ is dense in $A$, and $\phi(a) = 0$ for all $a \in \bar{A}$. For $k \in \{1, \cdots, s\}$ fix $a_k \in \bar{A}$, $\phi_k \in B$, such that $a = \sum_{k=1}^{s} a_k \otimes b_k$ is self-adjoint in $\bar{A} \otimes B \subset A \otimes B$. We shall show $\lambda(a) = 0$. We have

$a_k = \sum l \prod_{i_k} a_{k_i,l}$ with $a_{k_i,l} \in \bar{A}_{k_i}$

is a finite sum of finite products of operators in different $\bar{A}_i$'s. Let

$J = \{ j \in \mathbb{Z} : a_{k_i,j} \in \bar{A}_j \text{ for some } a_k \}$,

i.e. $J$ is the set of indices $i$ such that some $a_i' \in \bar{A}_i$ appears in the decomposition of $a$ into a finite sum of finite products of elements in the $\bar{A}_i$.

If we represent $(A, \phi)$ in its GNS-representation we may assume $(A, \phi)$ acts on the Hilbert space $\mathcal{H}$, where

$\mathcal{H} = \mathcal{C} \xi \bigoplus_{i_1 \neq i_2 \neq \cdots \neq i_r} \mathcal{H}_{i_1} \otimes \cdots \otimes \mathcal{H}_{i_r}$. 

For each $n \in \mathbb{Z}$ let

$\mathcal{H}(n) = \bigoplus_{i_1 = n}^{\infty} \mathcal{H}_{i_1} \otimes \cdots \otimes \mathcal{H}_{i_r}$

(Thus $\mathcal{H} = \mathcal{C} \xi \bigoplus_{n=-\infty}^{\infty} \mathcal{H}(n)$). Let

$\mathcal{H}_J = \bigoplus_{n \in J} \mathcal{H}(n) \subset \mathcal{H}$.

Then

$\mathcal{H}_J^\perp = \mathcal{C} \xi \bigoplus_{m \in J^c} \mathcal{H}(m)$,

where $J^c$ is the complement of $J$ in $\mathbb{Z}$.
If $\eta = \eta_i \otimes \cdots \otimes \eta_r \in \mathcal{H}(m)$, $m \in J^C$, and $i \in J$ then if $a_i \in \mathcal{A}_i$, we have (see section 2),
\[
\pi(a_i)\eta = \lambda_i(a_i)\eta = V_i(\pi_i(a_i) \otimes 1)V_i^*\eta = V_i(\pi_i(a_i) \otimes 1)\xi_i \otimes \eta = V_i(\pi_i(a_i)\xi_i \otimes \eta) = \pi_i(a_i)\xi_i \otimes \eta \in \mathcal{H}J.
\]

If $j \neq i, j \in J$, then similarly for $a_j \in \mathcal{A}_j$,
\[
\pi(a_ja_i)\eta = \pi(a_j)\pi(a_i)\eta = \pi(a_j)(\pi_i(a_i)\xi_i \otimes \eta) = \pi_j(a_j)\xi_j \otimes \pi_i(a_i)\xi_i \otimes \eta \in \mathcal{H}J.
\]

An easy induction argument shows that with $a_k \in \mathcal{A}$ as in the beginning of the proof, then $\pi(a_k)\eta \in \mathcal{H}J$, hence we have
\[
\pi(a_k)\mathcal{H}^J \subset \mathcal{H}J.
\]

Suppose $B$ acts on the Hilbert space $K$. Let
\[
M_J = \mathcal{H}J \otimes K.
\]

Then $M_J^J = \mathcal{H}^J \otimes K$, so $a = \sum a_k \otimes b_k$ satisfies
\[
\pi \otimes id(a)M_J^J \subset M_J.
\]

Since $\alpha$ is the free shift there exist integers $0 = n_1 < n_2 < \cdots < n_{20}$ such that if $a_0$ denotes the shift on $\mathbb{Z}$ then the sets $\alpha_0^{n_i}(J)$, $i = 1, \cdots , 20$, are all disjoint. Put
\[
b = \frac{1}{20} \sum_{i=1}^{20} (\alpha \otimes \beta)^{n_i}(a).
\]

Then $b \in \mathcal{A} \otimes B$. For simplicity of notation identify $a$ and $\pi \otimes id(a)$ and similarly for $(\alpha \otimes \beta)^{n_r}(a)$. Put
\[
\mathcal{H}_{J_r} = \bigoplus_{n \in \alpha_0^{n_r}(J)} \mathcal{H}(n),
\]
\[
M_{J_r} = \mathcal{H}_{J_r} \otimes K.
\]

Let $e_r$ denote the orthogonal projection of $H = \mathcal{H} \otimes K$ onto $M_{J_r}$. Since the sets $\alpha_0^{n_i}(J)$ are mutually disjoint, the projections $e_r$, $r = 1, \ldots , 20$, are mutually orthogonal. Furthermore
\[
(\alpha \otimes \beta)^{n_r}(a) : e_r^+(H) \to e_r(H).
\]
Corollary 2. The Sauvageot-Thouvenot entropy of a free shift is zero.


References


