On the Positivity of the Stochastic Heat Equation

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Abstract
We study the positivity preserving properties of the heat equation with random potential and initial condition. Moreover, we give a generalized Feynman–Kac formula for the solution of the problem using the $\mathcal{S}$–Transform. We show that this solution is positive, when the random initial condition is positive. For the case of just time–dependent random potential, we present a special representation of the solution together with regularity results.

1 Introduction

Within the White Noise Analysis, the $\mathcal{S}$– and Hermite–Transforms can be used to study stochastic partial differential equations. Given a problem of stochastic type, the transforms map the problem into a deterministic one where classical theory can be applied. This scheme has been used by several authors. We here mention some works in this direction: Holden et. al. [HLØUZ 1, HLØUZ 2, HLØUZ 3], Lindstrom et. al. [LØU 1, LØU 2], Potthoff [P 1, P 2], Benth [B 2], Gjerde et. al. [GHLØUZ] and Våge [V].

In this paper, we will consider models for stochastic heat transport. The equation

$$\frac{\partial u}{\partial t}(t, x) = \nu \frac{\partial^2 u}{\partial x^2}(t, x) + \sigma(t, x)u(t, x)$$

is known to model transport of heat, where we have a source or potential (given by $\sigma$) pumping heat into the system (or removing heat out of the system) proportional to the temperature $u$ at position $x$ and time $t$. If we don’t have complete knowledge about the source function $\sigma$, a possible modification of (1) could be the following stochastic model:

$$\frac{\partial u}{\partial t}(t, x, \omega) = \nu \frac{\partial^2 u}{\partial x^2}(t, x, \omega) + (\sigma(t, x) + h(t, x)W_t(x(\omega)))u(t, x, \omega),$$

1
where $W_{t,x}(\omega)$ is white noise in time and space. In section 3 we will give a precise interpretation of (2).

In nonlinear filtering theory, (2) with time-dependent noise $W_t$ appears as a model for the so-called unnormalized conditional density. In this situation, (2) is known as the Zakai-equation (see [Z]) and has been studied by — among others — Krylov and Rozovskii [KR] and Pardoux [Pa]. By methods from White Noise Analysis, (2) with time-dependent noise has been solved explicitly by Benth [B 2], Bertini and Cancrini [BC], Nualart and Zakai [NZ 2] and Holden et. al. [HLØUZ 2] have considered (2) with a noisy potential in both time and space.

The main object of this paper is to study problem (2) with random initial conditions. By using the $\mathcal{S}$–Transform, we show in section 3 that the unique solution of (2) can be represented by a Feynman–Kac type formula. With this explicit representation, we prove that the solution is positive (in a generalized sense) when the initial conditions are positive. In the deterministic case, it is well-known that (1) with positive initial conditions, give positive solutions. One says that the problem is positivity preserving, something which is natural considering the physical interpretation of (1). Also in the case of a random potential $W_{t,x}$, but with deterministic initial conditions, one sees that positivity is preserved (e.g. [B 2]). However, if the initial conditions are random, we meet difficulties. For example, consider

$$
\frac{\partial u}{\partial t}(t, x, \omega) = \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t, x, \omega) + u(t, x, \omega) W_t(\omega),
$$

$$
u(0, x, \omega) = u_0(x, \omega),
$$

interpreted in the Itô sense. In section 3 we show that the unique solution of this Cauchy problem is

$$u(t, x, \omega) = \left( \int_{\mathbb{R}} u_0(y, \omega) p_t(x, y) \, dy \right) \circ \exp(B_t(\omega) - \frac{1}{2} t),$$

where $p_t(x, y)$ is the heat kernel, $B_t = \int_0^t W_s \, ds$ is Brownian motion and $\circ$ is the Wick product (see section 2). Of course, if $u_0$ is positive, $\int u_0(y, \omega) p_t(x, y) \, dy$ is positive, and hence the solution $u(t, x, \omega)$ consists of a Wick product of two positive elements. Although in general the Wick product — unlike the pointwise ordinary product — does not preserve positivity (see [K]), we will in section 2 show that in some special cases it does. (We remark that the above example is worked out in more detail at the end of section 4.)

We derive some formulæ for the Wick product with the normalized exponential at the end of section 2. If we denote $\tau_\eta$ translation in direction $\eta \in \mathcal{S}'(\mathbb{R}^{d+1})$, and $\tau^{\ast}_\eta$ its adjoint, we prove that

$$\tau^{\ast}_\eta \Phi = \Phi \circ \text{Exp} W_\eta.$$  

(3)

This relation is easy to prove using the $\mathcal{S}$–Transform, and it says that if $\Phi$ is positive, then the Wick product with a Wick exponential is also positive. In the
special case of $d = 0$ and $\eta \in L^2(\mathbb{R})$, it can be shown that

$$(\tau_\eta \Phi) \cdot \text{Exp}_\eta = \Phi \circ \text{Exp}_\eta,$$

for a class of Hida distributions $\Phi$. Such a relation was first introduced by Gjessing in [G] in a Malliavin Calculus setting. In Benth and Gjessing [BG] the relation was generalized within the White Noise Analysis, and used to show existence of a solution to a stochastic nonlinear partial differential equation. In section 4 we will use this formula on the special case of (2) with a time-dependent white noise potential. In this case, we can also prove a regularity result for the solution dependent on the initial condition.

2 White Noise Preliminaries

In this section we present some aspects of the White Noise Analysis. For a complete account of the theory, the reader should confer the excellent book by Hida et. al. [HKPS]. At the end of the section, some new results concerning the translation operator on $(\mathcal{S})$ and its adjoint are proved.

Let $\mathcal{S}(\mathbb{R}^{d+1})$ denote the Schwartz functions on $\mathbb{R}^{d+1}$, and $\mathcal{S}'(\mathbb{R}^{d+1})$ its dual, the space of tempered distributions. We equip the space of tempered distributions with the weak-$*$-topology, and denote by $\mathcal{B}$ the $\sigma$-algebra of Borel subsets. By the Bochner–Minlos theorem we have a probability measure $\mu$ on $(\mathcal{S}'(\mathbb{R}^{d+1}), \mathcal{B})$, with characteristic functional

$$\int_{\mathcal{S}'(\mathbb{R}^{d+1})} e^{i\langle \omega, \xi \rangle} \, d\mu(\omega) = e^{-\frac{1}{2} ||\xi||^2},$$

where $\omega \in \mathcal{S}'(\mathbb{R}^{d+1})$ and $\xi \in \mathcal{S}(\mathbb{R}^{d+1})$. $|| \cdot ||_2$ denotes the $L^2(\mathbb{R}^{d+1})$-norm. We call the space

$$(\mathcal{S}'(\mathbb{R}^{d+1}), \mathcal{B}, \mu)$$

the white noise probability space.

By the Wiener–Itô–Segal decomposition theorem, every element $\Phi \in (L^2) := L^2(\mu)$ has the form

$$\Phi(\omega) = \sum_{n=1}^{\infty} (\omega^{\otimes n}, \Phi^{(n)}),$$

where the $\Phi^{(n)}$'s are symmetric functions of $L^2(\mathbb{R}^{(d+1)n})$. By $\omega^{\otimes n}$ : we mean the Wick ordering of $\omega \in \mathcal{S}'(\mathbb{R}^{d+1})$. Let $\mathcal{P}$ denote the subspace of polynomials in $(L^2)$, i.e. the space of elements on the form

$$\phi(\omega) = \sum_{n=1}^{N} (\omega^{\otimes n}, \phi^{(n)}).$$
Let $H$ be the Harmonic Oscillator

$$H = -\frac{d^2}{dx^2} + (x^2 + 1).$$

We define the space $(S)_p, \ p \in \mathbb{N}_0$ to be the completion of $\mathcal{P}$ under the norm

$$\|\phi\|_{2,p}^2 = \sum_{n=1}^{\infty} n! \| (H^{\otimes n(d+1)} \phi)_{1}^{2} \|_{L^2(\mathbb{R}^{d+1})}^2.$$

We will in the sequel denote the norms in $(L^p)$ by $\| \cdot \|_p$. Let $(S)_{-p}$ denote the dual of $(S)_p$. The space $(S)$ of Hida test functions is given by the projective limit of $(S)_p$. Furthermore, its dual $(S)^*$, the space of Hida distributions, is defined as the inductive limit of $(S)_{-p}$. We have the following triplet:

$$(S) \subset (L^2) \subset (S)^*.$$ 

The Hida test functions and distributions can be characterized through the $S$–Transform. (See [PS], [HKPS] and [KLPSW].) We discuss this more closely: Observe that for each $\xi \in S(\mathbb{R}^{d+1})$, the normalized exponential of the coordinate process $W_\xi(\omega) = (\omega, \xi)$

$$\text{Exp} W_\xi(\omega) := \exp(W_\xi(\omega) - \frac{1}{2} |\xi|^2)$$

is a Hida test function. This normalized exponential is frequently called the Wick exponential. The $S$–Transform is defined for $\Phi \in (S)^*$ as

$$S\Phi(\xi) = \langle (\Phi, \text{Exp} W_\xi) \rangle.$$

$\langle \cdot, \cdot \rangle$ denotes the dual pairing between $(S)^*$ and $(S)$. Observe that $S\Phi$ is a mapping from $S(\mathbb{R}^{d+1})$ into $\mathcal{C}$. The characterization of $(S)^*$ goes via the notion of $U$–Functionals: Let $F$ be a mapping

$$F : S(\mathbb{R}^{d+1}) \rightarrow \mathcal{C}.$$ 

$F$ is called a $U$–Functional if the following two properties are satisfied:

(i) For given $\xi, \eta \in S(\mathbb{R}^{d+1})$, the mapping

$$\lambda \in \mathbb{R} \rightarrow F(\xi + \lambda \eta) \in \mathcal{C}$$

has an entire extension to $\mathcal{C}$.

(ii) There exist a $\ p \in \mathbb{N}_0$ and constants $K_1, K_2 > 0$ such that for $z \in \mathcal{C}, \xi \in S(\mathbb{R}^{d+1})$ we have the bound:

$$|F(z\xi)| \leq K_1 \exp(K_2 |z|^2 |\xi|^2_{2,p}).$$

4
We remark that $| \cdot |_{2,p}$ denotes the norm of the space $S_p(\mathbb{R}^{d+1})$ given by the Harmonic Oscillator $H$, i.e:

$$|\xi|_{2,p} = |(H^{d+1})^p \xi|_2.$$ We have the following characterization theorem:

**Theorem 1** The $S$-Transform is a bijection onto the space of $U$-Functionals.

It is easy to see that the product of two $U$-Functionals again is a $U$-Functional. Hence, we can define the Wick product between two Hida distributions as follows: Let $\Phi, \Psi \in (S)^*$, then the Wick product of $\Phi, \Psi$, denoted $\Phi \circ \Psi$, is defined by

$$S(\Phi \circ \Psi)(\xi) = S\Phi(\xi)S\Psi(\xi).$$

By the characterization theorem, $\Phi \circ \Psi \in (S)^*$.

We can define the time–space white noise $W_{t,x}$ as an element of $(S)^*$ by using the characterization theorem: For each $\xi \in S(\mathbb{R}^{d+1})$, introduce $W_{t,x}$ as the Hida distribution with $S$-Transform

$$SW_{t,x}(\xi) := \xi(t,x).$$

We see that $W_{t,x}$ will have chaos expansion given as

$$W_{t,x}(\omega) = (\omega, \delta_{t,x}),$$

where $\delta_{t,x}$ is Dirac's $\delta$–function. We define the Wick exponential of $\Phi \in (S)^*$ as the $(S)^*$–element

$$\text{Exp} \Phi = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^{\otimes n},$$

whenever $e^{S\Phi(\xi)}$ is a $U$–Functional. Note that in the case $\Phi = W_{\xi}$, a coordinate process, this definition coincides with the normalized exponential mentioned above. I. e.

$$\text{Exp} W_{\xi} = \sum_{n=0}^{\infty} \frac{1}{n!} W_{\xi}^{\otimes n} = \exp(W_{\xi} - \frac{1}{2} |\xi|^2_2).$$

Strong convergence in $(S)^*$ is characterized through convergence of $U$–Functionals. (See theorem 4.41. in [HKPS]):

**Theorem 2** Assume that $(F_n)_n$ is a sequence of $U$–Functionals with the following properties:

(i) for every $\xi \in S(\mathbb{R}^{d+1})$, $(F_n(\xi))_n$ is a Cauchy sequence,

(ii) there exist $C_1, C_2 > 0, p \in \mathbb{N}_0$ so that for all $n \in \mathbb{N}, \xi \in S(\mathbb{R}^{d+1}), z \in \mathbb{C}$

$$|F_n(z\xi)| \leq C_1 e^{C_2 |z|^2} |\xi|_{2,p}.$$
Then there exists a unique element $\Phi \in (S)^*$ such that $S^{-1} F_n$ converges strongly to $\Phi$.

We study integration of elements in the Hida distribution space and its connection to stochastic integration. Consider the measure space $(X, \nu)$, and the mapping $x \to \Phi(x)$ from $X$ into $(S)^*$. If

$$\langle \langle \Phi(x), \phi \rangle \rangle \in L^1(X, \nu)$$

for each test function $\phi \in (S)$, we define the Pettis integral $\int_X \Phi(x) \, d\nu(x)$ as the unique $(S)^*$-element given by

$$\langle \langle \int_X \Phi(x) \, d\nu(x), \phi \rangle \rangle = \int_X \langle \langle \Phi(x), \phi \rangle \rangle \, d\nu(x).$$

Note that the $S$-Transform commutes with integration:

$$S(\int_X \Phi(x) \, d\nu(x))(\xi) = \int_X S\Phi(x)(\xi) \, d\nu(x).$$

Integrability in $(S)^*$-sense can be characterized through the $S$-Transform. In [KLPSW] and [HKPS] we find the following:

**Theorem 3** Let $(X, \nu)$ be a measure space, and $x \to \Phi(x)$ a mapping from $X$ to $(S)^*$. We assume that the $S$-Transform $F_x = S\Phi(x)$ satisfies the following conditions:

1. for every $\xi \in S(\mathbb{R}^{d+1})$ the mapping $x \to F_x(\xi)$ is measurable,
2. there exists a $p \in \mathbb{N}_0$ such that for all $x \in X$, $F_x$ satisfies the bound

$$|F_x(\xi)| \leq C_x e^{K_x|x|_p^2|x|_p^2},$$

where $x \to K_x$ is bounded $\nu$-a.e., and $x \to C_x$ is integrable with respect to $\nu$.

Then there exists a $q \in \mathbb{N}_0$ such that $\Phi(\cdot)$ is Bochner integrable on $(S)^{*-q}$. In particular,

$$\int_X \Phi(x) \, d\nu(x) \in (S)^*$$

and

$$S \left( \int_X \Phi(x) \, d\nu(x) \right)(\xi) = \int_X S\Phi(x)(\xi) \, d\nu(x),$$

for $\xi \in S(\mathbb{R}^{d+1})$.

If we consider $d = 0$, i.e. the probability space $(S'(\mathbb{R}), \mu)$, we have white noise $W_t$ with chaos decomposition

$$W_t(\omega) = \langle \omega, \delta_t \rangle,$$
where $\delta_t$ is the Dirac $\delta$–function. Brownian motion is given as

$$B_t(\omega) = (\omega, 1_{[0,t]}).$$

Moreover, we have

$$\int_0^t W_s \, ds = B_t.$$

If we assume $\Phi(t) \in (L^2)$ to be Skorohod integrable on $[0,T]$ equipped with Lebesgue measure, the following relation holds true:

$$\int_0^T \Phi(t) \, \delta B_t = \int_0^T \Phi(t) \circ W_t \, dt. \quad (6)$$

$\delta B_t$ denotes Skorohod integration. For the proof of this formula, see e.g [HKPS], [LÖU 2] and [B 1]. (The definition of the Skorohod integral can for instance be found in [NZ 1]).

In the rest of this section we are concerned with the translation operator and its adjoint on the Hida spaces: Let $\eta \in S'(R^{d+1})$ and define the translation operator

$$\tau_\eta : (S) \to (S)$$

by

$$\tau_\eta \phi(\omega) = \phi(\omega + \eta), \quad \phi \in (S).$$

By theorem 4.15 in [HKPS], we know that this is a linear continuous operator on $(S)$. The adjoint $\tau^*_\eta$ of the translation operator is a mapping from $(S)^*$ into itself defined by

$$\{\langle \tau^*_\eta \Phi, \phi \rangle \} = \{\langle \Phi, \tau_\eta \phi \rangle \}.$$  

We have the following proposition:

**Proposition 4** Assume $\eta \in S'(R^{d+1})$ and $\Phi \in (S)^*$. Then

$$\tau^*_\eta \Phi = \Phi \circ \text{Exp} W_\eta,$$

where $W_\eta$ is the Hida distribution with $S$–Transform

$$SW_\eta(\xi) = (\eta, \xi).$$  

(I.e. a generalized coordinate process.)

**Proof:** The $S$–Transform gives:

$$S(\tau^*_\eta \Phi)(\xi) = \langle \tau^*_\eta \Phi, \text{Exp} W_\xi \rangle = \langle \Phi, \tau_\eta \text{Exp} W_\xi \rangle = \langle \Phi, e^{(n, \xi)} \cdot \text{Exp} W_\xi \rangle = e^{(n, \xi)} \cdot S\Phi(\xi) = S(\text{Exp} W_\eta)(\xi) \cdot S\Phi(\xi).$$

Hence, the proposition follows. \hfill \blacksquare

A similar result can also be found in [BG].

A positive Hida distribution is defined as follows, (see def. 4.25 in [HKPS]):
Definition 5 We say that \( \Phi \in (S)^* \) is positive, if
\[
\langle \langle \Phi, \phi \rangle \rangle \geq 0
\]
for all \( \phi \in (S) \) with the property \( \phi(\omega) \geq 0 \) \( \mu \)-a.e.

Observe that if \( \Phi \in (L^2) \) is positive in the usual sense, it is positive in the sense of definition 5. This follows since
\[
\langle \langle \Phi, \phi \rangle \rangle = \int_{S'(\mathbb{R}^{d+1})} \Phi(\omega)\phi(\omega) \, d\mu(\omega).
\]

We have the following corollary of proposition 4:

Corollary 6 Let \( \eta \in S'(\mathbb{R}^{d+1}) \). If \( \Phi \in (S)^* \) is positive, then \( \Phi \circ \text{Exp}W_\eta \) is positive.

Proof: Assume \( \phi(\omega) \geq 0 \) \( \mu \)-a.e. It then follows:
\[
\langle \langle \Phi \circ \text{Exp}W_\eta, \phi \rangle \rangle = \langle \langle \Phi, \tau_\eta \phi \rangle \rangle \geq 0,
\]
since \( \tau_\eta \phi(\omega) = \phi(\omega + \eta) \geq 0 \) \( \mu \)-a.e. \[\blacksquare\]

3 A generalized Feynman–Kac solution and positivity

Recall relation (6) for Skorohod integrable elements \( \Phi_s \). This suggests the following interpretation of problem (2):
\[
\begin{align*}
    u(t, x, \omega) &= u_0(x, \omega) + \nu \int_0^t \Delta u(s, x, \omega) \, ds + \\
    &\quad \int_0^t \sigma(s, x)u(s, x, \omega) \, ds + \int_0^t h(s, x)u(s, x, \omega) \circ W_{s, \theta}(\omega) \, ds
\end{align*}
\] (7)

where \( (t, x) \in (0, T] \times \mathbb{R}^d \). The equation (7) is to be understood in \( (S)^* \)–sense, where the operator \( \nu \Delta \) is interpreted as the infinitesimal generator for the (modified) \( d \)-dimensional Brownian motion \( b_t = (b_{t}^1, \ldots, b_{t}^d) \).

We state the conditions which we need for existence of a generalized Feynman–Kac formula for (7):

(I) \( u_0(x, \cdot) \in (S)^* \) for all \( x \in \mathbb{R}^d \). Moreover, we assume that the mapping
\[
x \to S_{u_0}(x, \cdot)(\xi)
\]
is in \( C^2(\mathbb{R}^d) \) for each \( \xi \in S(\mathbb{R}^{d+1}) \).
(II) For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, where $|\alpha| := \alpha_1 + \ldots + \alpha_d \leq 2$, we assume that

$$\left| \frac{\partial^{\alpha}}{\partial^{\alpha_1}x_1 \ldots \partial^{\alpha_d}x_d} Su_0(x; z\xi) \right| \leq K_1^{(\alpha)}(x) \exp \left( K_2^{(\alpha)}(x) |z|^2 \xi_2^{(\alpha)} \right)$$

where $Q^{(\alpha)} \in \mathbb{N}_0$, $K_2^{(\alpha)}$ is bounded $x$-a.e. and $K_1^{(\alpha)}(\cdot) \in L^p(\mathbb{R}^d)$ for a $p \in [1, \infty]$.

(III) The functions $h, \frac{\partial h}{\partial t}$, $\sigma$ and $\frac{\partial \sigma}{\partial t}$ are continuous and bounded.

At first sight, condition II might seem strange, since it is imposed on the $S$-Transform of $u_0$. However, for a big class of initial data, this condition is satisfied. For example: Let $\tilde{u}_0(\cdot) \in C^2_0(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ and $X(\omega) \in (S)^*$. The initial data

$$u_0(x, \omega) = \tilde{u}_0(x)X(\omega)$$

satisfies condition II. This is easy to see since the $S$-Transform is given by

$$Su_0(x; \xi) = \tilde{u}_0(x)SX(\xi).$$

Note that we have not assumed any non-anticipativity conditions on $u_0$. In our White Noise setting, we can treat anticipating initial conditions as well as non-anticipating.

With

$$v(t, x; \xi) := Su(t, x, \cdot)(\xi),$$

informal $S$-Transformation of (7) gives the problem:

$$v(t, x; \xi) = v_0(x; \xi) + \int_0^t \nu \Delta v(s, x; \xi) \ ds +$$

$$\int_0^t (\sigma(s, x) + h(s, x)\xi(s, x))v(s, x; \xi) \ ds,$$

where $\xi \in S(\mathbb{R}^{d+1})$. We have the proposition:

**Proposition 7** Under the conditions I–III, the unique solution of problem (8) is given by the Feynman–Kac formula

$$v(t, x; \xi) = E \left[ v_0(b^\xi_0; \xi) \exp \left( \int_0^t \left\{ \sigma(t-s, b^\xi_s) + h(t-s, b^\xi_s)\xi(t-s, b^\xi_s) \right\} \ ds \right) \right].$$

$b^\xi_s$ means that $b^\xi_0 = x$.

**Proof:** Put $k(t, x) = \sigma(t, x) + h(t, x)\xi(t, x)$. The integrand in $E[\cdot]$ can be written as a stochastic process: Set

$$Y_t = v_0(b_t; \xi)$$
and

\[ Z_t = e^{\int_0^t k(t-s, b_s) \ ds}. \]

An application of Itô's formula then shows that

\[ dY_t = \sum_{i=1}^d \frac{\partial v_0}{\partial x_i}(b_t; \xi) \ dB^i_t + \frac{1}{2} \sum_{i=1}^d \frac{\partial^2 v_0}{\partial x_i^2}(b_t; \xi) \ dt. \]

By condition III we get

\[ dZ_t = Z_t(k(0, b_t) + \int_0^t \frac{\partial k}{\partial t}(t-s, b_s) \ ds) \ dt. \]

Hence

\[ v(t, x; \xi) = v_0(x; \xi) + \int_0^t E \left[ \left( v_0(b_s^x; \xi)(k(0, b_s^x) + \int_0^s \frac{\partial k}{\partial t}(s-\tau, b_s^x) \ d\tau \right) + \frac{1}{2} \sum_{i=1}^d \frac{\partial^2 v_0}{\partial x_i^2}(b_s^x; \xi) \right] e^{\int_0^s k(s-\tau, b_s^x) \ d\tau} \ ds. \] (10)

\[ v(t, x; \xi) \] will be differentiable with respect to \( t \).

Recall that we interpret the operator \( \nu \Delta \) as the infinitesimal generator to \( b_t \), i.e.

\[ \nu \Delta v(t, x; \xi) = \lim_{\tau \downarrow 0} \frac{1}{\tau} (E[v(t, b_{t+\tau}^x; \xi)] - v(t, x; \xi)). \] (11)

Using the Markov property, a straightforward calculation yields

\[ E[v(t, b_{t+\tau}^x)] = E \left[ v_0(b_{t+\tau}^x) \exp \left( \int_0^t k(t-s, b_{s+\tau}^x) \ ds \right) \right]. \]

After a change of variables, the integral \( \int_0^t k(t-s, b_{s+\tau}^x) \ ds \) becomes

\[ \int_0^t k(t-s, b_{s+\tau}^x) \ ds = \int_0^{t+\tau} k(t+\tau-s, b_s^x) \ ds - \int_0^\tau k(t+\tau-s, b_s^x) \ ds. \]

Inserted in (11), we have

\[ \nu \Delta v(t, x) = \lim_{\tau \downarrow 0} \frac{1}{\tau} \left( E \left[ v_0(b_{t+\tau}^x) \exp (-\int_0^\tau k(t+\tau-s, b_s^x) \ ds) \right] \times \exp \left( \int_0^{t+\tau} k(t+\tau-s, b_s^x) \ ds \right) - v(t, x) \right). \]
We calculate as follows

$$\nu \triangle v(t, x) = \lim_{\tau \to 0} \frac{1}{\tau} \left( v(t + \tau, x; \xi) - v(t, x; \xi) \right)$$

$$+ \lim_{\tau \to 0} \frac{1}{\tau} \left( E \left[ v_0(b_{t+\tau}; \xi) \exp \left( \int_0^\tau k(t + \tau - s, b_s^x) \, ds \right) \right] \right.$$ \(\exp \left( - \int_0^{t+\tau} k(t + \tau - s, b_s^x) \, ds \right) \times \left[ v_0(b_{t+\tau}; \xi) \exp \left( \int_0^{t+\tau} k(t + \tau - s, b_s^x) \, ds \right) \right] \right)$$

$$= \frac{\partial v}{\partial t} (t, x) + E \left[ v_0(b_t^x) \exp \left( \int_0^t k(t - s, b_s^x) \, ds \right) \right] \times$$

$$\lim_{\tau \to 0} \frac{1}{\tau} \left\{ \exp \left( - \int_0^\tau k(t + \tau - s, b_s^x) \, ds \right) - 1 \right\}.$$

By definition, the last limit is the derivative in zero of the function

$$g(\tau) = \exp \left( - \int_0^\tau k(t + \tau - s, b_s^x) \, ds \right).$$

Differentiation yields

$$\frac{dg}{d\tau}(0) = -k(t, x).$$

Uniqueness follows since $\sigma, h$ are bounded continuous, and $v_0(\cdot, \xi) \in C^2\beta$, see Friedman [F]. Hence, the proposition is proved.

A similar Feynman–Kac formula for equation (8) can be found in Karatzas and Shreve [KS].

Consider the element

$$u(t, x, \omega) = E \left[ v_0(b_t^x, \omega) \exp \left( \int_0^t \sigma(t - s, b_s^x) \, ds \right) \right] \times$$

$$\exp \left( \int_0^t h(t - s, b_s^x) \cdot W_{t-s, b_s^x}(\omega) \, ds \right). \quad (12)$$

In the rest of this section we are going to investigate $u(t, x, \omega)$. We will show that $u(t, x, \omega)$ is a Hida distribution and that its $S$–Transform is equal to $v(t, x; \xi)$ defined in proposition 7. Moreover, we prove that $u(t, x, \omega)$ solves (7). Finally we discuss positivity properties of $u(t, x, \omega)$.

We consider the integrand of the Wick exponential $\exp$ in (12): Recall that $W_{t,x}$ has chaos decomposition given by

$$W_{t,x} = \langle \omega, \delta_{t,x} \rangle.$$
Here $\delta_{t,x}$ is the Dirac $\delta$-function in $(t,x)$, which have a series expansion

$$
\delta_{t,x} = \sum_{\alpha \in \mathbb{N}^{d+1}} \xi_{\alpha_1}(x^1) \cdots \xi_{\alpha_d}(x^d) \xi_{\alpha_{d+1}}(t) \cdot e_{\alpha}.
$$

$\xi_{\alpha}$ is the Hermite function of order $n$, and

$$
e_{\alpha} := \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_{d+1}}.
$$

The representation for $W_{t,x}$ implies that

$$
\int_0^t h(t-s, b^x_s) W_{t-s, b^x_s}(\omega) \, ds = (\omega, \int_0^t h(t-s, b^x_s) \delta_{t-s, b^x_s} \, ds).
$$

Furthermore

$$
\int_0^t h(t-s, b^x_s) \delta_{t-s, b^x_s} \, ds = \sum_{\alpha \in \mathbb{N}^{d+1}} \int_0^t h(t-s, b^x_s) \xi_{\alpha_1}(b^1_s) \cdots \xi_{\alpha_{d+1}}(b^{d+1}_s)(t-s) \, ds \cdot e_{\alpha}.
$$

The Hermite functions are eigenfunctions to the Harmonic Oscillator $H$. Moreover,

$$(H^{d+1})^p e_{\alpha} = \left(2\left(\sum_{j=1}^{d+1} \alpha_j\right) + d + 2\right)^p e_{\alpha}.
$$

We estimate the norm of (13) for a $q \geq 0$:

$$
|\int_0^t h(t-s, b^x_s) \delta_{t-s, b^x_s} \, ds|^2_q = \sum_{\alpha \in \mathbb{N}^{d+1}} \left(\int_0^t h(t-s, b^x_s) \xi_{\alpha_1}(b^1_s) \cdots \xi_{\alpha_{d+1}}(b^{d+1}_s)(t-s) \, ds\right)^2 \cdot \left(2\left(\sum_{j=1}^{d+1} \alpha_j\right) + d + 2\right)^{-2q}.
$$

An estimate from Hille and Phillips [HP] says that $\sup_x |\xi_{\alpha}(x)| \leq Kn^{-1/12}$. Hence, using that $h$ is bounded, we get for a big enough $q > 0$

$$
|\int_0^t h(t-s, b^x_s) \delta_{t-s, b^x_s} \, ds|^2_q \leq T^2 C \sum_{\alpha \in \mathbb{N}^{d+1}} \left(\prod_{j=1}^{d+1} \alpha_j^{-1/6}\right) \times \left(\sum_{j=1}^{d+1} \alpha_j\right)^{-2q} < \infty.
$$
We conclude
\[ \int_0^t h(t-s, b^x_s) \delta_{t-s, b^x_s} \, ds \in \mathcal{S}_q(\mathbb{R}^{d+1}), \]
for all \((t, x) \in [0, T] \times \mathbb{R}^d\). Note that \(q\) is independent of \((t, x)\). Define the function
\[ \eta_{t, x, \omega}(\tau, y) := \int_0^\tau h(t-s, b^x_s(\omega)) \delta_{t-s, b^x_s(\omega)}(\tau, y) \, ds. \quad (14) \]
\(\eta_{t, x, \omega}(\cdot, \cdot)\) will be an element of \(\mathcal{S}'(\mathbb{R}^{d+1})\) for each \((t, x, \omega) \in [0, T] \times \mathbb{R}^{d+1} \times \mathcal{S}'(\mathbb{R}^{d+1})\). This yields that
\[ \int_0^t h(t-s, b^x_s(\omega)) W_{t-s, b^x_s(\omega)}(\omega) \, ds = \langle \omega, \eta_{t, x, \omega} \rangle \]
is a generalized coordinate process, and hence
\[ \text{Exp}(\int_0^t h(t-s, b^x_s(\omega)) W_{t-s, b^x_s(\omega)}(\omega) \, ds) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \omega, \eta_{t, x, \omega} \rangle^o \]
is an element of \((\mathcal{S})^*\) (or more precisely, \((\mathcal{S})_{-q}\)).

We show that the expectation in (12) makes sense in \((\mathcal{S})^*\): In view of theorem 3, we must show that the \(\mathcal{S}\)-Transform of the integrand in (12) satisfies the bound in theorem 3 for a.e. \((t, x)\): (For the sake of easiness in the argument, we have set \(\sigma = 0\).)
\[ |\mathcal{S}(u_0(b^x_t, \omega) \circ \text{Exp}W_{\eta_{t, x, \omega}}) (z\xi)| = \]
\[ |\mathcal{S}u_0(b^x_t; z\xi)| \exp(\int_0^t h(t-s, b^x_s)z\xi(t-s, b^x_s) \, ds)|. \]
By condition II and the boundedness of \(h\),
\[ \leq K_1(b^x_t(\omega)) \exp(K_2(b^x_t(\omega))) |z|^2 |\xi|^2_{2, q} \cdot \exp(|h|_{\infty} |z||\xi|_{\infty} t). \]
For a suitable \(r > 0\), we find
\[ \leq K_1(b^x_t(\omega)) \exp\left( K_2(b^x_t(\omega))) |z|^2 |\xi|^2_{2, q} + \frac{1}{2} |h|_{\infty} Ct + \frac{1}{2} |h|_{\infty} Ct |z|^2 |\xi|^2_{2, r} \right) \]
\[ \leq K_1(b^x_t(\omega)) \exp\left( \frac{1}{2} Ct |h|_{\infty} + (K_2(b^x_t(\omega))) + \frac{1}{2} Ct |h|_{\infty} |z|^2 |\xi|^2_{2, s} \right), \]
where \(s = \max(q, r)\). Since \(K_2\) is bounded for a.e. \(x\), one sees that the mapping
\[ \bar{\omega} \rightarrow K_2(b^x_t(\bar{\omega}))), + \frac{1}{2} Ct |h|_{\infty} \]

13
is bounded for a.e. \( \mathcal{O} \). Let \( p_t(x, y) \) denote the density for \( b^x_t \). Then estimation yields

\[
\int_{\mathbb{R}^{d+1}} K_1(b^x_t(\mathcal{O})) \, d\mu(\mathcal{O}) = \int_{\mathbb{R}^d} K_1(y)p_t(x, y) \, dy < \infty,
\]

for a.e. \((t, x) \in [0, T] \times \mathbb{R}^d\). This follows since by assumption \( K_1 \in L^p(\mathbb{R}^d) \).

Hence, the mapping

\[
\mathcal{O} \to K_1(b^x_t(\mathcal{O}))e^{-1/|\mathcal{O}|}
\]

belongs to \( L^1 \) for a.e \((t, x)\). From theorem 3 it now follows that \( u(t, x, \omega) \) given in (12) is an element of \((\mathcal{S})^*\). Furthermore, since \( \mathcal{S}\)-Transform commutes with integration, we see that the \( \mathcal{S}\)-Transform of \( u(t, x, \omega) \) coincides with \( v(t, x; \xi) \) given in (9).

We show that \( u(t, x, \omega) \) solves (7): By estimating as above, one sees that

\[
\int_0^t v(s, x; \xi)\xi(s, x) \, ds = \mathcal{S} \left( \int_0^t u(s, x, \omega) \circ W_{s, \xi} \, ds \right)(\xi).
\]

Hence, in order to establish \( u(t, x, \omega) \) as a solution of (7), we have to prove that the infinitesimal generator \( \nu \Delta \) commutes with the \( \mathcal{S}\)-Transform, i.e.

\[
\mathcal{S} \left( \lim_{\tau \to 0} \frac{1}{\tau} (E[u(t, b^x_t, \omega)] - u(t, x, \omega)) \right)(\xi) = \lim_{\tau \to 0} \frac{1}{\tau} (v(t, b^x_t; \xi) - v(t, x; \xi)).
\]

We have seen in proposition 7 that the right hand side of the above equality converges for each \( \xi \). In view of theorem 2, we must show a uniform (in \( \tau \)) exponential bound on

\[
\frac{1}{\tau} (E[v(t, b^x_t; z\xi)] - v(t, x; z\xi)).
\]

But from the proof of proposition 7, we know that

\[
\frac{1}{\tau} (E[v(t, b^x_t; z\xi)] - v(t, x; z\xi)) = \frac{1}{\tau} (v(t + \tau, x; z\xi) - v(t, x; z\xi))
\]

\[
+ E \left[ v_0(b^x_{t+\tau}; \xi) \int_0^{+\tau} k^x(t+\tau-s, b^x_s) \, ds \cdot \frac{1}{\tau} e^{-\int_0^{+\tau} k^x(t+\tau-s, b^x_s) \, ds} - 1 \right],
\]

with \( k^x(t, x) := \sigma(t, x) + z\mathcal{H}(t, x)\xi(t, x) \). By using the representation (10) for \( v(t, x; z\xi) \) together with the conditions II and III, the mean value theorem gives a uniform exponential bound. Hence, by inverting the \( \mathcal{S}\)-Transform, we see that \( u(t, x, \omega) \) solves (7) in \( (\mathcal{S})^* \).

The solution is unique in \( (\mathcal{S})^* \) since the \( \mathcal{S}\)-Transform is injective. We conclude our results:

**Theorem 8** Assume conditions I–III. Then \( u(t, x, \omega) \) given in (12) is the unique \((\mathcal{S})^*\)-solution of problem (7).
We state the positivity result for \( u(t, x, \omega) \) given in (12):

**Theorem 9** Assume conditions I–III of section 3. If \( u_0(x, \omega) \) is positive in the sense of definition 5, then \( u(t, x, \omega) \) given in (12) is a positive solution of problem (7).

**Proof:** Let \( n_{t,x,\tilde{\omega}}(\tau, y) \) be as in (14). In the preceding section we showed that \( n_{t,x,\tilde{\omega}} \in S'((R^d)^{d+1}) \) for each \((t, x, \tilde{\omega}) \in [0, T] \times R^d \times S'((R^d)^{d+1})\). Hence, by proposition 4, \( u(t, x, \omega) \) can be written

\[
 u(t, x, \omega) = E_{\tilde{\omega}} \left[ \exp \left( \int_0^t \sigma(t-s, b^*_x(\tilde{\omega})) \, ds \right) \cdot n_{t,x,\omega}^* u_0(b^*_x(\tilde{\omega}), \omega) \right]. \tag{15}
\]

\( E_{\tilde{\omega}} \) denotes expectation with respect to \( \tilde{\omega} \). Let \( \phi \in (S) \) be positive in the sense of definition 5. By Bochner integrability and corollary 6 we have

\[
 (\langle u(t, \cdot), \phi \rangle) = E_{\tilde{\omega}} \left[ (\exp \left( \int_0^t \sigma(t-s, b^*_x(\tilde{\omega})) \, ds \right) \cdot n_{t,x,\omega}^* u_0(b^*_x(\tilde{\omega}), \cdot), \phi) \right]
\]

\[
 = E_{\tilde{\omega}} \left[ \exp \left( \int_0^t \sigma(t-s, b^*_x(\tilde{\omega})) \, ds \right) \langle n_{t,x,\omega}^* u_0(b^*_x(\tilde{\omega}), \cdot), \phi \rangle \right]
\]

\[
 \geq 0.
\]

Hence, \( u(t, x, \omega) \) is positive in sense of definition 5, and theorem follows. ■

If \( u_0 \) is positive, we can associate to \( u(t, x, \omega) \) a finite measure on \((S'((R^d)^{d+1}), B)\):

**Corollary 10** Let \( u_0 \) be positive in sense of definition 5. Then for each \((t, x) \in [0, T] \times R^d \) there exists a finite measure \( \nu_{t,x} \) on \((S'((R^d)^{d+1}), B)\) such that for all \( \phi \in (S)\),

\[
 (\langle u(t, x, \cdot), \phi \rangle) = \int_{S'((R^d)^{d+1})} \tilde{\phi}(\omega) \, d\nu_{t,x}(\omega). \tag{16}
\]

\( \tilde{\phi} \) is the continuous version of \( \phi \) (see [HKPS]). ■

**Proof:** See theorem 4.26 in [HKPS]. ■

4 The case of white noise in time

In the rest of the paper, we will concentrate our discussion on white noise in time only, i.e. \( d = 0 \): The solution of (7), \( u(t, x, \omega) \) given in (12), will now be

\[
 u(t, x, \omega) = E \left[ u_0(b^*_x, \omega) \exp \left( \int_0^t \sigma(t-s, b^*_x) \, ds \right) \right].
\]

15
\[ \text{Exp} \left( \int_0^t h(t-s, b_s^r) W_{t-s}(\omega) \, ds \right) \]  

(17)

Recalling the connection between white noise and Itô–Skorohod integration, (see (6)), the following holds true

\[ \int_0^t h(t-s, b_s^r) W_{t-s} \, ds = \int_0^t h(s, b_s^r_{t-s}) W_s \, ds = \int_0^t h(s, b_s^r_{t-s}) \, dB_s \]

Define the function

\[ \gamma_{t,x,\omega}(s) := 1_{[0,t]}(s) h(s, b_s^r_{t-s}(\omega)). \]  

(18)

For each \((t,x,\omega), \gamma_{t,x,\omega} \in L^2(\mathbb{R})\). We can write

\[ \int_0^t h(s, b_s^r_{t-s}) \, dB_s(\omega) = \langle \omega, \gamma_{t,x,\omega} \rangle = W_{\gamma_{t,x,\omega}}(\omega). \]

We need to state some results from the papers by [G] and [BG] in order to do further studies of (17):

In [G] and [BG] the following relation is proved for \( \eta \in L^2(\mathbb{R}) \) and \( \Phi \) in a subspace of \((S)^*\):

\[(T_{-\theta}\Phi) \cdot \text{Exp}W_\eta = \Phi \circ \text{Exp}W_\eta.\]

The operator \( T_{-\theta} \) is a generalized translation. For the convinience of the reader, we will here indicate how such a relation is proved and for what kind of Hida distributions it is valid: Observe that \( \text{Exp}W_\eta \) is not a Hida test function when \( \eta \) is an element of \( L^2(\mathbb{R}) \). We need a test function space slightly bigger than \((S)\), in order to have the Wick exponentials of this type as test functionals. Such a space is considered in Potthoff and Timpel [PT]:

Recall the space of polynomials in \((L^2, \mathcal{P})\). If \( N \) denotes the number or the Ornstein–Uhlenbeck operator, we see that \( e^{\lambda N} \phi \in (L^2) \) for every \( \phi \in \mathcal{P} \) and \( \lambda \in \mathbb{R} \). Define \( \mathcal{G}_\lambda \) as the completion of \( \mathcal{P} \) under the norm

\[ ||\phi||_{(\lambda)} := ||e^{\lambda N} \phi||_2. \]

\( \mathcal{G} \) is the projective limit of \( \mathcal{G}_\lambda \). This test function space will be rich enough to contain the above mentioned Wick exponentials, i.e.

\[ \text{Exp}W_\eta \in \mathcal{G} \quad \text{for} \ \eta \in L^2(\mathbb{R}). \]

Denoting \( \mathcal{G}^* \) for the dual of \( \mathcal{G} \), we have the following chain of inclusions

\[ (S) \subset \mathcal{G} \subset (L^2) \subset \mathcal{G}^* \subset (S)^*. \]

Since the translation operator \( \tau \) is linear continuous from \( \mathcal{G} \) into \( \mathcal{G} \) (see [PT]) we can reformulate proposition 4: Given \( \eta \in L^2(\mathbb{R}) \) and \( \Phi \in \mathcal{G}^* \), then

\[ \tau_{\eta}^* \Phi = \Phi \circ \text{Exp}W_\eta. \]

A direct calculation, using the Cameron–Martin–Girsanov theorem, shows: (See [BG].)
Lemma 11 Assume $\Phi \in \mathcal{G}$. Then

$$(\tau_\eta \Phi) \cdot \text{Exp}W_\eta = \Phi \circ \text{Exp}W_\eta.$$  

Define generalized translation $T_\eta$ of $\mathcal{G}^*$-elements as follows:

$$\langle (T_\eta \Phi, \phi) \rangle = \langle (\Phi, \tau_\eta^* \phi) \rangle.$$  

For $\phi \in \mathcal{G}$, we have that $\tau_\eta^* \phi = \phi \circ \text{Exp}W_\eta \in \mathcal{G}$, since $\mathcal{G}$ is an algebra under the Wick product. Hence, $T_\eta$ is a well-defined linear operator on $\mathcal{G}^*$. Observe that $T_\eta = \tau_\eta$ on $\mathcal{G}$. We calculate the $S$-Transform of $T_\eta \Phi \cdot \text{Exp}W_\eta$ where $\Phi \in \mathcal{G}^*$ and $\eta \in L^2(\mathbb{R})$:

$$S(T_\eta \Phi \cdot \text{Exp}W_\eta)(\xi) = \langle (T_\eta \Phi, \text{Exp}W_\eta \cdot \text{Exp}W_\xi) \rangle$$

$$= e^{(\eta, \xi)} \langle (\Phi, \tau_\eta^* \text{Exp}W_{\eta+\xi}) \rangle$$

$$= e^{(\eta, \xi)} \langle (\Phi, \text{Exp}W_{-\eta} \circ \text{Exp}W_{\eta+\xi}) \rangle$$

$$= e^{(\eta, \xi)} \langle (\Phi, \text{Exp}(W_{-\eta} + W_{\eta+\xi})) \rangle$$

$$= S(\Phi \circ \text{Exp}W_\eta)(\xi).$$

Hence, we obtain the formula:

$$(T_\eta \Phi) \cdot \text{Exp}W_\eta = \Phi \circ \text{Exp}W_\eta,$$  

where $\Phi \in \mathcal{G}^*$ and $\eta \in L^2(\mathbb{R})$.

With formula (19), we study solution (17): In [BG] it is proved that $T_\eta = \tau_\eta$ on $(L^p)$ when $p > 2$. Hence, if we assume that for a given $p > 2$

$$u_0(x, \cdot) \in (L^p) \text{ for all } x \in \mathbb{R}^d,$$  

we can write solution (17) as follows:

$$u(t, x, \omega) = E \left[ u_0(b_t^x, \omega - \gamma_t, x, \omega) \cdot \text{Exp}W_{\gamma_t, x, \omega}(\omega) \right].$$  

(21)

Or, equivalently,

$$u(t, x, \omega) = E \left[ u_0(b_t^x, \omega - \gamma_t, x, \omega) \cdot \exp(W_{\gamma_t, x, \omega}(\omega) - \frac{1}{2} |\gamma_t, x, \omega|^2) \right].$$  

(22)

$\gamma_t, x, \omega$ is as in (18). With condition (20) on $u_0$, we study regularity properties of the solution (21): The following proposition is found in [BG]:

Proposition 12 Put

$$X = \text{Exp}W_\eta$$

for a $\eta \in L^2(\mathbb{R})$. Assume $\Phi \in (L^p)$ for a $p \in (2, \infty)$. Then for any $q \in [1, p/2)$ we have

$$X \circ \Phi \in (L^q).$$  

17
Moreover

\[ \|X \circ \Phi\|_q \leq \exp \left( \frac{1}{2} K(p, q) \|\eta\|_2^2 \right) \|\Phi\|_p, \]  

(23)

where \( K(p, q) = \frac{4p}{p - 2q} + \frac{4}{p} - 1 \).

If \( \Phi \in (L^\infty) \), then for each \( q \in [1, \infty) \), \( X \circ \Phi \in (L^q) \). Moreover, the following estimate holds

\[ \|X \circ \Phi\|_q \leq \exp \left( \frac{1}{2} (q - 1) \|\eta\|_2^2 \right) \|\Phi\|_\infty, \]  

(24)

**Proof:** Let \( q \in [1, p/2) \). Since \( \Phi \in (L^p) \) for \( p \in (2, \infty) \), we have \( T_\eta \Phi = \tau_\eta \Phi \).

Hence

\[ E[\|X \circ \Phi\|_q] = E[\|X\|_q \cdot \tau_{-\eta}[\Phi]^q]. \]

Put \( r_1 = \frac{p}{2q} \) and \( r_2 = \frac{p}{p - 2q} \). By Cauchy–Schwarz

\[ \leq \left( \int |X|^{\frac{p}{p - 2q}} \, d\mu \right)^{\frac{p - 2q}{p}} \left( \int \tau_{-\eta}[\Phi]^{p/2} \, d\mu \right)^{\frac{2q}{p}}. \]

The translation formula for Gaussian measures and Cauchy–Schwarz, yield

\[ \leq \left( \int |X|^{\frac{p}{p - 2q}} \, d\mu \right)^{\frac{p - 2q}{p}} \left( \int |X|^2 \, d\mu \right)^{\frac{2q}{p}} \|\Phi\|_p^2. \]

Using the identity

\[ X = \text{Exp}W_\eta(\omega) = \exp \left( \langle \omega, \eta \rangle - \frac{1}{2} \|\eta\|_2^2 \right), \]

we find that

\[ \int |X|^r \, d\mu = \exp \left( \frac{1}{2} \|\eta\|_2^2 (r^2 - r) \right). \]

Hence, the estimate (23) follows.

If \( \Phi \in (L^\infty) \), then \( \Phi \in (L^p) \) for each \( p \geq 1 \). Hence, for a fixed \( q \in [1, \infty) \) we have for every \( p > 2q \) that (23) holds. Moreover, since

\[ \|\Phi\|_p \leq \|\Phi\|_\infty \]

and letting \( p \) in \( K(p, q) \) tend to infinity, the estimate (24) follows. The proposition is then proved.

This proposition gives us regularity of \( u(t, x, \omega) \) given in (17) depending on the regularity of the initial condition:

**Proposition 13** Assume for \( p \in (2, \infty) \)

\[ \int_{\mathbb{R}^d} \| u_0(x, \cdot) \|_p \, dx < \infty. \]
Then for any \( q \in [1, p/2) \), \( u(t, x, \cdot) \in (L^q) \) and \( \|u(t, x, \cdot)\|_q \in L^1([0, T] \times \mathbb{R}^d) \). Moreover

\[
\int_0^T \int_{\mathbb{R}^d} \|u(t, x, \cdot)\|_q \, dx \, dt \leq C(p, q) \int_{\mathbb{R}^d} \|u_0(x, \cdot)\|_p \, dx,
\]

where

\[
C(p, q) = \left( \frac{1}{2} (\sup_{t, x} |h(t, x)|^2) \cdot K(p, q) \right)^{-1} \exp \left( \frac{1}{2} (\sup_{t, x} |h(t, x)|^2) \cdot K(p, q) T \right).
\]

\( K(p, q) \) is given in proposition 12. If

\[
\int_{\mathbb{R}^d} \|u_0(x, \cdot)\|_\infty \, dx < \infty,
\]

then \( u(t, x, \cdot) \in (L^q) \) and \( \|u(t, x, \cdot)\|_q \in L^1([0, T] \times \mathbb{R}^d) \) for all \( q \in [1, \infty) \). Moreover

\[
\int_0^T \int_{\mathbb{R}^d} \|u(t, x, \cdot)\|_q \, dx \, dt \leq C(q) \int_{\mathbb{R}^d} \|u_0(x, \cdot)\|_\infty \, dx,
\]

where

\[
C(q) = \left( \frac{1}{2} (\sup_{t, x} |h(t, x)|^2) (q - 1) \right)^{-1} \exp \left( \frac{1}{2} (\sup_{t, x} |h(t, x)|^2) (q - 1) T \right)
\]

for \( q > 1 \), and \( C(1) = T \).

**Proof:** In the proof, \( E_\tilde{\omega} \) and \( E_\omega \) denote expectation w.r.t. \( \tilde{\omega} \) and \( \omega \) respectively. Let \( q \in [1, p/2) \):

\[
E_\omega[|u(t, x, \cdot)|^q]^{1/q} \leq E_\tilde{\omega} \left[ E_\omega[|u_0 \circ \text{Exp} W_\gamma|^q]^{1/q} \right],
\]

where \( \gamma \) is the function defined in (18). By proposition 12,

\[
\leq E_\tilde{\omega} \left[ E_\omega[|u_0|^p]^{1/p} \exp \left( \frac{1}{2} K(p, q) |\gamma_{t, x, \omega}|^2 \right) \right] \leq \exp \left( \frac{1}{2} K(p, q) (\sup |h|^2) T \right) E_\sigma \left[ E_\omega[|u_0| b^\gamma_t(\tilde{\omega}, \omega)|^p]^{1/p} \right].
\]

Let \( p_t(x, y) \) denote the density for the modified Brownian motion \( b^\gamma_t \):

\[
= \exp \left( \frac{1}{2} (\sup |h|^2) K(p, q) T \right) \left( \int_{\mathbb{R}^d} |u_0(y, \omega)|^p \, d\mu(\omega) \right)^{1/p} p_t(x, y) \, dy.
\]
Hence,

\[
\int_{\mathbb{R}^d} \| u(t, x, \cdot) \|_q \, dx \leq \exp \left( \frac{1}{2} (\sup |h|^2) K(p, q) t \right) \times \\
\int_{\mathbb{R}^d} \left( \int_{S^1(\mathbb{R})} |u_0(y, \omega)|^p \, d\mu(\omega) \right)^{1/p} \left( \int_{\mathbb{R}^d} p_t(x, y) \, dx \right) \, dy \\
= \exp \left( \frac{1}{2} (\sup |h|^2) K(p) t \right) \int_{\mathbb{R}^d} \| u_0(y, \cdot) \|_p \, dy.
\]

This proves the first half of the proposition. The second half follows similarly using estimate (24) in proposition 12.

We end this paper with an example illustrating the results developed in this paper:

**Example:** We consider the heat transport problem with just a time-dependent noise potential \( W_t \),

\[
\frac{\partial u}{\partial t}(t, x, \omega) = \nu \Delta u(t, x, \omega) + u(t, x, \omega) \circ W_t(\omega),
\]

\[u(0, x, \omega) = u_0(x, \omega).\]

We have set \( \sigma = 0 \) and \( h = 1 \), and we assume that \( u_0(x, \omega) \) satisfies condition I and II in section 3. In view of equation (6), we remark that an Itô–Skorohod interpretation of the above problem is

\[u(t, x, \omega) = u_0(x, \omega) + \int_0^t \nu \Delta u(s, x, \omega) \, ds + \int_0^t u(s, x, \omega) \, dB_s(\omega).\]

Since no adaptability condition on \( u_0 \) is assumed, the stochastic integral has to be interpreted in Skorohod sense, and not in Itô sense. The function \( \gamma_{t, x, \omega}(\cdot) \) in (18) will be given by the characteristic function only:

\[\gamma_{t, x, \omega}(s) = 1_{[0, t]}(s).\]

If for \( p > 2, u_0(x, \cdot) \in (L^p) \) for all \( x \in \mathbb{R}^d \), we see from (21) that our solution can be written

\[u(t, x, \omega) = E \left[ u_0(b_t^x, \omega - 1_{[0, t]}) \right] \cdot \text{Exp} B_t(\omega),\]

or, equivalently

\[u(t, x, \omega) = E \left[ u_0(b_t^x, \omega - 1_{[0, t]}) \right] \cdot \exp(B_t(\omega) - \frac{1}{2} t).\]

Using the density \( p_t(x, y) \) of \( b_t^x \), we get the representation:

\[u(t, x, \omega) = \left( \int_{\mathbb{R}^d} u_0(y, \omega - 1_{[0, t]} p_t(x, y) \, dy \right) \cdot \exp(B_t(\omega) - \frac{1}{2} t).\]
Consider for $\phi \in L^2(\mathbb{R})$ the random variable $W_\phi(\omega) = (\omega, \phi)$. This is a Gaussian random variable with mean 0 and variance $|\phi|^2$. If $\text{supp}(\phi) \subset [0, t)$, $t \leq T$, then $W_\phi$ is anticipating. From this Gaussian random variable, we can construct two positive random variables, namely

$$W_\phi^2 \quad \text{and} \quad \exp W_\phi.$$ 

The $(L^p)$-norms, $p \geq 2$, are easily calculated:

$$\|W_\phi^2\|_p = \left(\frac{(2p)!}{p!2^p}\right)^{1/p} |\phi|^2,$$

and

$$\|\exp W_\phi\|_p = \exp\left(\frac{1}{2} |\phi|^2 \right) \left( \int |\exp W_\phi(\omega)|^p \ d\mu(\omega) \right)^{1/p} = \exp\left(\frac{1}{2} |\phi|^2 \right) \left( \exp\left(\frac{1}{2} |\phi|^2 (p^2 - p)\right) \right)^{1/p} = \exp\left(\frac{p}{2} |\phi|^2 \right).$$

Using these two positive random variables, we see that the two initial conditions

$$u_0(x, \omega) = \tilde{u}_0(x) \cdot W_\phi^2(\omega) \quad \text{and} \quad u_0(x, \omega) = \tilde{u}_0(x) \cdot \exp W_\phi(\omega),$$

are positive when $\tilde{u}_0$ is a positive function. Moreover, if $\tilde{u}_0 \in L^1(\mathbb{R}^d)$, we have a positive solution $u(t, x, \omega)$ with the bounds

$$\int_0^T \int_{\mathbb{R}^d} \|u(t, x, \cdot)\|_q \ dx \ dt \leq 2e^\frac{1}{2} K(p, q) |\phi|^2 \left( \frac{(2p)!}{p!2^p} \right)^{1/p} \int_{\mathbb{R}^d} |\tilde{u}_0(x)| \ dx$$

and

$$\int_0^T \int_{\mathbb{R}^d} \|u(t, x, \cdot)\|_q \ dx \ dt \leq 2e^\frac{1}{2} K(p, q) |\phi|^2 \exp\left(\frac{p}{2} |\phi|^2 \right) \int_{\mathbb{R}^d} |\tilde{u}_0(x)| \ dx,$$

respectively. Here, $1 \leq q < p/2$ and $K(p, q)$ is given in proposition (12).

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References


