SPINORS IN THE MINKOWSKI SPACE

VALENTIN LYCHAGIN AND LEV ZILBERGLEIT

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ABSTRACT. We propose a realization of spinors in the Minkowski space as exterior forms of a special type. It is well known that the symbol of the operator $d + \delta$ determines a Clifford module structure in the space of exterior forms. This module splits into a sum of two simple ones called a spinor space and a dual spinor space. The decomposition is determined by two projectors. We suggest a description of these projectors in terms of hyperbolic planes. This description produces two algebraic structures into the spinor space: (1) the canonically determined complex structure, and (2) a noncanonically determined quaternion structure. The choice of the latter structure depends on a basis in the elliptic plane orthogonal to the hyperbolic one. In this case the Clifford algebra is realized as $2 \times 2$ quaternion matrices algebra.

1. INTRODUCTION

This article arises from the attempt to understand a sense of Dirac equation solutions. From the formal point of view Dirac equation solutions are spinor fields. But such an approach gives no indications on a measurement of a such field. In this article we give realisation of spinors in the Minkowski space as a special type of exterior forms. One obtains two corollaries from this description. First, spinor fields can be measured like electromagnetic fields. And, second, a spinor structure is given by an measurable object which is an exterior decomposable 2-form of the unit length.

Recall that when introducing spinors the main Dirac’s idea was to compute a square root of the wave operator $[(1)|2]$. In differential geometry this problem resolves by means of the operator $d + \delta$. But $d + \delta$ is not the Dirac operator, because it is reducible. Namely, the algebra of differential forms can be represented as a sum of two submodules that invariant with respect to differential operator $d + \delta$ with possible addition of a zero order terms.

Considering symbols of these operators one obtains the following problem in linear algebra which is solved in this article. The symbol of operator $d + \delta$ determines

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a Clifford module structure in the exterior forms space. This module splits into a sum of two simple ones called a spinor space and a dual spinor space. We show that projectors corresponding to this decomposition are determined by hyperbolic planes. Moreover, the spinor space is equipped with a canonical complex structure and a noncanonically determined quaternion structure. The choice of the latter one depends on a basis in the elliptic plane orthogonal to the hyperbolic one. The Clifford algebra is realized as the $2 \times 2$ quaternion matrices algebra.

In conclusion note that the Dirac operator naturally arises as an operator satisfying the De Broglie principle. It means that singularities of solutions of corresponding equations move as material points. This remark together with the observation that the Schrödinger operator is the transport operator for special type singular solutions of the Dirac equation as well as the fact that spinors can be realized as special differential forms allow us to indicate geometrical sense of $\Psi$—function as a section of a quaternion fibre bundle.\footnote{The research described in this publications was made possible in part by Grant N2F00 from the International Science Foundation}

2. An Exterior Algebra

In this section we give a brief review of structures into the exterior algebra over vector space equipped with (pseudo)metric.

Let $E$ be a vector space over $\mathbb{R}$, $\dim E = n$, and $E^* = \text{Hom}(E, \mathbb{R})$ be the dual space.

Denote by $\Lambda^i(E^*)$ the space of $i$—dimensional skew symmetric forms on the space $E$, and by

$$\Lambda^*(E^*) = \bigoplus_{i=1}^{n} \Lambda^i(E^*)$$

the graded exterior algebra of the space $E^*$. This algebra is a direct sum of the subalgebra

$$\Lambda^{ev}(E^*) \subset \Lambda^*(E^*)$$

where

$$\Lambda^{ev}(E^*) = \bigoplus_{k \equiv 0 \mod 2} \Lambda^k(E^*)$$

and the $\Lambda^{ev}(E^*)$—module

$$\Lambda^{od}(E^*) \subset \Lambda^*(E^*)$$
where

$$\Lambda^0(E^*) = \bigoplus_{k=1 \mod 2} \Lambda^k(E^*).$$

There are two natural operators in the algebra $\Lambda^*(E^*)$:

1°. operator of exterior multiplication:

$$e_\theta : \Lambda^i(E^*) \longrightarrow \Lambda^{i+1}(E^*),$$

$$\omega \longrightarrow \theta \wedge \omega$$

by a covector $\theta \in E^*$, and

2°. operator of interior multiplication:

$$i_X : \Lambda^i(E^*) \longrightarrow \Lambda^{i-1}(E^*),$$

$$\omega \longrightarrow i_X \omega$$

by a vector $X \in E$.

We assumed here that

$$(i_X \omega)(X_1, \cdots, X_{i-1}) = \omega(X, X_1, \cdots, X_{i-1})$$

for all vectors $X_1, \cdots, X_{i-1} \in E$.

Operators $e$ and $i$ satisfy the following well known relations:

$$e_{\theta_1} \circ e_{\theta_2} + e_{\theta_2} \circ e_{\theta_1} = 0,$$

$$i_{X_1} \circ i_{X_2} + i_{X_2} \circ i_{X_1} = 0,$$

$$i_X \circ e_\theta + e_\theta \circ i_X = e_{i_X(\theta)},$$

for all $X \in E, \theta \in E^*$.

Therefore, we have the following representations of the graded algebra $\Lambda^*(E^*)$

(1)

$$e : \Lambda^*(E^*) \longrightarrow \text{End}(\Lambda^*(E^*)), \quad \theta \mapsto e_\theta$$

where $e_\theta$ has the form

$$e_\theta = e_{\theta_1} \circ \cdots \circ e_{\theta_k},$$

on decomposable elements $\theta = \theta_1 \wedge \cdots \wedge \theta_k, \theta_j \in E^*, j = 1, \cdots, k$ and
(2)

\[ i : \Lambda^k(E^*) \rightarrow \text{End}(\Lambda^k(E^*)), \]
\[ X \mapsto i_X \]

where the operator \( i_X \) has the form

\[ i_X = i_{X_1} \circ \cdots \circ i_{X_k}, \]

on decomposable elements \( X = X_1 \wedge \cdots \wedge X_k, \quad X_j \in E, j = 1, \cdots, k \).

Denote by \( \langle , \rangle \) the natural pairing between spaces \( \Lambda^k(E) \) and \( \Lambda^k(E^*) \):

\[ \langle , \rangle : \Lambda^k(E) \times \Lambda^k(E^*) \rightarrow \mathbb{R}, \]
\[ (a, b^*) \mapsto i_a b^*. \]

Let \( g \) be (pseudo)metric of the signature \((n - l, l)\) on the space \( E \).
In a usual way we will identify the metric \( g \) with the symmetric operator:

\[ A_g : E \rightarrow E^* \]

where \( \langle A_g(X), Y \rangle = g(X, Y), \) for all \( X, Y \in E \).
Presenting the metric \( g \) (or the operator \( A_g \)) allows us to determine in a natural way the metric \( g^* \) on the dual space \( E^* \):

\[ A_g^* = A_g^{-1} : E^* \rightarrow E, \]

as well as metrics \( g_k \) and \( g_k^* \) on the spaces \( \Lambda^k(E) \) and \( \Lambda^k(E^*) \) respectively.
Namely,

\[ A_{g_k} = \Lambda^k(A_g), A_{g_k^*} = \Lambda^k(A_g^{-1}). \]

In the sequel, to simplify our notations, we will omit indices \( k \) and \( * \) and denote metrics \( g_k \) and \( g_k^* \) by \( g \).

Denote by \( \tilde{\theta} \in \Lambda^*(E) \) and \( \tilde{X} \in \Lambda^*(E^*) \) elements \( A_g(\theta) \) and \( A_g^{-1}(X) \), for elements \( \theta \in \Lambda^*(E^*) \) and \( X \in \Lambda^*(E) \) respectively.
In these notations, one obtains the following relations:

\[ g(\theta_1, \theta_2) = \langle A_g(\theta_1), \theta_2 \rangle = i_{\tilde{\theta}_1} \theta_2. \]

Hence, the following relations between operators \( e, i \) and the metric \( g \):

\[ i_{\tilde{\theta}_1} \circ e_{\theta_2} + e_{\theta_2} \circ i_{\tilde{\theta}_1} = g(\theta_1, \theta_2) 1 \quad (1.1) \]

holds, for any \( \theta_1, \theta_2 \in E^* \).
Note also that the following equality is valid

\[ \text{sign}(g(\theta, \theta)) = (-1)^\nu, \]

for any nonvanishing form \( \theta \in \Lambda^n(E^*) \). Here \( \nu = \left[ \frac{n}{2} \right] + l \).

3. Clifford Modules

Assume that the space \( E \) is oriented and choose a volume form \( \Omega_g \in \Lambda^n(E^*) \) such that

\[ g(\Omega_g, \Omega_g) = (-1)^\nu. \]

The volume form \( \Omega_g \) determines the Hodge operator:

\[
\begin{align*}
# : & \quad \Lambda^k(E^*) \rightarrow \Lambda^{n-k}(E^*), \\
& \quad \theta \quad \mapsto \quad i_\Omega \Omega_g, \\
& \quad k = 1, \ldots, n. \quad k = 1, \ldots, n.
\end{align*}
\]

In other words one has the following formulae

\[ # (\theta_1 \wedge \cdots \wedge \theta_k) = i_{\theta_1} \circ \cdots \circ i_{\theta_k} (\Omega_g), \quad (2.1) \]

for the action of the operator \( # \) on decomposable forms.

Here \( \theta_1, \cdots, \theta_k \in E^* \).

**Proposition 3.1.** Let \( g \) be a metric of the signature \((n-l, l)\) on an oriented vector space \( E \), and \( \Omega_g \in \Lambda^n(E^*) \) be a volume form such that \( g(\Omega_g, \Omega_g) = (-1)^\nu, \nu = \left[ \frac{n}{2} \right] + l \).

Then the Hodge operator \( # \) satisfies following conditions:

1. \( \#^2 = (-1)^\nu 1, \)
2. \( \# \Omega_g = (-1)^\nu, \# 1 = \Omega_g, \)
3. \( \# \circ e_\theta = i_\Omega \circ \#, \)
   for all \( \theta \in \Lambda^n(E^*). \)
4. \( g(\theta_1, \theta_2) = \#^{-1} \circ e_{\theta_1} \circ \#(\theta_2), \)
   for all \( \theta_1, \theta_2 \in \Lambda^k(E^*). \)

From now on we assume that \( \nu \equiv 1 \mod 2. \)

In this case the Hodge operator \( # \) determines the complex structure on the space \( \Lambda^\bullet(E^*). \)

Consider morphisms:

\[
\begin{align*}
\sigma^+ : \quad E^* & \rightarrow \text{End}(\Lambda^\bullet(E^*)), \\
\theta & \mapsto e_\theta + i_\theta
\end{align*}
\]
and

\[ \sigma^- : E^* \rightarrow \text{End}(\Lambda^* (E^*)) , \]

\[ \theta \mapsto e_{\theta} - i_{\theta}. \]

Let \( \sigma^+ (\theta) = \sigma^+_{\theta}, \sigma^- (\theta) = \sigma^-_{\theta}. \)

Then one has

\[ \sigma^+_{\theta_1} \circ \sigma^+_{\theta_2} + \sigma^-_{\theta_2} \circ \sigma^+_{\theta_1} = 2g(\theta_1, \theta_2) 1, \]

(3.1)

and

\[ \sigma^-_{\theta_1} \circ \sigma^-_{\theta_2} + \sigma^+_{\theta_2} \circ \sigma^-_{\theta_1} = -2g(\theta_1, \theta_2) 1. \]

(3.2)

for any covectors \( \theta_1, \theta_2 \in E^*. \)

Due to these relations the mapping \( \sigma^+ \) defines a representation \( C(\sigma^+) \) of the Clifford algebra \( C(E, g) \) and the mapping \( \sigma^- \) determines the representation \( C(\sigma^-) \) of the Clifford algebra \( C(E, -g) \).

**Proposition 3.2.** The mappings \( \sigma^+ \) and \( \sigma^- \) anticommute. That is,

\[ \sigma^+_{\theta_1} \circ \sigma^-_{\theta_2} + \sigma^+_{\theta_2} \circ \sigma^-_{\theta_1} = 0 \]

for any covectors \( \theta_1, \theta_2 \in E^*. \)

**Proposition 3.3.** For any covector \( \theta \in E^* \) the mappings \( \sigma^+_{\theta} \) and \( \sigma^-_{\theta} \) are linear and antilinear ones with respect to the complex structure is given by the operator \#:

\[ \# \circ \sigma^+_{\theta} = \sigma^+_{\theta} \circ \#, \]

\[ \# \circ \sigma^-_{\theta} = -\sigma^-_{\theta} \circ \#. \]

4. **Clifford Modules over Minkowski Space**

From here on we assume that \((E, g)\) is the 4-dimensional Minkowski space with a metric \( g \) of the signature \((1, 3)\).

Recall that a spinor space is, by definition, an irreducible module over the Clifford algebra \( C(E, g) \). Using representation \( C(\sigma^+) \), one can realize this module as a submodule of the module \( \Lambda^* (E^*) \).

We will look for this submodule in a form \( P\Lambda^* (E^*) \) where \( P \in C^0 (\sigma^-) \) is a projector. Here we denoted by \( C^0 (\sigma^-) \) the image of the even subalgebra of the Clifford algebra \( C(E, -g) \) under the mapping \( \sigma^- \).

These projectors can be described in terms of hyperbolic planes.

**Definition 4.1.** An oriented 2-dimensional plane \( E_h \subset E \) is called hyperbolic, if the restriction of the metric \( g \) on this plane has the signature \((1, 1)\).
Any hyperbolic plane $E_h$ is uniquely determined by two isotropic directions that are intersection of the plane $E_h$ with the cone of isotropic vectors.

Let $v_1, v_2$ be isotropic vectors forming a basis in $E_h$.

Choose vectors $v_1, v_2$ in such a way, that

$$g(v_1, v_2) = -1.$$ 

Note that the choice of such vectors is determined up to a scale factor $v_1 \mapsto t \cdot v_1$, $v_2 \mapsto t^{-1} \times v_2$ where $t \in \mathbb{R} \setminus 0$.

By virtue of this remark the exterior 2-form $\omega = \hat{v}_1 \wedge \hat{v}_2$ is determined by the hyperbolic plane $E_h$ uniquely.

Therefore, one obtains the following statement.

**Proposition 4.1.** There exists a one-to-one correspondence between hyperbolic planes and exterior 2-forms $\omega \in \Lambda^2(E^*)$ such that

1. $\omega \wedge \omega = 0$, and
2. $\omega \wedge \# \omega = \Omega_g$.

**Proposition 4.2.** Any projector $P \in C^0(\sigma^-)$ is a uniquely determined by some hyperbolic plane $E_h \subset E$ and, conversely, any hyperbolic plane $E_h$, determined by an exterior 2-form $\omega \in \Lambda^2(E^*)$, corresponds to the projector

$$P_\omega = \frac{1 + S_{\omega}^-}{2}$$

where $\omega = \theta_1 \wedge \theta_2, \theta_1 = \hat{v}_1, \theta_2 = \hat{v}_2$, and

$$S_{\omega}^- = \frac{1}{2} \times \left[ \sigma_{\theta_1}^- \circ \sigma_{\theta_2}^- - \sigma_{\theta_2}^- \circ \sigma_{\theta_1}^- \right].$$

**Remark 4.1.** It is easy to check that

$$P_\omega = \frac{1}{2} \sigma_{\theta_1}^- \circ \sigma_{\theta_2}^-.$$ 

To prove proposition 4.2 we use the following statements.

**Lemma 4.3.** Operators $\sigma^-_\alpha \circ \sigma^-_\beta$ corresponding to pairs of orthonormal covectors $\alpha, \beta \in E^*$, are traceless:

$$tr_C \left( \sigma^-_\alpha \circ \sigma^-_\beta \right) = 0.$$ 

as linear operators in the complex structure defined by the operator $\#$. 
Proof. The spectrum of operator $\sigma_\beta$ coincides with $\pm 1$ or $\pm i$ depending on covector $\beta$. It is enough to note that the operator $\sigma_\alpha^-$ transposes proper subspaces $E(\lambda)$ of the operator $\sigma_\beta^-:$

$$\sigma^-_\alpha : E(\lambda) \rightarrow E(-\lambda).$$

**Lemma 4.4.** Introduce the operator

$$a_2 = \sum_{i<j} a_{ij} \sigma^-_i \circ \sigma^-_j,$$

where $\sigma^-_i = \sigma^-_{\theta_i}$ for some orthonormal basis $\theta_1, \cdots, \theta_4$ of the space $E^*$. Then

$$a_2^2 = Pf[v] \gamma + g(\nu, \nu), \quad (4.1)$$

where $\gamma = \sigma^-_1 \circ \sigma^-_2 \circ \sigma^-_3 \circ \sigma^-_4$, $\nu = \sum_{i<j} a_{ij} \theta_i \wedge \theta_j$ and $Pf[v]$ is the Pfaffian of 2-form $\nu$.

Proof. (Proposition 4.2) Represent the projector $P$ in the form

$$P = a_0 + a_2 + b\gamma,$$

where $a_0, b \in \mathbb{R}$.

Then, by using lemma 4.1, one obtains

$$tr_{\mathbb{C}}P = 8(a_0 + ib).$$

Since $P$ is a projector, $tr_{\mathbb{C}}P$ is a natural number. Hence $b = 0$. Using lemma 4.2 and the equality $P^2 = P$, one gets

$$a_0^2 + 2a_0a_2 + g(\nu, \nu) + Pf(\nu)\gamma = a_0 + a_2.$$

Therefore,

$$Pf(\nu) = 0, a_0 = \frac{1}{2}, g(\nu, \nu) = \frac{1}{4}.$$

Introducing an exterior 2-form $\omega = 2\nu$, we obtain the result.

**Remark 4.2.** A change of the plane $E_h$ orientation leads to the change $\omega \rightarrow \bar{\omega} = -\omega.$ In addition, the projector $P_\omega$ is a complementary to $P_\omega$:

$$P_\omega + P_\omega = 1,$$  

and

$$P_\omega \cdot P_\omega = 0.$$
Hyperbolic plane $E_h \subset E$ determines the following a direct sum decomposition of the space $E$

$$E = E_h \oplus E_e$$

where 2-dimensional (elliptic) plane $E_e$ is the orthogonal complement of the plane $E_h$ in $E$.

This decomposition leads to a decomposition of the exterior algebra

$$\Lambda^*(E^*) = \sum_i \Lambda^i(E^*_h) \otimes \Lambda^{*-i}(E^*_e).$$

In other words, any exterior form $\alpha \in \Lambda^k(E^*)$ can be represented in the form:

$$\alpha = \alpha_0 + \theta_1 \wedge \alpha_1 + \theta_2 \wedge \alpha_2 + \theta_1 \wedge \theta_2 \wedge \alpha_3$$

where

$$\alpha_0 \in \Lambda^k(E^*_e), \quad \alpha_1, \alpha_2 \in \Lambda^{k-1}(E^*_e), \quad \alpha_3 \in \Lambda^{k-2}(E^*_e).$$

We identified here forms $\beta \in \Lambda^*(E^*_e)$ with exterior forms on $E$ that satisfied to the following conditions:

$$i_{\delta_j} \beta = 0,$$

for all $j = 1, 2$.

**Proposition 4.5.**

$$\text{Im } P_\omega = (1 + e_\omega) \Lambda^*(E^*_e) + e_\theta_1 \Lambda^*(E^*_e)$$

In other words, any element $x \in \text{Im } P_\omega$ can be represented as a pair $(x_0, x_1)$ where $x_0, x_1 \in \Lambda^*(E^*_e)$, and

$$x = (1 + e_\omega)x_0 + e_\theta_1x_1.$$

**Proposition 4.6.** The operator $\sigma_{\theta_2}$ determines an isomorphism

$$\sigma_{\theta_2} : \text{Im } P_\omega \longrightarrow \text{Im } P_\omega.$$
Proof. By straightforward computations one gets

\[ \sigma_{\theta_2} [(1 + e_\omega)x_0 + e_{\theta_1}x_1] = 2e_{\theta_2}x_0 + (1 + e_\omega)x_1. \]

**Proposition 4.7.**

\[ \text{Ker } P_\omega = \text{Ker } \sigma_{\theta_2}. \]

Proof. The statement of the proposition follows from the equality:

\[ \sigma_{\theta_2} P_\omega = \sigma_{\theta_2}. \]

**Corollary 4.8.**

\[ \text{Im } P_\omega \approx \text{Coker } \sigma_{\theta_2}. \]

**Proposition 4.9.** Isotropic direction \( \theta_2 \) determines a pair of projectors \( P_\omega, P_\omega \) such that \( P_\omega + P_\omega = 1 \). That is, the direction determines a non-oriented hyperbolic plane \( E_h \subset E \).

5. **Spinors**

The restriction of the metric \( g \) onto the elliptic plane \( E_e \) is a negative definite metric \( g_e \).

Therefore, operators \( \sigma^+_\theta \) where \( \theta \in E^*_e \) induce an action of the Clifford algebra \( C(E_e, g_e) \) (isomorphic to the division ring of quaternions) on the exterior algebra \( \Lambda^*(E^*_e) \).

In addition, the Hodge operator \( \# \) induces the operator

\[ \#_e : \Lambda^i(E^*_e) \rightarrow \Lambda^{2-i}(E^*_e) \]

for all \( i = 1, 2, \) and

\[ \#^2_e = -1. \]

Hence, the space \( \Lambda^*(E^*_e) \) is a quaternion and complex space simultaneously.

Note also, that the operator \( \# \) admits the restriction onto the subspace \( \text{Im } P_\omega \)

and acts on in the following way:

\[ \# : (x_0, x_1) \rightarrow (\#_e x_0, -\#_e x_1). \]

Let's show that the module \( \text{Im } P_\omega \) is a simple.
Consider a cyclic vector \((x, y)\) in some irreducible \(\sigma^+\) module. Then one gets all covectors \((ax + by, cx + dy)\), for any real numbers \(a, b, c, d\) by acting \(\sigma^+_\theta\), for all \(\theta \in E^*_c\), and considering all linear combinations.

Consider now the subspace spanned in \(\Lambda^*(E^*_c)\) by covectors \(x\) and \(y\). Acting by elements \(\sigma^+_\theta, \theta \in E^*_c\), we obtain the whole space \(\Lambda^*(E^*_c)\). Thus the module under consideration coincides with the whole image \(\text{Im} \ P_\omega\).

Denote this module by \(S_\omega\). We call elements of the module \(S_\omega\) spinors with the orientation \(\omega\), while elements of the module \(S_\omega\) are called dual spinors with the orientation \(\omega\).

Note that there exists an isomorphism between spinors \(S_\omega\) and dual spinors \(S_\omega\) with the orientation \(\omega\) (Proposition 5.2) and the exterior algebra \(\Lambda^*(E^*)\) decomposes now into the direct sum

\[\Lambda^*(E^*) = S_\omega \oplus S_\omega.\]

Note also, that the module \(S_\omega\), in addition to the complex \#, carries a module structure over the Clifford algebra \(C(E_c, g_e)\). The latter algebra is isomorphic (but noncanonically) to the quaternion algebra.

Moreover, the complex structure \# is induced by the Clifford multiplication on the element \#\(\omega\).

In other words, we can consider the module \(S_\omega\) as a module over algebra \(C(E_c, g_e)\), while the Clifford algebra \(C(E, g)\) under the representation \(\sigma^+\) is mapped onto the \(2 \times 2\) matrices algebra over algebra \(C(E_c, g_e)\).

Namely,
\[
\begin{align*}
\sigma^+_{\theta^1} &: (x, y) \mapsto (0, x), \\
\sigma^+_{\theta^2} &: (x, y) \mapsto (-2y, 0), \\
\sigma^+_{\bar{\theta}^\omega} &: (x, y) \mapsto (\sigma^+_{\theta^\omega} x, -\sigma^+_{\theta^\omega} y), \theta^\omega \in E^*_c.
\end{align*}
\]

Therefore, the following theorem is proved.

**Theorem 5.1.** Let \((E, g)\) be the 4-dimensional Minkowski space with a metric \(g\) of signature \((1, 3)\).

Then the following statements hold.

1°. The spinor structure on the space \(E\) is determined by a hyperbolic plane \(E_h \subset E\), or by the exterior 2-form \(\omega \in \Lambda^2(E^*)\) such that

a. \(\omega \wedge \omega = 0\),

b. \(g(\omega, \omega) = 1\).

The hyperbolic plane \(E_h\) uniquely determined by two isotropic directions \(v_1, v_2\), such that \(g(v_1, v_2) = -1\).

The form \(\omega\) (one of two possible orientations of the plane \(E_h\)) coincides up to sign with the form \(\theta_1 \wedge \theta_2, \theta_1 = \bar{v}_1, \theta_2 = \bar{v}_2\). The pair of forms \((\omega, -\omega)\) corresponds to a non oriented plane determined by an isotropic direction.
2°. The spinor space $S_\omega$ corresponding to the hyperbolic plane $E_h \subset E$ with the orientation $\omega$ is isomorphic to the space $\text{Coker} \, \sigma_{\theta_\omega}$ or the space

$$(1 + e_\omega)\Lambda^*(E_e^*) + e_\theta, \Lambda^*(E_e^*)$$

where $E_e$ is the orthogonal complement to the plane $E_h$.

3°. There exists an isomorphism between the spinor space $S_\omega$ and the dual spinor space $S_{\overline{\omega}}, \overline{\omega} = -\omega$. This isomorphism is given by the mapping $\sigma_{\theta_\omega} : S_\omega \rightarrow S_{\overline{\omega}}$.

4°. The exterior algebra $\Lambda^*(E^*)$ splits into the direct sum:

$$\Lambda^*(E^*) = S_\omega \oplus S_{\overline{\omega}}.$$

5°. The spinor space $S_\omega$ is a module over the Clifford algebra $C(E_e, g_e)$ where $g_e$ is the restriction of the metric $g$ onto the elliptic plane $E_e$. This algebra possesses the canonically determined complex structure $\#$ and is noncanonically isomorphic to the division ring of quaternions.

6°. The representation $\sigma^\dagger$ of the Clifford algebra $C(E, g)$ in the spinor space $S_\omega$ is irreducible and defines the isomorphism of the algebra $C(E, g)$ with the $2 \times 2$ matrices algebra over the division ring of quaternions.

6. MISCELLANY

In conclusion we describe the kernel of the operator

$$\sigma^\dagger_{\theta} : S_\omega \rightarrow S_\omega.$$

It is easily seen that the nontriviality kernel condition implies isotropy of the covector $\theta$. Isotropy of the covector $\theta$ implies in turn an isomorphism

$$\sigma^\dagger_{\theta} : \text{Coker} \, \sigma^\dagger_{\theta} \rightarrow \text{Im} \, \sigma^\dagger_{\theta} \simeq \text{Ker} \, \sigma^\dagger_{\theta}.$$

Using the decomposition $E = E_h \oplus E_e$, it is possible to represent any covector $\theta$ in the form

$$\theta = \theta_e + \lambda_1 \theta_1 + \lambda_2 \theta_2,$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$, and $\theta_e \in E_e^*$.

In terms of this decomposition the isotropy condition takes the form:

$$2\lambda_1 \lambda_2 - g(\theta_e, \theta_e) = 0$$

while the action of the operator $\sigma^\dagger_{\theta}$ on an element $(x, y) \in S_\omega$ is given by
\( \sigma^+_\theta : (x, y) \mapsto (-2\lambda_2 y + \sigma^+_\theta x, \lambda_1 x - \sigma^+_\theta y). \)\) \( (7.1) \)

We set:

\[
\lambda_\theta = |g(\theta_e, \theta_e)|^{\frac{1}{2}},
\]

\[
\#_\theta = |g(\theta_e, \theta_e)|^{-\frac{1}{2}} \cdot \sigma^+_\theta,
\]

if \( \theta_e \neq 0 \).

Remark that the operator \( \#_\theta \) defines a complex structure in the module \( \Lambda^*(E_e^*) \).

In this situation we define the transformation

\[ \Xi^\theta : E_e^* \to \text{End} (\Lambda^*(E_e^*)) \]

as a composition of two transformations:

1) reflection in the plane \( E_e^* \) with respect to the line orthogonal to the covector \( \theta_e \), and

2) the transformation \( \sigma^+ \).

From the definition of the transformation \( \Xi^\theta \) it follows that the following relations is valid

\[ \sigma^+_\lambda \circ \Xi^\mu = \sigma^+_\mu \circ \sigma^+_\lambda, \]

for any elements \( \mu, \lambda \in E_e^* \).

Therefore, transformation \( \Xi^\theta \) determines a \( C(E_e, g_e) \) - module structure in the space \( \text{Ker} \sigma^+_\theta \).

Keeping in mind the description of the space \( \text{Ker} \sigma^+_\theta \), denote by \( S_{(w, 1)} \) the \( C(E_e, g_e) \) - module \( \text{Ker} \sigma^+_\theta |_{S_{(w, 1)}} \). The module \( S_{(w, 1)} \) is generated by spinors of the form \((0, y)\), where \( y \in \Lambda^*(E_e^*) \).

Denote also by \( S_{(w, 2)} \) the \( C(E_e, g_e) \) - module \( \text{Ker} \sigma^+_\theta |_{S_{(w, 2)}} \) generated by spinors of the type \((x, 0)\) where \( x \in \Lambda^*(E_e^*) \).

These remarks prove the following

**Proposition 6.1.** Let \( \theta \) be an isotropic covector in the space \((E^*, g)\). Then,

1) there exist an isomorphism

\[ \text{Coker} \sigma^+_\theta \to \text{Im} \sigma^+_\theta \simeq \text{Ker} \sigma^+_\theta, \]

2) \( \text{Ker} \sigma^+_\theta \) is a module over the Clifford algebra \( C(E, -g) \) with respect to the representation \( C(\sigma^-) \),

3) the kernel of the operator

\[ \sigma^+_\theta : S_\theta \to S_\theta \]
is nontrivial and have the following description:

a. if \( \theta_e = 0 \), then the subspace \( \text{Ker} \sigma_\theta^+ \) coincides with one of \( C(E_e, g_e) \) modules \( S(\omega_{ij}), i = 1, 2 \).

b. if \( \theta_e \neq 0 \), then the subspace \( \text{Ker} \sigma_\theta^+ \) is generated by spinors of the type

\[ (\lambda_1 \#_e x, \lambda_2 x), \]

where \( x \in \Lambda^*(E_e^*) \), \( \lambda_1, \lambda_2 \in \mathbb{R} \).

This space has an additional module structure over the Clifford algebra \( C(E_e, g_e) \) with respect to the representation \( \Xi^\theta \).

Proof. Prove the statement 1°. Note that for any isotropic covector \( \theta \) the embedding \( \text{Im} \sigma_\theta^+ \subset \text{Ker} \sigma_\theta^+ \) is obvious. To prove the embedding \( \text{Ker} \sigma_\theta^+ \subset \text{Im} \sigma_\theta^+ \), we consider a covector \( \theta_0 \subset E^* \) such that \( g(\theta_0, \theta_0) = -\frac{1}{2} \). If \( \varsigma \in \text{Ker} \sigma_\theta^+ \), then using the formula (3.1), one gets

\[ 0 = \sigma_{\theta_0}^+ \circ \sigma_{\theta_0}^+ \varsigma = [\sigma_{\theta_0}^+, \sigma_{\theta_0}^+] \varsigma = -\varsigma + \sigma_{\theta_0}^+ \circ \sigma_{\theta_0}^+ \varsigma. \]

Therefore, \( \varsigma \in \text{Im} \sigma_{\theta_0}^+ \).

Statement 2° follows from the anticommutativity of representations \( \sigma^+ \) and \( \sigma^- \) (proposition 3.1).

Statement 3° follows from description (7.1) of the operator \( \sigma_\theta^+ \) action, \( \theta \in E^* \) and from the definition of the representation \( \Xi^\theta \).

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P.Box 546, 119618, MOSCOW, RUSSIA

Current address: Centre for Advanced Study at The Norwegian Academy of Science and Letters
P.O.Box 7606 Skillebekk 0205 Oslo, Norway

GOLUBINSKAYA ST.7-2,APT.554,117574,MOSCOW,RUSSIA