Numerical Solution of the Pressure Equation for Fluid Flow in a Stochastic Medium

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Abstract

We solve the pressure equation numerically by applying a stochastic Ritz-Galerkin method.

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§1 Introduction

When modelling pressure in oil reservoirs one can interpret the permeability in the medium as a random field. Such a model has been suggested by Holden et al. in [HLØUZ]. They consider the pressure equation

$$\nabla_x (k(x, \omega) \circ \nabla_x p(x, \omega)) = -f(x, \omega) \quad x \in D$$
$$p(x, \omega) = g(x, \omega) \quad x \in \partial D$$

where $k(x, \omega)$ is the permeability and $D$ denotes the reservoir with $x \in D$. $\omega \in \Omega$ is a probability space and the $\circ$ stands for a renormalization product of functions on this probability space, called the Wick product. For the permeability, they use a lognormal random field, for which they are able to construct an explicit solution. Unfortunately, the solution for the stochastic pressure equation is singular and has to be understood in a distributional sense. This implies that it is difficult to study its stochastic properties.

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To understand the solution of (1) better, one may try to solve it numerically. We suggest in this paper a numerical method based on the Ritz–Galerkin approach. Our approach is a rather straightforward generalization of the standard finite element method technique for the deterministic pressure equation. We refer the interested reader to the paper [BG] for details about the theory behind the method used here. Our numerical scheme for problem (1) is rather computer extensive.

In the paper, we present some mathematical preliminaries in section 2, and proceed with a consideration of the stochastic permeability model in section 3. We look at some simulations of the lognormal permeability model of [HLÖUZ]. In section 4 we consider the numerical scheme for (1), based on the Ritz–Galerkin approach. Details around the numerical treatment together with a number of figures are presented.

§2 Mathematical Preliminaries

Let \( \mathcal{S} = \mathcal{S}(\mathbb{R}^d) \) be the Schwartz space of rapidly decreasing smooth functions on \( \mathbb{R}^d \), where \( d \in \mathbb{N} \) is the parameter dimension. The dual \( \mathcal{S}' = \mathcal{S}'(\mathbb{R}^d) \), equipped with the weak-star topology, is called the space of tempered distributions. By the Bochner-Minlos theorem there exists a probability measure \( \mu \) on the Borel subsets \( \mathcal{B} \) of \( \mathcal{S}' \) defined by the characteristic functional

\[
\int_{\mathcal{S}'} e^{i <\omega, \phi>} \, d\mu = e^{-\frac{1}{2} \|\phi\|_2^2} \quad \forall \phi \in \mathcal{S}
\]  

where \( \| \cdot \|_2 \) is the norm on \( L^2(\mathbb{R}^d) \) and \( \langle \cdot, \cdot \rangle \) is the dual pairing between \( \mathcal{S}' \) and \( \mathcal{S} \).

Define \( \mathcal{J} := (\mathbb{N}_0^d)^\infty \), i.e. the set of all sequences \( \alpha = (\alpha_1, \alpha_2, \cdots) \) with elements \( \alpha_i \in \mathbb{N}_0 \) and only finitely many \( \alpha_i \neq 0 \).

An orthogonal basis for \( L^2(\mu) \) is given by \( \{H_{\alpha}\}_{\alpha \in \mathcal{J}} \) where \( H_{\alpha}(\omega) \) is defined as

\[
H_{\alpha}(\omega) := \prod_{i=1}^{\ell(\alpha)} h_{\alpha_i}(\omega, \xi_i)
\]

\( \ell(\alpha) \) is the length of the multi-index \( \alpha \) and \( \{\xi_i\}_{i=1}^{\infty} \) is any orthonormal basis consisting of orthonormal tensor products of the Hermite functions in \( L^2(\mathbb{R}) \). \( h_n \) is the Hermite polynomial of order \( n \).

Let \( \mathcal{V} \) be any real Hilbert space. Then we define the Hilbert spaces \( \mathcal{S}^{\rho, k, \mathcal{V}} \) \( (\rho \in [-1, 1], k \in \mathbb{R}) \) as the set of formal sums

\[
\mathcal{F} = \sum_{\alpha \in \mathcal{J}} f_{\alpha} H_{\alpha} \quad (\forall \alpha \in \mathcal{J})
\]

with finite norm \( \|\mathcal{F}\|_{\rho, k, \mathcal{V}} \) induced from the inner product

\[
(\mathcal{F}, \mathcal{G})_{\rho, k, \mathcal{V}} := \sum_{\alpha \in \mathcal{J}} (f_{\alpha}, g_{\alpha})_{\mathcal{V}} (\alpha!)^{1+\rho} (2N)^{k\alpha}
\]

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\]
where

$$(2N)^{k\alpha} := \prod_j (2j)^{k\alpha_j}$$

Note that $(\rho > 0', (\mathcal{S})^{-\rho,-k,V}$ is the dual space of $(\mathcal{S})^{\rho,k,V}$, The dual action is defined by

$$\langle F, f \rangle := \sum_{\alpha \in J} (F_\alpha, f_\alpha)_V \alpha!$$

where $f \in (\mathcal{S})^{\rho,k,V}$ and $F \in (\mathcal{S})^{-\rho,-k,V}$.

When $V = H^m$ or $V = H_0^m$, we will use the notation $(\mathcal{S})^{\rho,k,m}$ and $(\mathcal{S})^{\rho,k,m}_0$ respectively. In the case when $V = \mathbb{R}$ we will use the notation $(\mathcal{S})^{\rho,k}$ and also define the familiar spaces $(\mathcal{S})^\rho := \cap_{k \geq 0} (\mathcal{S})^{\rho,k}$ and $(\mathcal{S})^{-\rho} := \cup_{k \geq 0} (\mathcal{S})^{-\rho,-k} ; \rho \in [0, 1]$.

More information can be found in [HØUZ], [HKPS] and [KLS].

For $N, K \in \mathbb{N}$, we will use the notation

$$J_{N,K} = \{0\} \cup \bigcup_{n=1}^N \bigcup_{k=1}^K A_{n,k}$$

where

$$A_{n,k} = \{\alpha \in \mathbb{N}_0^k | \alpha_k \neq 0, \alpha_1 + \cdots + \alpha_k = n\}$$

In [BG] the finite-dimensional approximation

$$F^{N,K} = \sum_{\alpha \in J_{N,K}} f_\alpha H_\alpha$$

of an element $F \in (\mathcal{S})^{\rho,k,V}$ (given by (3)) is defined. Particularly, convergence rates are found. These approximations are the basis of the following numerical considerations.

§3 Modelling the permeability process

A stochastic model for the permeability should have, at least approximately, the following properties

1. (Independence) If $x_1 \neq x_2$, then $k(x_1, \cdot)$ and $k(x_2, \cdot)$ are independent.

2. (Lognormality) For each $x$ the random variable $k(x, \cdot)$ is lognormal.

3. (Stationarity) For all $x_1, \cdots, x_n \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ the random variable

$$Y = (k(x_1 + y, \cdot), \cdots, k(x_n + y, \cdot))$$

has a distribution independent of $y$. 

3
A natural generalized process \( k(x, \omega) \) satisfying the above properties (although in a generalized way), is

\[
k(x, \omega) = \exp[W_x(\omega)]
\]

where

\[
\exp[W_x] = \sum_{n=1}^{\infty} \frac{1}{n!} W_x^\alpha = \sum_{\alpha} \frac{\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}}{\alpha!} H_\alpha
\]

This so-called positive noise is an element in \((\mathcal{S})^0 = (\mathcal{S})^*\), the space of Hida distributions.

It is also possible to define the so-called smooth positive noise,

\[
k_{\phi}(x, \omega) = \exp[W_{\phi_x}(\omega)] ; \phi \in L^2(\mathbb{R}^d)
\]

where

\[
\exp[W_{\phi_x}(\omega)] = e^{(\omega, \phi_x) - \frac{1}{2} \|\phi\|^2_{L^2(\mathbb{R}^d)}
\]

This is an element in \(L^2(\mu)\). Here \(\phi_x\) is the x-shift of \(\phi\). This element satisfies 1', 2 and 3 where

1' (Independence in a weak sense) If \(\text{supp} \phi_{x_1} \cap \text{supp} \phi_{x_2} = \emptyset\), then \(k(x_1, \cdot)\) and \(k(x_2, \cdot)\) are independent.

It can be shown that if \(\phi^k \to \delta\) in \(\mathcal{S}'\) then \(k_{\phi^k}(x, \cdot) \to k(x, \cdot)\) in \((\mathcal{S})^*\). \(\delta\) is the Dirac \(\delta\)-function.

We will in this paper only consider \(k(x, \omega)\).

\[\S3.1\] Plot of sample paths of positive noise

We will now look at some paths of positive noise. To do numerical simulations of \(\exp[W_x(\omega)]\), we consider the finite-dimensional approximation

\[
\exp[W_x]^{N,K} = \sum_{\alpha \in J_{N,K}} \frac{\xi_1^{\alpha_1}(x) \cdots \xi_n^{\alpha_n}(x)}{\alpha!} H_\alpha
\]

for given \(N, K \in \mathbb{N}\). The results for computer simulations can be seen on figures 1 to 18. To see how different choices of the cutting parameters \(N, K\) affects the sample path, we have fixed \(\omega\) in figure 1 to 15. In figure 1 to 6 we have chosen \(N = K\). In figure 7 to 15 we consider \(K\) bigger than \(N\). In the rest of the figures we show some sample paths for different choices of \(\omega\).
By increasing $K$ we obtain paths which look more irregular. Note that by allowing $N$ to be much less than $K$, we obtain paths which are sometimes negative. By increasing $N$, holding $K$ constant, it seems that we are actually lifting the curves. Note that the scaling of the $y$-axis is not constant.
§4 The pressure equation for fluid flow in a stochastic medium

Let \( D \) be an open, bounded domain in \( \mathbb{R}^d \). Assume further that we are given \( f \in (S)^{-1,-k,0} \). We will study the variation solution of the equation

\[
\nabla_x \cdot \{ \text{Exp}[W_x] \circ \nabla_x u \} = -f \quad \text{in } D
\]

(6)

We know from [V] that

\[
a(u,v) = \langle \text{Exp}[W_x] \circ \nabla u, \nabla v \rangle_{-1,-k,0}
\]

(7)

is a bounded bilinear elliptic form on \( (S)^{-1,-k,1} \times (S)_0^{-1,-k,1} \) whenever \( k \geq K \) for a constant \( K > 0 \).

Equation (6) on is variational form is given by

\[
a(u,v) = \langle f, v \rangle_{-1,-k,0}
\]

(8)

The existence and uniqueness of a solution \( u \in (S)_0^{-1,-k,1} \) follows from [V]. Note that we will not consider different boundary conditions.

§4.1 Numerical simulation of the pressure equation on a line segment

We will in this section assume that \( d = 1, f \equiv 1 \) and study a finite dimensional approximation of the variational solution to equation (8). We base our numerical treatment of the pressure equation on the Ritz-Galerkin method. See [H] for a thorough treatment of finite element methods.

Let \( D = [a,b] \) be an interval, \( D_i = (x_{i-1}, x_i), x_i = a + i h \) an equidistant partition of \( D \) with \( h = (b - a)/(M + 1) \) and \( M \in \mathbb{N} \). A suitable basis for the subspace \( V_M \subset H^1_0(a,b) \), the space of continuous functions, linear on each \( D_i \), is given by \( \{b_i \}_{i=1}^M \) where

\[
b_i(x) = \begin{cases} 
(x - x_{i-1})/(x_i - x_{i-1}) & \text{for } x_{i-1} < x \leq x_i \\
(x_{i+1} - x)/(x_{i+1} - x_i) & \text{for } x_i < x < x_{i+1}, \quad 1 \leq i \leq M \\
0 & \text{otherwise}
\end{cases}
\]

(9)

We are now interested in finding \( u_{RG}^{N,K,M} \in (S_{N,K,M})_0^{-1,-k,1} \) such that

\[
a(u_{RG}^{N,K,M}, v) = \langle f, v \rangle_{-1,-k,0}
\]

for all \( v \in (S_{N,K,M})_0^{-1,-k,1} \) where \( (S_{N,K,M})_0^{-1,-k,1} \) is the subspace of \( (S)_0^{-1,-k,1} \) consisting of those \( \Phi \) such that

\[
\Phi = \sum_{\alpha \in \mathcal{J}_{N,K}} c_\alpha H_\alpha \quad ; \quad c_\alpha \in V_M
\]

7
We now insert
\[ u_{RG}^{N,K,M} = \sum_{1 \leq i \leq M} \sum_{\beta \in \mathcal{J}_{N,K}} c_{i,\beta} b_i H_\beta ; \quad v = b_j H_\gamma \quad (1 \leq j \leq M, \gamma \in \mathcal{J}_{N,K}) \]
into equation (8). The equation now becomes
\[
\sum_{1 \leq i \leq M} \sum_{\beta \in \mathcal{J}_{N,K}} c_{i,\beta} \alpha(b_i H_\beta, b_j H_\gamma) = \begin{cases} 
(1, b_j)_{L^2([a,b])} & \text{if } \gamma = 0 \\
0 & \text{otherwise}
\end{cases} \quad (10)
\]
Equivalently, due to the properties of the Wick product and the structure of \( \alpha(u, v) \),
\[
\sum_{1 \leq i \leq M} \sum_{\beta \leq \gamma} c_{i,\beta} \alpha(b_i H_\beta, b_j H_\gamma) = \begin{cases} 
(1, b_j)_{L^2([a,b])} & \text{if } \gamma = 0 \\
0 & \text{otherwise}
\end{cases} \quad (11)
\]
Let now
\[
q_\gamma(u, v) = \alpha(u H_\gamma, v H_\gamma) \quad (12)
\]
be a bilinear form on \( H^1([a, b]) \times H^1_1([a, b]) \). Note that this form is also bounded and elliptic since
\[
q_\gamma(u, v) = \alpha(u H_\gamma, v H_\gamma) \\
\leq C_1^\alpha \|u H_\gamma\|_{-1,-k,1} \|v H_\gamma\|_{-1,-k,1} \\
= C_1^\alpha (2N)^{-\kappa \gamma} \|u\|_1 \|v\|_1
\]
and
\[
q_\gamma(u, u) = \alpha(u H_\gamma, u H_\gamma) \\
\geq C_2^\alpha \|u H_\gamma\|_{-1,-k,1}^2 \\
= C_2^\alpha (2N)^{-\kappa \gamma} \|u\|_1^2
\]
The existence of constants \( C_1^\alpha, C_2^\alpha \) follows since \( \alpha(\cdot, \cdot) \) is a bounded bilinear form. \( \| \cdot \|_1 \) is the norm of \( H^1([a, b]) \).

We may now rewrite equation (11) into
\[
\sum_{1 \leq i \leq M} c_{i,\gamma} q_\gamma(b_i, b_j) = \begin{cases} 
(1, b_j)_{L^2([a,b])} & \text{if } \gamma = 0 \\
- \sum_{1 \leq i \leq M} \sum_{\beta \leq \gamma} c_{i,\beta} \alpha(b_i H_\beta, b_j H_\gamma) & \text{otherwise}
\end{cases} \quad (13)
\]
Note that
\[ q_{\gamma}(b_i, b_j) = (b_i', b_j')_{L^2(a,b)}(2N)^{-k\gamma} \]  
and
\[ a(b_i H_\beta, b_j H_\gamma) = (g_\beta^0 b_i', b_j')_{L^2(a,b)}(2N)^{-k\gamma} \]
where
\[ g_\gamma^0(x) = \frac{\xi_{1}^{r_1-\beta} \cdots \xi_{n}^{r_n-\beta}(x)}{(\gamma - \beta)!} \]
This gives us the matrix-equation
\[ Lc_\gamma = f_\gamma \]
where
\[ L_{ii} = 2/h ; \quad L_{i,i+1} = -1/h ; \quad L_{ij} = 0 \text{ if } |i-j| > 1 \]
and
\[ \{f_\gamma \} = \begin{cases} \sum_{\beta < \gamma} \frac{1}{h^2} (c_{j+1,\beta} \int_{x_{j+1}}^{x_j} g_\beta^0(x) \, dx - c_{i,\beta} \int_{x_i}^{x_{j+1}} g_\gamma^0(x) \, dx + c_{j-1,\beta} \int_{x_{j-1}}^{x_j} g_\gamma^0(x) \, dx) & \text{if } \gamma = 0 \\
\text{otherwise} & \text{otherwise} \end{cases} \]
using the convention that \( c_{0,\gamma} = c_{M+1,\gamma} = 0 \).

### 4.2 Plots of sample paths for the pressure equation

We will now consider the case \( a = -5, b = 5 \) and \( M = 50 \) for different choices of \( N \) and \( K \). Note that we use different random paths \( \omega \) for all plots, and that we have also plotted the expectation of the solution which is seen to be \(-\frac{1}{2}x^2 + 12.5\). Figures 19 to 22 consists of plots for \( N = K = 5 \), and we observe that they are all relatively close to the expected value.

We obtain more irregular looking curves by increasing \( K \). If we also hold \( N \) constant, the curves seem to lowered. By holding \( K \) constant and increasing \( N \), the curves seem to be lifted up. Figures 23 to 41 should indicate such a behavior.
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