ARITHMETICALLY COHEN–MACAULAY CURVES IN $\mathbb{P}^4$
OF DEGREE 4 AND GENUS 0

MIREILLE MARTIN-DESHAMPS AND RAGNI PIENE

ABSTRACT. We show that the arithmetically Cohen–Macaulay (ACM) curves of degree 4 and genus 0 in $\mathbb{P}^4$ form an irreducible subset of the Hilbert scheme. Using this, we show that the singular locus of the corresponding component of the Hilbert scheme has dimension greater than 6. Moreover, we describe the structures of all ACM curves of $\text{Hilb}^{2n+1}(\mathbb{P}^4)$.

INTRODUCTION

Let $k$ be an algebraically closed field, and let $\mathbb{P}^n = \mathbb{P}^n_k$ denote projective $n$-space over $k$. If $C \subset \mathbb{P}^n$ is a subscheme, we denote by $\mathcal{I}_C$ its sheaf of ideals, and by $I_C$ its homogeneous saturated ideal in the ring $S = k[X_0, \ldots, X_n]$. We recall that $C$ is locally Cohen–Macaulay if all its local rings are Cohen–Macaulay (i.e., if their depth is equal to their dimension) and that $C$ is arithmetically Cohen–Macaulay (ACM) if its homogeneous coordinate ring $S/I_C$ is Cohen–Macaulay. If $C$ is ACM, then it is locally Cohen–Macaulay. By a curve we shall mean a locally Cohen–Macaulay scheme of pure dimension 1. Note that an ACM curve $C \subset \mathbb{P}^n$ is connected.

If $P(m)$ is a polynomial, we denote by $\text{Hilb}^{P(m)}(\mathbb{P}^n)$ the Hilbert scheme parametrizing subschemes of $\mathbb{P}^n$ with Hilbert polynomial $P(m)$. A 1-dimensional subscheme $C \subset \mathbb{P}^n$ of degree $d$ and (arithmetic) genus $g$ has Hilbert polynomial $dm + 1 - g$. We shall say that a curve of degree $d$ and arithmetic genus $g$ is a curve of type $(d, g)$ (or, for short, a $(d, g)$). The points of $\text{Hilb}^{P(m)}(\mathbb{P}^n)$ corresponding to ACM curves form an open subset.

A rational normal $(RN)$ curve in $\mathbb{P}^n$ is a $\mathbb{P}^1$ embedded in $\mathbb{P}^n$ by the sections of $\mathcal{O}_{\mathbb{P}^1}(n)$. The normal bundle of a RN curve is $\mathcal{O}_{\mathbb{P}^1}(n+2)\oplus^{n-1}$ (see e.g. [S]), and its Hilbert polynomial is $P(m) = nm + 1$. The RN curves correspond to the points of a smooth, irreducible open subscheme of $\text{Hilb}^{P(m)}(\mathbb{P}^n)$ of dimension $(n-1)(n+3)$. Moreover, the RN curves are the only reduced and irreducible ACM curves with Hilbert polynomial $nm + 1$.

For $n = 3$, the scheme $\text{Hilb}^{3m+1}(\mathbb{P}^3)$ was studied in [P–S], where it was shown that $\text{Hilb}^{3m+1}(\mathbb{P}^3)$ consists of two smooth components, intersecting transversally. The component containing the points corresponding to RN curves of degree 3 (the twisted cubics) has dimension 12, and the other component — consisting of plane cubics with an isolated or embedded point — has dimension 15. Moreover, in this

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case, every locally Cohen-Macaulay curve is also ACM ([E], [H2] Prop.3.5, [P-S]), and the saturated ideal of such a curve is generated by three independent quadratic forms. The fact that the component containing the twisted cubics is smooth, has been useful in enumerative questions concerning twisted cubics.

Except for the first fundamental results concerning existence and basic properties of Hilbert schemes ([G], [H1]), there are very few results describing Hilbert schemes in general. A. Reeves has shown that the diameter of the Hilbert scheme $\text{Hilb}^{P(m)}(\mathbb{P}^n)$ is $\leq 2d + 2$, where $d$ is the degree of $P(n)$ ([R], Thm. 8) — hence, in the case of curves ($d = 1$) the diameter is always $\leq 4$. She also describes points on various components, but unfortunately this is not sufficient to describe the different components, nor even to determine their number.

For $n \geq 4$ the scheme $\text{Hilb}^{n+m+1}(\mathbb{P}^n)$ has more than two components. For example, in addition to the component containing the RN curves, there is a component — also of dimension $(n-1)(n+3)$ — with general point corresponding to an elliptic curve of degree $n-1$ (in a $\mathbb{P}^{n-2}$ union a disjoint line; moreover, there is a component — of dimension $\frac{1}{2}(n^3 - 2n^2 + 11n - 12)$ — with general point corresponding to a plane smooth curve of degree $n$ union $\frac{1}{2}(n-1)(n-2)$ points, and so on. There seems to be no reason to expect the components to be nonsingular.

The easiest way to exhibit singularities of a component, is to find a curve such that the corresponding point on the Hilbert scheme belongs to only one component and such that the tangent space to the Hilbert scheme at the point is greater than the dimension of the component. When the ideal of the curve is given, it is often possible to compute the tangent space, but it is not always easy to determine the number of components containing the curve.

For $n \geq 4$ we shall see that the component of $\text{Hilb}^{n+m+1}(\mathbb{P}^n)$ containing the RN curves contains a large subvariety, isomorphic to the Grassmann variety $\text{Grass}(1,n)$ of lines in $\mathbb{P}^n$, where the tangent space is “too big”. These points correspond to an ACM $n$-fold structure on a line. Since the set of ACM curves is open, in order to show that the component is singular at these points, it suffices to show that the set of ACM curves is irreducible.

It is reasonable to believe that the set of ACM curves is irreducible if (and only if) $n \leq 7$, and hence that, in this case, any ACM curve can be smoothed (to a RN curve). For $n \geq 8$, however, the situation is different: the lowest length example of a fat point which is not smoothable is a point of degree 8 in 4-dimensional space, such that the ideal of the point is generated by seven quadratic forms (see [I], p.310). This point can be reembedded in $\mathbb{P}^7$ such that the cone over the point is an ACM curve (of genus 0) in $\mathbb{P}^8$ which is not smoothable ([C2]).

The aim of this paper is to show that in the case $n = 4$, the set of ACM curves is indeed irreducible. In particular, the possible non-reduced structures of an ACM curve with Hilbert polynomial $4m+1$ are limited, and we shall describe the possible structures.

In the first section we gather some known facts about ACM curves in $\mathbb{P}^n$ of degree $n$ and genus 0. In the next section we restrict to the case $n = 4$. The strategy for proving that the set of ACM curves is irreducible, is to show that any ACM curve is the limit of a family of (possibly degenerated) RN curves, and to do this by reducing to the case of curves of smaller degree (in a space of smaller dimension). The difficult cases are the non-reduced curves. When the underlying reduced curve
contains a line, we "remove" the line by intersecting with a hyperplane, to get a
degeneration of a RN curve of degree 3. In certain cases, we need to use projections
onto a hyperplane — the method is explained in Section 3.

As a byproduct of the methods used to prove irreducibility of the ACM curves,
we obtain a fairly explicit description of the possible ACM curves in \(Hilb^{4m+1}(\mathbb{P}^4)\).
These results are gathered in Section 4.

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1. ACM curves in \(\mathbb{P}^n\) with Hilbert polynomial \(nm + 1\)

The ACM curves in \(\mathbb{P}^n\) with Hilbert polynomial \(nm + 1\) have been studied by
various people ([W], [E-R-S], [C1]). We shall recall some results.

Proposition 1.1. Let \(C \subset \mathbb{P}^n\) be an ACM curve with Hilbert polynomial \(nm + 1\).
Then its homogeneous ideal \(I_C\) is generated by \(\frac{1}{2}n(n-1)\) independent quadratic
forms.

Proof. A connected curve \(C\) of genus 0 is ACM if and only if \(H^1(\mathbb{P}^n, \mathcal{I}_C(m)) = 0\)
for all \(m \geq 0\). Therefore, the sheaf of ideals \(\mathcal{I}_C\) is 2-regular. By the Castelnuovo–Mumford
regularity criterion it follows that \(I_C\) is generated by quadrics, and the number of independent quadrics is

\[
\dim H^0(C, \mathcal{I}_C(2)) = \binom{n+2}{2} - (2n+1) = \frac{1}{2}n(n-1). \quad \Box
\]

1.2 The \(n\)-fold line. By an \(n\)-fold line we shall mean any curve projectively
equivalent to \(L_n\), the line \(X_1 = \cdots = X_{n-1} = 0\) with \(n\)-fold structure defined
by \(I_{L_n} = (X_1, \ldots, X_{n-1})^2\). The curve \(L_n\) is ACM, and there is a 1-parameter
deformation of \(L_n\) to a RN curve: consider the matrix

\[
\begin{pmatrix}
 tX_0 & X_1 & \cdots & X_{n-2} & X_{n-1} \\
 X_1 & X_2 & \cdots & X_{n-1} & tX_n
\end{pmatrix}
\]

For \(t \neq 0\) the vanishing of the \((2 \times 2)\)-minors defines a RN curve, whereas for \(t = 0\),
the minors generate the ideal of \(L_n\).

The \(n\)-fold line is a particular multiple structure of degree \(n\) on a line. It can be
characterized in the following way:

Proposition 1.3. Let \(C \subset \mathbb{P}^n\) be an ACM curve of degree \(n\) and genus 0 which
contains a line \(L\). If the Zariski tangent space \(T_{C, x}\) to \(C\) at \(x\) has dimension \(n\) for
all \(x \in L\), then \(C\) is an \(n\)-fold line.

Proof. The dimension of the Zariski tangent space \(T_{C, x}\) is equal to the dimension
of the projective tangent space to \(C\) at \(x\), and the latter is equal to the dimension
of the null space of the Jacobian matrix of \(C\) at \(x\). We may assume that \(I = (X_1, \ldots, X_{n-1})\) is the ideal of the line. We know that \(I_C\) is generated by \(\frac{1}{2}n(n-1)\)
independent quadratic forms belonging to \(I\). All their partial derivatives with
respect to \(X_1, \ldots, X_{n-1}\) are also in \(I\), so \(I_C\) is contained in \(I^2\), therefore \(I_C = I^2\)
for degree reasons. \(\Box\)
Proposition 1.4. If $C \subset \mathbb{P}^n$ is an ACM curve with Hilbert polynomial $nm + 1$, then $C$ specializes to an $n$-fold line.

Proof. ([C1], [E–R–S] Ex.4.1, [W] Rem.2.2.1) The quadratic generators of the homogeneous ideal $I_C$ can be chosen in a particular way: $C$ is ACM if and only if $H^1(\mathbb{P}^n, \mathcal{I}_C(m)) = 0$ for all $m \geq 0$, hence if and only if $\dim(S/I_C)_m = nm + 1$ for all $m \geq 0$. Choose a regular sequence $L, M \in S_1$ for $S/I_C$ and put $A = S/(I_C + (L, M))$. One computes $\dim A_0 = 1$ and $\dim A_1 = n - 1$, while $\dim A_m = 0$ for $m \geq 2$. By a coordinate change, we may assume $L = X_0$ and $M = X_n$, so that $A$ is a quotient of $k[X_1, \ldots, X_{n-1}]$. But then $A$ must be $k[X_1, \ldots, X_{n-1}]/(X_1, \ldots, X_{n-1})^2$. It follows that the generators of $I_C$ may be written

$$X_iX_j - X_0L_{i,j} - X_nM_{i,j} - Q_{i,j},$$

where $1 \leq i \leq j \leq n - 1$, the $L_{i,j}$ and $M_{i,j}$ are linear forms in $k[X_1, \ldots, X_{n-1}]$ and the $Q_{i,j}$ are quadratic forms in $k[X_0, X_n]$, subject to certain conditions ensuring that $C$ is an ACM curve. Finally one checks that setting $L_{i,j} = M_{i,j} = Q_{i,j} = 0$ gives a flat specialization. \qed

Let $L \subset \mathbb{P}^n$ be a $n$-fold line, with normal bundle $\mathcal{N}_L$. Christophersen ([C1]) showed that $H^0(L, \mathcal{N}_L) = \text{Hom}(I_L/I_L^2, S/I_L)_0$ and has dimension $n(n - 1)^2$. Since $L$ lies on the component of $\text{Hilb}^{nm+1}(\mathbb{P}^n)$ containing the RN curves, which has dimension $(n - 1)(n + 3)$, it follows that $L$ is a singular point on $\text{Hilb}^{nm+1}(\mathbb{P}^n)$ whenever $n(n - 1)^2 \geq (n - 1)(n + 3)$, i.e., when $n \geq 4$. In order to determine whether the $n$-fold lines also are singular points on the component, one needs to check that they do not lie on any other component of the Hilbert scheme. We shall prove (in Section 3) that this holds for $n = 4$.

Proposition 1.5. ([W], Prop. 5.6, p. 257) Let $C \subset \mathbb{P}^n$ be a reduced curve with Hilbert polynomial $nm+1$, not contained in a hyperplane. Then $C$ can be smoothed in $\mathbb{P}^n$.

Note that it follows from this proposition, since any ACM curve is connected, that a reduced ACM curve with Hilbert polynomial $nm + 1$ can be smoothed to a RN curve.

2. The case $n = 4$

Let $H^0$ denote the open subscheme of $H = \text{Hilb}^{nm+1}(\mathbb{P}^4)$ consisting of ACM curves. We shall prove that $H^0$ is contained in the closure $H'$ of the open subscheme consisting of the RN curves in $H$.

Applying Propositions 1.1 and 1.5 to the case $n = 4$, we get that for any $C \in H^0$, the ideal $I_C$ is generated by 6 independent quadratic forms, and if $C \in H^0$ is reduced, then $C \in H'$.

When $C$ is ACM and not reduced, we have the following possibilities ($C_{\text{red}}$ is the underlying reduced curve of $C$):

Case I: $\deg C_{\text{red}} = 3$. Then $C$ is the union of a double line and a reduced curve of degree 2.

Case II: $\deg C_{\text{red}} = 2$. Then $C_{\text{red}}$ is a plane conic, and we have 2 subcases.
- II.1: $C$ is a double structure on a reduced conic;
- II.2: $C$ is the union of a reduced line and a triple structure on another line.

Case III: $\deg C_{\text{red}} = 1$. Then $C$ is a quadruple structure on a line.

In each case, there exists a hyperplane such that its intersection with $C$ contains a curve. This will give us, at least when the residual intersection is a line, a way of constructing the curve as a limit of a family of reduced ACM curves.

We shall need a technical result.

**Lemma 2.1.** Let $L$, $L_1$, $L_2$, $L_3$ be four independent linear forms, and $Q_1$, $Q_2$, $Q_3$ three quadratic forms contained in the ideal $(L_1, L_2, L_3)$. Then we have

$$(L, Q_1, Q_2, Q_3) \cap (L_1, L_2, L_3) = (L L_1, L L_2, L L_3, Q_1, Q_2, Q_3).$$

**Proof.** $aL + \sum \beta_i Q_i \in (L_1, L_2, L_3)$ if and only if $a \in (L_1, L_2, L_3)$. $\Box$

**Proposition 2.2.** Let $C \in H^0$ and let $L \in S$ be a linear form such that the corresponding hyperplane $H_L$ intersects $C$ in a curve. Denote by $C'$ the largest curve contained in $C \cap H_L$. We have an exact sequence of graded algebras:

$$0 \rightarrow S/I_{\Gamma}(-1) \rightarrow S/I_C \rightarrow S/I_C + (L) \rightarrow 0$$

where $I_{\Gamma}$ is the saturated ideal of a curve $\Gamma$ contained in $C$; as a set, $C$ is the union of $C'$ and $\Gamma$, and we have

$$\deg C' + \deg \Gamma = 4,$$

and

$$g(C') + g(\Gamma) \geq 1 - \deg \Gamma.$$

If $\Gamma$ is a line (or equivalently, if $C \cap H_L$ contains a curve of degree 3), then $I_{C'} = I_C + (L)$, $C'$ is a degeneration of a twisted cubic in $H_L$, and $C$ is in $H^1$.

**Proof.** Let $I_{\Gamma}$ be the kernel of the multiplication by $L$ from $S$ to $S/I_C$. It is the saturated ideal of a curve $\Gamma$ contained in $C$. Let $K = I_{C'}/I_C + (L)$. Since $C'$ and $C \cap H_L$ differ only at a finite set of points, the sheaf $\tilde{K}$ associated to $K$ has finite support, and we have, for all $m \in \mathbb{N}$, the equality:

$$\chi_{\mathcal{O}_C}(m) = \chi_{\mathcal{O}_\Gamma}(m - 1) + \chi_{\mathcal{O}_{C'}}(m) + lg \tilde{K}$$

from which we get the relations between the genera and degrees.

Suppose that $\Gamma$ is a line. Then we get $\deg C' = 3$ and $g(C') \geq 0$. (Conversely, if $C \cap H_L$ contains a curve of degree 3, this curve is equal to $C'$ — if not, $\deg C' = 4$, and $C = C'$ is contained in a hyperplane — and $\Gamma$ is a line.)

The ideal $I_{\Gamma}$ is generated by three independent linear forms $L_1$, $L_2$, $L_3$, so $I_C$ contains $(L L_1, L L_2, L L_3)$. Moreover, since $I_C$ is generated by six independent quadratic forms, we have necessarily (for simplicity assume $L = X_4$):

$$I_C = (X_4 L_1, X_4 L_2, X_4 L_3, Q_1, Q_2, Q_3)$$

where $Q_1, Q_2, Q_3$ are three quadratic forms contained in the ideal $I_{\Gamma} = (L_1, L_2, L_3)$. 
If \( g(C') = 1 \), \( C' \) is a plane cubic, and the \( Q_i \), modulo \( X_4 \), have a common linear factor \( M \). Hence \( I_C \) is contained in the ideal \((X_4, M)\), which is impossible.

It follows that \( g(C') = 0 \), so that \( C' \) is a curve of degree 3 and genus 0 in \( H_L \cong P^3 \). Since such a \( C' \) is ACM, its ideal \( I_{C'} \) in \( k[X_0, \ldots, X_3] \) is generated by three quadratic forms, which have to be the images of \( Q_1, Q_2, Q_3 \). Therefore \( I_{C'} = I_C + (X_4) = (Q_1, Q_2, Q_3, X_4) \).

1) Suppose that \( \Gamma \) is not in \( H_L \). Then \( \Gamma \) and \( H_L \) intersect in a point \( P \) and the rank of \( L_1, L_2, L_3, X_4 \) is 4. In this case \( C \) is the scheme theoretic union of the two curves \( C' \) and \( \Gamma \) that intersect in \( P \) (cf. Lemma 2.1). We shall show that \( C \) can be deformed into an ACM reduced curve through \( P \).

The set of curves of degree 3 and genus 0 in \( H_L \) passing through \( P \) is irreducible of dimension 9 (it is clear if we see it as a quotient of a space of matrices), and it contains smooth twisted cubics. One can deduce from ([E], Prop.1, p.424), that there exist, in the ring \( S[t] \), three polynomials \( Q_{1,t}, Q_{2,t}, Q_{3,t} \), which are quadratic forms in \( X_0, \ldots, X_3 \), and which define a flat family \( C' \) of curves contained in \( H_L \), parametrized by an affine open subscheme \( U \) of the line, containing the point \( t = 0 \), such that

- for \( t = 0 \), \( Q_{1,0} - Q_1 \in (X_4) \) (and \( C'_0 = C' \)),
- for \( t \neq 0 \), \( C'_t \) is a smooth twisted cubic passing through \( P \).

Since \( C'_t \) goes through \( P \), we have \( Q_{1,t} \in (X_4, L_1, L_2, L_3) \). Write \( Q_{1,t} = Q'_{1,t} + X_4 Q''_{1,t} \), where \( Q'_{1,t} \in (L_1, L_2, L_3) \).

Consider the following ideal in \( S[t] \):

\[
I_t = (X_4 L_1, X_4 L_2, X_4 L_3, Q'_{1,t}, Q''_{1,t}, Q'_{2,t}, Q''_{2,t}).
\]

By Lemma 2.1, for every \( t \),

\[
I_t = (L_1, L_2, L_3) \cap (X_4, Q'_{1,t}, Q''_{1,t}, Q'_{2,t}, Q''_{2,t}).
\]

This ideal defines a a flat family \( C \) of curves in \( P^4 \), parametrized by \( U \), such that

- for \( t = 0 \), \( C_0 \) is the curve defined by

\[
I_0 = (X_4 L_1, X_4 L_2, X_4 L_3, Q_{1,0}', Q''_{1,0}, Q'_{2,0}, Q''_{2,0})
\]

and since \( Q_i - Q''_i \in (X_4) \cap (L_1, L_2, L_3) \), we have \( C_0 = C \).

- for \( t \neq 0 \), \( C_t \) is ACM (since the limit \( C_0 = C \) is ACM) and is equal to the union of the line defined by \( (L_1, L_2, L_3) \) and the smooth twisted cubic \( C'_t \), which intersect at \( P \). Hence \( C_t \) is both ACM and reduced and therefore belongs to \( H' \).

2) Suppose that \( \Gamma \) is in \( H_L \), i.e., the rank of \( L_1, L_2, L_3, X_4 \) is 3. Let \( L' \) be a linear form independent of \( L_1, L_2, L_3, X_4 \), and suppose that \( L_1, L_2, X_4 \) are also independent. Since \( C \) contains \( \Gamma \), we have \( Q_i \in (X_4, L_1, L_2) \). Write \( Q_i = Q'_{i} + X_4 L''_i \), where \( Q'_{i} \in (L_1, L_2) \).

Consider the following ideal in \( S[t] \):

\[
I_t = (X_4 L_1, X_4 L_2, (X_4 + t L'), Q_1 + t L' L''_1, Q_2 + t L' L''_2, Q_3 + t L' L''_3).
\]

This ideal defines a flat family \( C \) of curves in \( P^4 \), parametrized by the affine line, such that
– for \( t = 0 \), \( C_0 = C \)
– for \( t \neq 0 \), \( C_t \) is ACM (since the limit \( C_0 = C \) is ACM) and is of the same type as the limit curve of the family studied in the preceding case, since the rank of \( L_1, L_2, X_4 + tL', X_4 \) is 4. Therefore, \( C_t \) belongs to \( H' \), and hence the limit curve \( C \) does too. \( \square \)

**Corollary 2.3.** If \( C \) is an ACM curve of \( H \), which is the union of a double line and a reduced curve of degree 2, then \( C \) is in \( H' \).

**Proof.** The curve \( C_{\text{red}} \) is either the connected union of three lines or the connected union of a line and a plane conic, hence is contained in a hyperplane. Apply Proposition 2.2 to this hyperplane. \( \square \)

We shall now go to the case II.1.

The idea is to deform the “double conic” to the union of two reduced conics intersecting in a point.

**Lemma 2.4.** Let \( L, M, L', M' \) be four independent linear forms, \( Q \in (L', M') \) and \( Q' \in (L, M) \) two quadratic forms. Then we have:

\[
(L, M, Q) \cap (L', M', Q') = (L, M) \cdot (L', M') + (Q, Q')
\]

**Proof.** Write \( Q = AL' + BM', Q' = A'L + B'M \). Then \( \alpha L' + \beta M' + \gamma (A'L + BM') \in (L, M, AL' + BM') \) if and only if there exists \( \lambda \) such that \((\alpha - \lambda A)L' + (\beta - \lambda B)M' \in (L, M)\). This holds if and only if there exist \( \lambda \) and \( \mu \) such that \( \alpha - \lambda A = \mu M' \) and \( \beta - \lambda B = \mu L' \) belong to \((L, M)\). In this case, we have

\[
\alpha L' + \beta M' + \gamma (A'L + B'M) - (\lambda Q + \gamma Q') \in (L, M) \cdot (L', M').
\]

**Proposition 2.5.** If \( C \) is an ACM curve of \( H \) which is a double structure on a reduced conic, then \( C \) is in \( H' \).

**Proof.** Suppose that the ideal of \( C_{\text{red}} \) is \((X_0, X_1, Q)\), where \( Q \) is a quadratic form. Since \( I_C \) is generated by quadratic forms, we may assume \( Q \in I_C \). Let \( L \) be a linear form in \( X_0, X_1 \). By Proposition 2.2 (and with the same notation) we may assume that \( \Gamma \) is not a line. Therefore we must have \( \Gamma = C_{\text{red}} \), and \( LI_\Gamma = (L_0, X_0, X_1) \subseteq I_C \). Since this holds for every \( L \), we have that \((X_0, X_1)^2 \subseteq I_C \).

Therefore we can write \( I_C = (X_0^2, X_0X_1, X_1^2, Q, X_0L + X_1M, X_0L' + X_1M') \), where \( L, L', M, M' \) are linear forms in \( X_2, X_3, X_4 \), and the six quadratic forms are independent.

If \( M'L - M'L = 0 \), we have either \( M' = aM \) and \( L' = aL \), with \( a \in k \), but then \( X_0L + X_1M \) and \( X_0L' + X_1M' \) are not independent; or \( M = aL \) and \( M' = aL' \), with \( a \in k \), and we get

\[
(X_0, X_1, L, L') \subseteq (I_C : (X_0 + aX_1)) = (X_0, X_1, Q),
\]

hence \( L = L' = M = M' = 0 \).
Hence \( ML' - M' \ell \neq 0 \). Since \( ML' - M' \ell \in (I_C : (X_0)) = (X_0, X_1, Q) \), we may write \( Q = AX_0 + BX_1 + ML' - M' \ell \).

We shall now see that \( C \) can be deformed into an ACM reduced curve.

If the rank of \( L, M \) is 2, consider the following ideal in \( S[t] \):

\[
I_t = (X_0, X_1) \cdot (X_0 - tM, X_1 + tL) + (Q, X_0L + X_1M, L'(X_0 - tM) + M'(X_1 + tL)).
\]

This ideal defines a flat family \( C \) of curves contained in \( \mathbf{P}^4 \) parametrized by the affine line, such that

- for \( t = 0 \), \( C_0 = C \),
- for \( t \neq 0 \), \( C_t \) is ACM and is defined by the ideal

\[
(X_0, X_1) \cdot (X_0 - tM, X_1 + tL) + (L'(X_0 - tM) + M'(X_1 + tL), X_0(L' + tA) + X_1(M' + tB)).
\]

It follows from Lemma 2.4 that \( C_t \) is the union of the two conics defined by the ideals \((X_0, X_1, L'(X_0 - tM) + M'(X_1 + tL))\) and \((X_0 - tM, X_1 + tL, X_0(L' + tA) + X_1(M' + tB))\). The first conic is \( C_{\text{red}} \), and the second is reduced for general \( t \), because it varies in a flat family of conics which is a deformation of \( C_{\text{red}} \).

If the rank of \( L', M' \) is 2, the situation is similar.

If the rank of \( L, M \) and the rank of \( L', M' \) are both 1, we can suppose that \( M = L' = 0 \) and that \( C \) is a double structure on the degenerated conic defined by \((X_0, X_1, LM')\). Choose a linear form \( L'' \), independent of \( M' \), and consider the following ideal in \( S[t] \):

\[
I_t = (X_0^2, X_0X_1, X_1^2, AX_0 + BX_1 - M'L, X_0L, tX_0L'' + X_1M')
\]

It defines a a flat family \( C \) of curves contained in \( \mathbf{P}^4 \) parametrized by the affine line, such that

- for \( t = 0 \), \( C_0 = C \),
- for \( t \neq 0 \), \( C_t \) is ACM, and it is of the same type (with \( L, M, L', M' \) replaced by \( L, 0, tL'', M' \)) as the limit curve of the family studied in the preceding case, since the rank of \( M', L'' \) is 2. Hence, \( C_t \) belongs to \( \mathcal{H} \), and so does the limit curve \( C \).  

For the remaining cases, II.2 (union of a reduced line and a triple structure on another line) and III (quadruple structure on a line), we need another approach, using projections.

3. Projection on a Hyperplane

Let \( P \) be a point of \( \mathbf{P}^4 \) and \( \pi \) the projection on a hyperplane \( \simeq \mathbf{P}^3 \) from the point \( P \). Let \( U = \mathbf{P}^4 - \{P\} \) denote the domain of definition of \( \pi \). If \( X \) is a closed subscheme of \( \mathbf{P}^4 \) not passing through \( P \), the scheme theoretic image \( X' \) of \( X \) is well defined: the ideal sheaf \( \mathcal{I}_{X'} \) is the kernel of the composed map \( \mathcal{O}_{\mathbf{P}^3} \to \pi_*\mathcal{O}_U \to \pi_*\mathcal{O}_X \). The closed subscheme \( X' \) of \( \mathbf{P}^3 \) is the projection of \( X \) and there is a morphism from \( X \) to \( X' \) induced by \( \pi \). Moreover, the projection of \( X_{\text{red}} \) is \( X'_{\text{red}} \).
Lemma 3.1. Let \( C \) be a (locally Cohen-Macaulay) curve in \( \mathbb{P}^4 \), not contained in a hyperplane, and \( \pi \) a projection on a hyperplane from a point not on \( C \). Then the projection \( C' \) of \( C \) is a (locally Cohen-Macaulay) curve not contained in a plane.

Proof. First, \( C' \) is not plane because \( C \) is not contained in a hyperplane.

There is a natural injection \( \mathcal{O}_{C'} \rightarrow \pi_*\mathcal{O}_C \). If \( C' \) is not locally Cohen-Macaulay, \( C' \) has an associated component of dimension 0, therefore \( C \) has a component which is a line contracted by \( \pi \), and the center of the projection is on \( C \), hence we get a contradiction. \( \square \)

Proposition 3.2. Let \( C \) be an ACM curve of \( H \), and let \( \pi \) be a projection with center \( P \notin C \). The projection \( C' \) of \( C \) is a curve, not contained in a plane, of genus 0 or 1, and degree 3 or 4. If \( C_{\text{red}} \) is a union of lines, then \( \pi : C \to C' \) is an isomorphism at every point \( x \in C \) such that \( P \notin T_{C,x} \).

Proof. From the exact sequence

\[
0 \to \mathcal{O}_{C'} \to \pi_*\mathcal{O}_C \to \mathcal{K} \to 0
\]

we obtain the following equality for all \( m \in \mathbb{Z} \)

\[
\chi_{\mathcal{K}}(m) = \chi_{\pi_*\mathcal{O}_C}(m) = \chi_{\mathcal{O}_C}(m),
\]

because \( \pi|_C \) is finite. Therefore the degree \( d' \) of \( C' \) is \( \leq 4 \), and its genus \( g' \) satisfies

\[
g' \leq \frac{(d' - 2)(d' - 3)}{2}.
\]

Now we have \( \dim H^0(C', \mathcal{O}_{C'}) \leq \dim H^0(C', \pi_*\mathcal{O}_C) = \dim H^0(C, \mathcal{O}_C) = 1 \). So we get \( \dim H^0(C', \mathcal{O}_{C'}) = 1 \), hence \( g' = \dim H^1(C', \mathcal{O}_{C'}) \geq 0 \). Since \( C' \) is not plane, we have \( d' \geq 3 \).

Suppose \( C_{\text{red}} \) is a union of lines. Since the projection of a line is a line, \( \pi \) is set theoretically bijective, therefore it is an isomorphism at a point \( x \) if and only if it is not ramified at \( x \), that is, if and only if \( P \notin T_{C,x} \). \( \square \)

Proposition 3.3. Let \( C \) be an ACM curve of \( H \), and let \( \pi \) be a projection with center \( P \notin C \). If the projection \( C' \) of \( C \) is a curve of degree 4 and genus 0, then the curve \( C \) belongs to \( H' \).

Proof. Suppose that \( C' \) is a (4,0). Then for all \( m \in \mathbb{Z} \), we have \( \chi(K(m)) = 0 \), hence \( K = 0 \) and \( \mathcal{O}_{C'} \simeq \pi_*\mathcal{O}_C \), so the restriction of \( \pi \) to \( C \) is an embedding. The curve \( C' \) is not linearly normal in \( \mathbb{P}^3 \): the sheaf \( \mathcal{O}_{C'}(1) \) has five sections which correspond to the embedding of \( C' \) as \( C \) in \( \mathbb{P}^4 \).

Now, we know from [M-D-P] that \( C' \) is a curve in the biliaison class of two disjoint lines: in particular, \( h^1(\mathcal{I}_{C'}(m)) = 0 \) for \( m \neq 1 \), and \( h^1(\mathcal{I}_{C'}(1)) = 1 \). It follows, on the one hand, that \( h^0(\mathcal{I}_{C'}(2)) = 1 \), so that \( C' \) is contained in a quadric, and on the other hand that \( h^1(\mathcal{I}_{C'}(2)) = h^2(\mathcal{I}_{C'}(1)) = 0 \), hence — by the Castelnuovo–Mumford regularity criterion — that \( C' \) is the intersection of cubic surfaces. In particular, one can make a linkage of type \((2,3)\), and one obtains a curve of degree 2 and genus \(-1\) which is either the union of two disjoint lines or a double structure on a line.

To conclude we need the following result.
Lemma 3.4. Let $C'$ be a curve of degree 4 and genus 0 in $\mathbb{P}^3$ such that $C'_{\text{red}}$ is a union of lines. Suppose $C'$ is contained in a complete intersection of type $(2,3)$. Then there exists a curve $C'_1$ of degree 3 and genus 0 and a line $D'$, both contained in $C'$, and an exact sequence

$$0 \to \mathcal{O}_{D'}(-1) \to \mathcal{O}_{C'} \to \mathcal{O}_{C'_1} \to 0.$$ 

Let us assume this result. Because $C'$ can be embedded in $\mathbb{P}^4$, so can $C'_1$, and its image is a curve $C_1$ of degree 3 contained in $C$. Moreover, the image of $\mathcal{O}_{D'}(-1) \to \mathcal{O}_{C'}$ corresponds to a section of $\mathcal{O}_C(1)$ vanishing on $C_1$, hence it comes from a linear form defining a hyperplane $H$ in $\mathbb{P}^4$ such that $H \cap C$ contains $C_1$. We can therefore apply Proposition 2.2.

**Lemma 3.5.** Let $\Gamma$ be a curve of degree 2 and genus $-1$ in $\mathbb{P}^3$. Assume that $\Gamma$ can be linked by a complete intersection $X$ of type $(2,3)$ to a curve supported by lines. Then there exists a curve $\Gamma_1$ of degree 3 and genus 0 containing $\Gamma$ and contained in $X$.

**Proof.** Let $X_0, X_1, X_2, X_3$ denote coordinates of $\mathbb{P}^3$. Recall (cf. [Mi]) that, up to projective equivalence, the equations of a surface of degree 2 and genus $-1$ on a line are $(X_0^2, X_0X_1, X_1^2, X_0X_2 + X_1X_3)$. One can then verify that the singular quadric surfaces which contain such a curve are unions of two planes or double planes.

We shall have to distinguish between several cases. Let $Q$ (resp. $S$) denote the quadric (resp. cubic) surface (and also its equation), and set $X = Q \cap S$.

1) If $Q$ is smooth, $\Gamma$ is linearly equivalent to two lines of one system on $Q$, and $X$ contains at least one line $D$ from the other system — hence $\Gamma_1 = \Gamma \cup D$ works.

2) If $Q$ is a cone, it does not contain two disjoint lines, nor a double line of genus $-1$.

3) If $Q$ is the union of two planes $H$ and $H'$, and $\Gamma$ is the union of two lines $D$ and $D'$, we may assume $D \subset H$ and $D' \subset H'$. Set $\Delta = H \cap H'$. If $X$ contains $\Delta$, $\Gamma_1 = \Gamma \cup \Delta$ works. If not, we have $S \cap H = D \cup D_1 \cup D_2$ and $S \cap H' = D' \cup D'_1 \cup D'_2$, where $D_1, D_2, D'_1, D'_2$ are (not necessarily distinct) lines. Then $D' \cap \Delta$ is contained in $(D \cup D_1 \cup D_2) \cap \Delta$, hence for example in $D_1 \cap \Delta$. So $\Gamma_1 = \Gamma \cup D_1$ works.

4) Suppose $Q$ is the union of two planes $H$ and $H'$ and $\Gamma$ is a double line. We may assume that

$$I_\Gamma = (X_0^2, X_0X_1, X_1^2, X_0X_2 + X_1X_3)$$

and $Q = X_0X_1$. Then

$$S \in (X_0, X_1 LL') \cap (X_1, X_0 MM'),$$

where $L, L'$ (resp. $M, M'$) are two linear forms independent of $X_0$ (resp. $X_1$), and hence

$$S = AX_0X_1 + \alpha X_0 MM' + \beta X_1 LL',$$

where $A$ is a linear form and $\alpha, \beta \neq 0$. Since $S \in I_\Gamma$, we must have $X_1 LL' \in (X_0^2, X_3), \text{ hence } LL' \in (X_1, X_3)$ and, say, $L \in (X_1, X_3)$. 


If $L \not\in (X_1)$, let $D$ denote the line defined by $(X_0, L)$ and set $\Gamma = \Gamma \cup D$. Then

$$I_{\Gamma \cap D} = I_\Gamma + (X_0, L) = (X_0, L, X_1, X_3) = (X_0, L, X_1^2),$$

hence $\Gamma$ is a curve of degree 3 and genus 0.

If $L \in (X_1)$ and $M \in (X_0)$, then $S \in (X_0^2, X_0X_1, X_1^2)$ and this ideal is the ideal of a curve $\Gamma$ that works.

5) If $Q$ is a double plane, then $\Gamma$ is a double line. We may assume $Q = X_0^2$ and

$$I_\Gamma = (X_0^2, X_0X_1, X_1^2, X_0X_2 + X_1X_3).$$

As in 4) we get $S \in (X_0, X_1, LL')$ and $LL' \in (X_1, X_3)$ and, say, $L \in (X_1, X_3)$.

If $L \notin (X_1)$, we conclude as in 4). If $L \in (X_1)$, then

$$S \in (X_0, X_1^2) \cap I_\Gamma = (X_0^2, X_0X_1, X_1^2)$$

and again we conclude as in 4). □

**Proof of 3.4.** By assumption, $C'$ is linked to a curve $\Gamma$ via a complete intersection $X$ of type $(2,3)$, so $\Gamma$ has degree 2 and genus $-1$. With the notations as in Lemma 3.5, the curve $\Gamma$ is linked to a curve $C'_1$ contained in $C'$. Because $\Gamma$ is of degree 3 and genus 0, so is $C'_1$.

Set $F = \mathcal{I}_{C'_1}/\mathcal{I}_{C'}$. For all $m$, we have $\chi(F(m)) = m$. Moreover, $h^0(F) = 0$, $h^0(F(1)) = 1$, $h^1(F) = 0$, so that $F(1)$ is generated by its sections and $F \simeq \mathcal{O}_{D'}(-1)$, for some $D'$ contained in $C$. It is then easy to see that $D'$ is a line. □

It remains to study the curves $C$ in the cases II.2 and III (where $C_{\text{red}}$ is then a union of lines) such that for every projection $\pi$ from a point not on $C$, the scheme theoretic image $C'$ of $C$ is a $(3,0)$ or a $(4,1)$. We may also assume (cf. Proposition 1.3) that the tangent spaces $T_{C,x}$ at the points $x$ of a component of $C$ are not all of dimension 4. We deduce from Propositions 3.2 and 3.3 the following result:

**Corollary 3.6.** Let $C$ be an ACM curve of $H$ such that $C_{\text{red}}$ is a union of lines, and such that for every projection $\pi$ from a point $P \notin C$, the scheme theoretic image $C'$ of $C$ is a $(3,0)$ or a $(4,1)$. Then the union of the tangent spaces $T_{C,x}$ at all points $x$ of $C$ is the whole space. Moreover, if $P \in T_{C,x} \setminus C$, then $C'$ is a $(3,0)$; if $P \notin T_{C,x}$, then there exists a unique point $x \in C$ such that $P \in T_{C,x}$, $\pi$ is an isomorphism outside of $x$, and $C'$ is a $(4,1)$.

**Proof.** In the exact sequence

$$0 \to \mathcal{O}_{C'} \to \pi_*\mathcal{O}_C \to \mathcal{K} \to 0$$

the support of $\mathcal{K}$ is the set of points of $C$ where $\pi : C \to C'$ is not an isomorphism. Proposition 3.2 shows that this is also the set of points $x$ of $C$ such that $P \in T_{C,x}$. If this set is empty, $C'$ is a $(4,0)$.

The degree of $C'$ is 4 if and only if the support of $\mathcal{K}$ is finite, and this is equivalent to $P \notin T_{C,x}$. In this case, for all $m \in \mathbb{Z}$, we have $\chi(\mathcal{K}(m)) = 1$ and the support of $\mathcal{K}$ is one point. □

We need now a description of the curves of type $(4,1)$ or $(3,0)$ in $\mathbb{P}^3$, supported by lines. The computations are elementary and will not be given.
Lemma 3.7. Let $\Gamma \subset \mathbb{P}^3$ be a curve of type $(4,1)$ which is the union of a reduced line $X_1 = X_3 = 0$ and a triple line with support $X_1 = X_2 = 0$. Then the ideal of $\Gamma$, up to a linear change of coordinates, is given as

$$(X_1 X_2, X_1^2 + X_2 X_3) \text{ or } (X_1 X_2, X_2 X_3, X_3^2) \text{ or } (X_1^2, X_1 X_2, X_1 Q + X_2 X_3)$$

where $Q$ is a quadratic form independent of $X_1, X_2$. In the first two cases, the tangent spaces to $\Gamma$ are constant along the triple line, equal to a plane. Moreover, in the first case, $\Gamma$ links the reduced line to the triple line given by the ideal $(X_1^2 + X_2 X_3, X_1 X_2, X_3^2)$, which is a curve of type $(3,0)$. In the third case, the tangent spaces to $\Gamma$ are also a constant plane, except at the points where $Q$ vanishes. If $Q = 0$, the tangent spaces are of dimension 3 (equal to the whole space) along the triple line.

Lemma 3.8. Let $\Gamma \subset \mathbb{P}^3$ be a curve of type $(4,1)$ which is a quadruple structure on the line $X_1 = X_2 = 0$, then the ideal of $\Gamma$, up to a linear change of coordinates, is given as

$$(X_1^2, X_2^2) \text{ or } (X_1^2, X_2^2 + X_1 X_3) \text{ or } (X_1^2, X_1 X_2, X_1 Q + X_2^2)$$

where $Q$ is a quadratic form. In the second and third cases (if $Q \neq 0$), (almost all) the tangent spaces are constant, equal to a plane. In the first and third cases (if $Q = 0$), the tangent spaces are the whole space. Moreover, in the first (resp. second) case, $\Gamma$ links the reduced line $X_1 = X_2 = 0$ to a triple structure of type $(3,0)$ on the same line, with equations $(X_1^2, X_1 X_2, X_2^2)$ (resp. $(X_1^2, X_1 X_2, X_2^2 + X_1 X_3)$).

As an easy consequence of these two results, we get:

Lemma 3.9. In $\mathbb{P}^3$, the only $(3,0)$ structures on a line are (up to a change of coordinates) the ones given by the ideal $(X_1^2, X_1 X_2, X_2^2 + \alpha X_1 X_3)$ ($\alpha = 0$ gives the 3-fold line).

Proof. A curve $\Gamma$ of type $(3,0)$ is the intersection of quadric surfaces, hence is contained in a curve $\Gamma'$ of type $(4,1)$, which is the complete intersection of two quadrics. If $\Gamma$ is supported by a line $D$, $\Gamma'$ is either a quadruple structure on $D$ or the union of a triple structure on $D$ with another reduced line $D'$. In the first (resp. the second) case, $\Gamma'$ links $D$ (resp. $D'$) to $\Gamma$ and the equations of $\Gamma$ are given by Lemmas 3.7 and 3.8. \qed

Remark 3.10. More generally, consider the $n$-uple structure on the line $X_1 = \cdots = X_{n-1} = 0$ in $\mathbb{P}^n$ given by the ideal generated by the $2 \times 2$-minors of the matrix

$$
\begin{pmatrix}
0 & X_1 & \cdots & X_{n-2} & X_{n-1} \\
X_1 & X_2 & \cdots & X_{n-1} & \alpha X_n
\end{pmatrix}
$$

This curve is ACM (and equal to the $n$-fold line if $\alpha = 0$). For $n = 4$, we computed (using the computer program Macaulay) the tangent space to the Hilbert scheme at such a point and found its dimension to be 24 when $\alpha \neq 0$. Therefore, these curves correspond to singular points on the Hilbert scheme if $n = 4$, and we conjecture the same is true for all $n \geq 4$. 


Proposition 3.11. Let $D \subset \mathbb{P}^4$ be a line and let $C$ be an ACM curve of $\mathbb{H}$ which is either the union of a triple structure on the line $D$ and a reduced line, or a quadruple structure on $D$, and such that for every projection $\pi$ from a point $P$ not on $C$, the scheme theoretic image $C'$ of $C$ is a $(3,0)$ or a $(4,1)$. If $\dim T_{C,x} = 3$ for a general point $x$ of $D$, then $C$ is in $\mathbb{H}'$.

Proof. Assume $D$ is the line $X_1 = X_2 = X_3 = 0$. Let $x_1$ and $x_2$ be two distinct points on $D$. Since $\dim T_{C,x_1} \cap T_{C,x_2} \geq 2$, one may choose a projection center $P \in T_{C,x_1} \cap T_{C,x_2} \setminus C$. It follows from Corollary 3.6 that $P \in T_{C,x}$ for every point $x$ of $D$, hence the tangent spaces along $D$ contain a fixed plane, say $X_1 = X_2 = 0$.

Let $F \in I_C$, $F = A_1 X_1 + A_2 X_2 + A_3 X_3$. The vector $(A_1 \ A_2 \ A_3 \ 0)$ satisfies $A_3 = 0$ along the line $D$, hence $A_3 \in (X_1, X_2, X_3)$, so that $I_C \subset (X_1, X_2, X_3)$.

We know that $I_C$ is generated by six quadrics $Q_1, \ldots, Q_6$. Set

$$Q_i = X_1 L_i + X_2 M_i + \lambda_i X_3^2.$$ 

Because the general tangent spaces to $C$ are of dimension 3, the matrix

$$\begin{pmatrix} L_1 & \cdots & L_6 \\ M_2 & \cdots & M_6 \end{pmatrix}$$

is of rank $\leq 1$ along the line $D$. The proof of the following lemma is easy:

Lemma 3.12. In the above situation, either

(i) $L_i = c M_i \mod (X_1, X_2, X_3)$ for some constant $c$ and for all $i$,

or

(ii) there exists $i_0$ and constants $\lambda_i$ such that $L_i = \lambda_i L_{i_0}$ and $M_i = \lambda_i M_{i_0}$, mod $(X_1, X_2, X_3)$.

Let us now return to the proof of Proposition 3.11. It follows from the lemma that we have two possibilities:

(i) $L_i = c M_i \mod (X_1, X_2, X_3)$.

Then $Q_i = M_i (c X_1 + X_2) \mod (X_1, X_2, X_3)^2$, so

$$I_C + (c X_1 + X_2) \subset (c X_1 + X_2) + (X_1, X_2, X_3)^2.$$ 

The hyperplane section of $C$ given by $c X_1 + X_2 = 0$ contains a curve of degree 3, and we can therefore conclude using Proposition 2.2.

(ii) We may assume $(L_i, M_i) \subset (X_1, X_2, X_3)$ for $i \neq 1$, hence $I_C \subset (Q_1) + (X_1, X_2, X_3)^2$, with $Q_1 = X_1 L_1 + X_2 M_1$, and where $L_1$ and $M_1$ are independent of $(X_1, X_2, X_3)$.

If $M_1 = 0$ (or $L_1 = 0$), or, more generally, if $M_1 = \lambda L_1$, then

$$I_C + (X_1) \subset (X_1) + (X_2, X_3)^2,$$

and we may conclude as above by Proposition 2.2.

Suppose $L_1$ and $M_1$ are independent. We can suppose $L_1 = X_0$ and $M_1 = X_4$. 
If $C$ contains a reduced line $D'$, which meets $D$ in a point $(a_0, 0, 0, 0, a_4)$, $D'$ is contained in the tangent space to $C$ at this point, which is defined by $a_0X_1 + a_4X_2 = 0$. Then

$$I_C + (a_0X_1 + a_4X_2) \subset (X_1, X_2, X_3^2) \cup I_{D'},$$

and we may conclude as above by Proposition 2.2.

If $C$ is a quadruple structure on the line $D$, project $C$ from the point $(0, 0, 0, 1, 0)$ (which belongs to all the tangent spaces to $C'$) into the plane $X_3 = 0$. We obtain a curve of type $(3,0)$ supported on the line $X_1 = X_2 = 0$, such that its ideal is generated by three quadratic forms and is contained in

$$(((Q_1) + (X_1, X_2, X_3)^2) \cap k[X_0, X_1, X_2, X_4] = (Q_1) + (X_1, X_2)^2.$$  

From what we have seen concerning curves of type $(3,0)$ supported on a line, and because $Q_1$ is irreducible, the image curve must be the 3-fold line. Hence $(X_1, X_2)^2 \subset I_C$.

If instead we project from the point $(0, 0, 1, 0, 0)$ (which does not belong to all the tangent spaces) into the plane $X_2 = 0$, the image curve is of type $(4,1)$, supported on a line, and the general tangent space has dimension 3:

- if its ideal has the form $(X_1^2, X_1L, L^3)$, where $L$ is a linear form in $X_1, X_3$, linearly independent of $X_1$, we see that $(I_C : X_1)$ contains $X_1, X_2, L$ and we apply Proposition 2.2, intersecting $C$ with the hyperplane $X_1 = 0$.
- if its ideal is generated by the squares of two linear forms, we can suppose that the intersection $I_C \cap k[X_0, X_1, X_3, X_4]$, which already contains $X_1^2$, also contains $X_2^2$. Then $(X_1, X_2)^2 + (X_3)^2 \subset I_C \subset (X_1, X_2, X_3^2)$. The two “missing” generators are $Q_1 + Q_1^2$ and $Q_2^2$, with $(Q_1^2, Q_2^2) \subset (X_1, X_2, X_3)^2$. We may therefore choose $Q_5 = \alpha X_1 X_3 + \beta X_2 X_3$ and apply Proposition 2.2, intersecting $C$ with the hyperplane $\alpha X_1 + \beta X_2 = 0$. □

**Proposition 3.13.** Let $D \subset \mathbb{P}^4$ be a line, and let $C$ be an ACM curve of $H$ which is either the union of a triple structure on the line $D$ and a reduced line, or a quadruple structure on $D$, and such that for every projection $\pi$ from a point $P \notin C$, the scheme theoretic image $C'$ of $C$ is a $(3,0)$ or a $(4,1)$. If $\dim T_{C,x} = 2$ for a general point $x$ of $D$, then $C$ is in $H'$.

**Proof.** Assume $D$ is the line $X_1 = X_2 = X_3 = 0$. A general projection of $C$ is of type $(4,1)$, with 2-dimensional tangent spaces along the multiple line. By Corollary 3.8, these tangent planes are equal.

If the tangent planes to $C$ are not constant, the hyperplane they span must contain the center of projection (since they all project to the same plane) — but this cannot happen for a general projection. Therefore they are constant, equal to, say, $X_1 = X_2 = 0$.

As in the proof of Proposition 3.11, one has $I_C \subset (X_1, X_2, X_3^2)$. We can therefore write the six quadratic generators of $I_C$ on the form $Q_i = X_1 L_i + X_2 M_i + Q_i$, where $Q_i \in (X_1, X_2, X_3^2)$ and $L_i, M_i \in (X_0, X_4)$.

If the rank of $(L_1, \ldots, M_6)$ is equal to 2, then we see from the equations that there is no point $x$ on $C$ where the tangent space has dimension 4, and only a finite number of points where the tangent space has dimension 3, so the union of
the tangent spaces to $C$ along the multiple line is different from $\mathbb{P}^4$, and we get a contradiction. So we may assume $Q_i = X_0(\alpha_i X_1 + \beta_i X_2) + Q'_i$.

If the rank of $(Q'_1, \ldots, Q'_6)$ is $\leq 5$, we may assume that for some $i$, we have $Q'_i = 0$, hence $X_0(\alpha_i X_1 + \beta_i X_2) \in I_C$. Then $X_0$ is a zero divisor in $S/I_C$, and necessarily, $C$ is the union of a triple line $A$ and a reduced line $D$ contained in $X_0 = 0$. Moreover, $\alpha_i X_1 + \beta_i X_2 \in I_A$, hence $I_C + (\alpha_i X_1 + \beta_i X_2) \subseteq I_A$. When we intersect $C$ by $\alpha_i X_1 + \beta_i X_2 = 0$, we get a curve of degree $\geq 3$, hence again Proposition 2.2 applies.

If not, we may assume that $I_C$ is generated by $Q_{ij} = X_0 L_{ij} + X_i X_j$, where $L_{ij} = \alpha_{ij} X_1 + \beta_{ij} X_2$, for $i, j \in \{1, 2, 3\}$. Hence $I_C$ contains $X_0(X_i L_{ij} - X_j L_{ii})$. Since $I_C + (X_0) = (X_0) + (X_1, X_2, X_3)^2$, $X_0$ is not a zero divisor in $S/I_C$. Hence $I_C$ contains $X_i L_{ij} - X_j L_{ii}$.

If the rank of $L_{11}, L_{12}, L_{22}$ is $\leq 1$, then $I_C$ contains two quadratic forms in $X_1, X_2$, and hence the square of some linear form, and we may assume, e.g., that $L_{11} = 0$. Then $(X_1, L_{12}, X_1 L_{13}, X_1^2) = (X_1^2, \beta_{12} X_1 X_2, \beta_{13} X_1 X_3) \subset I_C$. Hence $I_C$ contains $X_1(X_2 + \lambda_2 X_0)$ where $\lambda_2 = 0$ if $\beta_{12} \neq 0$, and $\lambda_2 = \alpha_{12}$ if $\beta_{12} = 0$. For the same reason, $I_C$ contains $X_1(X_3 + \lambda_3 X_0)$. So we may intersect with $X_1 = 0$ and apply Proposition 2.2.

If the rank of $L_{11}, L_{12}, L_{22}$ is $2$, then $I_C$ contains a quadratic form in $X_1, X_2$. If this is a square, we conclude as in the preceding case. If it is not a square, we may assume $L_{12} = 0$, hence $(L_{11}, L_{22}, X_1, X_2) = (\beta_{11} X_1^2, \alpha_{22} X_2^2, X_1 X_2) \subset I_C$. But because $I_C$ does not contain a square in $X_1, X_2$, we have $\beta_{11} = \alpha_{22} = 0$. Hence

$$Q_{11} = X_1(X_1 + \alpha_{11} X_0), \quad Q_{22} = X_2(X_2 + \beta_{22} X_0),$$

so that $X_1 + \alpha_{11} X_0$ and $X_2 + \beta_{22} X_0$ are zero divisors in $S/I_C$ (note that $\alpha_{11} \beta_{22} \neq 0$ since the rank of $L_{11}, L_{12}, L_{22}$ is 2). It follows that $C$ must contain a reduced line $D'$, and that $(X_1, X_2, X_1 + \alpha_{11} X_0, X_2 + \beta_{22} X_0) \subseteq I_D$. But this implies $X_0 \in I_D$, contradicting the fact that $X_0$ is not a zero divisor in $S/I_C$. $\square$

We have now proved the following.

**Proposition 3.14.** If $C$ is an ACM curve of $\mathbb{H}$ and $C$ is equal either to the union of a reduced line and a triple structure on another line, or to a quadruple structure on a line, then $C$ is in $\mathbb{H}'$.

Putting Proposition 3.14 together with Corollary 2.3 and Proposition 2.5, we obtain the following result.

**Theorem 3.15.** The ACM curves form an irreducible open subscheme of the Hilbert scheme $\text{Hilb}^{4n+1}(\mathbb{P}^4)$.

As noted in Section 1, the 4-fold lines correspond to points on the 21-dimensional scheme $\text{Hilb}^6$ where the tangent space has dimension 36. Since these points are not contained in any other component of the Hilbert scheme, they must be singular points on $\text{Hilb}^6$. Hence the singular locus of $\text{Hilb}^6$ contains a variety isomorphic to $\text{Grass}(1, 4)$. In fact, in Remark 3.10 we described more general ACM $n$-fold structures on a line; for $n = 4$ we checked that these curves also correspond to singular points on the Hilbert scheme. Hence we obtain the following corollary.
Corollary 3.16. The irreducible component $H'$ of $\text{Hilb}^{4m+1}(\mathbb{P}^4)$ containing the rational normal curves is singular. More precisely, the open subscheme $H^0 \subset H'$ corresponding to ACM curves has a singular locus that contains (strictly) a variety isomorphic to Grass(1,4).

4. ACM curves in $\mathbb{P}^4$ with Hilbert polynomial $4m + 1$

The study made in the previous section allows us to give a precise description of the curves corresponding to points of $H^0$.

Proposition 4.1. Let $C$ be a curve of $H^0$. Then one of the following holds:

(i) $C$ is a rational normal curve.
(ii) $C$ is the union of two smooth conics intersecting in one point.
(iii) $C$ is a double structure on a conic.
(iv) There exists a hyperplane $H$ such that $C \cap H$ contains a curve of degree 3.

Proof. If $C$ is reduced and irreducible, $C$ is a RN curve. If $C$ is reduced, but is neither irreducible nor the union of two smooth conics, $C$ is the union of a line $D$ and a connected reduced (possibly reducible) curve $C'$ of degree 3 that intersects $D$. Hence we’re in the case (iv).

Assume $C$ is not reduced and not equal to a double structure on a conic. The $C$ is either the union of a reduced conic with a double structure on a line, or the union of a line and a triple structure on another line, or a quadruple structure on a line. By the proofs of Corollary 2.3 and Proposition 3.14, (iv) holds in these cases. □

We shall now give the corresponding structures, i.e., describe the ideals of these curves. The following result — apart from (i), which is classical — is a direct consequence of Sections 2 and 3.

Proposition 4.2. With the notations of Proposition 4.1, the ideal $I_C$ of the curve $C$ is given, up to projective equivalence, as

(i) $I_C$ is the ideal generated by the $(2 \times 2)$-minors of the matrix

$$
\begin{pmatrix}
X_0 & X_1 & X_2 & X_3 \\
X_1 & X_2 & X_3 & X_4
\end{pmatrix}
$$

(ii) $I_C = (X_0, X_1) \cdot (X_2, X_3) + (Q, Q')$, where $Q$ and $Q'$ are quadratic forms such that $Q \in (X_2, X_3)$, $Q \not\in (X_0, X_1)$, and $Q' \in (X_0, X_1)$, $Q' \not\in (X_2, X_3)$.

(iii) $I_C = (X_0, X_0X_1, X_1^2, X_0X_2 + X_1M, X_0L + X_1M', AX_0 + BX_1 + ML' - M'L')$, where $A, B, L, M, L', M'$ are linear forms in $X_2, X_3, X_4$ such that $ML' - M'L \neq 0$.

(iv) $I_C = (LX_1, LX_2, LX_3, Q_1, Q_2, Q_3)$, where $L$ is linear and $Q_1, Q_2, Q_3$ are quadratic, $(Q_1, Q_2, Q_3) \subset (X_1, X_2, X_3)$, and the ideal $(L, Q_1, Q_2, Q_3)$ defines a $(3,0)$ in the hyperplane $L = 0$.

In order to complete the description of the points of $H^0$ we must verify that all the above structures define ACM curves with Hilbert polynomial $4m + 1$.

The cases (i), (ii), (iii) are verified by considering the generators of the ideal $I_C$ (this can be done directly, or by using the computer program Macaulay). To treat the last case, we need the following result.
Lemma 4.3. Let $L_1, L_2, L_3$ (resp. $Q_1, Q_2, Q_3$) be independent linear (resp. quadratic) forms. Assume that $(Q_1, Q_2, Q_3) \subset (L_1, L_2, L_3)$, and let $L$ be a linear form such that the ideal $(L, Q_1, Q_2, Q_3)$ defines a curve $C'$ of degree 3 and genus 0. Then the following are equivalent:

(a) The curve $C$ defined by the ideal $I_C = (LL_1, LL_2, LL_3, Q_1, Q_2, Q_3)$ is ACM (of degree 4 and genus 0).

(b) The ideal $((Q_1, Q_2, Q_3) : L)$ is contained in $(L_1, L_2, L_3)$.

Proof. Let $D$ denote the line defined by the ideal $I_D = (L_1, L_2, L_3)$. We note that (b) is equivalent to the equality $(I_C : L) = I_D$. Hence (a) implies (b) by Proposition 2.2.

Conversely, if $(I_C : L) = I_D$, then there is an exact sequence

$$0 \to S/I_D(-1) \to S/I_C \to S/I_{C'} \to 0,$$

where $S/I_D$ and $S/I_{C'}$ are Cohen–Macaulay of dimension 2 — hence so is $S/I_C$. □

Remark 4.4. If $L, L_1, L_2, L_3$ are linearly independent, (b) holds. In fact, we then have

$$((Q_1, Q_2, Q_3) : L) \subset ((L_1, L_2, L_3) : L) = (L_1, L_2, L_3).$$

Geometrically, we can state this as follows: Let $H \subset \mathbb{P}^4$ be a hyperplane, $D$ a line not contained in $H$ and $C' \subset H$ a curve of degree 3 and genus 0 passing through the point $H \cap D$. Then the scheme theoretic union $C$ of $D$ and $C'$ is an ACM curve of $H$.

To sum up, we have shown that the ACM curves can be described as follows:

Theorem 4.5. The ACM curves of $\mathbb{P}^4$ with Hilbert polynomial $4m + 1$ are the curves of the following four types:

(i) a rational normal curve

(ii) the union of two smooth conics intersecting in one point and not contained in a hyperplane

(iii) a double structure on a conic, as described in Proposition 4.2 (ii).

(iv) a curve obtained from a curve of degree 3 and genus 0 contained in a hyperplane, and a line, satisfying the conditions of Lemma 4.3.

REFERENCES

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Département de Mathématiques et d'Informatique, École Normale Supérieure, 45 rue d'Ulm, F-75230 Paris Cédex 05, France
E-mail address: Mireille.Deschamps@ens.fr

Matematisk institutt, Universitetet i Oslo, P.B. 1053 Blindern, N-0316 Oslo, Norway
E-mail address: ragnip@math.uio.no