ON INFINITESIMAL DEFORMATIONS AND OBSTRUCTIONS
FOR RATIONAL SURFACE SINGULARITIES

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INTRODUCTION

The purpose of this paper is to prove dimension formulas for $T^1$ and $T^2$ for rational surface singularities. These modules play an important role in the deformation theory of isolated singularities in analytic and algebraic geometry. The first may be identified as the Zariski tangent space of the versal deformation of the singularity; i.e. it is the space of infinitesimal deformations. The second contains the obstruction space — in all known cases it is the whole obstruction space for rational surface singularities.

The dimension formulas for $T^1_X$ and $T^2_X$ relate these dimensions to similar invariants on the blow up, $\bar{X}$, of $X$. An important result of Tjurina, which we state below (Theorem 1.1), shows that the minimal resolution $\bar{X}$ may be gotten by a series of blow-ups. Thus, in principle, the formulas allow one to compute these dimensions via blowing up. In fact, the nature of the formulas allows one in many cases to compute these dimensions from the graph.

Computing $T^1$ and $T^2$ for rational surface singularities has a history which we briefly recall. (The terms involved here are explained in Section 1.) Of course for the rational double points $T^2 = 0$ and $T^1$ is easily computed. We will from now on assume that singularities are not hypersurfaces; i.e. the embedding dimension $e$ is not 3. In the 80’ s much work was done in Hamburg on computing $T^1$ for quotient surface singularities, a sub-set of the rationals, and the general form turned out to be $\dim T^1_X = (e - 4) + \dim H^1(\bar{X}, \Theta_X)$ ([BKR88]). Behnke and Knörrer ([BK87]) where able to prove the same formula for a larger, but still very restricted class of rational surface singularities. In 1987, J. Arndt and the first author proved independently that for a cyclic quotient singularity $\dim T^2_X = (e - 2)(e - 4)$. Later, using hypersurface sections, Behnke and the first author proved this formula for rational surface singularities with reduced fundamental cycle and $T^2_X = 0$ and for all quotient singularities ([BC91]). Finally, de Jong and van Straten ([dJvS94]), gave the correct formulas for all rational surface singularities with reduced fundamental cycle.

Let $I$ be an index set for all singularities $X_\nu$ (including $X$ itself) that appear in the process of resolving a rational surface singularity with reduced fundamental cycle by blowing up points. Let $e(\nu)$ be the embedding dimension of $X_\nu$ and set $I_4 = \{ \nu \in I : e(\nu) \geq 4 \}$; i.e. the indices of non-hypersurface singularities. What de
Jong and van Straten proved ([dJvS94, Theorem 3.16]) that
\[
\dim T^1_X = \sum_{\nu \in I_4} (e(\nu) - 4) + \dim H^1(\tilde{X}, \Theta_{\tilde{X}})
\]
\[
\dim T^2_X = \sum_{\nu \in I_4} (e(\nu) - 2)(e(\nu) - 4).
\]

On $\tilde{X}$ we have the groups $T^1_{\tilde{X}}$ and $T^2_{\tilde{X}}$ (see Section 1.2). Since rational surface singularities are absolutely isolated, $\dim T^1_{\tilde{X}} = \sum_{p \in \tilde{X}} \dim T^1_{\tilde{X},p} + \dim H^1(\tilde{X}, \Theta_{\tilde{X}})$ and $\dim T^2_{\tilde{X}} = \sum_{p \in \tilde{X}} \dim T^2_{\tilde{X},p}$. It is well known that for a rational double point $\dim T^1_{\tilde{X}} = \dim H^1(\tilde{X}, \Theta_{\tilde{X}})$. Thus, using the Leray spectral sequence for $p : \tilde{X} \to \tilde{X}$ and that $p_* \Theta_{\tilde{X}} \simeq \Theta_{\tilde{X}}$ we see that the de Jong–van Straten result is equivalent to saying that for all rational surface singularities with reduced fundamental cycle
\[
\dim T^1_X = (e - 4) + \dim T^1_{\tilde{X}}
\]
\[
\dim T^2_X = (e - 2)(e - 4) + \dim T^2_{\tilde{X}}.
\]

The results in this paper originated from a wish to find a direct relationship between the $T^i$ and blowing up for rational singularities. This is described in Section 1.2. This allows us to compute the $T^i$ in terms of the cohomology of certain sheaves on $\tilde{X}$. What we get (Theorem 3.11 and Theorem 3.8) is that for all rational surface singularities (with $e \geq 4$)
\[
\dim T^1_X = (e - 4) + \dim T^1_{\tilde{X}} + c(X)
\]
\[
\dim T^2_X = (e - 2)(e - 4) + \dim T^2_{\tilde{X}} + c(X)
\]
where $c(X)$ is the dimension of the $H^1$ of a certain sheaf (in fact several) on $\tilde{X}$ (Definition 3.7). We give some partial results on $c(X)$ in Section 4, in particular we show that $c(X) = 0$ when the fundamental cycle is reduced, reproving the de Jong–van Straten result.

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## 1. Preliminaries

### 1.1. Results on rational singularities

The singularities we study are algebraic over $\mathbb{C}$, i.e. of the form $X = \text{Spec } A$ where $A = P/I$ and $P$ is a regular local $\mathbb{C}$ algebra. A normal surface singularity $X$ with minimal resolution $f : \tilde{X} \to X$ is rational if $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ ([Art66]). The exceptional divisor $E \subset \tilde{X}$ is a union of irreducible components $E_i \simeq \mathbb{P}^1$. There is a fundamental cycle $Z$, supported on $E$, defined by $m\mathcal{O}_{\tilde{X}}$. Here $m$ is the maximal ideal in $\mathcal{O}_X$. This divisor may be constructed as the unique smallest positive divisor $Z = \sum r_i E_i$ satisfying $Z \cdot E_i \leq 0$ for all irreducible components $E_i$. The embedding dimension of $X$, $e = \dim_{\mathbb{C}} m/m^2$, equals $-Z^2 + 1$ and the multiplicity $m(X) = e - 1 = -Z^2$.

There are three theorems on rational surface singularities which are essential for our results. We collect them and partially rephrase them here. The first is a result from [Tju68] which shows how the blow up $\tilde{X}$ may be obtained from $X$. 

Theorem 1.1 (Tjurina). If $X$ is a rational surface singularity, then the blow up of $X$ is isomorphic to the surface obtained from $\bar{X}$ by contracting all components $E_i$ with $\mathcal{Z} \cdot E_i = 0$.

In [Wah77] we find the basic algebraic property of rational surface singularities that we will need.

Theorem 1.2 (Wahl). Let $A = P/I$ be the local ring of a rational surface singularity, where $P$ is a regular local algebra of dimension $e$ over an algebraically closed field $k$. Let $\bar{P}$ and $\bar{A}$ be the associated graded rings with respect to the maximal ideal. Then there exist minimal projective resolutions:

$$
0 \to P^{b_1} \to \ldots \to P^{b_0} \xrightarrow{\phi} P \xrightarrow{\phi_1} A \to 0,
$$

$$
0 \to \bar{P}^{\bar{b}_1} \to \ldots \to \bar{P}^{\bar{b}_0} \xrightarrow{\bar{\phi}} \bar{P} \xrightarrow{\bar{\phi}_1} \bar{A} \to 0,
$$

so that

(i) the second resolution is the associated graded complex attached to the first;
(ii) $\phi_i$ is homogeneous, of degree 1 ($i > 1$) or 2 ($i = 1$);
(iii) $\bar{b}_i = b_i^{(i-1)}$.

Actually we will only need part (i) and (ii) for $i = 1, 2, 3$. These imply that the ring of a rational surface singularity is something we call a QL ring (see Section 2.2), making it easy to compare the equations and relations defining the blow up locally with those of $X$.

The only result from previous work on $T^2$ we need is about the module structure. This is [BC91, Theorem 5.1.1 (1)], but the statement in that paper about annihilators of $T^2$ is incorrect. What actually is proven is

Theorem 1.3 (Behnke–Christophersen). If $X$ is a rational surface singularity with $e \geq 4$ and $x \in m \setminus m^2$ is generic (i.e., projects onto a general element of $m/m^2$), then $\dim T_X^2/mT_X^2 = (e-2)(e-4)$ and $xT_X^2 = mT_X^2$.

It follows that not only the minimal number of generators of $T_X^2$ equals $(e - 2)(e - 4)$, but also the socle, $\{[\phi] \in T_X^2 : m[\phi] = 0\}$, has this dimension. This will be important in Section 3.2.

1.2. Cotangent cohomology. We review some properties of the cotangent complex. For our later use it is enough to assume that we have a noetherian ring $S$ and an $S$-algebra $A$ of essentially finite type. There exists a complex of free $A$ modules; the cotangent complex $\mathbb{L}^{A/S}$. See [And74, p. 34] for a definition. For an $A$ module $M$ we get the cotangent cohomology modules $T^n(A/S; M) := H^n(\text{Hom}(\mathbb{L}^{A/S}, M))$. If $S$ is the ground field we abbreviate $T^n(A/S; M) =: T^n_A(M)$ and $T^n_A(A) =: T^n_X =: T^n_X$ if $X = \text{Spec} A$.

The first three modules are important in deformation theory and we could have given an ad hoc definition as follows. Let $P$ be a polynomial $S$ algebra (or the localization of such an algebra) mapping onto $A$ so that $A \simeq P/I$ for an ideal $I$. Let

$$
0 \to R \to F \xrightarrow{j} P \to A \to 0
$$

be an exact sequence presenting $A$ as a $P$ module with $F \simeq P^m$ free. We have $T^0(A/S; M) = \text{Der}_S(A, M)$, the module of $S$ derivations into $M$. The cokernel of
the natural map $\text{Der}_S(P, M) \to \text{Hom}_A(I/I^2, M)$ is $T^1(A/S; M)$. Let $R_0$ be the submodule of $R$ generated by the trivial relations; i.e. those of the form $j(x) y - j(y) x$. Then $R/R_0$ is an $A$ module and we have an induced map $\text{Hom}_A(F/R_0 \otimes_P A, M) \to \text{Hom}_A(R/R_0, M)$. The cokernel is $T^2(A/S; M)$. Notice that $\text{Hom}_A(F/R_0 \otimes_P A, M)$ is just the sum of $m$ copies of $M$ and the map is

$$\langle \alpha_1, \ldots, \alpha_m \rangle \mapsto [\bar{f} \mapsto \sum r_i \alpha_i]$$

where $r \in F$ represents $r \in R/R_0$.

If $A$ is a smooth $S$ algebra then $T^i(A/S; M) = 0$ for $i \geq 1$ and all $A$ modules $M$. As usual a short exact sequence of $A$ modules induces a long exact sequence in the $T^i(A/S; \ast )$. More importantly, two ring homomorphisms $S \to A \to B$ induce a long exact sequence called the Zariski–Jacobi long exact sequence:

$$\cdots \to T^i(B/A; M) \to T^i(B/S; M) \to T^i(A/S; M) \to T^{i+1}(B/A; M) \to \cdots$$

where $M$ is a $B$ module. (See [And74, Théorème 5.1].)

If $Y$ is a scheme we may globalise the above local construction. (See for example [And74, Appendix], [Buc81, 2.2.3] and [Lau79, 3.2] for details and proofs.) If $S$ is a sheaf of rings and $A$ an $S$ algebra we set $L^{A/S}$ to be the complex of sheaves associated with the presheaves $U \mapsto L^{A(U)/S(U)}$. Let $F$ be an $A$ module. We get the cotangent cohomology sheaves $T^i_{A/S}(F)$ as the cohomology sheaves of $\text{Hom}_A(L^{A/S}, F)$ and the cotangent cohomology groups $T^i_{A/S}(F)$ as the cohomology of $\text{Hom}_A(L^{A/S}, F)$.

Because of the functoriality of these constructions: $T^i_{A/S}(F)$ is the sheaf associated to the presheaf $U \mapsto T^i(A(U)/S(U); F(U))$ and $T^i_{A/S}(F)$ is the hypercohomology of $\text{Hom}_A(L^{A/S}, F)$. In particular there is a “local-global” spectral sequence

$$H^p(Y, T^q_{A/S}(F)) \Rightarrow T^p_{A/S}(F). \tag{1.1}$$

If $A$ is the structure sheaf $\mathcal{O}_Y$ and $S$ is the ground field, then (abbreviating as above) the $T^q_Y$ play a role in the deformation theory of $Y$ similar to the local case. The spectral sequence (1.1) becomes $H^p(Y, T^q_Y) \Rightarrow T^p_Y$ and shows how local and global deformations contribute to the total.

1.3. Cotangent cohomology and modifications of rational singularities. Consider first any morphism of schemes $f : Y \to X$. Let $f^{-1}$ be the sheaf theoretical adjoint functor of $f_*$ as defined in for example [Har77, II.1]. We have the for us very important result in [And74, Appendix. Proposition 56] which we translate to our notation. (Notice that the $f^*$ in [And74] is $f^{-1}$ and not the $f^*$ in standard algebraic geometry notation).

**Proposition 1.4.** If $f : Y \to X$ is a morphism of schemes, $A$ is an $S$ algebra on $X$ and $\mathcal{F}$ is an $f^{-1}A$ module on $Y$ such that $R^k f_* (\mathcal{F}) = 0$ for $k \geq 1$, then there are natural isomorphisms

$$T^i_{f^{-1}A/f^{-1}S}(\mathcal{F}) \cong T^i_{A/S}(f_* \mathcal{F})$$

for all $i \geq 0$.

Assume now that $X = \text{Spec } A$ is a normal singularity and $f : Y \to X$ is a modification; i.e. $f$ is proper and birational. We will slightly abuse notation and
write $f^{-1}A$ for $f^{-1}\mathcal{O}_X$. From the spectral sequence (1.1) and Proposition 1.4 we get immediately

**Theorem 1.5.** If $f : Y \rightarrow X = \text{Spec} A$ is a modification and $\mathcal{F}$ is a coherent sheaf on $Y$ with $R^kf_*(\mathcal{F}) = 0$ for $k \geq 1$, then there is a spectral sequence $\{E^{p,q}_2\}$ with $E^{p,q}_2 = H^p(Y, \mathcal{T}_{f^{-1}A}(\mathcal{F}))$ such that $E^{p,q}_2 = T^p_A(f_*, \mathcal{F})$.

**Remark.** The results we have compiled from the literature to get Theorem 1.5 involve injective resolutions to compute hyper-cohomology etc. In the computational part of this paper it will be important to know some of the maps from the spectral sequence explicitly, and therefore in terms of Čech cohomology. We will state these descriptions without proof. For a proof of Theorem 1.5 using Čech cohomology (done before we found the relevant known results) and explaining the maps see [SG94].

**Corollary 1.6.** If $f : Y \rightarrow X = \text{Spec} A$ is a modification of a rational surface singularity then there are exact sequences

$$0 \rightarrow H^1(Y, \mathcal{T}_{f^{-1}A}(\mathcal{O}_Y)) \rightarrow T^1_X \rightarrow H^0(Y, \mathcal{T}_{f^{-1}A}(\mathcal{O}_Y)) \rightarrow 0$$

for all $i \geq 0$.

**Proof.** The spectral sequence in Theorem 1.5 is derived from a double complex for computing the hyper-cohomology $T^p_{f^{-1}A}(\mathcal{O}_Y)$. On the other hand, since $X$ is affine, $H^i(Y, G) = 0$ for $i \geq 2$ for any coherent $G$. Thus the $E^{p,q}_2$ consists of two adjacent non-zero rows. The result follows from standard arguments. Notice that $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$ by normality. \[\square\]

We will use Corollary 1.6 when the modification is the blow-up $\pi : \tilde{X} \rightarrow X$ to prove our formulas. To shorten notation we define the sheaves on $\tilde{X}$

$$\mathcal{F}^i := \mathcal{T}^i_{\pi^{-1}A}(\mathcal{O}_{\tilde{X}}).$$

Notice that there are natural maps $\mathcal{T}^i_Y \rightarrow \mathcal{T}^i_{\pi^{-1}A}(\mathcal{O}_Y)$ that induce, with the help of the exact sequences, natural maps $\mathcal{T}^i_Y \rightarrow T^i_{\tilde{X}}$. If $i = 1$ these are the tangent maps to the contraction of deformations of $Y$ to deformations of $X$. They behave very sporadically, and we have not found them useful for proving the formulas. Instead we will have to make some unnatural maps relating the $\mathcal{F}^i$ to the $T^i_{\tilde{X}}$.

2. **QL-RINGS AND BLOWING UP**

2.1. **Associated graded rings and standard bases.** We recall some facts regarding associated graded rings and standard bases.

**Definition 2.1.** Let $R$ be a noetherian local ring with maximal ideal $\mathfrak{m}$ and let $M$ be a finitely generated $R$-module. Let $N \subset M$ be a submodule. We set

$$G(\mathfrak{m}, M) := \bigoplus_{i \geq 0} \mathfrak{m}^i M / \mathfrak{m}^{i+1} M$$

$$G(\mathfrak{m}, N \subset M) := \bigoplus_{i \geq 0} \mathfrak{m}^i M \cap N + \mathfrak{m}^{i+1} M / \mathfrak{m}^{i+1} M$$

$$B(\mathfrak{m}, M) := \bigoplus_{i \geq 0} \mathfrak{m}^i M.$$
Also, for any nonzero \( m \in M \) we put
\[
\text{ord}(m, M)(m) = \sup \{ n | m \in m^n M \},
\]
and if \( m \neq 0 \) and \( \text{ord}(m, M)(m) = d \) we define the initial form
\[
\text{in}(m, M)(m) = m + m^d M/m^{d+1} M
\]
Let \( m_1, \ldots, m_k \) be generators for \( N \). Then \( m_1, \ldots, m_k \) is called a standard basis for \( N \) if the submodule \( G(m, N \subset M) \) is generated by \( \text{in}(m, M)(m_1), \ldots, \text{in}(m, M)(m_k) \).

We will write \( \text{ord}(m) \) for \( \text{ord}(m, M)(m) \) and \( \text{in}(m) \) for \( \text{in}(m, M)(m) \) when no misunderstanding is likely to occur.

We will need the following result – see e.g. [HIO88, Theorem 13.7].

**Theorem 2.2.** Let \( R \) be a noetherian local ring with maximal ideal \( m \), let \( M \) be a finitely generated \( R \)-module and let \( N \subset M \) be a submodule. Then \( m_1, \ldots, m_k \) is a standard basis for \( N \) if and only if for any \( z \in N \) there are \( a_1, \ldots, a_k \in R \) such that \( z = a_1 m_1 + \cdots + a_k m_k \) and \( \text{ord}(z) \leq \text{ord}(a_i) + \text{ord}(m) \) for all \( i \).

2.2. **QL-rings.** Let \( P = \mathbb{C}[x_1, \ldots, x_e]_m \) be the polynomial ring with \( e \) generators localized in \( m = (x_1, \ldots, x_e) \). We let \( m \) denote both the maximal ideal in \( P \) and quotients of \( P \) unless this causes confusion.

**Definition 2.3.** We will say that \( A \) is a QL-ring (quadratic generators and linear relations) if \( A = P/I \) where \( I \subset P \) is a prime ideal such that

(i) The ideal \( I \) has a standard basis \( f_1, \ldots, f_m \) with \( \text{ord}(f_i) = 2 \) for \( i = 1, \ldots, m \).

(ii) The relation module \( R = \{ (p_1, \ldots, p_l) \in P^m | \sum p_i f_i = 0 \} \) has a standard basis \( r^1, \ldots, r^s \) with \( \text{ord}(r^i) = 1 \) for \( i = 1, \ldots, s \).

(iii) The \( \text{in}(f_i) \) and \( \text{in}(r^i) \) are linearly independent.

We say that \( X = \text{Spec} A \) is a QL singularity if \( A \) is a QL-ring.

Rational surface singularities with \( e \geq 4 \) are QL singularities by Theorem 1.2. Another example is the class of minimal elliptic surface singularities with \( e \geq 5 \) ([Wah77, Theorem 2.8]).

QL-singularities have an algebraic property that will be very important for us in the proof of the formulas. We state it here for future reference.

**Lemma 2.4.** Suppose \( A \) is a QL-ring and assume \( f_1, \ldots, f_m \) and \( r^1, \ldots, r^s \) are standard bases as in Definition 2.3. Let \( x \in m \setminus m^2 \).

(i) Every \( f_j \) is involved in some relation \( r = \langle r_1, \ldots, r_m \rangle \) with \( r_j \in m \setminus m^2 \).

(ii) Every \( f_j \) is involved in some relation \( r = \langle r_1, \ldots, r_m \rangle \) with \( r_j \notin \langle x \rangle \).

**Proof.** The first statement is proven in [Wah87, 2.5] and the second one follows from the same argument, so we repeat it here. We may assume \( j = 1 \) and consider the trivial relation \( \langle f_2, -f_1, 0, \ldots, 0 \rangle = \sum q_i r^i \). Thus all \( q_i \in m \) and \( f_2 = \sum q_i r^i \notin m^3 \). Also \( f_2 \notin \langle x \rangle \) since \( x \notin m^2 \) and \( A \) is a domain. So some \( r_j^i \notin m^2 \) and some \( r_j^i \notin \langle x \rangle \). \( \square \)

As a consequence we get a slight generalization of [Wah87, Lemma 2.6].

**Lemma 2.5.** If \( A = P/I \) is a QL-ring and \( x \in (m \setminus m^2) \cup \{ 0 \} \), then every \( \phi \in \text{Hom}_A(I/I^2, A/(x)) \) has \( \text{Im}(\phi) \subseteq m_{A/(x)} \).
Proof. Assume $\phi(f_i) = 1$ and let $h_i \in P$ represent $\phi(f_i) \in A/(x)$. After changing $f_i$ to $f_i - h_i f_1$ we may assume $\phi(f_i) = 0$ for $i \geq 2$. (Remember that the in($f_i$) are independent.) This contradicts Lemma 2.4, since for all relations $\sum r_i \phi(f_i) = r_1 \equiv 0 \mod (x). \qed$

2.3. QL singularities and blowing up. Let $A = P/I$ be a QL-ring. Let $\pi : \hat{X} = \text{Proj} B(m, A) \to \text{Spec} A$ be the blow up of Spec $A$. Recall that the blow-up $\hat{X}$ is covered by Spec $B(m, A)_{(x^i)}$ for $x \in A$.

For $x \in P, x \not\in I$ we denote by $P[m/x]$ the subring of $C(x_1, \ldots, x_e)$ generated by the image $P$ and the elements of the form $\frac{a}{x}$ where $a \in m$. Similarly we denote by $A[m/x]$ the subring of $A$'s quotient field generated by the image of $A$ and the elements of the form $\frac{a}{x}$ where $a \in m$. For the covering of the blow-up we have $B(m, A)_{(x^i)} \cong A[m/x]$ and the restriction of $\pi$ to Spec $B(m, A)_{(x^i)}$ is induced by the inclusion $A \subset A[m/x]$. 

Lemma 2.6. Suppose $A = P/I$ is a QL-ring and assume $f_1, \ldots, f_m$ and $r^1, \ldots, r^s$ are standard bases as in Definition 2.3. Then

(i) $A[m/x] \cong P[m/x]/I_B$ where $I_B$ is generated by $f_1/x^2, \ldots, f_m/x^2$.

(ii) The relation module

$$S = \left\{ (p_1, \ldots, p_m) \in P[m/x] \mid \sum p_i f_i/x^2 = 0 \text{ in } P[m/x] \right\}$$

is generated by $r^1/x, \ldots, r^s/x$.

Proof. The first statement is a special case of a well known property of blow-ups, see e.g. [HIO88, Proposition 13.13].

We prove the second statement for lack of reference. Let $R$ be the relation module for the $f_i$, and set $R/x$ to be the $P[m/x]$ module generated by $r^1/x, \ldots, r^s/x$. Clearly $R/x \subset S$. Choose some $p = (p_1, \ldots, p_m) \in S$. We may find an $n$ such that $x^{a-5} p_i \in P$ for all $i = 1, \ldots, m$. Thus $\sum x^n p_i f_i = 0$, so $x^np \in R$. Since $x^np \in R$, we are by Theorem 2.2 able to find $q_1, \ldots, q_s$ such that $x^np = \sum q_i r^j$ and $\text{ord}(m, P)(q_j) + \text{ord}(m, P^m)(r^j) \geq \text{ord}(m, P^m)(x^np)$. Now $\text{ord}(r^j) = 1$ and $\text{ord}(x^np) \geq n$, hence $\text{ord}(q_j) \geq n - 1$. We end up with $p = \sum (q_j/x^{n-1}) (r^j/x)$ with $q_j/x^{n-1} \in P[m/x]$, which shows that $p \in R/x$. \qed

3. The formulas

3.1. Computation of the $\mathcal{F}^i$. Let $A = P/I$ be a QL-ring, where $P$ is as above with $e = \dim m_A/m_A^2$. Let $\hat{X} \to X$ be the blow up of $X = \text{Spec} A$ and $M$ the blowup of Spec $P$, so that we may view $\hat{X}$ as the strict transform of $X$ in $M$. Let $C \subset \hat{X}$ be the exceptional divisor defined by $mO_{\hat{X}}$. Thus $C = \text{Proj} B(m, A)$.

Notation. Throughout the following proofs we will be working locally on $\hat{X}$ with affine charts Spec $B$ with $B = A[m/x]$ as in Lemma 2.6. Set $P_B = P[m/x]$. We use variables $x, x_1, \ldots, x_n (n - e - 1)$ for $P$, so $t_i := x_i/x$ and $x$ generate $P_B$. Generators for $I$ will be denoted $f_1, \ldots, f_m$ and $g_1, \ldots, g_m$ will be generators of $I_B$ as in Lemma 2.6; i.e. $x^2 g_i = f_i(x, t_1, \ldots, t_n)$. We view the $x_i = xt_i$ as elements of $P_B$ as well.

Let $\mathcal{N}_{\hat{X}/M}$ be the normal sheaf of $\hat{X}$ in $M$ and set $N_A(O_{\hat{X}}) = T_{\pi^{-1}A/\pi^{-1}P} (O_{\hat{X}})$. Let $\Theta_{\hat{X}}$ be the tangent sheaf on $\hat{X}$. Let Der$C(\hat{X})$ be the subsheaf of $\Theta_{\hat{X}}$ consisting
of derivations $D$ with $D(\mathcal{I}_C) \subseteq \mathcal{I}_C$. Finally define $A^1_{C/\hat{X}}$ to be the cokernel of the map $\Theta_{\hat{X}} \to \mathcal{O}_C(C)$ defined locally – where $C$ is defined by $x$ as $D \mapsto D(x) \otimes \frac{1}{x}$ mod $(x)$. Notice that there is an exact sequence

$$0 \to \text{Der}_C(\hat{X}) \to \Theta_{\hat{X}} \to \mathcal{O}_C(C) \to A^1_{C/\hat{X}} \to 0$$

with the maps as above.

First we compare the $\mathcal{F}^i$ with the $\mathcal{T}^i_{\hat{X}}$.

**Proposition 3.1.** If $X$ is a QL-singularity, then

(i) $\mathcal{F}^0 \simeq \text{Der}_C(\hat{X})(C)$.

(ii) $N_A(\mathcal{O}_{\hat{X}}) \simeq \mathcal{N}_{\hat{X}/M}(2C)$.

(iii) There is an exact sequence $0 \to A^1_{C/\hat{X}}(C) \to m\mathcal{F}^1 \to \mathcal{T}^1_{\hat{X}}(C) \to 0$.

(iv) $m\mathcal{F}^2 \simeq \mathcal{T}^2_{\hat{X}}$.

The isomorphisms and maps are non-canonical.

**Proof.** Consider an affine chart Spec $B$ of $\hat{X}$ with $B = A[\frac{m}{x}]$ as above. The isomorphism in (i) is given by the map $\text{Der}_C(\hat{X})(C) \to \mathcal{F}^0$ which is locally $D \otimes \frac{1}{x} \mapsto \frac{1}{x} D|_A$.

This is easily checked to be injective, and if $\delta \in \text{Der}(A, B)$ then it comes from a derivation $D$ determined by $D(x) = x\delta(x)$ and $D(t_i) = \delta(x_i) - t_i\delta(x)$.

Let $\mathcal{J}$ be the ideal sheaf of $\hat{X}$ in $M$ and $V$ the exceptional divisor of $\pi : M \to \text{Spec } P$. Lemma 2.6 implies that $\pi^*I \simeq \mathcal{J}(-2V)$; this induces the isomorphism in (ii).

The chain rule and the fact that $x$ is not a zero divisor, yields the following equalities in $B$:

$$\frac{\partial f_j}{\partial x} = x(\frac{\partial g_j}{\partial x} - \sum_i t_i \frac{\partial g_j}{\partial t_i})$$

(3.2a)

$$\frac{\partial f_j}{\partial x_i} = x \frac{\partial g_j}{\partial t_i}$$

(3.2b)

The isomorphism (ii) takes $\phi \in \text{Hom}_P(I, B)$ to the morphism determined by $g_j \mapsto \phi(f_j)$. In particular (3.2) shows that it induces (after a twist) a map $\mathcal{F}^1(-C) \to \mathcal{T}^1_{\hat{X}}(C)$ which must be surjective. We claim that this factors through $m\mathcal{F}^1$. Indeed, if $x[\phi] = 0$ in $\mathcal{T}^1_{\hat{X}}(B)$, then $x\phi(f_j) = b\frac{\partial f_j}{\partial x} + \sum b_i \frac{\partial f_j}{\partial t_i}$, so by (3.2), $\phi(f_j) = b(\frac{\partial g_j}{\partial x} - \sum t_i \frac{\partial g_j}{\partial t_i}) + \sum b_i \frac{\partial g_j}{\partial t_i}$. Thus $[\phi] \otimes x \mapsto 0$. This gives the right surjection in (iii).

Let $K$ be the kernel of this map locally, i.e. of $x\mathcal{T}^1_{\hat{X}}(B) \to \mathcal{T}^1_{\hat{X}}$. We have $x[\phi] \in K$ iff $\phi(f_j) = b \frac{\partial g_j}{\partial x} + \sum b_i \frac{\partial g_j}{\partial t_i}$. But then $x\phi(f_j) = bx \frac{\partial g_j}{\partial x} + \sum b_i \frac{\partial g_j}{\partial t_i}$, so $x[\phi]$ equals the class of the map $f_j \mapsto bx \frac{\partial g_j}{\partial x}$ in $\mathcal{T}^1_{\hat{X}}(B)$. In particular $K$ is a cyclic $B$ module generated by the class of the map $f_j \mapsto x \frac{\partial g_j}{\partial x}$.

This yields a surjection $B \twoheadrightarrow K$. The kernel of this map is

$$\{b \in B : bx \frac{\partial g_j}{\partial x} = b_0 \frac{\partial f_j}{\partial x} + \sum b_i \frac{\partial f_j}{\partial x_i} \text{ for some } b_i \in B, j = 1, \ldots, m\}$$

$$= \{b \in B : b \frac{\partial g_j}{\partial x} = b_0 x \frac{\partial g_j}{\partial x} + \sum (b_i - b_0 x_i) \frac{\partial g_j}{\partial t_i} \} = a + (x)$$
where \( a = \{ b : b \partial y_j / \partial x = \sum b_i \partial y_j / \partial t_i \} \). But clearly this last condition is the same as \( b = D(x) \) for some \( D \in \text{Der}(B) \). This gives an exact sequence

\[ 0 \to B / a + (x) \to xT_A^1(B) \to T_B^1 \to 0 \]

which globalizes to the one in (iii).

Let \( R \) and \( S \) be as in Lemma 2.6 and let \( R_0 \) and \( S_0 \) be the submodules of Koszul relations. Thus \( R \otimes_P P_B \simeq x \cdot S \) and \( R_0 \otimes_P P_B \simeq x^2 \cdot S_0 \). Now \( \text{Hom}_A(R/R_0, B) \) is the kernel of the natural map \( \text{Hom}_A(R/I(R), B) \to \text{Hom}_A(R_0/I(R), B) \), so it is isomorphic to the kernel of \( \text{Hom}_{P_B}(x \cdot S, B) \to \text{Hom}_{P_B}(x^2 \cdot S_0, B) \). This kernel is again isomorphic to \( \text{Hom}_B(S/S_0, B) \) since \( S_0/xS_0 \) is annihilated by the non-zero divisor \( x \).

This isomorphism induces a surjection \( T_A^2(B) \to T_B^2 \). One checks that the kernel is \( \{ [\phi] \in T_A^2(B) : x \cdot [\phi] \equiv 0 \} \) which is also the kernel of the multiplication map \( T_A^3(B) \to T_A^2(B) \). This induces the isomorphism (iv) locally.

\[ \square \]

### 3.2. The \( T^2 \) formula.

**Proposition 3.2.** If \( X \) is a rational surface singularity, then \( H^0(\tilde{X}, F^1|_C) = H^0(\tilde{X}, F^2|_C) = 0 \).

**Proof.** From the quotient map \( O_X \to O_C \) and Theorem 1.5 we get the following commutative diagram with surjective horizontal maps:

\[
\begin{array}{ccc}
T_x^0 & \longrightarrow & H^0(\tilde{X}, F^2) \\
\alpha \downarrow & & \beta \\
T_x^0(\pi, O_C) & \longrightarrow & H^0(\tilde{X}, T^2_{x-1}(O_C))
\end{array}
\]

Now \( \pi_* O_C \simeq A/m \). For a rational singularity with \( e \geq 5 \) the “relations among relations” are generated by independent linear ones (Theorem 1.6). We may argue as in Lemma 2.5 to show that the images of all \( \phi \in \text{Hom}_A(R/R_0, A) \) are in \( m \). So \( \alpha \) is the zero-map and therefore \( \beta \) is the zero-map. On the other hand \( \beta \) factors \( H^0(\tilde{X}, F^2) \to H^0(\tilde{X}, F^2|_C) \to H^0(T^2_{x-1}(O_C)) \). The second map is injective and the cokernel of the first map is contained in \( H^1(m, F^2) \) which is zero by Proposition 3.1. This proves that \( H^0(\tilde{X}, F^2|_C) = 0 \).

In the case of \( F^1|_C \) we can make a direct calculation relying only on the QL property. As above we consider the injective map \( H^0(F^1|_C) \hookrightarrow H^0(T^2_{x-1}(O_C)) \). Since the \( \pi^{-1}A \) module structure on \( O_C \) is defined by \( A \to A/m \simeq C \to O_C \), we have \( T^2_{x-1}(O_C) \simeq mO_C \) where \( m \) is the minimal number of generators for \( I \). In particular \( H^0(T^2_{x-1}(O_C)) \simeq C^m \). A global section of \( H^0(F^1|_C) \) must therefore be locally represented by a homomorphism that looks like \( f_j \mapsto \lambda_j + I_C \) with \( \lambda_j \in C \).

We claim that for every \( f_j \) there exists a chart with coordinate ring \( \tilde{B} = A \left[ \frac{m}{x} \right] \), such that there are no \( \phi \in \text{Hom}_A(I/I^2, B) \) with \( \phi(f_j) \equiv \lambda \mod (x) \) and \( \lambda \neq 0 \) a constant. To prove this consider for \( f_j \) a relation as in Lemma 2.4 and set \( x = r_j \).

If any of the other \( r_k \in (x) \), say \( r_k = h_kx \), change \( f_j \) to \( f_j + \sum h_kf_k \). Thus we may assume all other \( r_k \in m \setminus (x) \). We must have \( \sum r_i \phi(f_i) = 0 \in B \). So \( \phi(f_j) = -\sum_{i \neq j} (r_i/x) \phi(f_i) \) in \( B \), but by the assumption on these \( r_i \), none of the \((r_i/x)\) are constants.

\[ \square \]

The following result follows immediately from Proposition 3.1 and Proposition 3.2.
Corollary 3.3. If $X$ is a rational surface singularity then
$$H^0(\tilde{X}, \mathcal{F}^2) \simeq H^0(\tilde{X}, T^2_{\tilde{X}})$$
and the sequence
$$0 \to H^1(\tilde{X}, \mathfrak{m}\mathcal{F}^1) \to H^1(\tilde{X}, \mathcal{F}^1) \to H^1(\tilde{X}, \mathcal{F}^1|_{\mathfrak{m}}) \to 0$$
is exact.

Let us now concentrate on $H^1(\tilde{X}, \mathcal{F}^1)$. Using Theorem 1.3 we will prove via two lemmas that $\dim_{\mathbb{C}} H^1(\tilde{X}, \mathcal{F}^1|_{\mathfrak{m}}) = (e-2)(e-4)$. (We view cohomology groups on $\tilde{X}$ as $\mathcal{A}$ modules by their isomorphisms with $R^i\pi_*$'s).

Lemma 3.4. Suppose $X$ is a rational surface singularity and $K$ is a submodule of $T^2_{\tilde{X}}$ containing the socle of $T^2_{\tilde{X}}$. If $x \in \mathfrak{m}$ is generic, then $\dim_{\mathbb{C}} K/xK = (e-2)(e-4)$.

Proof. Recall that Theorem 1.3 tells us that the socle of $T^2_{\tilde{X}}$ equals the set of elements annihilated by $x$. In particular its dimension is also $(e-2)(e-4)$. Since this set is contained in $K$, it must be the kernel of the multiplication map $K \xrightarrow{x} K$ and have the same dimension as the cokernel. \hfill \Box

Lemma 3.5. If $X$ is a rational surface singularity and we identify $H^1(\tilde{X}, \mathcal{F}^1)$ with the kernel of $T^2_{\tilde{X}} \to H^0(\tilde{X}, \mathcal{F}^2)$, then $H^1(\tilde{X}, \mathcal{F}^1)$ contains the socle of $T^2_{\tilde{X}}$.

Proof. We need to show that the local maps $T^2_{\tilde{X}} \to T^2_{\tilde{X}}(B)$ map a socle element to zero for all $B = A [\frac{m}{x}]$. If $[\phi]$ in the socle, then we may find $a_1, \ldots, a_m \in A$ such that $x\phi(r) = \sum a_i r_i$ in $A$ for all relations $r = (r_1, \ldots, r_m)$. We claim that $a_1, \ldots, a_m \in \mathfrak{m}$. This is because the application $f_j \mapsto a_j$ defines an element of $\text{Hom}_A (I/I^2, A/(x))$, so Lemma 2.5 applies. But then $a_i/x \in B$ and $\phi(r) = \sum (a_i/x)r_i$ in $B$, thus $[\phi] \mapsto 0$. \hfill \Box

Proposition 3.6. If $X$ is a rational surface singularity, then $\dim_{\mathbb{C}} H^1(\tilde{X}, \mathcal{F}^1|_{\mathfrak{m}}) = (e-2)(e-4)$.

Proof. Since $x$ is generic, the cokernel of $\mathcal{F}^1 \xrightarrow{x} \mathfrak{m}\mathcal{F}^1$ has support at points. Since $H^1(\tilde{X}, -)$ is right exact this proves that $xH^1(\mathcal{F}^1) \simeq H^1(\mathfrak{m}\mathcal{F}^1)$. The result now follows from Corollary 3.3, Lemma 3.4 and Lemma 3.5. \hfill \Box

Putting all of this together we get the formula for $\dim_{\mathbb{C}} T^2_{\tilde{X}}$. We first define the “correction term”. We will see several other ways of defining this number in Section 4.

Definition 3.7. If $X$ is a rational surface singularity, we define the invariant
$$c(X) := \dim_{\mathbb{C}} H^1(\tilde{X}, \mathfrak{m}\mathcal{F}^1).$$

Theorem 3.8. If $X$ is a rational surface singularity of embedding dimension $e$ and $\tilde{X}$ is the blow up of $X$, then
$$\dim_{\mathbb{C}} T^2_{\tilde{X}} = (e-2)(e-4) + \sum_{p \in \tilde{X}} \dim_{\mathbb{C}} T^2_{\tilde{X},p} + c(X).$$

Proof. This follows from Corollary 1.6, Corollary 3.3 and Proposition 3.6. \hfill \Box
3.3. The $T^1$ formula. First notice that Proposition 3.1 implies that the exact sequence (3.1) extends (after a twist) to an exact sequence
\begin{equation}
0 \to \mathcal{F}^0 \to \Theta_{\bar{X}}(C) \to \mathcal{O}_C(2C) \to \mathfrak{m}\mathcal{F}^1 \to T^1_{\bar{X}}(C) \to 0.
\end{equation}

Only the two first sheaves have support outside $C$; i.e. have an infinite dimensional $H^0$. On the other hand, the sequence induces an exact sequence
\begin{equation}
0 \to H^0(\bar{X}, \mathcal{F}^0) \to H^0(\bar{X}, \Theta_{\bar{X}}(C)) \to H^0(\bar{X}, \mathcal{O}_C(2C)).
\end{equation}

Now $C$ is an arithmetically Cohen-Macaulay curve in $\mathbb{P}^{n-1}$, (see e.g. [Wah77]). In particular $H^0(\bar{X}, \mathcal{O}_C(2C)) = H^0(C, \mathcal{O}_C(-2)) = 0$. Thus the sequence (3.3) induces an isomorphism $H^0(\bar{X}, \mathcal{F}^0) \cong H^0(\bar{X}, \Theta_{\bar{X}}(C))$. (They are actually isomorphic to $\text{Der}(A)$, which is seen immediately from Theorem 1.5.)

Proposition 3.2 tells us that $h^0(\mathcal{F}^1) = h^0(\mathfrak{m}\mathcal{F}^1)$. Since $T^1_{\bar{X}}$ is a skyscraper sheaf $h^0(T^1_{\bar{X}}) = h^0(T^1_{\bar{X}}(C))$ and $h^1(T^1_{\bar{X}}(C)) = 0$. Using all this information and Corollary 1.6 the sequence (3.3) yields the formula
\begin{equation}
\dim_C T^1_{\bar{X}} = h^1(\mathcal{F}^0) + h^0(\mathcal{F}^1)
= h^1(\Theta_{\bar{X}}(C)) + \chi(\mathcal{O}_C(2C)) + h^0(T^1_{\bar{X}}) + c(X).
\end{equation}

Consider now the minimal resolution $\bar{X}$ of $X$ which factors $\bar{X} \overset{p}{\to} \tilde{X} \overset{\pi}{\to} X$. Clearly $\mathcal{O}_{\tilde{X}}(-Z) \simeq p^*\mathcal{O}_{\bar{X}}(-C)$, so we also have $\mathcal{O}_{\tilde{X}}(kZ) \simeq p^*\mathcal{O}_{\bar{X}}(kC)$.

If we use the projection formula on $\Theta_{\tilde{X}}(Z)$ we find that
\begin{equation}
R^lp_{\ast}\Theta_{\tilde{X}}(Z) \simeq R^lp_{\ast}\Theta_{\bar{X}} \otimes \mathcal{O}_{\bar{X}}(C).
\end{equation}

It is also true for rational surface singularities that $p_{\ast}\Theta_{\bar{X}} \simeq \Theta_{\tilde{X}}$ ([BW74]), so in particular, $p_{\ast}\Theta_{\tilde{X}}(Z) \simeq \Theta_{\tilde{X}}(C)$. Since $H^2$‘s vanish, the Leray spectral sequence gives in our situation, an exact sequence
\begin{equation}
0 \to H^1(\tilde{X}, \Theta_{\tilde{X}}(C)) \to H^1(\bar{X}, \Theta_{\bar{X}}(Z)) \to H^0(\bar{X}, R^1p_{\ast}\Theta_{\tilde{X}}(Z)) \to 0.
\end{equation}

Also by (3.5) we see that $h^0(R^1p_{\ast}\Theta_{\tilde{X}}(Z)) = h^0(R^1p_{\ast}\Theta_{\bar{X}})$.

Consider the exact sequence
\begin{equation}
0 \to \Theta_{\tilde{X}} \to \Theta_{\bar{X}}(Z) \to \Theta_{\bar{X}} \otimes \mathcal{O}_Z(Z) \to 0.
\end{equation}

We state and prove for lack of reference the following

**Lemma 3.9.** If $X$ is a rational surface singularity, then the induced map
\begin{equation}
H^0(\bar{X}, \Theta_{\bar{X}}) \to H^0(\bar{X}, \Theta_{\bar{X}}(Z))
\end{equation}
is an isomorphism.

**Proof.** There is a well known exact sequence on the resolution of a normal singularity
\begin{equation}
0 \to \text{Der}_E(\bar{X}) \to \Theta_{\bar{X}} \to \bigoplus \mathcal{O}_{E_i}(E_i) \to 0
\end{equation}(see [Wah76, Proposition 2.2]). After tensoring this sequence with $\mathcal{O}_{\bar{X}}(Z)$ and applying $H^0$ we get a commutative diagram
\begin{equation}
\begin{array}{ccc}
H^0(\bar{X}, \text{Der}_E(\bar{X})) & \longrightarrow & H^0(\bar{X}, \Theta_{\bar{X}}) \\
\alpha \downarrow & & \downarrow \beta \\
H^0(\bar{X}, \text{Der}_E(\bar{X})(Z)) & \longrightarrow & H^0(\bar{X}, \Theta_{\bar{X}}(Z))
\end{array}
\end{equation}
where all the maps are injective. The sheaves $\mathcal{O}_{E_i}(E_i)$ and $\mathcal{O}_{E_i}(E_i + Z)$ on $E_i \simeq \mathbb{P}^1$ have negative degree, so the horizontal maps are also surjective. The cokernel of $\alpha$ sits in $H^0(\tilde{X}, \mathcal{D}er_E(\tilde{X}) \otimes \mathcal{O}(Z))$ which is trivial by a vanishing result $– H^1_k(\mathcal{D}er_E(\tilde{X})) = 0$ – of Wahl. See [BK87, Corollary 2.6] for an argument. So $\alpha$, and therefore $\beta$, is an isomorphism.

Remark. There is something to prove, since $H^0(\tilde{X}, \Theta_{\tilde{X}} \otimes \mathcal{O}(Z))$ is in general non-trivial. In fact, if $Z$ is reduced, then it has dimension equal to $\dim_c H^0_k(\Theta_{\tilde{X}})$ which again equals the number of $-2$ components of $E$ ([Wahl75, Theorem 6.1]).

In any case we now get from the sequence (3.6), the equality $h^1(\Theta_{\tilde{X}}(Z)) = h^1(\Theta_{\tilde{X}}) - \chi(\Theta_{\tilde{X}} \otimes \mathcal{O}(Z))$. From the Leray spectral sequence for $\Theta_{\tilde{X}}$, we get $h^1(\Theta_{\tilde{X}}) = h^1(\Theta_{\tilde{X}}) - h^0(R^1p_*\Theta_{\tilde{X}})$. So finally

$h^1(\Theta_{\tilde{X}}(Z)) = h^1(\Theta_{\tilde{X}}(Z)) - h^0(R^1p_*\Theta_{\tilde{X}}(Z))$

$= h^1(\Theta_{\tilde{X}}) - \chi(\Theta_{\tilde{X}} \otimes \mathcal{O}(Z)) - h^0(R^1p_*\Theta_{\tilde{X}})$

$= h^1(\Theta_{\tilde{X}}) - \chi(\Theta_{\tilde{X}} \otimes \mathcal{O}(Z))$

Since $p_*\mathcal{O}_{\tilde{X}}(kZ) = \mathcal{O}_{\tilde{X}}(kC)$ and $R^1p_*\mathcal{O}_{\tilde{X}}(kZ) = 0$ by the projection formula, we have $p_*\mathcal{O}_{\tilde{X}}(2Z) = \mathcal{O}_{\tilde{X}}(2C)$ and $R^1p_*\mathcal{O}_{\tilde{X}}(2Z) = 0$. Thus $\chi(\mathcal{O}_{\tilde{X}}(2Z)) = \chi(\mathcal{O}_{\tilde{X}}(2Z))$. Putting all this into formula (3.4) we get a new version

$\dim_c T^1_{\tilde{X}} = h^1(\Theta_{\tilde{X}}) - \chi(\Theta_{\tilde{X}} \otimes \mathcal{O}(Z)) + \chi(\mathcal{O}_{\tilde{X}}(2Z)) + h^0(\mathcal{H}^1_{\tilde{X}}) + c(X)$

(3.7)

Lemma 3.10. If $X$ is a rational surface singularity, then

$\chi(\mathcal{O}_{\tilde{X}}(2Z)) = \chi(\Theta_{\tilde{X}} \otimes \mathcal{O}(Z)) = e - 4$.

Proof. We have $\chi(\mathcal{O}_{\tilde{X}}(2Z)) = 2Z^2 + 1 = -2e + 3$ by Riemann–Roch.

We compute $\chi(\Theta_{\tilde{X}} \otimes \mathcal{O}(Z))$ in a standard manner (see e.g. [BK87, page 109] for another example). Since $X$ is rational we may construct a “computation sequence” $Z_0, \ldots, Z_l = Z$ where $Z_0 = E_{i_0}, Z_1 = Z_0 + E_{i_1}, \ldots, Z_l = Z_{l-1} + E_{i_l}$ with the property $Z_{k+1} \cdot E_{i_k} = 1$. If we tensor the exact sequence

$0 \rightarrow \mathcal{O}_{E_{i_k}}(Z_k) \rightarrow \mathcal{O}_{Z_{k+1}}(Z_{k+1}) \rightarrow \mathcal{O}_{E_{i_k+1}}(Z_{k+1}) \rightarrow 0$

with $\Theta_{\tilde{X}}$, we may compute recursively if we know $\chi(\Theta_{\tilde{X}} \otimes \mathcal{O}_{E_{i_k+1}}(Z_{k+1}))$. To compute this consider the standard exact sequence

$0 \rightarrow \Theta_{E_{i_k}} \rightarrow \Theta_{\mathcal{O}_{E_{i_k}}} \otimes \mathcal{O}_{E_{i_k}} \rightarrow \Theta_{E_{i_k}}(E_{i_k}) \rightarrow 0$.

If $k > 0$, then after twisting with $Z_{k-1}$, we get this sequence on $E_{i_k} \simeq \mathbb{P}^1$:

$0 \rightarrow \mathcal{O}_{E_{i_k}}(3 - b_{i_k}) \rightarrow \Theta_{\mathcal{O}_{E_{i_k}}}(Z_k) \rightarrow \mathcal{O}_{E_{i_k}}(-2b_{i_k} + 1) \rightarrow 0$

where $b_i = -E_i^2$. If $k = 0$ subtract 1 from the degrees of the left and right sheaves.

After adding everything up we get

$\chi(\Theta_{\tilde{X}} \otimes \mathcal{O}(Z)) = 3 \sum_{k=0}^l (2 - b_{i_k}) - 2$.

If $K$ is a canonical divisor, then by the adjunction formula we find that $-e - 1 = Z^2 = -2 - K \cdot Z = -2 + \sum_{k=0}^l (2 - b_{i_k})$. So $\chi(\Theta_{\tilde{X}} \otimes \mathcal{O}(Z)) = -3e + 7$ and we have proven the lemma.
Remark. It may be just a curiosity, but the number $e - 4$ comes from sheaves of more deformation theoretical interest. Notice that

$$
\chi(O_Z(2Z)) - \chi(\Theta X \otimes O_Z(Z)) = \chi(T^1 \otimes (Z)) - \chi(\Theta Z(Z)).
$$

This follows from the standard sequence for $T^1$.

If we plug the result of Lemma 3.10 into formula 3.7 we get

**Theorem 3.11.** If $X$ is a rational surface singularity of embedding dimension $e$ and $X$ is the blow up of $X$, then

$$\dim \mathcal{C} T^1_X = (e - 4) + \dim \mathcal{C} T^1_X + c(X).$$

4. About the correction term $c(X)$.

4.1. **Alternative definitions.** We have not been able to compute $c(X)$ in general, though there are partial results which we present here. First let us list several other $H^1$'s which have dimension $c(X)$.

**Proposition 4.1.** If $X$ is a rational surface singularity, then $c(X)$ equals the dimension of

(i) $H^1(\tilde{X}, m\mathcal{F}^1/m^2\mathcal{F}^1)$

(ii) $H^1(\tilde{X}, N_{A}(\mathcal{O}_{\tilde{X}})(-C))$

(iii) $H^1(\tilde{X}, \mathcal{F}^1(-C))$

(iv) $H^1(\tilde{X}, N_{\tilde{X}/M}(C)).$

**Proof.** To prove (i) it is enough to show that $m^2\mathcal{F}^1$ has support at points. We claim that the isomorphism in Proposition 3.1 (iii) induces locally a surjection $T^1_B \twoheadrightarrow x^2 T^1_B(B)$. Indeed if $|\phi|$ is in the kernel $K$ of $T^1_B(B) \twoheadrightarrow T^1_B$, then $x^2 \phi(f_j) = bx^2 \partial g_j/\partial x + \sum b_j x^2 \partial g_j/\partial x + \sum t_i \partial f_j/\partial x + \sum b_i \partial f_j/\partial x_l$ by (3.2). Thus $x^2 \phi = 0$ in $T^1\otimes K$ and $K$ is contained in the kernel of the multiplication map $T^1\otimes x^2 \twoheadrightarrow T^1\otimes (B)$.

Consider the commutative diagram with exact rows and surjective vertical maps.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N_{A}(\mathcal{O}_{\tilde{X}})(-C) & \longrightarrow & N_{A}(\mathcal{O}_{\tilde{X}}) & \longrightarrow & N_{A}(\mathcal{O}_{\tilde{X}})|_{C} & \longrightarrow & 0 \\
& & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \longrightarrow & m\mathcal{F}^1 & \longrightarrow & \mathcal{F}^1 & \longrightarrow & \mathcal{F}^1|_{C} & \longrightarrow & 0
\end{array}
$$

The argument in the proof of Proposition 3.2 shows that $H^0(N_{A}(\mathcal{O}_{\tilde{X}})|_{C}) = 0$ as well, so $H^0(\text{Ker}(\gamma)) = 0$ and $H^1(\text{Ker}(\alpha))$ injects into $H^1(\text{Ker}(\beta))$. But $\text{Ker}(\beta)$ is an image of $\pi^* \text{Der}(P) \simeq e\mathcal{O}_{\tilde{X}}$ so $H^1(\text{Ker}(\beta)) = 0$. This proves (ii).

On the other hand $\alpha$ factors surjectively through $\mathcal{F}^1(-C)$, which gives (iii). We have $H^1(N_{\tilde{X}/M}(C)) \simeq H^1(N_{A}(\mathcal{O}_{\tilde{X}})(-C))$ by Proposition 3.1. \qed
4.2. **Partial results.** The exact sequence in Proposition 3.1 sits in the following large commutative diagram of exact rows and columns.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & m^2\mathcal{F}^1 & \rightarrow & m\mathcal{T}_{A}^{1}(C) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & A_{C/\hat{X}}^{1}(C) & \rightarrow & m\mathcal{F}^1 & \rightarrow & \mathcal{T}_{A}^{1}(C) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{T}_{C/\hat{X}}^{1}(C) & \rightarrow & \mathcal{T}_{A}^{1}(C) & \rightarrow & \mathcal{T}_{A}^{1}(\mathcal{O}_C)(C) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{T}_{\hat{X}}^{2} & \rightarrow & \mathcal{T}_{\hat{X}}^{2} & \rightarrow & 0 & & \\
\end{array}
\]

(4.1)

We will not prove this, only explain the sheaves and some of the sequences. We use the notation of Section 3.1.

The sheaf \( \mathcal{T}_{C/\hat{X}}^{1} \) is the cokernel of \( \text{Der}(\mathcal{O}_{\hat{X}}, \mathcal{O}_C) \rightarrow \mathcal{O}_C(C) \) defined locally by \( D \rightarrow D(x) \). The sheaf \( \mathcal{T}_{\hat{X}}^{1}(\mathcal{O}_C) \) is locally \( T_{B}^{1}(B/(x)) \). The right vertical sequence is induced from the exact sequence \( 0 \rightarrow \mathcal{O}_{\hat{X}} \rightarrow \mathcal{O}_{\hat{X}}(C) \rightarrow \mathcal{O}_C(C) \rightarrow 0 \) and the next to bottom row is from the Zariski–Jacobi sequence for \( \mathcal{C} \rightarrow \mathcal{O}_{\hat{X}} \rightarrow \mathcal{O}_C \). The map \( m\mathcal{F}^1 \rightarrow \mathcal{T}_{A}^{1}(C) \) is locally the map \( xT_{A}^{1}(B) \rightarrow T_{B}^{1}(C) \) which takes \( x/\phi \) to the class of the homomorphism \( \tilde{g}_j \rightarrow \phi(f_j) + (x) \). Here \( \tilde{g}_j \) is the image of \( g_j \) in \( P_B/(x) \).

With the help of this diagram we can prove

**Proposition 4.2.** If \( X \) is a rational surface singularity, then

(i) \( c(X) = 0 \) if the fundamental cycle is reduced.

(ii) \( c(X) \geq \text{dim}_C H^1(\hat{X}, \mathcal{T}_{A}^{1}(C)) \).

(iii) If \( \mathcal{T}_{\hat{X}}^{2} = 0 \), i.e. the singularities on \( \hat{X} \) all have embedding dimension \( \leq 4 \), then \( c(X) = \text{dim}_C H^1(\hat{X}, \mathcal{T}_{A}^{1}(C)) \).

**Proof.** If the fundamental divisor is reduced, then \( C \) is reduced (\cite[Proof of Corollary 3.6]{Wah77}). In this case \( A_{C/\hat{X}}^{1} \) has support at points so \( H^1(A_{C/\hat{X}}^{1}(C)) = 0 \).

From the diagram we get an exact sequence \( 0 \rightarrow m\mathcal{F}^1/m^2\mathcal{F}^1 \rightarrow \mathcal{T}_{A}^{1}(C) \rightarrow \mathcal{T}_{\hat{X}}^{2} \), and \( c(X) = h^1(m\mathcal{F}^1/m^2\mathcal{F}^1) \) by Proposition 4.1. This proves the last two statements. \( \square \)

Proposition 4.2 allows us to generate examples where \( c(X) > 0 \).

**Proposition 4.3.** If \( X \) is a rational surface singularity, \( \hat{X} \) is smooth and \( C \) is non-reduced, then \( c(X) > 0 \).

**Proof.** In this case \( \hat{X} = \hat{X}, C = Z \) is the fundamental divisor and \( c(X) = h^1(T_{\hat{X}}^{2}(Z)) \). It follows from \cite[2.6]{Wah79} that \( H^1(T_{\hat{X}}^{2}(Z)) \cong H^1(\mathcal{O}_{\hat{X}-E}(2Z)) \) where \( E = Z_{\text{red}} \).

We claim that \( H^0(\mathcal{O}_{\hat{X}-E}(2Z)) = 0 \). In \cite{Wah75} Wahl proves that \( H^1_{\hat{X}}(\mathcal{O}_{\hat{X}}(E)) = 0 \). The proof actually shows that \( H^1_{\hat{X}}(\mathcal{O}_{\hat{X}}(E + Z')) = 0 \) for any cycle \( Z' \) with \( Z':E_i \leq 0 \) for all irreducible components \( E_i \) of \( E \). In particular \( H^1_{\hat{X}}(\mathcal{O}_{\hat{X}}(E+Z)) = 0 \),
and we may use [Wahl76, Lemma B.2] to conclude that $H^0(\mathcal{O}_Z(E + 2Z)) = 0$. But $H^0(\mathcal{O}_{Z-E}(2Z))$ injects into this last $H^0$.

Thus $h^1(T^2_2(Z)) = -\chi(\mathcal{O}_{Z-E}(2Z)) = (Z - E) \cdot (K - Z)$ using Riemann–Roch. (See [Wahl76, Proof of Proposition 2.15] for a similar argument.) If we set $b_i = -E_i^2$, $Z = \sum n_i E_i$ and $r_i = -Z \cdot E_i$ we find $(Z - E) \cdot (K - Z) = \sum (n_i - 1)(b_i - 2 + r_i)$.

Now we have assumed $X = X$, so all the $r_i > 0$ by Theorem 1.1. Thus $c(X) > 0$ in this case if $Z \neq E$; i.e. at least one $n_i > 2$.

It is a purely combinatorial problem to make dual graphs for rational singularities satisfying the conditions in Proposition 4.3. The one with lowest multiplicity is the “standard counter example” to the $T^1$ and $T^2$ formulas appearing before this paper – see e.g. [BK87]. Here is the dual graph:

![Diagram]

where $\bullet \simeq \mathbb{P}^1$ with self-intersection $-3$.

In fact any exceptional configuration of 4 components with this type of intersection will have $c(X) > 0$, as long as the central curve has self-intersection $-2$ and the other self-intersections are $\leq -3$.

If one extends the three arms off the central $-2$ curve, then these singularities will also have $c(X) > 0$ as long as neighbors of the $-2$ curve have self-intersection $\leq -4$ if the arm has length $> 1$ and non-end nodes have self-intersection $\leq -3$.

Here is an example with $e = 7$.

![Diagram]

where $\times \simeq \mathbb{P}^1$ with self-intersection $-4$.

References


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