Second order differential equations associated with the contact geometry: a divergence type and invariant solutions

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Abstract

Using the set of Lie structures over 1- jet manifold the symmetries structure is analyzed for Monge-Ampere equations. A method is proposed to determine invariant solutions with respect to symmetries algebras which preserve a divergence type. Moreover, an example of the von Karman equation is considered.

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Chapter 1

Introduction

Physical interest to the stable states of processes generates interest to the invariant solutions of differential equations.From the mathematical point of view the theory of invariant solutions is based on the fundamental ideas of S.Lie [1]. We propose the technique of constructing such solutions for the Monge-Ampere equations (in Lychagin's sense). We confine ourself with the class of divergence type equations as a very important type of equations in physical applications. Note in particular that equations such kind have the selected conservation laws. Using the set of Lie structures over 1-jet manifold we propose conditions on the symmetries algebras under which an equation preserves its conformal divergence type after the reduction procedure. This conditions generalize the classical integrating factor theory.

This result allows us to reduce the search of invariant solutions associated with symmetries algebras which preserve a divergence type for divergence type Monge-Ampere equations to the integration of divergence type Monge-Ampere equations over a manifold with a smaller number of independent variables. In some important cases the last equation reduces to the 1-order ordinary equation. As an example we realize our technique for the search invariant solutions such kind for the von Karman equation in gas and hydrodynamics. [1]

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Chapter 2

A set of the Lie structures over 1-jet manifold

Let M be a smooth manifold, $\dim M = n$. Let μ_x be the ideal of the ring $C^\infty(M)$ associated with a point $x \in M$: $\mu_x = \{f \in C^\infty(M) | f(x) = 0\}$. A smooth fiber bundle $\pi_k : J^k M \to M$ with a fiber $J^k_x M = C^\infty(M)/\mu_x^{k+1} C^\infty(M)$ over a point $x \in M$ is called the k-jets fiber bundle. The image of a function $f \in C^\infty(M)$ in a fiber $J^k_x M$ we denote by $j_k(f)_x = [f]^k_x$. Denote by $\mathcal{J}^k(M)$ the module of smooth sections of the fiber bundle $J^k M$. By $S_{j_k(f)} \subset \mathcal{J}^k(M)$ we denote the section $S_{j_k(f)}(m) = j_k(f)_m, m \in M$.

Any smooth map $F : M_1 \to M_2$ generates a module homomorphism

$$\begin{array}{rcccl} \mathcal{J}^k(F) & : & \mathcal{J}^k(M_2) & \to & \mathcal{J}^k(M_1) \\ & & [f]^k_{m_2} & \mapsto & [F^*(f)]^k_{m_1} \end{array}$$

where $f \in C^\infty(M_2)$, $m_1 \in M_1$, $m_2 \in M_2$, $m_2 = F(m_1)$.

Recall some basic results on the geometric structure on $J^1 M$ [2].

Proposition 1 *There exists a unique element $\rho_1 \in \mathcal{J}^1(J^1 M)$ such that for any $\theta \in \mathcal{J}^1(M)$ one has $\mathcal{J}^1(\theta)(\rho_1) = \theta$.*

The module $\mathcal{J}(M)$ is the direct sum of the module of 1-forms $\Lambda^1(M)$ and the ring $C^\infty(M) : \mathcal{J}^1(M) = \Lambda^1(M) \oplus C^\infty(M)$. Therefore any element $s \in \mathcal{J}^1(M)$ may be understood as a pair (ω, f), $\omega \in \Lambda^1(M)$, $f \in C^\infty(M)$.

Define an operator

$$\begin{array}{rcccl} D & : & \mathcal{J}^1(M) & \to & \Lambda^1(M) \\ & & (\omega, f) & \mapsto & df - \omega \end{array}$$

where d is the de Rham operator.

Proposition 2 *The fiber bundle J^1M possesses a natural contact structure defined by the universal Cartan's 1-form $U_1 = D\rho_1$.*

Distribution $K : x_1 \mapsto Ker\, U_{1,x_1}$, $x_1 \in J^1M$ is called the Cartan distribution.

Denote by $C^\infty_*(J^1M)$ the set of functions $\alpha \in C^\infty(J^1M)$, $\alpha \neq 0$ at any point $x_1 \in J^1M$. We also define $U_1^\alpha = \frac{1}{\alpha}U_1$, $\alpha \in C^\infty_*(J^1M)$.

Definition 3 *Differential ideal $\mathcal{C}$ in the algebra $\Lambda^\bullet(J^1M)$ generated by the form U_1 is called the Cartan ideal.*

It is obvious that any form U_1^α, $\alpha \in C^\infty_*(J^1M)$ is a generator of the Cartan ideal $\mathcal{C}$ as well.

Definition 4 *A vector field X on the manifold J^1M is called a contact vector field if it preserves the Cartan ideal: $L_X(\mathcal{C}) \subset \mathcal{C}$.*

Here L_X is the Lie derivative along the vector field X.

It is determined a vector field $X_{\langle 1|\alpha\rangle} = X_{\langle\alpha|1\rangle} = X_\alpha$ [2]. By this reason one has the following

Proposition 5 *Let $\alpha \in C^\infty_*(J^1M)$. Any contact vector field X on the manifold J^1M is uniquely determined by its generating function $f = U_1^\alpha(X)$.*

For $\alpha \in C^\infty_*(J^1M)$ denote by $X_{\langle f|\alpha\rangle}$ and $L_{\langle f|\alpha\rangle}$ a contact vector field with a generating function f with respect to U_1^α and the Lie derivative along $X_{\langle f|\alpha\rangle}$ respectively.

Proposition 6 *Let $\alpha, \beta \in C^\infty_*(J^1M)$. Then $X_{\langle f|\alpha\rangle} = X_{\langle \frac{\alpha}{\beta}f|\beta\rangle}$.*

Proof. By the Definition we obtain

$$X_{\langle f|\alpha\rangle}\rfloor U_1^\beta = \frac{\alpha}{\beta}X_{\langle f|\alpha\rangle}\rfloor U_1^\alpha = \frac{\alpha}{\beta}f.$$

From the Definition one has following

Proposition 7 *The bracket*

$$[f_1,\, f_2]_\alpha = U_1^\alpha([X_{\langle f_1|\alpha\rangle},\, X_{\langle f_2|\alpha\rangle}])$$

determines a Lie algebra structure on $C^\infty(J^1M)$ for any $\alpha \in C_*^\infty(J^1M)$.
Here $[X,\, Y]$ is the commutator of vector fields X and Y:

$$[X, Y] = X \circ Y - Y \circ X.$$

Proposition 8 *Let $\alpha,\, \beta \in C_*^\infty(J^1M)$, $f_1, f_2, f \in C_*^\infty(J^1M)$. Then*

$$\begin{array}{lll} 1^\circ & [f_1,\, f_2]_\alpha & = \frac{\beta}{\alpha}\left[\frac{\alpha}{\beta}f_1,\, \frac{\alpha}{\beta}f_2\right]_\beta \\ 2^\circ & L_{\langle f|\alpha\rangle}U_1^\beta & = \left[1,\, \frac{\alpha}{\beta}f\right]_\beta U_1^\beta \end{array}$$

Proof. The statement 1° is a consequence of Proposition 6.
From the Definition 4 we obtain

$$L_{\langle f|\alpha\rangle}U_1^\beta = g(f)U_1^\beta \ , \ \ g(f) \in C^\infty(J^1M).$$

Using Proposition 6 and the interior product $X_{\langle 1|\beta\rangle}\rfloor$ we obtain

$$U_1^\beta\left(\left[X_{\langle 1|\beta\rangle},\, X_{\langle \frac{\alpha}{\beta}|\beta\rangle}\right]\right) = \left[1,\, \frac{\alpha}{\beta}f\right]_\beta = g(f).$$

As a consequence we obtain

Proposition 9 *Let $\alpha \in C_*^\infty(J^1M)$, $f_1, f_2 \in C^\infty(J^1M)$. Then*

$$[f_1,\, f_2]_\alpha = L_{\langle f_1|\alpha\rangle}(f_2) - [1,\, f_1]_\alpha f_2.$$

Proof. Applying the Lie derivative $L_{\langle f_1|\alpha\rangle}$ to the equality

$$X_{\langle f_2|\alpha\rangle}\rfloor U_1^\alpha = f_2$$

we obtain:

$$\left[X_{\langle f_1|\alpha\rangle},\, X_{\langle f_2|\alpha\rangle}\right]\rfloor U_1^\alpha + X_{\langle f_2|\alpha\rangle}\rfloor\left[L_{\langle f_1|\alpha\rangle}U_1^\alpha\right] = L_{\langle f_1|\alpha\rangle}(f_2).$$

Using statement 2° of Proposition 8 we obtain

$$\left[X_{\langle f_1|\alpha\rangle},\ X_{\langle f_2|\alpha\rangle}\right]\rfloor U_1^\alpha = [f_1, f_2]_\alpha =$$
$$L_{\langle f_1,\alpha\rangle}(f_2) - X_{\langle f_2|\alpha\rangle}\rfloor([1, f_1]\, U_1^\alpha) = L_{\langle f_1|\alpha\rangle}(f_2) - [1, f_1]_\alpha f_2.$$

Let $\alpha, \beta \in C^\infty_*(J^1M)$. Introduce the mapping

$$N_\alpha^\beta : C^\infty(J^1M) \to C^\infty(J^1M)$$
$$f_\alpha \mapsto f_\beta = \tfrac{\alpha}{\beta} f_\alpha$$

From statement 1° of Proposition 8 one obtains

Proposition 10 *Let $\alpha, \beta \in C^\infty_*(J^1M)$, $f_1, f_2 \in C^\infty(J^1M)$. The mapping N_α^β is consistent with Lie algebra structures in $C^\infty(J^1M)$:*

$$\left[N_\alpha^\beta(f_1),\ N_\alpha^\beta(f_2)\right]_\beta = N_\alpha^\beta\left([f_1, f_2]_\alpha\right).$$

If $q_1, \ldots, q_n$ are the local coordinates on the manifold M in a neighbourhood of a point $x \in M$ and $u, p_1, \ldots, p_n$ are the local coordinates in fibres J^1_xM over this neighbourhood, then the form U_1 may be described as

$$U_1 = du - \sum_{i=1}^n p_i dq_i.$$

In these coordinates the mapping $f \mapsto X_{\langle f|\alpha\rangle}$ has the form

$$X_{\langle f|\alpha\rangle} = \sum_{i=1}^n \left(-\frac{\partial f}{\partial p_i}\cdot\frac{d^{(\alpha)}}{dq_i} + \frac{d^{(\alpha)} f}{dq_i}\cdot\frac{\partial}{\partial p_i}\right) + f X_{\langle 1|\alpha\rangle}.$$

Here $\frac{d^{(\alpha)}}{dq_i} = \alpha\frac{\partial}{\partial q_i} + p_i X_{\langle 1|\alpha\rangle}$, $i = 1, 2, \ldots, n$.

Let Δ_1, Δ_2 be scalar differential operators in the ring $C^\infty(J^1M)$.

For any $f_1, f_2 \in C^\infty(J^1M)$ denote

$$\langle(\Delta_1, \Delta_2)|(f_1, f_2)\rangle = \det\begin{pmatrix}\Delta_1(f_1) & \Delta_1(f_2)\\ \Delta_2(f_1) & \Delta_2(f_2)\end{pmatrix}.$$

As a consequence of Proposition 9 we obtain

Proposition 11 *Let $\alpha \in C^\infty_*(J^1M)$, $f_1, f_2 \in C^\infty(J^1M)$. Then*

$$[f_1, f_2]_\alpha = -\sum_{i=1}^n \left\langle\left(\frac{\partial}{\partial p_i}, \frac{d^{(\alpha)}}{dq_i}\right)\middle|(f_1, f_2)\right\rangle + \langle(1, [1,\ ]_\alpha)|(f_1, f_2)\rangle.$$

Remark 1 *The form $U_1 \in \Lambda^1(J^1M)$ has the next basic property: a section $\theta \in \mathcal{J}^1(M)$ is equal to $S_{j1(f)}$ for some $f \in C^\infty(M)$ if and only if $\theta^*(U_1) = 0$.*

Chapter 3

Differential forms over 1-jet manifold: an $\mathrm{sl}(2,\mathbb{R})$-representation and a divergence type

Denote by $\Lambda^\bullet_\alpha(K^*) \subset \Lambda^\bullet(J^1M)$ the differential forms degenerating along the vector field $X_{\langle 1|\alpha\rangle}$, $\alpha \in C^\infty_*(J^1M)$.

The contact field $X_{\langle 1|\alpha\rangle}$, $\alpha \in C^\infty_*(J^1M)$ defines a decomposition of the space $T_{x_1}(J^1M)$ in a direct sum at any point $x_1 \in J^1M$:

$$T_{x_1}(J^1M) = K_{x_1} \oplus \mathbb{R}X_{\langle 1|\alpha\rangle, x_1}.$$

This decomposition generates a projection

$$\begin{array}{llll} P_\alpha & : & \Lambda^s(J^1M) & \to & \Lambda^s_\alpha(K^*) \\ & & \omega & \mapsto & X_{\langle 1|\alpha\rangle} \rfloor (U^\alpha_1 \wedge \omega) \end{array}$$

The restriction of the 2-form dU^α_{1,x_1} defines a symplectic structure on the space K_{x_1}.

Define operators

$$\begin{array}{lllll} T_\alpha & : & \Lambda^s_\alpha(K^*) & \to & \Lambda^{s+2}_\alpha(K^*) \\ & & \omega & \mapsto & dU^\alpha_1 \wedge \omega \\ \perp_\alpha & : & \Lambda^s_\alpha(K^*) & \to & \Lambda^{s-2}_\alpha(K^*) \\ & & \omega & \mapsto & \widehat{dU^\alpha_1} \rfloor \omega \end{array}$$

Here $\widehat{dU^\alpha_1}$ is a bivector dual to the 2-form dU^α_1 due to the symplectic structure. Denote by Π_r a projection on r-homogeneous component of $\Lambda^\bullet_\alpha(K^*)$

$$\Pi_r \ : \ \Lambda^\bullet_\alpha(K^*) \ \to \ \Lambda^r_\alpha(K^*)$$

Let Π denote the mapping

$$\begin{array}{rcl} \Pi \ : \ \Lambda^\bullet_\alpha(K^*) & \to & \Lambda^\bullet_\alpha(K^*) \\ \omega & \mapsto & \sum_{k=0}^{2n}(n-k)\Pi_k\omega \end{array}$$

Proposition 12 *[[3],[4],[5]] Let $\alpha \in C^\infty_*(J^1M)$. Then operators T_α, $\perp_\alpha$ and Π define an* $\mathrm{sl}(2,\mathbb{R})$*- representation on module $\Lambda^\bullet_\alpha(K^*)$.*

Denote by $\Lambda^\bullet_{\varepsilon,\alpha}(J^1M)$ primitive elements of this representation.

Using the structure of $\mathrm{sl}(2,\mathbb{R})$ - primitive elements we obtain

Proposition 13 *[[3],[4],[5],[6]] Let $\alpha \in C^\infty_*(J^1M)$. A k-form $\omega \in \Lambda^k(J^1M)$, $k \leq n$ belongs to the module $\Lambda^k_{\varepsilon,\alpha}(J^1M)$ if and only if*

1°. $X_{\langle 1|\alpha\rangle} \rfloor \omega = 0$

2°. $T_\alpha^{n-k+1}\omega = 0$.

A form $\omega \in \Lambda^k_{\varepsilon,\alpha}(J^1M), k \leq n$ is called an effective form.

We see at once that primitive elements do not depend upon the generator $\alpha \in C^\infty_*(J^1M)$ modulo $\mathcal{C}$. Denote $\Lambda^\bullet_\varepsilon(J^1M) = \Lambda^\bullet_{\varepsilon,\alpha}(J^1M) \bmod \mathcal{C}$.

As a consequence of the Definition we obtain

Proposition 14 *Let $\alpha, \beta \in C^\infty_*(J^1M)$, $\omega \in \Lambda^{n-k}_{\varepsilon,\beta}(J^1M)$.*

Then $P_\alpha(\omega) \in \Lambda^{n-k}_{\varepsilon,\alpha}(J^1M)$.

Proof. It is sufficient to prove that $(dU_1^\alpha)^{k+1} \wedge \left(X_{\langle 1|\alpha\rangle} \rfloor [U_1^\alpha \wedge \omega]\right) = 0$, if $(dU_1^\beta)^{k+1} \wedge \omega = 0$. It is easily seen that

$$\begin{aligned} (dU_1^\alpha)^{k+1} \wedge \left(X_{\langle 1|\alpha\rangle} \rfloor [U_1^\alpha \wedge \omega]\right) &= X_{\langle 1|\alpha\rangle} \rfloor \left([dU_1^\alpha]^{k+1} \wedge U_1^\alpha \wedge \omega\right) \\ &\quad - \left(X_{\langle 1|\alpha\rangle} \rfloor [dU_1^\alpha]^{k+1}\right) \wedge U_1^\alpha \wedge \omega. \end{aligned}$$

But it is easy to check that

$$\begin{aligned} [dU_1^\alpha]^{k+1} \wedge U_1^\alpha \wedge \omega &= \left(\frac{\beta}{\alpha}\right)^{k+2} [dU_1^\beta]^{k+1} \wedge U_1^\beta \wedge \omega = 0, \\ X_{\langle 1|\alpha\rangle} \rfloor dU_1^\alpha &= L_{\langle 1|\alpha\rangle} U_1^\alpha = 0. \end{aligned}$$

Proposition 15 *[[6],[7]] Let $\alpha \in C_*^\infty(J^1M)$. For any form $\omega \in \Lambda^k(J^1M)$ the form $P_\alpha(\omega)$ admits a unique Hodge-Lepage decomposition*

$$P_\alpha(\omega) = \omega_0^\alpha + T_\alpha\omega_1^\alpha + T_\alpha^2\omega_2^\alpha + \dots .$$

Here $\omega_i^\alpha \in \Lambda_{\varepsilon,\alpha}^{k-2i}(J^1M)$.

By analogy with [6] define the operator

$$\begin{array}{cccc} d_p^\alpha & : & \Lambda_\alpha^s(K^*) & \to & \Lambda_\alpha^s(K^*) \\ & & \omega & \mapsto & P_\alpha d\omega \end{array}$$

and the Euler operator

$$\begin{array}{ccccc} \mathcal{E}_\alpha & : & \Lambda_{\varepsilon,\alpha}^n(J^1M) & \to & \Lambda_{\varepsilon,\alpha}^n(J^1M) \\ & & \omega & \mapsto & (L_{\langle 1|\alpha\rangle} + d_p^\alpha \circ \perp_\alpha \circ d_p^\alpha)\omega \end{array}$$

It is easy to check that following equalities hold:

1°. $\mathcal{E}_\alpha L_{\langle 1|\alpha\rangle} = L_{\langle 1|\alpha\rangle}\mathcal{E}_\alpha$

2°. $d_p^\alpha \mathcal{E}_\alpha = 0$

3°. $\mathcal{E}_\alpha \circ \mathcal{E}_\alpha = L_{\langle 1|\alpha\rangle}\mathcal{E}_\alpha$

4°. $\mathcal{E}_\alpha(f\omega) = f\mathcal{E}_\alpha(\omega) + L_{\langle 1|\alpha\rangle}(f)\omega + d_p^\alpha f \wedge (\perp_\alpha d_p^\alpha\omega) + d_p^\alpha(X_{\langle f|\alpha\rangle}\rfloor\omega)$

5°. $P_\alpha(L_{\langle f|\alpha\rangle}\omega) = f\mathcal{E}_\alpha(\omega) + d_p^\alpha\left[X_{\langle f|\alpha\rangle}\rfloor\omega - f\perp_\alpha d_p^\alpha\omega\right] + T_\alpha\left(X_{\langle f|\alpha\rangle}\rfloor\perp_\alpha d_p^\alpha\omega\right)$.

Here $f \in C^\infty(J^1M)$, $\omega \in \Lambda_{\varepsilon,\alpha}^n(J^1M)$.

Keeping in mind the notion of a divergence type differential operator we recall the notion of a divergence type form [6].

Definition 16 *A form $\omega \in \Lambda^n(J^1M)$ is of a divergence type, if there exists a form $\theta \in \mathcal{C}$ such that $d(\omega + \theta) = 0$.*

From Proposition 14 we obtain that a divergence type does not depend upon generator α.

Proposition 17 *[6] Let $\alpha \in C_*^\infty(J^1M)$. A form $\omega \in \Lambda_{\varepsilon,\alpha}^n(J^1M)$ has a divergence type if and only if $\mathcal{E}_\alpha\omega = 0$.*

Chapter 4

Monge-Ampere equations: contact symmetries

Recall basic notions of Monge-Ampere equations geometric theory [6].

Definition 18 *Let $\omega \in \Lambda^n_\varepsilon(J^1M)$. The Monge-Ampere operator Δ_ω associated with ω is defined by*

$$\begin{array}{rccl} \Delta_\omega & : & C^\infty(M) & \to & \Lambda^n(M) \\ & & f & \mapsto & S^*_{j1(f)}\omega \end{array}$$

Recall that any point $x_2 \in J^2M,\ x_2 = [f]^2_x,\ f \in C^\infty(M)$ defines a plane $L(x_2) = T_{x_1}[S_{j1(f)}(M)] \subset T_{x_1}(J^1M)$ at a point $x_1 \in J^1M,\ x_1 = [f]^1_x$.

The Monge-Ampere equation associated with a form $\omega \in \Lambda^n_\varepsilon(J^1M)$ is called a submanifold $E_\omega \subset J^2M$:

$$E_\omega = \{x_2 \in J^2M| \ \ \omega|_{L(x_2)} = 0\}.$$

Denote by $\mathrm{Ct}(J^1M)$ a group of (contact) diffeomorphisms preserving an ideal $\mathcal{C}$. An action of $\mathrm{Ct}(J^1M)$ on Monge-Ampere operators is defined by

$$a(\Delta_\omega) = \Delta_{a^*(\omega)},\ a \in \mathrm{Ct}(J^1M).$$

These transformations preserving the Monge-Ampere equation E_ω form a group $\mathrm{Sym}\,\omega$. A group $\mathrm{Sym}\,\omega$ is define by

$$\mathrm{Sym}\ \omega = \{a \in Ct(J^1M) | a^*\omega - h_a \cdot \omega \in \mathcal{C},\ h_a \in C^\infty(J^1M)\}.$$

It is possible to characterize the group $\mathrm{Sym}\,\omega$ by its infinitesimal algebra $\mathrm{sym}\,\omega$.

Definition 19 *The algebra*

$$\mathrm{sym}\,\omega = \{f \in C^\infty(J^1M) | L_f(\omega) - \lambda(f)\omega \in \mathcal{C}, \lambda(f) \in C^\infty(J^1M)\}$$

is called the algebra of contact symmetries of the Monge-Ampere equation E_ω.

Proposition 20 *Let $\alpha \in C_*^\infty(J^1M)$, $f \in \mathrm{sym}\,\omega$, $\omega \in \Lambda^n_{\varepsilon,\alpha}(J^1M)$. Then*

$$L_{\langle f|\alpha\rangle}\omega = \lambda(f)\omega + U_1^\alpha \wedge (X_{\langle [1,f]|\alpha\rangle}\rfloor\omega).$$

Proof. From Definition 19 we obtain

$$L_{\langle f|\alpha\rangle}\omega = \lambda(f)\omega + U_1^\alpha \wedge \theta + dU_1^\alpha \wedge \Xi, \quad \theta \in \Lambda^{n-1}_\alpha(K^*),\ \Xi \in \Lambda^{n-2}_\alpha(K^*).$$

Since $X_{\langle 1|\alpha\rangle}\rfloor \circ L_{\langle f|\alpha\rangle} - L_{\langle f|\alpha\rangle}\circ X_{\langle 1|\alpha\rangle}\rfloor = X_{\langle [1,f]|\alpha\rangle}\rfloor$ then applying the interior multiplication $X_{\langle 1|\alpha\rangle}\rfloor$ to the last equality one gets $\theta = X_{\langle [1,f]|\alpha\rangle}\rfloor\omega$. It is easily seen that $P_\alpha L_{\langle f|\alpha\rangle}\omega = \lambda(f)\omega + dU_1^\alpha \wedge \Xi$. Multiplying by the form dU_1^α and using an effectiveness of ω we obtain

$$\begin{aligned} & \lambda(f)dU_1^\alpha \wedge \omega + [dU_1^\alpha]^2 \wedge \Xi = [dU_1^\alpha]^2 \wedge \Xi = dU_1^\alpha \wedge P_\alpha L_{\langle f|\alpha\rangle}(\omega) \\ = \ & P_\alpha L_{\langle f|\alpha\rangle}(dU_1^\alpha \wedge \omega) = 0. \end{aligned}$$

Therefore $\Xi = 0$.

A majority of second order differential operators in mathematical physics is the divergence type Monge-Ampere operators [6].

Definition 21 *The Monge-Ampere operator Δ_ω (respectively $\Delta_{\psi\omega}$, $\psi \in C^\infty(J^1M)$), $\omega \in \Lambda^n_\varepsilon(J^1M)$ is called to be of a divergence type (respectively a conformal divergence type), if the form ω has a divergence type. Corresponding equation is called a divergence type Monge-Ampere equation.*

We confine ourself with the representative class of the divergence type Monge-Ampere equations E_ω associated with forms $\omega \in \Lambda^n_{\varepsilon,\alpha}(J^1M)$, $d\omega = 0$ for some $\alpha \in C_*^\infty(J^1M)$.

As a consequence of Proposition 20 we obtain

Proposition 22 *Let $\alpha \in C_*^\infty(J^1M)$, $\omega \in \Lambda_{\varepsilon,\alpha}(J^1M)$, $d\omega = 0$, $f \in \mathrm{sym}\,\omega$. Then $d_p^\alpha[\lambda(f) + [1,f]_\alpha] \wedge \omega = 0$.*

Proof. From Proposition 20 it follows that $L_{\langle f|\alpha\rangle}(U_1^\alpha \wedge \omega) = [\lambda(f) + [1,f]_\alpha]U_1^\alpha \wedge \omega$. Applying the operator d and using an effectiveness of the from ω we obtain the statement.

Define Monge-Ampere equations "indivisible" by first order differential equations. This notion is somewhat similar to absence of intermediate integrals [8].

Definition 23 *Let $\alpha \in C_*^\infty(J^1M)$. We call a form $\omega \in \Lambda_{\varepsilon,\alpha}^n(J^1M)$ the indivisible one if the rank of ω is equal to $2n$ almost everywhere.*

Denote by $\Lambda_{*,\alpha}^n(J^1M)$ the set of indivisible forms.

Proposition 24 *Let $\alpha \in C_*^\infty(J^1M)$, $\omega \in \Lambda_{*,\alpha}^n(J^1M)$, $d\omega = 0$, $f \in \mathrm{sym}\,\omega$. Then $\lambda(f) + [1,f]_\alpha = const$.*

Proof. Using effectiveness of the form ω and multiplying the result of Proposition 22 by $\perp_\alpha$ we obtain that $X_{\langle\lambda(f)+[1,f]_\alpha|\alpha\rangle}\rfloor\omega = 0$. Since $\mathrm{rk}(\omega) = 2n$ almost everywhere and $X_{\langle 1|\alpha\rangle}\rfloor\omega = 0$ then $\lambda(f) + [1,f]_\alpha = const$.

It is easily seen that if $\alpha \in C_*^\infty(J^1M)$, $\omega \in \Lambda_{*,\alpha}^n(J^1M)$, $d\omega = 0$, $f \in \mathrm{n}(1, \mathrm{sym}\,\omega)$ then $\lambda(f) = const$. Here $\mathrm{n}(1, \mathrm{sym}\,\omega)$ is a normalizer of 1 in the algebra $\mathrm{sym}\,\omega$.

Define $\mathrm{sym}_0\omega = \{f \subset \mathrm{sym}\,\omega | \lambda(f) + [1,f]_\alpha = 0\}$.

Proposition 25 *Let $\alpha \in C_*^\infty(J^1M)$, $\omega \in \Lambda_{*,\alpha}^n$, $d\omega = 0$. Then*

$$[\mathrm{sym}\,\omega, \mathrm{sym}\,\omega] \subset \mathrm{sym}_0\omega.$$

Proof. Using Proposition 24 and the Jacoby identity one gets

$$\begin{aligned}&\lambda([f_1,f_2]_\alpha) = L_{\langle f_1|\alpha\rangle}[\lambda(f_2)] - L_{\langle f_2|\alpha\rangle}[\lambda(f_1)] = \\ &-L_{\langle f_1|\alpha\rangle}([1,f_2]_\alpha) + L_{\langle f_2|\alpha\rangle}([1,f_1]_\alpha) = -[1,[f_1,f_2]]_\alpha.\end{aligned}$$

Corollary 26 *Under the conditions of Proposition 25, $\mathrm{sym}_0\,\omega$ is an ideal of the algebra $\mathrm{sym}\,\omega$.*

Chapter 5

Invariant solutions

Recall basic facts on invariant solutions [2].

Definition 27 *A submanifold $j : \mathcal{L} \hookrightarrow J^1M, \dim\mathcal{L} = \dim M$, is called a solution of the Monge-Ampere equation E_ω, $\omega \in \Lambda^n_\varepsilon(J^1M)$, if*

$$\begin{aligned} 1^\circ. \quad j^*U_1 &= 0 \\ 2^\circ. \quad j^*\omega &= 0 \end{aligned}$$

Let g be some finite dimensional and complete subalgebra of the Lie algebra $\operatorname{sym}\omega$. Using the Lie algebra g we construct connected Lie group $G \subset \operatorname{sym}\,\omega$ whose Lie algebra coincides with the algebra g.

Denote by $G(x_1)$ the G-orbit passing through the point $x_1 \in J^1M$.

Definition 28 *A solution $\mathcal{L}$ of the equation E_ω, $\omega \in \Lambda^n_\varepsilon(J^1M)$ is called invariant with respect to action of $G \subset \operatorname{sym}\omega$ (G-invariant), if $G(\mathcal{L}) = \mathcal{L}$.*

Let $f_1, \ldots, f_k \in C^\infty(J^1M)$ be a basis of the algebra g. It is easy to check

Proposition 29 *A solution $\mathcal{L}$ of the equation E_ω, $\omega \in \Lambda^n_\varepsilon(J^1M)$, is G-invariant if and only if $f_i|_\mathcal{L} = 0$, $i = 1, 2, \ldots, k$.*

As a consequence of Proposition 29 we obtain that all G-invariant solutions of equations E_ω lie on a 1-order equation

$$E_{f_1,\ldots,f_k} = \{x_1 \in J^1M | f(x_1) = \ldots = f_k(x_1) = 0\}.$$

Suppose that $t : E_{f_1,\ldots,f_k} \hookrightarrow J^1M$ is a smooth submanifold in J^1M.

Denote by $L^\perp$ skew orthogonal complement to subspace $L \subset K_{x_1}$.

Definition 30 *A subspace $L \subset K_{x_1}$ is called involute if $L^\perp \subset L$. Equation $E \subset J^1M$ is called involute at the point $x_1 \in J^1M$ if subspace $T_{x_1}(E) \cap K_{x_1}$ is involute in K_{x_1}.*

Using this Definition one immediately gets

Proposition 31 *An equation $E_{f_1,\ldots,f_k}$ is involute at the point $x_1 \in E_{f_1,\ldots,f_k}$ if and only if $[f_i, f_j](x_1) = 0$ for all $i, j = 1, 2, \ldots, k$.*

Definition 32 *An involute equation $E \subset J^1M$ is called regular at the point $x_1 \in E$, if the spaces $T_{x_1}(E)$ and K_{x_1} are transversal at the point $x_1 \in E$.*

As a consequence of the fields $X_{f_1}, \ldots, X_{f_k}$ Definition one obtains

Proposition 33 *The equation $E_{f_1,\ldots,f_k}$ is involute and regular. The contact fields $X_{f_1}, \ldots, X_{f_k}$ generate a k-dimensional completely integrable distribution on an equation $E_{f_1,\ldots,f_k}$.*

Suppose that there exists a submanifold $s : S(G) \hookrightarrow E_{f_1,\ldots,f_k}$ transversal to G-orbits and identified with the factor-set $E_{f_1,\ldots,f_k}$. Denote

$$i = t \circ s : S(G) \to J^1M.$$

From Proposition 33 we obtain

Proposition 34 *Let $\alpha \in C^\infty_*(J^1M)$. There exists a skew orthogonal decomposition with respect to 2-form $dt^*U_1^\alpha$:*

$$T_{x_1}E_{f_1,\ldots,f_k} = T_{x_1}S(G) \oplus T_{x_1}G(x_1)$$

and, moreover, $\ker dt^*U_1^\alpha|_{T_{x_1}E_{f_1,\ldots,f_k}} = T_{x_1}G(x_1)$.

Proof. From the definition one obtains that the annihilator of $T_{x_1}E_{f_1,\ldots,f_k}$ in $T^*_{x_1}J^1M$ is dU_1^α-dual to the distribution $X_{f_1}, \ldots, X_{f_k}$.

Proposition 35 *Let $\alpha \in C^\infty_*(J^1M)$. The form $i^*U_1^\alpha$ defines a contact structure on the manifold $S(G)$.*

Proof. Since the equation $E_{f_1,\ldots,f_k}$ is regular (Proposition 33) and

$$\ker dt^*U_1^\alpha|_{T_{x_1}E_{f_1,\ldots,f_k}} = T_{x_1}G(x_1)$$

(see Proposition 34), then the form $di^*U_1^\alpha$ is nondegenerate.

Chapter 6

The reduction procedure and a divergence type

A contact manifold $S(G)$ can locally be regarded as a manifold J^1M_G for some $(n-k)$-dimensional manifold M_G. Any G-invariant solution $\mathcal{L}$ of the equation E_ω is uniquely determined by a factor set $\mathcal{L}_G = \mathcal{L}/G \subset S(G) = J^1M_G$. This manifold (may be, with singularities) is tangent to the Cartan distribution on J^1M_G. A manifold $\mathcal{L}_G$ is a solution of the (reduced) Monge-Ampere equation. This equation is the result of the equation E_ω reduction with respect to G.

Our purpose is to find conditions guaranteeing preservation of a divergence type of the Monge-Ampere operator after the reduction procedure.

Keeping in mind the reduction procedure we define a k-vector $|g| = X_{f_1} \wedge \ldots \wedge X_{f_k}$ on the Lie algebra g.

Definition 36 *The reduction of the form $V \in \Lambda^r(J^1M)$ with respect to the group G is called the (reduced) form $(V|G) \in \Lambda^{r-k}S(G)$, $(V|G) = i^*(|g| \rfloor V)$.*

Note that the reduced form is defined up to an $\mathbb{R}$-multiplication. It is easy to check

Proposition 37 *Let $\alpha \in C^\infty_*(J^1M)$, $\omega \in \Lambda^n_{\varepsilon,\alpha}(J^1M)$ and the group G satisfy the above assumptions, M_G be a smooth manifold. Then $(\omega|G) \in \Lambda^{n-k}_{\varepsilon,i^*\alpha}(J^1M_G)$.*

In spite of $\mathbb{R}$-indeterminacy of the reduced form $(\omega|G)$, the reduced equation $E_{(\omega|G)}$ is well-defined.

Below we denote by $\mathrm{n}_\beta(g, \mathrm{sym}\ \omega)$ the normalizer of the algebra g in sym ω with respect to the β-Lie algebra structure, $\beta \in C^\infty_*(J^1M)$. We also use the natural notation :

$$\begin{array}{rcccl} \mathrm{ad}_\beta f & : & \mathrm{sym}\ \omega & \to & \mathrm{sym}\ \omega \\ & & g & \mapsto & [g, f]_\beta \end{array}$$

$g,\ f \in \mathrm{sym}_\beta\omega$. For $\omega \in \Lambda^n_{*,\beta}(J^1M)$, $d\omega = 0$ we define a functional

$$\begin{array}{rcccl} \nu_\beta & : & \mathrm{n}_\beta(g, \mathrm{sym}\,\omega) & \to & \mathbb{R} \\ & & f & \mapsto & \lambda(f) + [1, f]_\beta - \mathrm{tr}\,\mathrm{ad}\, f\ |_g \end{array}$$

Proposition 38 *Let α, $\beta \in C^\infty_*(J^1M)$, $\omega \in \Lambda^n_{*,\beta}(J^1M)$, $d\omega = 0$, the group G satisfy the above conditions, $1_\beta \in \mathrm{n}_\beta(g, \mathrm{sym}\,\omega)$, $\nu(g) = 0$ for any $g \in g$. Then $d^\beta_p(\omega|G) = 0$.*

Proof. Let $C^k_{i,j,}$ be the structure constants of the Lie algebra g. Then

$$\begin{aligned} & d^\beta_p(X_{\langle f_1|\beta\rangle}\rfloor \ldots \rfloor X_{\langle f_k|\beta\rangle}\rfloor\omega) = P_\beta[L_{\langle f_1|\beta\rangle}(X_{\langle f_2|\beta\rangle}\rfloor \ldots X_{\langle f_k|\beta\rangle}\rfloor\omega) \\ & -P_\beta X_{\langle f_1|\beta\rangle}\rfloor d(X_{\langle f_2|\beta\rangle}\rfloor \ldots X_{\langle f_k|\beta\rangle}\rfloor\omega) \\ = \; & [-\mathrm{tr}\,\mathrm{ad}\,\langle f_1|\beta\rangle\ |_g + \lambda(f_1)]P_\beta(X_{\langle f_2|\beta\rangle}\rfloor \ldots X_{\langle f_k|\beta\rangle}\rfloor\omega) \\ & +\sum_{i=2}^{k}(-1)^i C^1_{1i} P_\beta(X_{\langle f_1|\beta\rangle}\rfloor X_{\langle f_3|\beta\rangle}\rfloor \ldots X_{\langle f_{i-1}|\beta\rangle}\rfloor X_{\langle f_{i+1}|\beta\rangle}\rfloor \ldots X_{\langle f_k|\beta\rangle}\rfloor\omega) \\ & -P_\beta X_{\langle f_1|\beta\rangle}\rfloor L_{\langle f_2|\beta\rangle}(X_{\langle f_3|\beta\rangle}\rfloor \ldots X_{\langle f_k|\beta\rangle}\rfloor\omega) \\ & +P_\beta X_{\langle f_1|\beta\rangle}\rfloor X_{\langle f_2|\beta\rangle}\rfloor d(X_{\langle f_3|\beta\rangle}\rfloor \ldots X_{\langle f_k|\beta\rangle}\rfloor\omega) = \ldots \\ = \; & \sum_{i=1}^{k}(-1)^{i-1}[-\mathrm{tr}\,\mathrm{ad}\,\langle f_i|\beta\rangle|_g + \lambda(f_i)] \\ & \times P_\beta(X_{\langle f_1|\beta\rangle}\rfloor \ldots X_{\langle f_{i-1}|\beta\rangle}\rfloor X_{\langle f_{i+1}|\beta\rangle}\rfloor \ldots X_{\langle f_k|\beta\rangle}\rfloor\omega). \end{aligned}$$

We use the following relations:

1°. $[L_X, Y\rfloor] = L_{[X,Y]}$.

2°. $[X_{\langle f_i|\beta\rangle}, X_{\langle f_j|\beta\rangle}]\rfloor X_{\langle f_1|\beta\rangle}\rfloor \ldots X_{\langle f_{i-1}|\beta\rangle}\rfloor X_{\langle f_{i+1}|\beta\rangle}\rfloor \ldots X_{\langle f_{j-1}|\beta\rangle}\rfloor X_{\langle f_{j+1}|\beta\rangle}\rfloor \ldots X_{\langle f_k|\beta\rangle}\rfloor\omega = (C^i_{ij}X_{\langle f_i|\beta\rangle} + C^j_{ij}X_{\langle f_j|\beta\rangle})\rfloor X_{\langle f_1|\beta\rangle}\rfloor \ldots X_{\langle f_{i-1}|\beta\rangle}\rfloor X_{\langle f_{i+1}|\beta\rangle}\rfloor \ldots X_{\langle f_{j-1}|\beta\rangle}\rfloor X_{\langle f_{j+1}|\beta\rangle}\rfloor \ldots X_{\langle f_k|\beta\rangle}\rfloor\omega$.

Here $X, Y, X_{\langle f_l|\beta\rangle}$ are vector fields.

Proposition 39 *Let $\alpha, \beta \in C^\infty_*(J^1M)$, $\omega \in \Lambda^n_{*,\beta}(J^1M)$, $d\omega = 0$, the group G satisfy the above conditions, $\nu_\alpha(g) = 0$ for any $g \in g$, $(X_{\langle 1|\alpha\rangle}\rfloor\omega|G)_\alpha = 0$. Then $d^\alpha_p(\omega|G) = 0$.*

Proof. It is easy to check that $(\omega|G) = (P_\alpha\omega|G) + i^*U_1^\alpha(X_{\langle 1|\alpha\rangle}\rfloor\omega|G) = (P_\alpha\omega|G)$. It is sufficient to prove that $i^*U_1^\alpha \wedge d(P_\alpha\omega|G) = 0$. Since the form $(P_\alpha\omega|G) \in \Lambda^n_{\varepsilon,\alpha}(S(G))$, it suffices to prove that $d(U_1^\alpha \wedge P_\alpha(\omega)|G) = 0$. Using computations similar to the ones from Proposition 38 we obtain

$$d(U_1^\alpha \wedge P_\alpha\omega|G) = i^*d[X_{\langle f_1|\alpha\rangle}\rfloor \dots X_{\langle f_k|\alpha\rangle}\rfloor(U_1^\alpha \wedge P_\alpha\omega)] = i^*\{\textstyle\sum_{j=1}^k \\ (-1)^{j-1}[-\operatorname{tr}\operatorname{ad}_\alpha f_j|_g + \lambda(f_j) + [1, f_j]_\alpha][X_{\langle f_1|\alpha\rangle}\rfloor \dots X_{\langle f_{j-1}|\alpha\rangle}\rfloor X_{\langle f_{j+1}|\alpha\rangle}\rfloor \dots \\ X_{\langle f_k|\alpha\rangle}\rfloor(U_1^\alpha \wedge P_\alpha\omega) + (-1)^k(|g|\rfloor[dU_1^\alpha \wedge P_\alpha\omega - U_1^\alpha \wedge dP_\alpha\omega])\}.$$

Since $P_\alpha\omega \in \Lambda^n_{\varepsilon,\alpha}(J^1M)$ then $dU_1^\alpha \wedge P_\alpha\omega = 0$.

Using the expression $P_\alpha\omega = \omega - U_1^\alpha \wedge X_{\langle 1|\alpha\rangle}\rfloor\omega$ we obtain

$i^*(|g|\rfloor[U_1^\alpha \wedge dP_\alpha\omega]) = -i^*(|g|\rfloor[U_1^\alpha \wedge dU_\alpha \wedge (X_{\langle 1|\alpha\rangle}\rfloor\omega)]) = -i^*U_1^\alpha \wedge di^*U_1^\alpha \wedge (X_{\langle 1|\alpha\rangle}\rfloor\omega|G) = 0$.

Let $\mathrm{h}(\omega, g)$ denote the set $\mathrm{n}_\alpha(g, \operatorname{sym}\omega)/g$.

Theorem 40 *Let α, $\beta \in C^\infty_*(J^1M)$, $\omega \in \Lambda^n_{*,\beta}(J^1M)$, $d\omega = 0$, the group G satisfy the above conditions, $(X_{\langle 1|\alpha\rangle}\rfloor\omega|G) = 0, \nu_\alpha(g) = 0$ for any $g \in g$. Then $(h\omega|G)$ has a divergence type for some function $h \in \mathrm{h}(\omega, g)$ if and only if $\nu_\alpha(h) = 0$.*

Proof. Below we shall use the notation k both for the function $k \in C^\infty(J^1M)$ and for $i^*k \in C^\infty(S(G))$. At first using the formula $P_\alpha L_{\langle f|\alpha\rangle} = fL_{\langle 1|\alpha\rangle} + d_p^\alpha X_{\langle f|\alpha\rangle}\rfloor + X_{\langle f|\alpha\rangle}\rfloor d_p^\alpha$ we compute

$$\begin{aligned} &\mathcal{E}_\alpha[h(\omega|G)] = [L_{\langle 1|\alpha\rangle} + d_p^\alpha \perp_\alpha d_p^\alpha][h(\omega|G)] = L_{\langle 1|\alpha\rangle}(h)(\omega|G) + hL_{\langle 1|\alpha\rangle}(\omega|G) \\ &+hd_p^\alpha \perp_\alpha d_p^\alpha(\omega|G) + d_p^\alpha \perp_\alpha(d_p^\alpha h \wedge (\omega|G)) + d_p^\alpha h \wedge (\perp_\alpha d_p^\alpha(\omega|G)) \\ = \ &L_{\langle 1|\alpha\rangle}(h)(\omega|G) + P_\alpha L_{\langle h|\alpha\rangle}(\omega|G) + hd_p^\alpha \perp_\alpha d_p^\alpha(\omega|G) - X_{\langle h|\alpha\rangle}\rfloor d_p^\alpha(\omega|G) \\ &+d_p^\alpha h \wedge (\perp_\alpha d_p^\alpha(\omega|G)). \end{aligned}$$

From Proposition 39 it follows that

$$\mathcal{E}_\alpha(h(\omega|G)) = i^*[1, h]_\alpha(\omega|G) + P_\alpha L_{\langle h|\alpha\rangle}(\omega|G).$$

Using the computations similar to the ones from Proposition 38 we obtain

$$\mathcal{E}_\alpha(h\omega|G) = \nu(h)(\omega|G).$$

Note that under Theorem conditions the function h exists if the condition $\dim \mathrm{h}(\omega,\, g) > 1$ is valid. Namely,it is possible two cases:

1°. The algebra is $\mathrm{h}(\omega, g)$ non-commutative. Then an existence of the function h is a consequence of Proposition 25:

$$\nu([f_1, f_2]_\alpha) = 0$$

for any $f_1, f_2 \in \mathrm{h}(\omega, g)$.

2°. If the condition 1° is not valid, the equation $\nu(h) = 0$ determines a hyperspace in the vector space with the dimension > 1.

Remark 2 *Using the reduction procedure to define invariant solutions with 1-symmetry group of the Monge-Ampere equation allows us to reduce the dimension by one. If the dimension of a G-orbits is equal to $n-1$, then from the theorem we obtain that the problem of a determination of the divergence type Monge-Ampere equation G-invariant solutions associated with symmetries algebras which preserve a divergence type reduces to the integration of the set of ordinary differential first order equations. If the dimension of a G-orbit is smaller than $n-1$, then it is necessary to have additional information about the structure of the reduced equation [[8],[10]].*

Chapter 7

Example: the von Karman equation

Apply our results to the equation

$$\frac{\partial\varphi}{\partial q_1}\cdot\frac{\partial^2\varphi}{\partial q_1^2}-\frac{\partial^2\varphi}{\partial q_2^2}-\frac{\partial^2\varphi}{\partial q_3^2}=0.$$

This equation describes a behaviour of the velocity potential φ in the transonic approximation of gas and hydrodynamics. It also defines a spreading of rays in a neighbourhood of a caustic in a homogeneous atmosphere [11].

This equation is defined by the effective form ω over manifold $J^1\mathbf{R}^3$:

$$\omega = p_1 dp_1\wedge dq_2\wedge dq_3 + dp_2\wedge dq_1\wedge dq_3 + dp_3\wedge dq_2\wedge dq_1$$

in canonic coordinates $(q_1,\ q_2,\ q_3,\ u,\ p_1,\ p_2,\ p_3)$. It is easy to check that E_ω is the indivisible Monge-Ampere equation: $\omega\in\Lambda^3_{*,1}(\mathbb{R}^3)$. In the following table all generators of infinite-dimensional contact symmetry group of von Karman equation are represented [[8],[9]]:

N	Symmetry f	$\lambda(f)$	A contact transformation image of a point $(q_1,\ q_2,\ q_3,\ u,\ p_1,\ p_2,\ p_3)$
1	$q_2p_2 + q_3p_3 + 2u$	2	$(q_1,\ e^{-t}q_2,\ e^{-t}q_3,\ e^{2t}u,\ e^{2t}p_1,\ e^{3t}p_2,\ e^{3t}p_3)$
2	$2q_1p_1 + 3(p_2q_2 + p_3q_3)$	-2	$(e^{-2t}q_1,\ e^{-3t}q_2,\ e^{-3t}q_3,\ u,\ e^{2t}p_1,\ e^{3t}p_2,\ e^{3t}p_3)$
3	$q_3p_2 - q_2p_3$	0	$(q_1,\ q_2\cos t - q_3\sin t,\ q_2\sin t + q_3\cos t,\ u,\ p_1,$ $p_2\cos t - p_3\sin t,\ p_2\sin t + p_3\cos t)$
4	p_1	0	$(q_1 - t,\ q_2,\ q_3,\ u,\ p_1,\ p_2,\ p_3)$
5	p_2	0	$(q_1,\ q_2 - t,\ q_3,\ u,\ p_1,\ p_2,\ p_3)$
6	p_3	0	$(q_1,\ q_2,\ q_3 - t,\ u,\ p_1,\ p_2,\ p_3)$
7	$H(q_2,\ q_3)$ $\frac{\partial^2 H}{\partial q_2^2} + \frac{\partial^2 H}{\partial q_3^2} = 0$	0	$(q_1,\ q_2,\ q_3,\ u + tH(q_2, q_3),\ p_1,$ $p_2 + t\frac{\partial H}{\partial q_2}(q_2,\ q_3),\ p_3 + t\frac{\partial H}{\partial q_3}(q_2, q_3))$

Our goal is to find invariant solutions decreasing on ∞. By this reason we except symmetry H. We obtain a 6-dimensional algebra g_6 whose structure in the following table is represented:

	f_1	f_2	f_3	f_4	f_5	f_6
f_1	0	0	0	0	$-f_5$	$-f_6$
f_2	0	0	0	$-2f_4$	$-3f_5$	$-3f_6$
f_3	0	0	0	0	f_6	$-f_5$
f_4	0	$2f_4$	0	0	0	0
f_5	f_5	$3f_5$	$-f_6$	0	0	0
f_6	f_6	$3f_6$	f_5	0	0	0

It is easy to see that g_6 is a solvable algebra: it contains a commutative ideal $g_3 = (f_4,\ f_5,\ f_6)$ and factor-algebra $g_3^{\perp} = g_6/g_3$ is commutative.

Keeping in mind invariant solution construction we find 2-dimensional subalgebras of the algebra g_6:

1°.2-dimensional subalgebras of the algebra g_3. These algebras correspond to trivial invariant solutions.

2°.2-dimensional subalgebras of the algebra $g_3^{\perp}$.

3°.2-dimensional subalgebras spanned by $(e_1,\ e_2)$, $e_1 \in g_6$, $e_2 \in g_3$. We except symmetries f_5, f_6,because of the decreasing condition on ∞. Therefore these algebras have the form $(af_4 + g_1,\ g_2)$, $a \in \mathbb{R}$; $g_1,\ g_2 \in g_3^{\perp}$.

For example we consider the spiral rotation algebra g_2. This algebra is commutative and generated by

$$\begin{aligned} f_I &= p_1 + a(q_3p_2 - q_2p_3) \\ f_{II} &= q_2p_2 + q_3p_3 + 2u. \end{aligned}$$

An easy computation shows that this algebra satisfies the conditions of Theorem with $\beta = 1$ and $\alpha = (q_2^2 + q_3^2)^{-2}$. By this reason this algebra preserves a divergence type. The submanifold $S(G_2)$ is described by

$$\begin{aligned} q_2 &= \frac{1}{2}, \; p_2 = [2au + p_1]/a, \\ q_3 &= \frac{1}{2}, \; p_3 = [2au - p_1]/a. \end{aligned}$$

with the contact form

$$i^* U_1^\alpha = \frac{1}{2}(du - p_1 dq_1).$$

The function h is equal to p_1, the reduced form is

$$(h\omega|G_2) = d[-2au^2 + \frac{a}{6}p_1^3 - \frac{1}{2a}p_1^2] \operatorname{mod} \mathcal{C}$$

and the reduced equation is

$$\frac{a}{6}\left[\frac{\partial \Psi}{\partial q_1}\right]^3 - \frac{1}{2a}\left[\frac{\partial \Psi}{\partial q_1}\right]^2 - 2a\Psi^2 = C,$$

where the function Ψ is the reduction of the solution φ onto the manifold $S(G_2)$, $C = const.$

Finally we obtain G_2-invariant solutions in the form

$$\varphi(q) = (q_2^2 + q_3^2)^{-1}\chi\left(q_1 - \frac{1}{a}\arctan\frac{q_2}{q_3}\right),$$

where

$$\chi(m) = [\frac{1}{12}P(m)^3 - \frac{1}{4a^2}P(m)^2 - \frac{C}{2a}]^{1/2},$$

the function $P(m)$ is defined from the equation

$$\zeta[v(P(m) - a^{-2})] + a^{-2}v(P(m) - a^{-2}) + 8\sqrt{3}m = 0$$

in which

$$\zeta' = \wp, \; \wp(v(x)) = x$$

and $\wp$ is the Wierstrass function with invariants

$$g_2 = 12a^{-4}, \; g_3 = 8a^{-6} + 24Ca^{-1}.$$

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