LOW-DIMENSIONAL PSEUDO-RIEMANNIAN HOMOGENEOUS SPACES

BORIS DOUBROV AND BORIS KOMRAKOV

1. Introduction

The aim of this paper is to describe all pseudo-Riemannian homogeneous spaces in dimensions 2 and 3. The case of Riemannian homogeneous spaces is well-known (see [1, 2, 3]). We divide the solution of this problem into two parts: local and global. The local classification of pseudo-Riemannian homogeneous spaces is equivalent to the description of effective pairs of of Lie algebras supplied with an invariant non-degenerate symmetric bilinear form on the isotropy module. This classification for two- and three-dimensional homogeneous spaces is completed in section 2. Then in sections 3 and 4 we construct all global pseudo-Riemannian homogeneous spaces in dimensions 2 and 3 respectively. In the three-dimensional case we restrict ourselves to the case of a non-trivial stationary subgroup. All other pseudo-Riemannian homogeneous spaces in this dimension are just Lie groups with a left-invariant metric.

Definition 1. A *pseudo-Riemannian homogeneous space* is a triple $(\overline{G}, M, \mathbf{g})$, where $\overline{G}$ is a connected Lie group, M is a connected smooth manifold supplied with a transitive action of $\overline{G}$, and $\mathbf{g}$ is an invariant pseudo-Riemannian metric on M. The *dimension* of $(\overline{G}, M, \mathbf{g})$ is the dimension of M.

We assume that $\overline{G}$ acts on M *effectively* (i.e. the identity is the only element that acts trivially on M). This allows us to identify $\overline{G}$ with a subgroup of the Lie group $\operatorname{Aut}(M, \mathbf{g})$.

Example 1.1. Let us describe all one-dimensional pseudo-Riemannian homogeneous spaces $(\overline{G}, M, \mathbf{g})$. If $\dim M = m$, then $\dim \operatorname{Aut}(M, \mathbf{g}) \leq m(m+1)/2$. So, in our case $\dim \overline{G} \leq 1$, and since $\overline{G}$ is transitive on M, we have $\dim \overline{G} = 1$. Hence, $\overline{G} = \mathbb{R}$ or $\overline{G} = S^1$. In both cases $\overline{G}$ is abelian and acts on M effectively. This follows that $M = \overline{G}$, the action of $\overline{G}$ on M is simply the left action of $\overline{G}$ on itself, and $\mathbf{g}$ is a left-invariant metric on $\overline{G}$. So, we see that all one-dimensional pseudo-Riemannian homogeneous spaces have the form:

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(1) $\overline{G} = M = \mathbb{R}$, $\mathbf{g} = a\,dx^2$, where $a \neq 0$;
(2) $\overline{G} = M = S^1$, $\mathbf{g} = a\,d\phi^2$, where $a \neq 0$.

In both cases these pseudo-Riemannian homogeneous spaces are maximal. It is easy to see that up to the isomorphism of pseudo-Riemannian manifolds we can assume in case (1) that $a = \pm 1$, while in case (2) all pseudo-Riemannian manifolds are non-isomorphic. It follows from the following fact: all automorphisms of the Lie group $\mathbb{R}$ have the form $x \mapsto \lambda x$, $\lambda \neq 0$, while the only automorphisms of S^1 are $\phi \mapsto \pm\phi$.

2. Simply connected and local pseudo-Riemannian homogeneous spaces

Let $(\overline{G}, M, \mathbf{g})$ be a pseudo-Riemannian homogeneous space. We fix an arbitrary point $a \in M$ and let $G = \overline{G}_a$ be the stationary subgroup at the point a. The *isotropy action* of G on $T_a(M)$ is a linear action defined by $g.v = d_a g(v)$ for $g \in G$, $v \in T_aM$. It supplies the tangent space T_aM with a G-module structure. Let $\bar{\mathfrak{g}}$ be the algebra of the Lie group $\overline{G}$, and $\mathfrak{g}$ the subalgebra of $\bar{\mathfrak{g}}$ corresponding to the subgroup G. Since the action of $\overline{G}$ on M is effective, the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has the following property: subalgebra $\mathfrak{g}$ contains no non-zero ideals of $\bar{\mathfrak{g}}$. We shall call these pairs of Lie algebras *effective* too.

The manifold M can be identified with the set $\overline{G}/G$ of left cosets, and the action of $\overline{G}$ on M becomes simply the left action on $\overline{G}/G$:

$$g.(hG) = (gh)G, \quad g, h \in \overline{G}.$$

Moreover, the tangent space T_aM can be identified with the quotient space $\bar{\mathfrak{g}}/\mathfrak{g}$ and the isotropy action of G on T_aM with the adjoint action of G on $\bar{\mathfrak{g}}/\mathfrak{g}$:

$$g.(x + \mathfrak{g}) = (\operatorname{Ad} g)(x) + \mathfrak{g}, \quad g \in G,\ x \in \bar{\mathfrak{g}}.$$

Invariant pseudo-Riemannian metrics $\mathbf{g}$ on M are in the one-to-one correspondence with invariant symmetric non-degenerate bilinear forms B on the G-module $\bar{\mathfrak{g}}/\mathfrak{g}$.

The $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$ corresponding to the isotropy action of G on $\bar{\mathfrak{g}}/\mathfrak{g}$ has the form:

$$x.(y + \mathfrak{g}) = [x, y] + \mathfrak{g}, \quad x, y \in \bar{\mathfrak{g}},$$

and the bilinear form B is also an invariant bilinear form on the $\mathfrak{g}$ -module $\bar{\mathfrak{g}}/\mathfrak{g}$, i.e.

$$B(x.v_1, v_2) + B(v_1, x.v_2) = 0 \quad \text{for all } x \in \mathfrak{g},\ v_1, v_2 \in \bar{\mathfrak{g}}/\mathfrak{g}.$$

Moreover, if G is connected, then the converse is also true.

Summarizing all this, we see that to each pseudo-Riemannian homogeneous space $(\overline{G}, M, \mathbf{g})$ there corresponds a triple $(\bar{\mathfrak{g}}, \mathfrak{g}, B)$, where $(\bar{\mathfrak{g}}, \mathfrak{g})$ is an effective pair of Lie algebras and B is an invariant symmetric non-degenerate bilinear form on the $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$. We shall call these triples *local pseudo-Riemannian homogeneous spaces.*

(The reason for this notation is that, locally, a triple $(\bar{\mathfrak{g}}, \mathfrak{g}, B)$ uniquely determines the corresponding pseudo-Riemannian homogeneous space.)

Let us describe all pseudo-Riemannian homogeneous spaces corresponding to a given local pseudo-Riemannian homogeneous space $(\bar{\mathfrak{g}}, \mathfrak{g}, B)$.

Theorem 1. *Let $(\bar{\mathfrak{g}}, \mathfrak{g}, B)$ be a local pseudo-Riemannian homogeneous space such that $\operatorname{codim}_{\bar{\mathfrak{g}}} \mathfrak{g} \leq 4$. Then there exists a unique (up to the equivalence) pseudo-Riemannian homogeneous space $(\overline{G}^*, M^*, \mathbf{g}^*)$, corresponding to $(\bar{\mathfrak{g}}, \mathfrak{g}, B)$, such that M^* is simply connected and the stationary subgroup G^* is connected.*

Proof. It follows from [5] that there exists a unique (up to the equivalence) effective homogeneous space $(\overline{G}^*, M^*)$ such that M^* is simply connected and the stationary subgroup G^* is connected. Let us show that this homogeneous space admits a unique invariant pseudo-Riemannian metric $\mathbf{g}^*$ corresponding to the bilinear form B. Let $m = eG^* \in M^*$, where e is the identity element of $\overline{G}^*$. Then for the existence of $\mathbf{g}^*$, it is sufficient for B to be invariant with respect to the isotropy action of G^* on $T_m M^* \cong \bar{\mathfrak{g}}/\mathfrak{g}$. But this condition is satisfied, since G^* is connected and B is an invariant bilinear form on the $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$. □

Remark 1. This result is no longer true when $\operatorname{codim}_{\bar{\mathfrak{g}}} \mathfrak{g} \geq 5$. The simplest counterexample (see [5]) has the following form. Let

$$\bar{\mathfrak{g}} = \mathfrak{su}(2) \times \mathfrak{su}(2), \text{ and } \mathfrak{g} = \left\{ \left(\begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix}, \begin{pmatrix} i\alpha x & 0 \\ 0 & -i\alpha x \end{pmatrix} \right) \middle| \, x \in \mathbb{R} \right\},$$

where α is an arbitrary irrational number. Since $\bar{\mathfrak{g}}$ is compact, there exists an invariant positive bilinear form B on the $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$, and hence $(\bar{\mathfrak{g}}, \mathfrak{g}, B)$ is a local pseudo-Riemannian homogeneous space. But the corresponding virtual subgroup $G \subset SU(2) \times SU(2)$ is not closed, and, therefore, there are no global pseudo-Riemannian homogeneous spaces corresponding to $(\bar{\mathfrak{g}}, \mathfrak{g}, B)$.

Let $N = \operatorname{Norm}(G^*)/G^*$. Then N is a Lie group, and we can define the action of N on M^* in the following way:

$$(nG^*).(h.a) = (hn).a \quad \text{for all } nG^* \in N,\ h \in \overline{G}^*.$$

Lemma 1. *The action of N on M^* is well-defined and has the following properties:*

(1) *it is effective and free;*
(2) *it commutes with the action of $\overline{G}^*$ on M^*;*
(3) *the mapping $N \to M^*$, $nG^* \mapsto n.a$ is injective and its image is the set of all points in M^* whose stationary subgroups are equal to G^*.*

Proof. If $h_1.a = h_2.a$ for $h_1, h_2 \in \overline{G}^*$, then $h_2 = h_1 g$ for a certain $g \in G^*$ and

$$(h_2 n).a = (h_1 g n).a = (h_1 n)(n^{-1} g n).a = (h_1 n).a,$$

since $n^{-1}gn \in G^*$. This means that the action of N to M^* is well-defined.

If $(hn).a = h.a$ for some $h \in \overline{G}^*$, then $n = h^{-1}(hn) \in G^*$, and hence nG^* is the identity element in N. So, we see that the action of N on M^* is effective and free. The proof of (2) and (3) is similar. □

Consider the subgroup N_0 of N that consists of all elements leaving the metrics $\mathbf{g}^*$ stable. It follows from Lemma 1 that we can identify N with the subset of M^* consisting of all points whose stationary subgroups are equal to G^*. If $x \in M^*$ is any of these points, then we can identify the tangent space T_xM^* with $\bar{\mathfrak{g}}/\mathfrak{g}$, and it is easy to show that N_0 can be identified with the following subset in M^*:

$$N_0 = \{x \in M^* \mid \overline{G}^*_x = G^*,\ g_x = B\}.$$

Let D be any discrete subgroup in N_0. Then we can consider the manifold $M = M^*/D$ and the canonical projection $\pi\colon M^* \to M$. From item (3) of Lemma 1 it follows that we can define the action of $\overline{G}^*$ on M and the pseudo-Riemannian metric $\mathbf{g}$ on M that will be invariant under this action:

$$g.\pi(x) = \pi(h.x),\ \mathbf{g}_{\pi(x)} = d\pi.(\mathbf{g}^*_x), \quad g \in \overline{G}^*,\ x \in M^*. \tag{1}$$

Notice that the action of $\overline{G}^*$ on M is not necessarily effective. Let H be the subgroup in G^* consisting of all elements that act identically on M. Then H is a normal discrete subgroup. Moreover, it is possible to show that $H = D \cap Z(\overline{G}^*)$, where $Z(\overline{G}^*)$ is the center of $\overline{G}^*$, and both $\overline{G}^*$ and D are considered as subgroups in $\operatorname{Aut}(M^*, \mathbf{g}^*)$. We can consider, instead of $\overline{G}^*$, the group $\overline{G} = \overline{G}^*/H$ and the induced action of $\overline{G}$ on M. So, by means of the discrete subgroup $D \subset N_0$ we have constructed a new pseudo-Riemannian homogeneous space $(\overline{G}, M, \mathbf{g})$ which is locally equivalent to $(\overline{G}^*, M^*, \mathbf{g}^*)$.

Theorem 2. *The procedure described below gives a one-to-one correspondence between discrete subgroups $D \subset N_0$ and all pseudo-Riemannian homogeneous spaces corresponding to $(\bar{\mathfrak{g}}, \mathfrak{g}, B)$.*

Proof. Let $(\overline{G}, M, \mathbf{g})$ be an arbitrary pseudo-Riemannian homogeneous space corresponding to $(\bar{\mathfrak{g}}, \mathfrak{g}, B)$. It follows from [5] that the homogeneous space $(\overline{G}, M)$ is equivalent to $(\overline{G}^*/(Z(\overline{G}^*) \cap D), M^*/D)$, where D is a certain discrete subgroup in N. Let $\pi\colon M^* \to M$ be the canonical projection, $m^* = eG^*$, and $m = \pi(m^*)$. The projection π induces the identical transformation of the tangent spaces $T_{m^*}M^*$ and T_mM, which are both identified with $\bar{\mathfrak{g}}/\mathfrak{g}$. Therefore, it is compatible with the pseudo-Riemannian structures on M^* and M. This implies that the discrete transformation group D must preserve the metric $\mathbf{g}^*$ on M^*, and hence D lies in N_0. □

Theorems 1 and 2 allow us to divide the classification of all pseudo-Riemannian homogeneous spaces in dimensions 2 and 3 into the following parts:

(1) The classification of all local pseudo-Riemannian homogeneous spaces $(\bar{\mathfrak{g}}, \mathfrak{g}, B)$ such that $\operatorname{codim}_{\bar{\mathfrak{g}}} \mathfrak{g} = 2, 3$.
(2) For each triple $(\bar{\mathfrak{g}}, \mathfrak{g}, B)$, the construction of the corresponding simply connected pseudo-Riemannian homogeneous space $(\overline{G}^*, M^*, \mathrm{g}^*)$.
(3) Description of all discrete subgroups $D \subset N_0$.

3. Local classification

3.1. Two-dimensional case.

Theorem 3. *Let* $(\bar{\mathfrak{g}}, \mathfrak{g}, B)$ *be a local pseudo-Riemannian homogeneous space such that* $\operatorname{codim} \mathfrak{g} = 2$. *Then it is equivalent to one and only one of the following triples:*

1.1 $\bar{\mathfrak{g}} = \mathbb{R}^2$, $\mathfrak{g} = \{0\}$:

	u_1	u_2
u_1	0	0
u_2	0	0

, $$B = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

1.2 $\bar{\mathfrak{g}} = \mathbb{R}^2$, $\mathfrak{g} = \{0\}$:

	u_1	u_2
u_1	0	0
u_2	0	0

, $$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

2.1 $\bar{\mathfrak{g}} = \mathbb{R} \leftthreetimes \mathbb{R}$, $\mathfrak{g} = \{0\}$:

	u_1	u_2
u_1	0	u_1
u_2	$-u_1$	0

, $$B = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad a \neq 0;$$

2.2 $\bar{\mathfrak{g}} = \mathbb{R} \leftthreetimes \mathbb{R}$, $\mathfrak{g} = \{0\}$:

	u_1	u_2
u_1	0	u_1
u_2	$-u_1$	0

, $$B = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \quad a \neq 0;$$

2.3 $\bar{\mathfrak{g}} = \mathbb{R} \leftthreetimes \mathbb{R}$, $\mathfrak{g} = \{0\}$:

	u_1	u_2
u_1	0	u_1
u_2	$-u_1$	0

, $$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

3.1 $\bar{\mathfrak{g}} = \mathfrak{so}(1,1) \rtimes \mathbb{R}^2$, $\mathfrak{g} = \mathfrak{so}(1,1)$,

	e_1	u_1	u_2
e_1	0	u_2	$-u_1$
u_1	$-u_2$	0	0
u_2	$-u_1$	0	0

,

$$B = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

4.1 $\bar{\mathfrak{g}} = \mathfrak{so}(2) \rtimes \mathbb{R}^2$, $\mathfrak{g} = \mathfrak{so}(2)$,

	e_1	u_1	u_2
e_1	0	u_1	$-u_2$
u_1	$-u_1$	0	0
u_2	u_2	0	0

,

$$B = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

5.1 $\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{g} = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$:

	e_1	u_1	u_2
e_1	0	$2u_1$	$-2u_2$
u_1	$-2u_1$	0	e_1
u_2	$2u_2$	$-e_1$	0

,

$$B = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \quad a \neq 0;$$

6.1 $\bar{\mathfrak{g}} = \mathfrak{su}(2)$, $\mathfrak{g} = \left\{ \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$:

	e_1	u_1	u_2
e_1	0	u_2	$-u_1$
u_1	$-u_2$	0	e_1
u_2	u_1	$-e_1$	0

,

$$B = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad a \neq 0;$$

7.1 $\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{g} = \left\{ \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$:

	e_1	u_1	u_2
e_1	0	u_2	$-u_1$
u_1	$-u_2$	0	$-e_1$
u_2	u_1	e_1	0

,

$$B = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad a \neq 0.$$

Proof. The local classification of all two-dimensional homogeneous spaces was provided by S. Lie [7]. To extend this classification to the case of local pseudo-Riemannian homogeneous spaces one needs

(1) to choose those pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ from Lie's classification for which the $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$ admits an invariant symmetric non-degenerate bilinear form B;
(2) to describe all these forms B up to the induced action of $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g})$.

The first item was completed in [6].

The second part is trivial in most cases except the following: $\bar{\mathfrak{g}} = \mathbb{R} \rightthreetimes \mathbb{R}$, $\mathfrak{g} = \{0\}$. Let $\{u_1, u_2\}$ be a basis of $\bar{\mathfrak{g}}$, such that $[u_1, u_2] = u_1$. In this basis, the group $\operatorname{Aut}(\bar{\mathfrak{g}}, \mathfrak{g}) = \operatorname{Aut}(\bar{\mathfrak{g}})$ has the following form:

$$\left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \middle| \; x \in \mathbb{R}^*, \, y \in \mathbb{R} \right\}.$$

This group induces the following transformations on the set of all symmetric bilinear forms on $\bar{\mathfrak{g}}$:

$$B = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \longmapsto \begin{pmatrix} ax^2 & x(ay+b) \\ x(ay+b) & ay^2 + 2by + c \end{pmatrix}.$$

We see that if $a \neq 0$, then the bilinear form B can be transformed to one of the forms given in items 2.1 and 2.2 of the theorem. If $a = 0$, then $b \neq 0$ (otherwise, the form B would be degenerate), and the form B is given in item 2.3 of the theorem. □

3.2. Three-dimensional case.

Here we restrict our attention to the case of a non-trivial stationary subalgebra.

Theorem 4. *Let $(\bar{\mathfrak{g}}, \mathfrak{g}, B)$ be a local pseudo-Riemannian homogeneous space such that $\mathfrak{g} = 3$ and $\mathfrak{g} \neq \{0\}$. Then it is equivalent to one and only one of the following triples:*

1.1 $\bar{\mathfrak{g}} = (\mathfrak{so}(1,1) \rightthreetimes \mathbb{R}^2) \times \mathbb{R}$, $\mathfrak{g} = \mathfrak{so}(2)$:

	e_1	u_1	u_2	u_3
e_1	0	u_1	$-u_2$	0
u_1	$-u_1$	0	0	0
u_2	u_2	0	0	0
u_3	0	0	0	0

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix};$$

1.2 $\bar{\mathfrak{g}} = \mathfrak{a} \rightthreetimes \mathbb{R}^2$, $\mathfrak{g} = \mathfrak{so}(1,1) \subset \mathfrak{a}$, *where* $\mathfrak{a} = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| \; x, y \in \mathbb{R} \right\}$:

	e_1	u_1	u_2	u_3
e_1	0	u_1	$-u_2$	0
u_1	$-u_1$	0	0	0
u_2	u_2	0	0	u_2
u_3	0	0	$-u_2$	0

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} \quad a \neq 0;$$

1.3 $\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathbb{R},\ \mathfrak{g} = \mathfrak{so}(1,1)$:

	e_1	u_1	u_2	u_3
e_1	0	$2u_1$	$-2u_2$	0
u_1	$-2u_1$	0	e_1	0
u_2	$2u_2$	$-e_1$	0	0
u_3	0	0	0	0

$$B = \begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \quad a \neq 0;$$

1.4 $\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes \mathfrak{n}_3,\ \mathfrak{g} = \mathfrak{a}$, *where* $\mathfrak{n}_3$ *is the three dimensional Heisenberg algebra with basis* $\{u_1, u_2, u_3\}$, $[u_1, u_2] = u_3$, *and* $\mathfrak{a} = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & -x & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \ x \in \mathbb{R} \right\}$:

	e_1	u_1	u_2	u_3
e_1	0	u_1	$-u_2$	0
u_1	$-u_1$	0	u_3	0
u_2	u_2	$-u_3$	0	0
u_3	0	0	0	0

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} \quad a \neq 0;$$

1.5 $\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathbb{R},\ \mathfrak{g} = \left\{ \left(\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, x \right) \middle| \ x \in \mathbb{R} \right\}$:

	e_1	u_1	u_2	u_3
e_1	0	u_1	$-u_2$	0
u_1	$-u_1$	0	$e_1 + u_3$	0
u_2	u_2	$-e_1 - u_3$	0	0
u_3	0	0	0	0

$$B = \begin{pmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & b \end{pmatrix} \quad ab \neq 0;$$

2.1 $\bar{\mathfrak{g}} = (\mathfrak{so}(2) \leftthreetimes \mathbb{R}^2) \times \mathbb{R},\ \mathfrak{g} = \mathfrak{so}(2)$:

	e_1	u_1	u_2	u_3
e_1	0	$-u_2$	u_1	0
u_1	u_2	0	0	0
u_2	$-u_1$	0	0	0
u_3	0	0	0	0

$$B = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_2 \end{pmatrix} \quad \epsilon_1, \epsilon_2 = \pm 1;$$

2.2 $\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes \mathbb{R}^2,\ \mathfrak{g} = \mathfrak{so}(2) \subset \mathfrak{a}$, *where* $\mathfrak{a} = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \middle| \ x, y \in \mathbb{R} \right\}$:

	e_1	u_1	u_2	u_3
e_1	0	$-u_2$	u_1	0
u_1	u_2	0	0	u_1
u_2	$-u_1$	0	0	u_2
u_3	0	$-u_1$	$-u_2$	0

$$B = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & a \end{pmatrix} \quad \epsilon = \pm 1,\ a \neq 0;$$

2.3 $\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathbb{R}$, $\mathfrak{g} = \left\{ \left(\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}, x \right) \middle| \; x \in \mathbb{R} \right\}$:

	e_1	u_1	u_2	u_3
e_1	0	$-u_2$	u_1	0
u_1	u_2	0	$e_1 + u_3$	0
u_2	$-u_1$	$-e_1 - u_3$	0	0
u_3	0	0	0	0

, $B = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$ $ab \neq 0$;

2.4 $\bar{\mathfrak{g}} = \mathfrak{su}(2) \times \mathbb{R}$, $\mathfrak{g} = \left\{ \left(\begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix}, x \right) \middle| \; x \in \mathbb{R} \right\}$:

	e_1	u_1	u_2	u_3
e_1	0	$-u_2$	u_1	0
u_1	u_2	0	$-e_1 + u_3$	0
u_2	$-u_1$	$e_1 - u_3$	0	0
u_3	0	0	0	0

, $B = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$ $ab \neq 0$;

2.5 $\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathbb{R}$, $\mathfrak{g} = \mathfrak{so}(2)$:

	e_1	u_1	u_2	u_3
e_1	0	$-u_2$	u_1	0
u_1	u_2	0	e_1	0
u_2	$-u_1$	$-e_1$	0	0
u_3	0	0	0	0

, $B = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$ $a \neq 0$;

2.6 $\bar{\mathfrak{g}} = \mathfrak{su}(2) \times \mathbb{R}$, $\mathfrak{g} = \mathfrak{so}(2) = \left\{ \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} \middle| \; x \in \mathbb{R} \right\}$:

	e_1	u_1	u_2	u_3
e_1	0	$-u_2$	u_1	0
u_1	u_2	0	$-e_1$	0
u_2	$-u_1$	e_1	0	0
u_3	0	0	0	0

, $B = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$ $a \neq 0$;

2.7 $\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes \mathfrak{n}_3$, $\mathfrak{g} = \mathfrak{a}$, *where* $\mathfrak{n}_3$ *is the three dimensional Heisenberg algebra with basis* $\{p, q, h\}$, $[p, q] = h$, *and* $\mathfrak{a} = \left\{ \begin{pmatrix} 0 & x & 0 \\ -x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \; x \in \mathbb{R} \right\}$*:*

	e_1	u_1	u_2	u_3
e_1	0	$-u_2$	u_1	0
u_1	u_2	0	u_3	0
$-u_1$	u_2	$-u_3$	0	0
u_3	0	0	0	0

, $\quad B = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & a \end{pmatrix} \quad \epsilon = \pm 1,\ a \neq 0;$

3.1 $\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes \mathbb{R}^3$, $\mathfrak{g} = \mathfrak{a}$, *where* $\mathfrak{a} = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} \middle| \; x \in \mathbb{R} \right\}$*:*

	e_1	u_1	u_2	u_3
e_1	0	0	u_1	u_2
u_1	0	0	0	0
u_2	$-u_1$	0	0	0
u_3	$-u_2$	0	0	0

, $\quad B = \pm \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$

3.2 $\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{R}) \times \mathbb{R}$, $\mathfrak{g} = \left\{ \left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, x \right) \middle| \; x \in \mathbb{R} \right\}$*:*

	e_1	u_1	u_2	u_3
e_1	0	0	u_1	u_2
u_1	0	0	u_1	u_2
u_2	$-u_1$	$-u_1$	0	u_3
u_3	$-u_2$	$-u_2$	$-u_3$	0

, $\quad B = \begin{pmatrix} 0 & 0 & a \\ 0 & -a & 0 \\ a & 0 & \epsilon \end{pmatrix} \quad a \neq 0,\ \epsilon = \pm 1, 0;$

3.3 $\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes \mathfrak{n}_3$, $\mathfrak{g} = \mathbb{R}q \subset \mathfrak{n}_3$, *where* $\mathfrak{n}_3$ *is the three dimensional Heisenberg algebra with basis* $\{p, q, h\}$, $[p, q] = h$, *and* $\mathfrak{a} = \left\{ \begin{pmatrix} x & x & 0 \\ 0 & x & 0 \\ 0 & 0 & 2x \end{pmatrix} \middle| \; x \in \mathbb{R} \right\}$*:*

	e_1	u_1	u_2	u_3
e_1	0	0	u_1	u_2
u_1	0	0	0	$2u_1$
u_2	$-u_1$	0	0	$2u_2 - e_1$
u_3	$-u_2$	$-2u_1$	$-2u_2 + e_1$	0

, $\quad B = \pm \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(here $e_1 = q$, $u_1 = h$, $u_2 = p + q$*;* u_3 *is a non-zero element in* $\mathfrak{a}$*);*

3.4 $\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes \mathfrak{n}_3$, $\mathfrak{g} = \mathbb{R}(p+q) \subset \mathfrak{n}_3$, *where* $\mathfrak{n}_3$ *is the three dimensional Heisenberg algebra with basis* $\{p,q,h\}$, $[p,q]=h$, *and* $\mathfrak{a} = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & \alpha x & 0 \\ 0 & 0 & (1+\alpha)x \end{pmatrix} \middle| \; x \in \mathbb{R} \right\}$, $-1 \le \alpha < 1$:

	e_1	u_1	u_2	u_3
e_1	0	0	u_1	u_2
u_1	0	0	0	$(\alpha+1)u_1$
u_2	$-u_1$	0	0	$-\alpha e_1 + (\alpha+1)u_2$
u_3	$-u_2$	$-(\alpha+1)u_1$	$\alpha e_1 - (\alpha+1)u_2$	0

,

(here $e_1 = p+q$, $u_1 = h$, $u_2 = p + \alpha q$; u_3 *is a non-zero element in* $\mathfrak{a}$*),*

$$B = \pm \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

3.5 $\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes \mathfrak{n}_3$, $\mathfrak{g} = \mathbb{R}p \subset \mathfrak{n}_3$, *where* $\mathfrak{n}_3$ *is the three dimensional Heisenberg algebra with basis* $\{p,q,h\}$, $[p,q]=h$, *and* $\mathfrak{a} = \left\{ \begin{pmatrix} \alpha x & x & 0 \\ -x & \alpha x & 0 \\ 0 & 0 & 2\alpha x \end{pmatrix} \middle| \; x \in \mathbb{R} \right\}$, $0 \le \alpha$:

	e_1	u_1	u_2	u_3
e_1	0	0	u_1	u_2
u_1	0	0	0	$2\alpha u_1$
u_2	$-u_1$	0	0	$-(\alpha^2+1)e_1 + 2\alpha u_2$
u_3	$-u_2$	$-2\alpha u_1$	$(\alpha^2+1)e_1 - 2\alpha u_2$	0

,

(here $e_1 = p$, $u_1 = h$, $u_2 = \alpha p - q$; u_3 *is a non-zero element in* $\mathfrak{a}$*),*

$$B = \pm \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

4.1 $\bar{\mathfrak{g}} = \mathfrak{a} \leftthreetimes \mathbb{R}^3$, $\mathfrak{g} = \mathfrak{a}$, *where* $\mathfrak{a} = \left\{ \begin{pmatrix} x & y & 0 \\ 0 & 0 & y \\ 0 & 0 & -x \end{pmatrix} \middle| \; x, y \in \mathbb{R} \right\}$:

	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	0	$-u_3$
e_2	$-e_2$	0	0	u_1	u_2
u_1	$-u_1$	0	0	0	0
u_2	0	$-u_1$	0	0	0
u_3	u_3	$-u_2$	0	0	0

,

$$B = \pm \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

4.2 $\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{st}(2,\mathbb{R})$, $\mathfrak{g} = \{(x,x) \mid x \in \mathfrak{st}(2,\mathbb{R})\}$, *where* $\mathfrak{st}(2,\mathbb{R})$ *is the subalgebra of* $\mathfrak{sl}(2,\mathbb{R})$ *consisting of all upper trangular matices:*

	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	0	$-u_3$
e_2	$-e_2$	0	0	u_1	u_2
u_1	$-u_1$	0	0	u_1	u_2
u_2	0	$-u_1$	$-u_1$	0	u_3
u_3	u_3	$-u_2$	$-u_2$	$-u_3$	0

, $B = \pm \begin{pmatrix} 0 & 0 & a \\ 0 & -a & 0 \\ a & 0 & 0 \end{pmatrix}$ $a \neq 0;$

5.1 $\bar{\mathfrak{g}} = \mathfrak{so}(2,1) \leftthreetimes \mathbb{R}^3$, $\mathfrak{g} = \mathfrak{so}(2,1)$*:*

	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$-e_3$	u_1	0	$-u_3$
e_2	$-e_2$	0	e_1	0	u_1	u_2
e_3	e_3	$-e_1$	0	u_2	u_3	0
u_1	$-u_1$	0	$-u_2$	0	0	0
u_2	0	$-u_1$	$-u_3$	0	0	0
u_3	u_3	$-u_2$	0	0	0	0

, $B = \pm \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$

5.2 $\bar{\mathfrak{g}} = \mathfrak{so}(2,2) \cong \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{g} = \mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$*:*

	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$-e_3$	u_1	0	$-u_3$
e_2	$-e_2$	0	e_1	0	u_1	u_2
e_3	e_3	$-e_1$	0	u_2	u_3	0
u_1	$-u_1$	0	$-u_2$	0	e_2	$-e_1$
u_2	0	$-u_1$	$-u_3$	$-e_2$	0	$-e_3$
u_3	u_3	$-u_2$	0	e_1	e_3	0

, $B = \pm \begin{pmatrix} 0 & 0 & a \\ 0 & -a & 0 \\ a & 0 & 0 \end{pmatrix}$ $a \neq 0;$

5.3 $\bar{\mathfrak{g}} = \mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$, $\mathfrak{g} = \mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$*:*

	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$-e_3$	u_1	0	$-u_3$
e_2	$-e_2$	0	e_1	0	u_1	u_2
e_3	e_3	$-e_1$	0	u_2	u_3	0
u_1	$-u_1$	0	$-u_2$	0	$-e_2$	e_1
u_2	0	$-u_1$	$-u_3$	e_2	0	e_3
u_3	u_3	$-u_2$	0	$-e_1$	$-e_3$	0

, $B = \pm \begin{pmatrix} 0 & 0 & a \\ 0 & -a & 0 \\ a & 0 & 0 \end{pmatrix}$ $a \neq 0;$

6.1 $\bar{\mathfrak{g}} = \mathfrak{so}(3) \rightthreetimes \mathbb{R}^3$, $\mathfrak{g} = \mathfrak{so}(3)$:

	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_3	$-e_3$	$-u_3$	0	u_1
e_2	$-e_3$	0	e_1	$-u_2$	u_1	0
e_3	e_2	$-e_1$	0	0	$-u_3$	u_2
u_1	u_3	u_2	0	0	0	0
u_2	0	$-u_1$	u_3	0	0	0
u_3	$-u_1$	0	$-u_2$	0	0	0

, $$B = \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

6.2 $\bar{\mathfrak{g}} = \mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$, $\mathfrak{g} = \mathfrak{so}(3) \cong \mathfrak{su}(2)$:

	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_3	$-e_3$	$-u_3$	0	u_1
e_2	$-e_3$	0	e_1	$-u_2$	u_1	0
e_3	e_2	$-e_1$	0	0	$-u_3$	u_2
u_1	u_3	u_2	0	0	e_2	e_1
u_2	0	$-u_1$	u_3	$-e_2$	0	e_3
u_3	$-u_1$	0	$-u_2$	$-e_1$	$-e_3$	0

, $$B = \pm \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \quad a \neq 0;$$

6.3 $\bar{\mathfrak{g}} = \mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$, $\mathfrak{g} = \mathfrak{so}(3) \cong \mathfrak{su}(2)$:

	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_3	$-e_3$	$-u_3$	0	u_1
e_2	$-e_3$	0	e_1	$-u_2$	u_1	0
e_3	e_2	$-e_1$	0	0	$-u_3$	u_2
u_1	u_3	u_2	0	0	$-e_2$	$-e_1$
u_2	0	$-u_1$	u_3	e_2	0	$-e_3$
u_3	$-u_1$	0	$-u_2$	e_1	e_3	0

, $$B = \pm \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \quad a \neq 0;$$

Proof. Since each invariant pseudo-Riemannian metric defines an invariant affine connection, the $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$ is faithful. All these pairs for $\operatorname{codim}_{\bar{\mathfrak{g}}} \mathfrak{g}$ where classified in [8]. The rest of the proof is similar to the two-dimensional case. □

4. Global classification. Two-dimensional case.

In this section for each triple $(\bar{\mathfrak{g}}, \mathfrak{g}, B)$ from Theorem 3 we describe:

(1) the corresponding pseudo-Riemannian homogeneous space $(\overline{G}^*, M^*, \mathbf{g}^*)$ such that M^* is simply connected and the stationary subgroup $G^* = \overline{G}^*_a$ at an arbitrary point $a \in M^*$ is connected;

(2) the subgroup $N_0 \subset \operatorname{Norm}(G^*)/G^*$ and its action on M^*;

(3) all discrete subgroups in N_0 (up to the group $\mathrm{Aut}(\overline{G}^*, M^*, \mathbf{g}^*)$) and the corresponding pseudo-Riemannian homogeneous spaces.

This gives us a complete description of all pseudo-Riemannian homogeneous spaces in dimensions 2 and 3.

1.1. Here $\overline{G}^* = M^* = \mathbb{R}^2$, $\overline{G}^*$ acts on M^* as the group of all parallel translations, and in the standard coordinates (x_1, x_2) on M^*, we have $\mathbf{g}^* = \pm(dx_1^2 + dx_2^2)$. Since the stationary subgroup G^* is trivial at each point of M^*, we obtain that $N_0 = \overline{G}^*$, and its action on M^* coincides with the action of $\overline{G}^*$. We identify N_0 with the vector space $\mathbb{R}^2$ considered as an abelian group. Each discrete subgroup D in N_0 has one of the following forms:

(1) $D = \{0\}$;
(2) $D = \mathbb{Z}e$, $e \in \mathbb{R}^2$, $e \neq 0$;
(3) $D = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, where $\{e_1, e_2\}$ is a basis in $\mathbb{R}^2$.

The group $\mathrm{Aut}(\overline{G}^*, M^*, \mathbf{g}^*)$ is equal to $O(2) \rightthreetimes \mathbb{R}^2$, and its action on N_0 is equivalent to the natural action of $O(2)$ on $\mathbb{R}^2$. So, up to the equivalence, we can assume in (2) that $e = (a, 0)$, $a > 0$. In (3), two discrete subgroups D_1 and D_2 are equivalent if and only if they can be transformed to each other by means of the elements of $O(2)$.

The corresponding homogeneous spaces have the form $\overline{G} = M = \overline{G}^*/D$. Topologically, M is homeomorphic to $\mathbb{R}^2$, $\mathbb{R} \times S^1$, and $S^1 \times S^1$ in cases (1), (2), and (3) respectively.

1.2. This case is similar to the previous one. The only difference is that $\mathbf{g}^* = dx_1^2 - dx_2^2$, and the group $\mathrm{Aut}(\overline{G}^*, M^*, \mathbf{g}^*)$ is equal to $O(1,1) \times \mathbb{R}^2$. So, up to the equivalence, we can assume in (3) that $e = (a, 0)$ or $e = (a, a)$, where $a > 0$. In case (3), two discrete subgroups D_1 and D_2 are equivalent if and only if they can be transformed to each other by means of the elements of $O(2)$.

2.1–2.3. Here $\overline{G}^* = M^* = \mathbb{R}^2$, where the multiplication in $\overline{G}^*$ has the form:

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1 + y_1, x_2 + e^{x_1} y_2),$$

and the action of $\overline{G}^*$ on M^* is the left action of $\overline{G}^*$ on itself. The pseudo-Riemannian metric has the form:

2.1 $\mathbf{g}^* = a(dx_1^2 + e^{-2x_1} dx_2^2)$;
2.2 $\mathbf{g}^* = a(dx_1^2 - e^{-2x_1} dx_2^2)$;
2.3 $\mathbf{g}^* = e^{-x_1} dx_1 dx_2$.

In all these cases $N_0 = \mathbb{R}$, and $p.(x_1, x_2) = (x_1 + p, x_2)$ for all $p \in N_0$, $(x_1, x_2) \in M$. All non-trivial discrete subgroups of N_0 have the form $D = p\mathbb{Z}$, $(p > 0)$, and the corresponding homogeneous spaces $M = \overline{G}^*/D$ are cylinders.

3.1. In this case $M^* = \mathbb{R}^2$ and $\overline{G}^* = SO(2) \ltimes \mathbb{R}^2$ is the group of all Eucledian transformations of the plane preserving orientation. In the standart coordinates on $\mathbb{R}^2$ the metric $\mathrm{g}^*$is equal to $\pm(dx_1^2 + dx_2^2)$. Moreover, it is trivial to check that the origin is the only point on M^* with stationary subgroup $SO(2) \times \{0\}$. So, from Lemma 1 it follows that N_0 is trivial and there are no other pseudo-Riemannian spaces locally equivalent to $(\overline{G}^*, M^*, \mathrm{g}^*)$.

4.1. In this case, $M^* = \mathbb{R}^2$ and $\overline{G}^* = SO(1,1)_0 \ltimes \mathbb{R}^2$ is the connected component of the unit in the group of all pseudo-Euclidian transformations of the plane. The metric g^* has the form $dx_1^2 - dx_2^2$. As in the previous case we obtain no other pseudo-Riemannian homogeneous spaces, locally equivalent to $(\overline{G}^*, M^*, \mathrm{g}^*)$.

5.1. In this case $\overline{G} = \widetilde{SL(2,\mathbb{R})}$ is the simply connected covering group of $SL(2,\mathbb{R})$. It can be described, for example, as follows (see [6]): $\widetilde{SL(2,\mathbb{R})} = \mathbb{R}_+^* \times \mathbb{R}^2$, where the projection $\pi\colon \widetilde{SL(2,\mathbb{R})} \to SL(2,\mathbb{R})$ has the form:

$$\pi\colon (x,y,z) \mapsto \begin{pmatrix} x^{-1/2} & -yx^{-1/2} \\ 0 & x^{1/2} \end{pmatrix} \cdot \begin{pmatrix} \cos z & \sin z \\ -\sin z & \cos z \end{pmatrix},$$

and the multiplication in $\widetilde{SL(2,\mathbb{R})}$ is given by

$$(x_1,y_1,z_1)\cdot(x_2,y_2,z_2) = \\ (X(x_2,y_2,z_1)x_1,\ Y(x_2,y_2,z_1) + X(x_2,y_2,z_1)y_1,\ z_2 + Z(x_2,y_2,z_1)),$$

where

$$X(x,y,z) = \frac{(x\cos z + y\sin z)^2 + \sin^2 z}{x},$$

$$Y(x,y,z) = \frac{(x\cos z + y\sin z)(-x\sin z + y\cos z) + \sin z\cos z}{x},$$

$$Z(x,y,z) = \int_0^z \frac{x\,dt}{(x\cos t + y\sin t)^2 + \sin^2 t}.$$

Let $G^* = \{(x,0,0) \mid x \in \mathbb{R}_+^*\}$ be the connected subgroup of $\overline{G}^*$. Then the manifold $M^* = \overline{G}^*/G^*$ can be identified with the plane and the action of $\overline{G}^*$ on M^* can be written as:

$$(x,y,z).(p,q) = (Y(x,y,q) + X(x,y,q)p,\ z + Z(x,y,q)).$$

The subgroup N_0, in this case, is isomorphic to $\mathbb{Z}$ and acts on M^* in the following way: $n.(p,q) = ((-1)^n p, q + \pi n/2)$. Every non-trivial subgroup D of N_0 has one of the following forms:

(i) $2n\mathbb{Z}$;

(ii) $(2n+1)\mathbb{Z}$,

where $n \in \mathbb{N}$. The corresponding manifolds $M = M^*/D$ are cylinder and Möbius strip respectively.

In the cases $D = 2\mathbb{Z}$ and $D = \mathbb{Z}$, these homogeneous spaces can be described in classical terms. Namely, let $\overline{G} = PSL(2,\mathbb{R})$ be the group of all projective transformations of $\mathbb{R}P^1$. Then we can consider the following action of $\overline{G}$ on $\mathbb{R}P^1 \times \mathbb{R}P^1$:

$$X.(p_1, p_2) = (X.p_1,\ {}^tX^{-1}.p_2), \quad X \in \overline{G},\ p_1, p_2 \in \mathbb{R}P^1.$$

It is easy to see that this action has two orbits:

(1) $O_1 = \{\, ([x_1 : x_2],\ [y_1 : y_2]) \mid x_1y_1 + x_2y_2 = 0 \,\}$;
(2) $O_2 = (\mathbb{R}P^1 \times \mathbb{R}P^1) \backslash O_1$.

Moreover, the orbit O_2 is open, and the mapping

$$([x_1 : x_2],\ [y_1 : y_2]) \mapsto (x = x_1/x_2, y = y_1/y_2)$$

introduces a local coordinate system on O_2. There exists an invariant pseudo-Riemannian metric $\mathbf{g}$ on O_2, given in these local coordinates as

$$\mathbf{g} = \frac{dx\,dy}{(1+xy)^2}.$$

The triple $(\overline{G}, O_2, \mathbf{g})$ is exactly the pseudo-Riemannian homogeneous homogeneous space corresponding to the case $D = 2\mathbb{Z}$.

Consider the action of the group $\mathbb{Z}_2$ on O_2 generated by the mapping

$$\phi\colon ([x_1 : x_2],\ [y_1, y_2]) \mapsto ([y_2 : -x_2],\ [-y_1 : x_1]),$$

or, in the introduced local coordinates,

$$\phi\colon (x, y) \mapsto (-1/y, -1/x).$$

It is easy to check that this action is free and commutes with the action of $\overline{G}$. It follows that we can construct the quotient manifold $O_2/\mathbb{Z}_2$ and introduce a well-defined action of the group $\overline{G}$ on it. Thus we obtain the homogeneous space $(\overline{G}, M = O_2/\mathbb{Z}_2)$. Moreover, the transformation ϕ preserves the form $\mathbf{g}$. This allows to introduce a well-defined pseudo-Riemannian metric on M, and the constructed pseudo-Riemannian homogeneous space corresponds exactly to the case $D = \mathbb{Z}$.

6.1. In this case $\overline{G}^* = SO(3)$, $M^* = S^2$. We assume that S^2 is imbedded into $\mathbb{R}^3$ as the set of all vectors of length 1. Then $\overline{G}^*$ acts on M^* by rotations in space.

The metric $\mathbf{g}^*$ is the restriction of the metric $\pm(dx_1^2 + dx_2^2 + dx_3^2)$ in $\mathbb{R}^3$ to S^2. Let $a = (0, 0, 1)$. Then

$$G^* = \overline{G}_a^* = \left\{ \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; \alpha \in \mathbb{R} \right\}.$$

It is easy to check that $N_0 = \mathbb{Z}_2$ and that N_0 acts on S^2 in the following way:

$$(\pm 1).v = \pm v, \qquad v \in S^2.$$

So, in the case $D = N_0$ we obtain a new pseudo-Riemannian homogeneous space $(\overline{G}, M, \mathbf{g})$, where $\overline{G} = SO(3)$ and $M = M^*/D = \mathbb{R}\mathrm{P}^2$. In a certain local coordinate system (x_1, x_2) on M, the metric $\mathbf{g}$ has the form

$$\mathbf{g} = \pm \frac{dx_1^2 + dx_2^2}{(1 + x_1^2 + x_2^2)^2}.$$

7.1. In this case the corresponding homogeneous space $(\overline{G}^*, M^*, \mathbf{g}^*)$ is Lobachevsky plane:

$$\overline{G}^* = PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\{\pm E_2\}, \qquad M^* = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\};$$

the action of $\overline{G}^*$ on M^* has the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = \frac{az + b}{cz + d},$$

and in local coordinates $(x, y) \in \mathbb{R} \times \mathbb{R}_+^*$, $z = x + iy$ the metric $\mathbf{g}^*$ is equal to $\pm y^{-2}(dx_1^2 + dx_2^2)$. Fix the point $a = i$ in M^*. Straightforward computations shows that N_0 is trivial. So, in this case we have no other pseudo-Riemannian homogeneous spaces.

5. Global classification. Three-dimensional case.

Here we give a global description of pseudo-Riemannian homogeneous spaces corresponding to the pairs listed in Theorem 4.

1.1, 1.3, 2.1, 2.5, 2.6 (extension of two-dimensional spaces). Let $(\overline{G}^*, M^*, \mathbf{g}^*)$ be a two-dimensional pseudo-Riemannian homogeneous space such that $M^* = \overline{G}^*/G^*$ is simply connected and G^* is connected. We consider the manifold $M^* \times \mathbb{R}$ and introduce the pseudo-Riemannian metric on it equal to $\mathbf{g}^* \pm dt^2$. (Here t is a coordinate on $\mathbb{R}$.) The group $\overline{G}^* \times \mathbb{R}$ acts transitively and effectively on $M^* \times \mathbb{R}$:

$$(g, a).(m, t) = (g.m, t + a), \quad g \in \overline{G}^*,\ a, t \in \mathbb{R}, m \in M^*.$$

The stationary subgroup of this action is G^*, which is a connected subgroup.

The simply connected pseudo-Riemannian homogeneous spaces corresponding to the pairs 1.1, 1.3, 2.1, 2.5, and 2.6 of Theorem 4 can be derived in this way from the spaces 3.1, 5.1, 4.1, 7.1, and 6.1, respectively, of the two-dimensional case. Those homogeneous spaces which are not simply connected are considered in much the same way as in the two-dimensional case.

1.2, 2.2. Let $CO(1,1)$ be the Lie group of all linear transformations of $\mathbb{R}^2$ preserving (up to a constant) a pseudo-scalar product, and let $CO_0(1,1)$ be its identity component:

$$CO_0(1,1) = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| \; x, y \in \mathbb{R}_+^*, \right\}.$$

Similarly, define $CO(2)$ as the group of all linear transformations of $\mathbb{R}^2$ preserving (up to a constant) a scalar product, and let $CO_0(2)$ be its identity component.

The simply connected pseudo-Riemannian homogeneous spaces corresponding to the pairs 1.2 and 2.2 of Theorem 4 can be described as follows. The group $\overline{G}^*$ is equal to $H \leftthreetimes \mathbb{R}^2$, where H is equal to $CO_0(1,1)$ for the pair 1.2, and to $CO_0(2)$ for 2.2. The manifold M^* is equal to $\mathbb{R}^3$, and the action of $\overline{G}^*$ to M^* has the form:

$$(A, v).(w, x_3) = (Aw + v, x_3 + \ln\det(A))$$
$$A \in H,\ v, w = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \mathbb{R}^2,\ x_1, x_2, x_3 \in \mathbb{R}.$$

The metric g^* equals $e^{-z}dx_1dx_2 + adx_3^2$ for the pair 1.2, and $\pm e^{-z}(dx_1^2 + dx_2^2) + adx_3^2$ for 2.2. In both cases the group N_0 is trivial and we have no other pseudo-Riemannian homogeneous spaces.

The homogeneous space $(\overline{G}^*, M^*)$ can also be described in the following way. The group $\overline{G}^*$ acts transitively on $\mathbb{R}^2$ as a subgroup of the affine group. Consider the natural prolongation of this action to the one-dimensional vector bundle $\Lambda^2(T^*\mathbb{R}^2)$ over $\mathbb{R}^2$,. Then M^* can be considered as an open orbit of this action given by an arbitrary orientation of $\mathbb{R}^2$.

1.4, 2.7. Let N_3 be a simply connected Lie group whose Lie algebra is isomorphic to the three-dimensional Heisenberg algebra $\mathfrak{n}_3$. Then we can identify N_3 with $\mathfrak{n}_3 = \mathbb{R}^3$ (as vector spaces), multiplication being defined by the Campbell–Hausdorff formula:

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + (x_1y_2 - x_2y_1)/2).$$

Then the group $\overline{G}^*$ can be identified with the semidirect product $A \leftthreetimes N_3$, where A is the following subgroup of $\mathrm{Aut}(N_3)$:

$$1.4 \quad \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & 1/x & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \; x \in \mathbb{R}_+^* \right\};$$

2.7 $\left\{ \begin{pmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}.$

The manifold M^* can also be identified with N_3, and the action of $\overline{G}^*$ on M^* has the form: $(a, g).x = g \cdot a(x)$, $g, x \in N_3$, $a \in A$. In our coordinates on M^*, the metric $\mathbf{g}^*$ is equal to

1.4 $dx_1 dx_2 + a(dx_3 + (x_2 dx_1 - x_1 dx_2)/2)^2$;
2.7 $\pm(dx_1^2 + dx_2^2) + a(dx_3 + (x_2 dx_1 - x_1 dx_2)/2)^2$.

The group N_0 in this case is equal to $\mathbb{R}$ and acts on M^* by shifts of the third coordinate:

$$t.(x_1, x_2, x_3) = (x_1, x_2, x_3 + t), \; t \in N_0, \; (x_1, x_2, x_3) \in N_3.$$

All non-trivial discrete subgroups in N_0 are equivalent to $\mathbb{Z}$, and the corresponding homogeneous space is diffeomorphic to $\mathbb{R}^2 \times S^1$.

3.3, 3.4, 3.5. In these cases the manifold M^* is equal to $\mathbb{R}^3$, the Lie group $\overline{G}^*$ is diffeomorphic to $\mathbb{R}^4$ and can be defined as a following transformation group on M^*:

$$\{(x_1, x_2, x_3) \mapsto (x_1 + a, \; x_2 + f(x_1 + a), \; x_3 + g(x_1 + a) + f'(x_1 + a)(y + 1/2 f(x_1 + a))), \quad a \in \mathbb{R}, \; f \in V, \; g \in W, \}$$

where V and W are the following subspaces in $C^\infty(\mathbb{R})$:

3.3 $V = \langle e^x, xe^x \rangle$, $\quad W = \langle e^{2x} \rangle$;
3.4 $V = \langle e^x, e^{\alpha x} \rangle$, $\quad W = \langle e^{(\alpha+1)x} \rangle$;
3.5 $V = \langle e^{\alpha x} \sin x, e^{\alpha x} \cos x \rangle$, $\quad W = \langle e^{2\alpha x} \rangle$.

The metric $\mathbf{g}^*$ is equal to

3.3 $\pm(2dx_1 dx_3 + (x_2^2 - 4x_3) dx_1^2 - dx_2^2)$;
3.4 $\pm(2dx_1 dx_3 + (\alpha x_2^2 - 2\alpha(\alpha + 1)x_3) dx_1^2 - dx_2^2)$;
3.5 $\pm(2dx_1 dx_3 + ((2\alpha^2 + 1)x_2^2 - 4\alpha x_3) dx_1^2 - dx_2^2)$.

In the cases 3.3, 3.4, the group N_0 is isomorphic to $\mathbb{R}$ and acts on M^* as shifts of the third coordinate. In the case 3.5, the group N_0 is isomorpohic to $\mathbb{R} \times \mathbb{Z}$ and acts on M^* in the following way:

$$(y, n).(x_1, x_2, x_3) = (x_1 + \pi n, (-1)^n e^{\alpha \pi n} x_2, e^{2\alpha \pi n}).$$

The analysis of discrete subgroups in N_0 is trivial.

1.5, 2.3, 2.4, 3.2, 4.2, 5.3, 6.3. Let G be a simple three-dimensional simply connected Lie group (i.e., G is isomorphic either to $SU(2)$ or to the simply connected covering of $SL(2,\mathbb{R})$), H the following closed subgroup in G:

$$\mathbf{G = SU(2)}$$

2.4 $H = \left\{ \begin{pmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}$;
6.2 $H = G$.

$$\mathbf{G = \widetilde{SL(2,\mathbb{R})}}$$

Here we use the description of $\widetilde{SL(2,\mathbb{R})}$ given in the previous section and denote by π the natural surjection $\widetilde{SL(2,\mathbb{R})} \to SL(2,\mathbb{R})$.

1.5 $H = \pi^{-1}\big(SO_0(1,1)\big) = \{(x,0,0) \mid x \in \mathbb{R}^*_+\}$;
2.3 $H = \pi^{-1}\big(SO(2)\big) = \{(1,0,z) \mid z \in \mathbb{R}\}$;
3.2 $H = \pi^{-1}\left(\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}\right) = \{(1,y,0) \mid y \in \mathbb{R}\}$;
4.2 $H = \pi^{-1}\big(ST(2,\mathbb{R})\big) = \{(x,y,0) \mid x \in \mathbb{R}^*_+, y \in \mathbb{R}\}$;
5.2 $H = G$.

Consider the following action of $G \times H$ on G: $(g,h).x = gxh^{-1}$, $g, x \in G$, $h \in H$. It is not necessarily effective, and the element (g,h) acts trivially on G if and only if $g = h$ and g belongs to the center of G. Denote by Z the set of all such elements in $G \times H$. Then we can put $M^* = G$, $\overline{G}^* = (G \times H)/Z$, and the action of $\overline{G}^*$ on M^* can be derived from that on $G \times H$.

The metric $\mathbf{g}^*$ on M^* is just a left-invariant metric on the Lie group G uniquely determined by its value at the identity. In the cases 4.2, 5.2, and 6.2 this is a bi-invariant metric on G uniquely determined (up to a constant factor) by the Killing form on $\mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{su}(2)$. Notice that in the other cases the metric $\mathbf{g}^*$ is not right-invariant in general.

The subgroup N_0 in the all cases is equal to the centralizer $\mathcal{Z}_G(H)$ of H in G and acts on M^* by means of right shifts. In the cases 4.2, 5.2, and 6.2, it coincides with the center $\mathcal{Z}(G)$ of G and is isomorphic to $\mathbb{Z}_2$ for $G = SU(2)$ and $\mathbb{Z}$ for $G = \widetilde{SL(2,\mathbb{R})}$. In the other cases it is equal to $H\mathcal{Z}(G)$ which is isomorphic to S^1 for 2.4, to $\mathbb{R}$ for 2.3, and to $\mathbb{R} \times \mathbb{Z}$ for 1.5 and 3.2. The analysis of discrete subgroups in N_0 is trivial in all these cases.

As an example consider the case 2.4. Then all non-trivial discete subgroups in N_0 have the form

$$D_n = \left\{ \begin{pmatrix} e^{2\pi ik/n} & 0 \\ 0 & e^{2\pi ik/n} \end{pmatrix} \middle| \, k = 0, \dots, n-1 \right\}, \quad n \geq 2.$$

The corresponding manifolds $M = M^*/D_N$ are linse spaces $L^3_{(n)}$.

6.1. In this case the simply connected pseudo-Riemannian homogeneous space is the usual three-dimensional Euclidean plane: $\overline{G}^* = SO(3) \leftthreetimes \mathbb{R}^3$, $M^* = \mathbb{R}^3$, $\mathbf{g}^* = \pm(dx_1^2+dx_2^2+dx_3^2)$. The group N_0 is trivial, and we have no other pseudo-Riemannian homogeneous spaces corresponding to this pair.

3.1, 4.1, 5.1. The simply connected pseudo-Riemannian homogeneous space corresponding to the pair 5.1 is the usual pseudo-Euclidean plane: $\overline{G}^* = SO_0(2,1) \leftthreetimes \mathbb{R}^3$, $M^* = \mathbb{R}^3$. We choose a coordinate system (x_1, x_2, x_3) on $\mathbb{R}^3$ in such a way that $\mathbf{g}^* = \pm(2dx_1dx_3 - dx_2^2)$. The simply connected pseudo-Riemannian homogeneous spaces corresponding to the pairs 3.1 and 4.1 can be realized as restrictions of the action of $\overline{G}^*$ on M^* to a subgroup $H \leftthreetimes \mathbb{R}^3$, where

$$3.1\ H = \left\{ \begin{pmatrix} 1 & y & y^2/2 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, y \in r \right\}.$$

$$4.1\ H = \left\{ \begin{pmatrix} x & xy & xy^2/2 \\ 0 & 1 & y \\ 0 & 0 & 1/x \end{pmatrix} \middle| \, x \in \mathbb{R}^*_+, y \in r \right\}.$$

In the cases 4.1 and 5.1, the group N_0 is trivial and we have no other pseudo-Riemannian homogeneous spaces corresponding to these pairs. In the case 3.1, the group N_0 is isomorphic to $\mathbb{R}$ and acts on $\mathbb{R}^3$ as $t.(x_1, x_2, x_3) = (x_1, x_2, x_3+t)$, $t \in N_0$. All non-trivial discrete subgroups in N_0 are equivalent to $\mathbb{Z}$.

5.3, 6.3. Consider the natural action of the group $SO(3,1)$ on $\mathbb{R}^4$. We choose coordinates (x_1, x_2, x_3, x_4) on $\mathbb{R}^4$ in such a way that the group $SO(3,1)$ preserves the form $x_1^2 + x_2^2 + x_3^2 - x_4^2$. Then this action leaves stable quadrics the $Q_\pm = \{x_1^2 + x_2^2 + x_3^2 - x_4^2 = \pm 1\}$. Put $\overline{G}^* = SO(3,1)_0$, the identity component of $SO(3,1)$, and let $M^* = Q_+$ in the case 5.3, and $M^* = \{(x_1, x_2, x_3, x_4) \in Q_- \mid x_4 > 0\}$ in the case 6.3. It is easy to see that these manifolds are homogeneous with respect to the action of $\overline{G}^*$, and diffeomorphic to $S^2 \times \mathbb{R}$ and $\mathbb{R}^3$ respectively. The metric $\mathbf{g}^*$ is equal to the restriction of the metric $\pm(dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2)$ on $\mathbb{R}^4$ to M^*.

In the case 6.3, the group N_0 is trivial. In the case 5.3, the group N_0 is equal to $\mathbb{Z}_2$ and acts on M^* in the following way:

$$(\pm 1).(x_1, x_2, x_3, x_4) = (\pm x_1, \pm x_2, \pm x_3, \pm x_4).$$

The corresponding manifold M^*/N_0 is diffeomorphic to the canonical vector bundle over $\mathbb{R}P^3$.

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International Sophus Lie Centre, P.O. Box 70, Minsk, 220123, Belarus
Current address: Centre of Advanced Study, P.O. Box 7606, Skillebekk, 0205, Oslo, Norway
E-mail address: islc@islc.minsk.by, islc@shs.no