INVARIANT AFFINE CONNECTIONS ON THREE-DIMENSIONAL HOMOGENEOUS SPACES

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INTRODUCTION

This work is devoted to the classification of affine connections with transitive symmetry groups. In particular, we describe all affine connections on three-dimensional manifolds whose symmetry group is transitive and at least five-dimensional.

Symmetry groups of affine connections on two-dimensional manifolds were described in the works of I. Egorov [1] and J. Levine [2]. The same problem on three-dimensional manifolds was studied in a series of papers by V. Dumitraş [3, 4, 5, 6].

Since every homogeneous space $(\overline{G}, M)$ with an invariant affine connection is isotropy faithful, the description of transitive symmetry groups of affine connections can be divided into the following parts:

Part I: Classification of all isotropy faithful homogeneous spaces $(\overline{G}, M)$.

Part II: For each $(\overline{G}, M)$, description of all invariant affine connections on it.

Part III: For a given invariant affine connection on $(\overline{G}, M)$, finding out if $\overline{G}$ is the group of all symmetries of this connection.

The classification of all three-dimensional isotropy faithful homogeneous spaces was done in [11]. The description of invariant affine connections on homogeneous spaces was obtained by K. Nomizu [8]. In the present paper we translate Part III into the language of Lie algebras and give an effective algorithm for solving it.

Then we apply this algorithm to invariant connections on three-dimensional homogeneous spaces and obtain a complete classification of those affine connections on three-dimensional manifolds whose symmetry group is transitive and at least five-dimensional.

Computational results presented in this work were obtained with the help of the supercomputer IBM 6000/590 of the University of Oslo.

1. Connections on pairs of Lie algebras

In the sequel, by a pair of Lie algebras $(\bar{\mathfrak{g}}, \mathfrak{g})$ we mean a finite-dimensional real Lie algebra $\bar{\mathfrak{g}}$ together with a subalgebra $\mathfrak{g} \in \bar{\mathfrak{g}}$. Two pairs $(\bar{\mathfrak{g}}_1, \mathfrak{g}_1)$ and $(\bar{\mathfrak{g}}_2, \mathfrak{g}_2)$ are said to be equivalent if there exists a Lie algebra isomorphism $f\colon \bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}_2$ such that $f(\mathfrak{g}_1) = \mathfrak{g}_2$.

Let $(\overline{G}, M)$ be a homogeneous space, o an arbitrary point in M, and $G = \overline{G}_o$ the stationary subgroup of o. Then we can assign to $(\overline{G}, M)$ a pair of Lie algebras $(\bar{\mathfrak{g}}, \mathfrak{g})$, where $\bar{\mathfrak{g}}$ is the Lie algebra of $\overline{G}$ and $\mathfrak{g}$ is the subalgebra of $\bar{\mathfrak{g}}$ corresponding to the subgroup $G \subset \overline{G}$. This pair uniquely determines the local structure of $(\overline{G}, M)$: two homogeneous spaces are locally isomorphic if and only if the corresponding pairs of Lie algebras are equivalent.

Definition 1. A pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is *effective* if the only ideal of $\bar{\mathfrak{g}}$ contained in $\mathfrak{g}$ is $\{0\}$.

It is clear that a homogeneous space $(\overline{G}, M)$ is locally effective if and only if the corresponding pair of Lie algebras is effective. In the following we shall be concerned only with effective pairs of Lie algebras.

The isotropic representation of the stationary subgroup G on the tangent space T_oM can also be described in terms of Lie algebras.

Definition 2. Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be a pair of Lie algebras. The $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$ with

$$x.(y + \mathfrak{g}) = [x, y] + \mathfrak{g}$$

is called the *isotropic* $\mathfrak{g}$*-module* and denoted by $\mathfrak{m}$ The corresponding representation $\lambda\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$ is called the *isotropic representation* of $(\bar{\mathfrak{g}}, \mathfrak{g})$. A pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is said to be *isotropy faithful* if its isotropic representation is injective.

Notation. For $x \in \bar{\mathfrak{g}}$, let $x_{\mathfrak{m}}$ denote the image of x under the canonical projection $\bar{\mathfrak{g}} \to \mathfrak{m}$. And for any $\mathfrak{g}$-module M, let $M^{\bar{\mathfrak{g}}}$ denote the space of all $\mathfrak{g}$-invariant elements in M; in other words,

$$M^{\mathfrak{g}} = \{u \in M \mid \mathfrak{g}.u = \{0\}\}$$

.

The description of all invariant affine connections on homogeneous spaces was first obtained by K. Nomizu and has the following form:

Theorem 1 ([8]). *Let $(\overline{G}, M = \overline{G}/G)$ be a homogeneous space. Suppose that the stationary subgroup G is connected. Then invariant affine connections on $(\overline{G}, M)$ are in one-to-one correspondence with linear mappings $\Lambda\colon \bar{\mathfrak{g}} \to \mathfrak{gl}(\mathfrak{m})$ such that:*

1. $\Lambda|_{\mathfrak{g}} = \lambda$;
2. Λ *is* $\mathfrak{g}$*-invariant, i.e.,*

$$\Lambda([x,y]) = [\lambda(x), \Lambda(y)] \quad \textit{for all } x \in \mathfrak{g}, y \in \bar{\mathfrak{g}}.$$

The mappings $\Lambda\colon \bar{\mathfrak{g}} \to \mathfrak{gl}(\mathfrak{m})$ which satisfy the conditions of Theorem 1 will be called *(invariant) affine connections* on the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$. The curvature and torsion tensors of the invariant affine connection Λ can be described in terms of Lie algebras as follows:

- the curvature tensor:

$$R\colon \mathfrak{m} \wedge \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m}),$$
$$(x_1 + \mathfrak{g}) \wedge (x_2 + \mathfrak{g}) \mapsto \Lambda([x_1, x_2]) - [\Lambda(x_1), \Lambda(x_2)];$$

- the torsion tensor:

$$T\colon \mathfrak{m} \wedge \mathfrak{m} \to \mathfrak{m},$$
$$(x_1 + \mathfrak{g}) \wedge (x_2 + \mathfrak{g}) \mapsto [x_1, x_2]_{\mathfrak{m}} - \Lambda(x_1)(x_2 + \mathfrak{g}) + \Lambda(x_2)(x_1 + \mathfrak{g}).$$

It is easy to check that the mappings R and T are both well-defined and $\mathfrak{g}$-invariant.

The following result is well-known and can be found in a global version, for example, in [9, 10]. The proof given here is purely algebraic.

Lemma 1. *Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be an effective pair of Lie algebras. Suppose there exists at least one invariant affine connection on $(\bar{\mathfrak{g}}, \mathfrak{g})$. Then this pair is isotropy faithful.*

Proof. Let Λ be an arbitrary affine connection on $(\bar{\mathfrak{g}}, \mathfrak{g})$, and $\mathfrak{a} = \ker \Lambda \cap \mathfrak{g} = \ker \lambda$. Let us proof that $\mathfrak{a}$ is an ideal of $\bar{\mathfrak{g}}$. For any $x \in \mathfrak{a}$ and $y \in \bar{\mathfrak{g}}$, we have

$$\Lambda([x,y]) = [\lambda(x), \Lambda(y)] = 0.$$

Hence $[x,y] \in \ker \Lambda$. On the other hand,

$$\lambda(x)(y + \mathfrak{g}) = [x,y] + \mathfrak{g} = 0.$$

Therefore $[x,y] \in \mathfrak{g}$ and $[\mathfrak{a}, \bar{\mathfrak{g}}] \subset \mathfrak{a}$. Since the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is effective, it follows that $\mathfrak{a} = \{0\}$. □

It turns out that for $(\bar{\mathfrak{g}}, \mathfrak{g})$ to have invariant connections, it is not sufficient that $(\bar{\mathfrak{g}}, \mathfrak{g})$ be isotropy faithful. The simplest example can be given for $\operatorname{codim}_{\bar{\mathfrak{g}}} \mathfrak{g} = 2$. Let $\bar{\mathfrak{g}}$ be given by the following commutation table:

	e_1	e_2	u_1	u_2
e_1	0	e_2	$2u_1$	$e_2 + u_2$
e_2	$-e_2$	0	0	u_1
u_1	$-2u_1$	0	0	0
u_2	$-e_2 - u_2$	$-u_1$	0	0

,

and let $\mathfrak{g}$ be the subalgebra spanned by e_1 and e_2. A straightforward calculation shows that this pair has no affine connections. Moreover, by examining all isotropy-faithful effective pairs of codimension 2 (a classification of such pairs can be found in [7]), we find that this is the only example of this kind for $\operatorname{codim}_{\bar{\mathfrak{g}}} \mathfrak{g} = 2$.

Let $(\bar{\mathfrak{g}}, \mathfrak{g})$ be an arbitrary effective pair of Lie algebras. Denote by $\mathcal{C}(\bar{\mathfrak{g}}, \mathfrak{g})$ the set of all affine connections on $(\bar{\mathfrak{g}}, \mathfrak{g})$.

Lemma 2. *The set $\mathcal{C}(\bar{\mathfrak{g}}, \mathfrak{g})$ forms the affine space associated with the vector space $(\mathfrak{m} \otimes \mathfrak{m} \otimes \mathfrak{m}^*)^{\mathfrak{g}}$.*

Proof. Let Λ_1, Λ_2 be two affine connections on $(\bar{\mathfrak{g}}, \mathfrak{g})$. Then $\Lambda_1 - \Lambda_2$ is zero on $\mathfrak{g}$ and, therefore, can be identified with a mapping $\Delta\colon \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$. Since the mappings Λ_1 and Λ_2 are $\mathfrak{g}$-invariant, it follows that Δ is also $\mathfrak{g}$-invariant. But the set of all $\mathfrak{g}$-invariant mappings from $\mathfrak{m}$ to $\mathfrak{gl}(\mathfrak{m})$ is canonically isomorphic to $(\mathfrak{m} \otimes \mathfrak{m} \otimes \mathfrak{m}^*)^{\mathfrak{g}}$.

On the other hand, if Λ is an affine connection on $(\bar{\mathfrak{g}}, \mathfrak{g})$ and $\Delta\colon \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ is $\mathfrak{g}$-invariant, then $\Lambda + \Delta$ is also an affine connection on $(\bar{\mathfrak{g}}, \mathfrak{g})$ (here we identify Δ with a mapping $\bar{\mathfrak{g}} \to \mathfrak{gl}(\mathfrak{m})$ which is zero on $\mathfrak{g}$). □

Definition 3. We say that two connections $\Lambda_1, \Lambda_2 \in \mathcal{C}(\bar{\mathfrak{g}}, \mathfrak{g})$ are *g-equivalent* if $\Lambda_1 - \Lambda_2 \in (\Lambda^2\mathfrak{m} \otimes \mathfrak{m}^*)^{\mathfrak{g}}$.

This definition has the following geometric meaning: two invariant affine connections on a homogeneous space are g-equivalent if and only if their geodesics coincide (see [9]).

It follows immediately from the definition that g-equivalence classes of affine connections are just parallel affine subspaces in $\mathcal{C}(\bar{\mathfrak{g}}, \mathfrak{g})$ corresponding to the subspace $(\Lambda^2\mathfrak{m} \otimes \mathfrak{m}^*)^{\mathfrak{g}} \subset (\mathfrak{m} \otimes \mathfrak{m} \otimes \mathfrak{m}^*)^{\mathfrak{g}}$.

Lemma 3. *Each g-equivalence class of affine connections on $\bar{\mathfrak{g}}$ contains a unique torsion-free connection.*

Proof. Let $\Lambda \in \mathcal{C}(\bar{\mathfrak{g}}, \mathfrak{g})$. Let $T \in (\Lambda^2\mathfrak{m} \otimes \mathfrak{m}^*)^{\mathfrak{g}}$ be the torsion tensor of Λ. Consider the affine connection $\Lambda' = \Lambda + 1/2T$. Its torsion tensor T' has the form:

$$T'\colon (x_1 + \mathfrak{g}) \wedge (x_2 + g) \mapsto [x_1, x_2]_{\mathfrak{m}} - \Lambda'(x_1)(x_2 + \mathfrak{g}) + \Lambda'(x_2)(x_1 + \mathfrak{g})$$
$$= T(x_1+\mathfrak{g})\wedge(x_2+\mathfrak{g}) - \frac{1}{2}T(x_1+\mathfrak{g})\wedge(x_2+\mathfrak{g}) + \frac{1}{2}T(x_2+\mathfrak{g})\wedge(x_1+\mathfrak{g}) = 0$$

for all $x_1, x_2 \in \bar{\mathfrak{g}}$.

The uniqueness of this torsion-free connection can be proved in the same way. □

Lemma 4. *All torsion-free affine connections on $(\bar{\mathfrak{g}}, \mathfrak{g})$ form the affine subspace in $\mathcal{C}(\bar{\mathfrak{g}}, \mathfrak{g})$ corresponding to the subspace $(S^2\mathfrak{m} \otimes \mathfrak{m}^*)^{\mathfrak{g}} \subset (\mathfrak{m} \otimes \mathfrak{m} \otimes \mathfrak{m}^*)^{\mathfrak{g}}$.*

Proof. Let Λ_1, Λ_2 be torsion-free affine connections on $(\bar{\mathfrak{g}}, \mathfrak{g})$, and let $\Delta = \Lambda_1 - \Lambda_2$. We need to prove that

$$\Delta(u_1)u_2 = \Delta(u_2)u_1 \quad \text{for all } u_1, u_2 \in \mathfrak{m}.$$

But we have

$$\Delta(u_1)u_2 - \Delta(u_2)u_1 = T_1(u_1 \wedge u_2) - T_2(u_1 \wedge u_2) = 0.$$

The rest of the proof is similar to the proof of Lemma 2. □

2. Inclusions between affine triples

By an *affine triple* $(\bar{\mathfrak{g}}, \mathfrak{g}, \Lambda)$ we mean an effective pair of Lie algebras $(\bar{\mathfrak{g}}, \mathfrak{g})$ together with an affine connection Λ. This triple determines locally a certain homogeneous space with an invariant affine connection on it.

Consider an arbitrary affine triple $(\bar{\mathfrak{g}}, \mathfrak{g}, \Lambda)$. Let V be an arbitrary complement of $\mathfrak{g}$ in $\bar{\mathfrak{g}}$. For $x \in \bar{\mathfrak{g}}$, denote by $x_{\mathfrak{g}}$ and x_V the natural projections of x to $\mathfrak{g}$ and to V, respectively. We can identify the $\mathfrak{g}$-module $\mathfrak{m}$ with V by letting $x.v = [x, v]_V$ for $x \in \mathfrak{g}$, $v \in V$. Since the isotropic representation of $\mathfrak{g}$ is faithful, we can identify $\mathfrak{g}$ with a certain subalgebra in $\mathfrak{gl}(V)$. Then $\Lambda|_{\mathfrak{g}}$ is the canonical embedding of $\mathfrak{g}$ into $\mathfrak{gl}(V)$, and Λ is uniquely determined by its restriction $\overline{\Lambda}$ to V.

Lemma 5.

1. With the above identifications, for any $x \in \mathfrak{g}$, $v \in V$ we have

$$[x, v]_V = xv, \quad [x, v]_{\mathfrak{g}} = [x, \overline{\Lambda}(v)] - \overline{\Lambda}(xv). \tag{1}$$

2. Suppose that the connection Λ has zero torsion. Then for any $v_1, v_2 \in V$ we have

$$[v_1, v_2]_V = \overline{\Lambda}(v_1)v_2 - \overline{\Lambda}(v_2)v_1.$$

Proof.

1. The equality $[x, v]_V = xv$ is evident. Then from the definition of an affine connection on $(\bar{\mathfrak{g}}, \mathfrak{g})$ it follows that

$$[x, \overline{\Lambda}(v)] = \Lambda([x, v]) = \Lambda([x, v]_{\mathfrak{g}} + [x, v]_V) = [x, v]_{\mathfrak{g}} + \Lambda(xv),$$

and we get the required formula for $[x, v]_{\mathfrak{g}}$.

2. This follows immediately from the formula for the torsion tensor:

$$0 = T(v_1, v_2) = [v_1, v_2]_V - \overline{\Lambda}(v_1)v_2 + \overline{\Lambda}(v_2)v_1.$$

□

Remark. The natural action of $\mathfrak{gl}(V)$ on $\mathcal{L}(V, \mathfrak{gl}(V))$ is given by:

$$(x.\phi)(v) = [x, \phi(v)] - \phi(xv), \quad x \in \mathfrak{gl}(V),\ v \in V,\ \phi \in \mathcal{L}(V, \mathfrak{gl}(V)).$$

Therefore, formula (1) can be rewritten as $[x, v] = (x.\overline{\Lambda})(v)$.

Let $\bar{\mathfrak{h}}$ be a subalgebra of $\bar{\mathfrak{g}}$ such that $\bar{\mathfrak{h}}+\mathfrak{g}=\bar{\mathfrak{g}}$. Define $\mathfrak{h}=\bar{\mathfrak{h}}\cap\mathfrak{g}$. It is easy to check that $(\bar{\mathfrak{h}},\mathfrak{h})$ is an effective pair of Lie algebras. Moreover, the quotient space $\bar{\mathfrak{h}}/\mathfrak{h}$ is canonically isomorphic to $\mathfrak{m}=\bar{\mathfrak{g}}/\mathfrak{g}$. Therefore we can define

$$\Theta=\Lambda|_{\bar{\mathfrak{h}}}\colon \bar{\mathfrak{h}}\to\mathfrak{gl}(\bar{\mathfrak{h}}/\mathfrak{h}).$$

It is clear that Θ is an invariant affine connection on $(\bar{\mathfrak{h}},\mathfrak{h})$.

Definition 4. The affine triple $(\bar{\mathfrak{h}},\mathfrak{h},\Theta)$ is called a *restriction* of $(\bar{\mathfrak{g}},\mathfrak{g},\Lambda)$, and the triple $(\bar{\mathfrak{g}},\mathfrak{g},\Lambda)$ is called an *extension* of $(\bar{\mathfrak{h}},\mathfrak{h},\Theta)$.

Our nearest aim is to show that for every affine triple $(\bar{\mathfrak{h}},\mathfrak{h},\Theta)$ there exists the greatest extension and to give an algorithm for how this affine triple can be constructed.

Consider an arbitrary extension $(\bar{\mathfrak{g}},\mathfrak{g},\Lambda)$ of an affine triple $(\bar{\mathfrak{h}},\mathfrak{h},\Theta)$. As above, let V be a complement of $\mathfrak{h}$ in $\bar{\mathfrak{h}}$. Then V is at the same time a complement of $\mathfrak{g}$ in $\bar{\mathfrak{g}}$. Hence both $\mathfrak{g}$ and $\mathfrak{h}$ can be identified with subalgebras of $\mathfrak{gl}(V)$, and $\mathfrak{h}\subset\mathfrak{g}$. Moreover, $\overline{\Theta}=\Theta|_V=\overline{\Lambda}$. But then Lemma 5 implies that the subalgebra $\mathfrak{g}$ uniquely determines the whole triple $(\bar{\mathfrak{g}},\mathfrak{g},\Lambda)$. Indeed, the Lie algebra $\bar{\mathfrak{g}}$, viewed as a vector space, can be identified with $\mathfrak{g}\oplus V$, the connection Λ is identical on $\mathfrak{g}$ and is equal to $\overline{\Theta}$ on V, the multiplication for the elements of V in $\bar{\mathfrak{g}}$ is derived from that in $\bar{\mathfrak{h}}$, and finally, the multiplication between the elements of $\mathfrak{g}$ and V is determined by (1).

Lemma 6. *Let $(\bar{\mathfrak{h}},\mathfrak{h},\Theta)$ be an affine triple, V a subspace in $\bar{\mathfrak{h}}$ complementary to $\mathfrak{h}$, and let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}(V)$ containing $\mathfrak{h}$. Put $\bar{\mathfrak{g}}=\mathfrak{g}\times V$ and define the multiplication*

$$\bar{\mathfrak{g}}\wedge\bar{\mathfrak{g}}\to\mathfrak{gl}(V)\times V,\ \ x\wedge y\mapsto[x,y]$$

in the following way:

$$\begin{aligned}
[(x_1,0),(x_2,0)] &= ([x_1,x_2],0) && \text{for all } x_1,x_2\in\mathfrak{g};\\
[(x,0),(0,v)] &= ((x.\overline{\Theta})(v),xv) && \text{for all } x\in\mathfrak{g}, v\in V;\\
[(0,v_1),(0,v_2)] &= ([v_1,v_2]_{\mathfrak{h}},[v_1,v_2]_V) && \text{for all } v_1,v_2\in V.
\end{aligned}$$

Then $\bar{\mathfrak{g}}$ is closed with respect to this multiplication and is a Lie algebra if and only if the following two conditions are satisfied:

(A) $(x.\overline{\Theta})(V)\subset\mathfrak{g}$ *for all* $x\in\mathfrak{g}$;

(B) *the Jacobi identity on elements* $(x,0)$, $(0,v_1)$, *and* $(0,v_2)$ *is satisfied for all* $x\in\mathfrak{g}$, $v_1,v_2\in V$.

Proof. It is clear that $\bar{\mathfrak{g}}$ is closed with respect to the multiplication defined above if and only if the condition (A) is satisfied.

Then $\bar{\mathfrak{g}}$ is a Lie algebra if and only if the Jacobi identity is satisfied. It is sufficient to check it either for the elements of $\mathfrak{g}$ or those of V.

Since $\mathfrak{g}$ is a Lie algebra, it follows that the Jacobi identity is satisfied for any three elements of $\mathfrak{g}$. The multiplication obtained extends the multiplication in $\bar{\mathfrak{h}}$. Therefore the Jacobi identity holds for any triple from $V \subset \bar{\mathfrak{h}}$.

Simple calculations show that the Jacobi identity is satisfied when two elements lie in $\mathfrak{g}$ and one in V. Hence $\bar{\mathfrak{g}}$ is a Lie algebra if and only if the condition (B) is true. □

Theorem 2. *There exists the greatest extension of* $(\bar{\mathfrak{h}}, \mathfrak{h}, \Theta)$.

Proof. Assuming in Lemma 6 that $\mathfrak{g} = \mathfrak{gl}(V)$, we can define the multiplication on $\mathfrak{gl}(V) \times V$. Then denote by $\mathfrak{a}$ the set of all $x \in \mathfrak{gl}(V)$ for which the condition (B) of Lemma 6 is satisfied (for all $v_1, v_2 \in V$). It is easy to check that $\mathfrak{a}$ is a subalgebra in $\mathfrak{gl}(V)$. Obviously, for any extension $(\bar{\mathfrak{g}}, \mathfrak{g}, \Lambda)$ we have $\mathfrak{g} \subset \mathfrak{a}$.

Let us show that there exists the greatest subalgebra $\mathfrak{g}$ of $\mathfrak{a}$ such that the condition (A) of Lemma 6 is fulfilled. Let W_1, W_2 be two subspaces of $\mathfrak{a}$ satisfying (A). Then their sum also satisfies this condition:

$$((x_1 + x_2).\overline{\Theta})(V) \subset (x_1.\overline{\Theta}(V)) + (x_2.\overline{\Theta})(V) \subset W_1 + W_2$$

for all $x_1 \in W_1$, $x_2 \in W_2$.

Moreover, if a certain subspace $W \subset \mathfrak{a}$ satisfies (A), then $W + [W, W] \subset \mathfrak{a}$ also satisfies this condition:

$$\begin{aligned}([x, y].\overline{\Theta})(V) = (x.y.\overline{\Theta} - y.x.\overline{\Theta})(V) &\subset [x, (y.\overline{\Theta})(V)] - [y, (x.\overline{\Theta})(V)] \\ &\quad - (x.\overline{\Theta})(V) + (y.\overline{\Theta})(V) \subset W + [W, W]\end{aligned}$$

for all $x, y \in W$. Therefore, the subalgebra generated by W satisfies (A).

But this immediately implies the existence of the greatest subalgebra $\mathfrak{g}$. For example, we can take as $\mathfrak{g}$ the subalgebra in $\mathfrak{a}$ generated by the sum of all subalgebras in $\mathfrak{a}$ satisfying (A). □

3. Maximal affine triples

The aim of this section is to describe an algorithm that allows to find the greatest extension $(\bar{\mathfrak{g}}, \mathfrak{g}, \Lambda)$ for a given affine triple $(\bar{\mathfrak{h}}, \mathfrak{h}, \Theta)$ (that is to construct the greatest subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ satisfying the conditions of Lemma 6).

Let $\mathfrak{a} \subset \mathfrak{gl}(V)$ be as in the proof of Theorem 2. Then $\mathfrak{g}$ is the greatest subalgebra of $\mathfrak{a}$ satisfying the condition (A) of Lemma 6.

Theorem 3. *Let*

$$\mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \cdots \supset \mathfrak{a}_n \supset \mathfrak{a}_{n+1} \supset \ldots$$

be the decreasing sequence of subalgebras in $\mathfrak{a}$ *defined by:*

$$\mathfrak{a}_0 = \mathfrak{a}, \quad \mathfrak{a}_{n+1} = \{x \in \mathfrak{a}_n \mid (x.\overline{\Theta})(V) \subset \mathfrak{a}_n\}. \tag{2}$$

Then

$$\mathfrak{g} = \mathfrak{a}_\infty = \cap_{n=1}^{\infty} \mathfrak{a}_n.$$

Proof. It is easy to check that sequence (2) is well-defined (i.e., that all sets $\mathfrak{a}_n$ are indeed subalgebras). Since for all $x \in \mathfrak{a}_\infty$

$$(x.\overline{\Theta})(V) \subset \mathfrak{a}_n \quad \text{for all } n \in \mathbb{N},$$

it follows that $(x.\overline{\Theta})(V) \subset \mathfrak{a}_\infty$, and $\mathfrak{a}_\infty$ satisfies the condition (A). Hence $\mathfrak{g} \supset \mathfrak{a}_\infty$.

Let us proof by induction that $\mathfrak{g} \subset \mathfrak{a}_n$ for all $n \in \mathbb{N} \cup \{0\}$. It is obvious for $n = 0$. Suppose that $\mathfrak{g} \subset \mathfrak{a}_n$. Then for all $x \in \mathfrak{g}$ we have

$$(x.\overline{\Theta})(V) \subset \mathfrak{g} \subset \mathfrak{a}_n.$$

Hence $\mathfrak{g} \subset \mathfrak{a}_{n+1}$.

This implies that $\mathfrak{g} \subset \mathfrak{a}_\infty$ and therefore $\mathfrak{g} = \mathfrak{a}_\infty$. □

Theorem 3 gives us a straightforward algorithm for finding the greatest extension of $(\bar{\mathfrak{h}}, \mathfrak{h}, \Theta)$:

Step 1: As in Lemma 6, construct the multiplication on $\mathfrak{gl}(V) \times V$.
Step 2: Find the subalgebra $\mathfrak{a} \subset \mathfrak{gl}(V)$ that consists of all $x \in \mathfrak{gl}(V)$ satisfying the condition (B).
Step 3: Construct sequence (2) and find the least $n \in \mathbb{N}$ such that $\mathfrak{a}_n = \mathfrak{a}_{n+1}$.
Step 4: Put $\mathfrak{g} = \mathfrak{a}_\infty$ and construct the whole triple $(\bar{\mathfrak{g}}, \mathfrak{g}, \Lambda)$.

Definition 5. We say that an affine triple $(\bar{\mathfrak{g}}, \mathfrak{g}, \Lambda)$ is *maximal* if it has no extensions except itself.

It is clear that $(\bar{\mathfrak{g}}, \mathfrak{g}, \Lambda)$ is maximal if and only if its greatest extension coincides with it. In particular, the algorithm described above gives a constructive way to check if the triple $(\bar{\mathfrak{g}}, \mathfrak{g}, \Lambda)$ is maximal.

4. Maximal affine triples of codimension 3

4.1. **Preliminaries.** Below we use the following notation:

- the sign $\leftthreetimes$ denotes a semidirect product of Lie algebras;
- by $\mathfrak{r}_2$ we denote the two-dimensional subalgebra with basis $\{p, q\}$ and $[p, q] = p$;
- by $\mathfrak{co}(p, q)$, $p \geq q$, we denote the subalgebra in $\mathfrak{gl}(p + q, \mathbb{R})$ given by:

$$\mathfrak{co}(p, q) = \{A + \lambda E_{p+q} \mid A \in \mathfrak{so}(p, q),\ \lambda \in \mathbb{R}\}.$$

The subalgebra $\mathfrak{co}(p, 0)$ is denoted by $\mathfrak{co}(p)$.

In the following sections we give the list of all maximal triples $(\bar{\mathfrak{g}}, \mathfrak{g}, \Lambda)$, where Λ is torsion-free, $\operatorname{codim}_{\bar{\mathfrak{g}}} \mathfrak{g} = 3$, and $\dim g \geq 2$.

In subsection 2 we list all maximal triples $(\bar{\mathfrak{g}}, \mathfrak{g}, \Lambda)$ which satisfy the following conditions:

1. the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is symmetric;
2. Λ is the canonical connection on $(\bar{\mathfrak{g}}, \mathfrak{g})$ (see [9]);
3. Λ is the only torsion-free affine connection on $(\bar{\mathfrak{g}}, \mathfrak{g})$.

In subsections 3–5 we list all other triples $(\bar{\mathfrak{g}}, \mathfrak{g}, \Lambda)$. We give the commutation table for $\bar{\mathfrak{g}}$ relative to a basis $\{e_1, \ldots, e_k, u_1, u_2, u_3\}$ such that $\{e_1, \ldots, e_k\}$ is a basis for $\mathfrak{g}$ ($\dim \mathfrak{g} = k \geq 2$) and $\{u_1, u_2, u_3\}$ is a basis for V. Then we describe the connection Λ by the matrix of $\overline{\Lambda}$ relative to the bases $\{u_1, u_2, u_3\}$ and $E_{11}, E_{12}, E_{13}, E_{21}, \ldots, E_{33}$ of V and $\mathfrak{gl}(V)$ respectively. (Here by E_{ij}, $1 \leq i, j \leq 3$, we denote the 3×3 matrix with 1 in the ij-place and 0 elsewhere.)

The numeration of pairs listed in subsection 3–5 is derived from the list of all isotropy faithful pairs of codimension 2 in [11].

4.2. Canonical connections on reductive pairs.

$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{r}_2, \qquad \mathfrak{g} = \mathfrak{so}(1,1) \times \langle p \rangle;$$
$$\bar{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{r}_2, \qquad \mathfrak{g} = \mathfrak{so}(2) \times \langle p \rangle;$$
$$\bar{\mathfrak{g}} = \mathfrak{su}(2) \times \mathfrak{r}_2, \qquad \mathfrak{g} = \mathfrak{so}(2) \times \langle p \rangle;$$
$$\bar{\mathfrak{g}} = \mathfrak{so}(2,2), \qquad \mathfrak{g} = \mathfrak{so}(2,1);$$
$$\bar{\mathfrak{g}} = \mathfrak{so}(3,1), \qquad \mathfrak{g} = \mathfrak{so}(2,1);$$
$$\bar{\mathfrak{g}} = \mathfrak{so}(3,1), \qquad \mathfrak{g} = \mathfrak{so}(3);$$
$$\bar{\mathfrak{g}} = \mathfrak{so}(4), \qquad \mathfrak{g} = \mathfrak{so}(3);$$
$$\bar{\mathfrak{g}} = \mathfrak{gl}(2) \leftthreetimes \mathbb{R}^2, \qquad \mathfrak{g} = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} + \begin{pmatrix} z \\ 0 \end{pmatrix} \,\middle|\, x, y, z \in \mathbb{R} \right\};$$
$$\bar{\mathfrak{g}} = \mathfrak{co}(2,1) \leftthreetimes \mathbb{R}^3, \qquad \mathfrak{g} = \mathfrak{co}(1,1) \leftthreetimes \mathbb{R}^2;$$
$$\bar{\mathfrak{g}} = \mathfrak{co}(2,1) \leftthreetimes \mathbb{R}^3, \qquad \mathfrak{g} = \mathfrak{co}(2) \leftthreetimes \mathbb{R}^2;$$
$$\bar{\mathfrak{g}} = \mathfrak{co}(3) \leftthreetimes \mathbb{R}^3, \qquad \mathfrak{g} = \mathfrak{co}(2) \leftthreetimes \mathbb{R}^2;$$
$$\bar{\mathfrak{g}} = \left\{ \begin{pmatrix} 2x & t & z & 0 \\ 0 & x+y & t & z \\ 0 & 0 & 2y & t \\ 0 & 0 & 0 & 3y-x \end{pmatrix} \right\} \leftthreetimes \mathbb{R}^4, \quad \mathfrak{g} = \left\{ \begin{pmatrix} 2x & 0 & z & 0 \\ 0 & x+y & 0 & z \\ 0 & 0 & 2y & 0 \\ 0 & 0 & 0 & 3y-x \end{pmatrix} + \begin{pmatrix} u \\ 0 \\ v \\ 0 \end{pmatrix} \right\};$$
$$\bar{\mathfrak{g}} = \mathfrak{gl}(3,\mathbb{R}) \leftthreetimes \mathbb{R}^3, \qquad \mathfrak{g} = \mathfrak{gl}(3,\mathbb{R}).$$

4.3. General case, $\dim \mathfrak{g} = 2$.

2.8.7. ($\lambda = 0$)

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	e_1	0	u_1
e_2	0	0	0	u_2	0
u_1	$-e_1$	0	0	0	u_3
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	0	$-u_3$	0	0

The connection with zero torsion has the form:

$$\begin{pmatrix} -1/2 & 0 & \frac{\eta_4}{2} & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & \eta_5 & 0 & 0 & 0 \\ \frac{\eta_4}{2} & 0 & \eta_2 & 0 & \eta_5 & 0 & -1/2 & 0 & \eta_4 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_4 = 2\,\eta_5$, $\eta_2 = 0$.

2.8.7. ($\lambda = -1$)

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	$-e_1$	e_1	0	u_1
e_2	e_1	0	0	u_2	$-u_3$
u_1	$-e_1$	0	0	0	u_3
u_2	0	$-u_2$	0	0	0
u_3	$-u_1$	u_3	$-u_3$	0	0

The connection with zero torsion has the form:

$$\begin{pmatrix} -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & \eta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta_2 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_2 = 0$.

2.9.1. ($\lambda = \mu = 0$)

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	0	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	0
u_2	0	0	0	0	0
u_3	0	$-u_1$	0	0	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_6 & \eta_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_6 & 0 & 0 & 0 & \eta_1 & \eta_3 & 0 & 0 & \eta_6 \\ \eta_8 & 0 & 0 & 0 & \eta_3 & \eta_4 & 0 & \eta_6 & 2\,\eta_8 \end{pmatrix}$$

The riple is not maximal if and only if one of the following conditions is satisfied:

1. $\eta_4\,(\eta_1 - 2\,\eta_6) - (\eta_8 - \eta_3)^2 = 0$.
2. $\eta_6 = 0$.

2.9.1. ($\lambda = 1/2$, $\mu = 0$)

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	$\frac{u_2}{2}$	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	0
u_2	$-\frac{u_2}{2}$	0	0	0	0
u_3	0	$-u_1$	0	0	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \frac{\eta_5}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta_1 & 0 & 0 & 0 & \eta_4 & 0 & 0 & 0 \\ \frac{\eta_5}{2} & 0 & 0 & 0 & \eta_4 & 0 & 0 & 0 & \eta_5 \end{pmatrix}$$

The riple is not maximal if and only if one of the following conditions is satisfied:

1. $\eta_1 = 0$.
2. $\eta_5 = 2\,\eta_4$.

2.9.2. ($\mu = -1/2$)

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	$\frac{3\,e_2}{2}$	u_1	$\frac{u_2}{2}$	$-\frac{u_3}{2}$
e_2	$-\frac{3\,e_2}{2}$	0	0	0	u_1
u_1	$-u_1$	0	0	0	u_2
u_2	$-\frac{u_2}{2}$	0	0	0	0
u_3	$\frac{u_3}{2}$	$-u_1$	$-u_2$	0	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & \eta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_1 = 0$.

2.9.4. ($\mu = 0$)

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	0	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	u_1	0
u_2	0	0	$-u_1$	0	$-u_3$
u_3	0	$-u_1$	0	u_3	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_3 - 1 & \eta_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & \eta_6 & \eta_7 & 0 & 0 & \eta_3 \\ \eta_8 & 0 & 0 & 0 & \eta_7 & \eta_5 & 0 & \eta_3 - 1 & 2\,\eta_8 \end{pmatrix}$$

The riple is not maximal if and only if one of the following conditions is satisfied:

1. $\eta_8 = \eta_7$, $\eta_3 = 1 + \eta_6/2$.
2. $\eta_8 = \eta_7$, $\eta_5 = 0$.

2.9.5. ($\lambda = 0$)

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	0	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	e_2
u_2	0	0	0	0	$\alpha\, u_2$
u_3	0	$-u_1$	$-e_2$	$-\alpha\, u_2$	0

where $\alpha \geq 0$.

$$\begin{pmatrix} 0 & \eta_8 & \frac{\eta_6}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_8 & 0 & 0 & 0 & \eta_2 & \eta_4 - \alpha & 0 & 0 & \eta_8 \\ \frac{\eta_6}{2} & 0 & 0 & 0 & \eta_4 & \eta_5 & 0 & \eta_8 & \eta_6 \end{pmatrix}$$

The riple is not maximal if and only if one of the following conditions is satisfied:

1. $\alpha > 0$, $\eta_8 = \eta_2 = 0$.
2. $\alpha = 0$, $\eta_8 = 0$.
3. $\alpha = 0$, $4\,(\eta_2 - 2\eta_8)\,\eta_5 = (\eta_6 - 2\,\eta_4)^2 - 4$.

2.9.5. ($\lambda = 1/2$)

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	$\frac{u_2}{2}$	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	e_2
u_2	$-\frac{u_2}{2}$	0	0	0	$\alpha\, u_2$
u_3	0	$-u_1$	$-e_2$	$-\alpha\, u_2$	0

where $\alpha \geq 0$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \frac{\eta_5}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta_1 & 0 & 0 & 0 & \eta_4 & 0 & 0 & 0 \\ \frac{\eta_5}{2} & 0 & 0 & 0 & \eta_4 + \alpha & 0 & 0 & 0 & \eta_5 \end{pmatrix}$$

The riple is not maximal if and only if one of the following conditions is satisfied:

1. $\alpha > 0$, $\eta_1 = 0$.
2. $\alpha = 0$, $(\eta_5 - 2\,\eta_4)^2 = 4$.
3. $\alpha = 0$, $\eta_1 = 0$.

2.9.6. ($\lambda = 0$)

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	0	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	$-e_2$
u_2	0	0	0	0	$\alpha\, u_2$
u_3	0	$-u_1$	e_2	$-\alpha\, u_2$	0

where $\alpha \geq 0$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_7 & \frac{\eta_8}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_7 & 0 & 0 & 0 & \eta_2 & \eta_4 & 0 & 0 & \eta_7 \\ \frac{\eta_8}{2} & 0 & 0 & 0 & \eta_4 + \alpha & \eta_5 & 0 & \eta_7 & \eta_8 \end{pmatrix}$$

The riple is not maximal if and only if one of the following conditions is satisfied:

1. $\alpha > 0$, $\eta_7 = \eta_2 = 0$.
1. $\alpha = 0$, $\eta_5 = -\frac{4+(\eta_8 - 2\eta_4)^2}{4(2\,\eta_7 - \eta_2)}$, $\eta_2 \neq 2\,\eta_7$.
2. $\alpha = 0$, $\eta_7 = 0$.

2.9.6. ($\lambda = 1/2$)

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	$\frac{u_2}{2}$	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	$-e_2$
u_2	$-\frac{u_2}{2}$	0	0	0	$\alpha\, u_2$
u_3	0	$-u_1$	e_2	$-\alpha\, u_2$	0

where $\alpha \geq 0$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \frac{\eta_4}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta_1 & 0 & 0 & 0 & \eta_5 & 0 & 0 & 0 \\ \frac{\eta_4}{2} & 0 & 0 & 0 & \eta_5+\alpha & 0 & 0 & 0 & \eta_4 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_1 = 0$.

2.9.7. ($\lambda = 0$)

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	0	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	0
u_2	0	0	0	0	u_2
u_3	0	$-u_1$	0	$-u_2$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_7 & \frac{\eta_8}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_7 & 0 & 0 & 0 & \eta_1 & \eta_3 & 0 & 0 & \eta_7 \\ \frac{\eta_8}{2} & 0 & 0 & 0 & \eta_3+1 & \eta_4 & 0 & \eta_7 & \eta_8 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_7 = \eta_1 = 0$.

2.9.7. ($\lambda = 1/2$)

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	e_2	u_1	$\frac{u_2}{2}$	0
e_2	$-e_2$	0	0	0	u_1
u_1	$-u_1$	0	0	0	0
u_2	$-\frac{u_2}{2}$	0	0	0	u_2
u_3	0	$-u_1$	0	$-u_2$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta_5 & 0 & 0 & 0 & \eta_3-1 & 0 & 0 & 0 \\ \eta_4 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 & 2\,\eta_4 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_5 = 0$.

2.13.2.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	u_2
e_2	0	0	0	0	u_1
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	0	e_1
u_3	$-u_2$	$-u_1$	0	$-e_1$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \frac{\eta_5}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\eta_4}{2} & 0 & 0 & \frac{\eta_5}{2} & 0 & 0 & 0 \\ \frac{\eta_5}{2} & \frac{\eta_4}{2} & \eta_2 & 0 & \frac{\eta_5}{2} & \eta_4 & 0 & 0 & \eta_5 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_4 = 0$.

2.13.3.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	u_2
e_2	0	0	0	0	u_1
u_1	0	0	0	0	0
u_2	$-u_1$	0	0	0	$-e_1$
u_3	$-u_2$	$-u_1$	0	e_1	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_5 & 0 & 0 & \eta_2 & 0 & 0 & 0 \\ \eta_2 & \eta_5 & \eta_4 & 0 & \eta_2 & 2\,\eta_5 & 0 & 0 & 2\,\eta_2 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_5 = 0$.

2.13.6.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	u_2
e_2	0	0	0	0	u_1
u_1	0	0	0	0	u_1
u_2	$-u_1$	0	0	0	$\alpha\, e_1 + u_2$
u_3	$-u_2$	$-u_1$	$-u_1$	$-\alpha\, e_1 - u_2$	0

where $\alpha \neq 0$.

$$\begin{pmatrix} 0 & 0 & \frac{\eta_5}{2} - 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\eta_4}{2} & 0 & 0 & \frac{\eta_5}{2} - 1/2 & 0 & 0 & 0 \\ \frac{\eta_5}{2} + 1/2 & \frac{\eta_4}{2} & \eta_2 & 0 & \frac{\eta_5}{2} + 1/2 & \eta_4 & 0 & 0 & \eta_5 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_4 = 0$.

2.13.7.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	u_2
e_2	0	0	0	0	$e_2 + u_1$
u_1	0	0	0	0	$\alpha\, u_1$
u_2	$-u_1$	0	0	0	$(1-\alpha)\, e_1 + e_2 + \alpha\, u_2$
u_3	$-u_2$	$-e_2 - u_1$	$-\alpha\, u_1$	*	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \frac{\eta_5}{2} - \frac{\alpha}{2} - 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & \frac{\eta_4}{2} & 0 & 0 & \frac{\eta_5}{2} - \frac{\alpha}{2} & 0 & 0 & 0 \\ \frac{\eta_5}{2} + \frac{\alpha}{2} - 1/2 & \frac{\eta_4}{2} & \eta_2 & 0 & \frac{\eta_5}{2} + \frac{\alpha}{2} & \eta_4 & 0 & 0 & \eta_5 \end{pmatrix}$$

The triple is not maximal if and only if $\alpha \neq 3/2$.

2.13.8.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	u_2
e_2	0	0	0	0	e_2+u_1
u_1	0	0	0	0	$\alpha\, u_1$
u_2	$-u_1$	0	0	0	$\beta\, e_1+\alpha\, u_2$
u_3	$-u_2$	$-e_2-u_1$	$-\alpha\, u_1$	$-\beta\, e_1-\alpha\, u_2$	0

where $\alpha,\ \beta\in\mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_2-1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & \eta_4 & 0 & 0 & \eta_2 & 0 & 0 & 0 \\ \eta_2+\alpha-1/2 & \eta_4 & \eta_5 & 0 & \eta_2+\alpha & 2\,\eta_4 & 0 & 0 & 2\,\eta_2+\alpha \end{pmatrix}$$

The riple is not maximal if and only if one of the following conditions is satisfied:

1. $\alpha=\frac{1-4\,\beta}{2}$.
2. $\alpha=0,\ \eta_4=0$.

2.20.3.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	e_1+u_1	0
e_2	0	0	0	e_2	u_1
u_1	0	0	0	$2\,u_1$	0
u_2	$-e_1-u_1$	$-e_2$	$-2\,u_1$	0	e_2-u_3
u_3	0	$-u_1$	0	$-e_2+u_3$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_7-1 & \frac{\eta_8}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_7-1 & \eta_1 & \eta_3 & 0 & 2\,\eta_7-1 & \frac{\eta_8}{2} & 0 & 0 & \eta_7 \\ \frac{\eta_8}{2} & \eta_3 & \eta_4 & 0 & \frac{\eta_8}{2} & 0 & 0 & \eta_7-1 & \eta_8 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_8=0$.

2.20.4.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	e_1+u_1	0
e_2	0	0	0	e_2	u_1
u_1	0	0	0	$2\,u_1$	0
u_2	$-e_1-u_1$	$-e_2$	$-2\,u_1$	0	e_1-u_3
u_3	0	$-u_1$	0	$-e_1+u_3$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_5 & \eta_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_5 & \eta_2 & \eta_8 & 0 & 2\,\eta_5+1 & \eta_6 & 0 & 0 & \eta_5+1 \\ \eta_6 & \eta_8 & \eta_7 & 0 & \eta_6 & 0 & 0 & \eta_5 & 2\,\eta_6 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_6=0$.

2.20.5.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	e_1+u_1	0
e_2	0	0	0	e_2	u_1
u_1	0	0	0	u_1	0
u_2	$-e_1-u_1$	$-e_2$	$-u_1$	0	0
u_3	0	$-u_1$	0	0	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_7 & \frac{\eta_8}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_7-1 & \eta_1 & \eta_3 & 0 & 2\,\eta_7 & \frac{\eta_8}{2} & 0 & 0 & \eta_7 \\ \frac{\eta_8}{2} & \eta_3 & \eta_4 & 0 & \frac{\eta_8}{2} & 0 & 0 & \eta_7 & \eta_8 \end{pmatrix}$$

The riple is not maximal if and only if one of the following conditions is satisfied:

1. $\eta_8 = 0$.
2. $\eta_4 = \eta_3 = 0$.

2.20.6.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	0
e_2	0	0	0	e_1	u_1
u_1	0	0	0	0	0
u_2	$-u_1$	$-e_1$	0	0	e_2
u_3	0	$-u_1$	0	$-e_2$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_7 & \frac{\eta_8}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_7 & \eta_2 & \eta_4 & 0 & 2\,\eta_7 & \frac{\eta_8}{2} & 0 & 1 & \eta_7 \\ \frac{\eta_8}{2} & \eta_4 & \eta_5 & 0 & \frac{\eta_8}{2} & 0 & 0 & \eta_7 & \eta_8 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_8 = \eta_7 = 0$.

2.20.7.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	0
e_2	0	0	0	e_1	u_1
u_1	0	0	0	0	0
u_2	$-u_1$	$-e_1$	0	0	0
u_3	0	$-u_1$	0	0	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_7 & \frac{\eta_8}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_7 & \eta_1 & \eta_3 & 0 & 2\,\eta_7 & \frac{\eta_8}{2} & 0 & 1 & \eta_7 \\ \frac{\eta_8}{2} & \eta_3 & \eta_4 & 0 & \frac{\eta_8}{2} & 0 & 0 & \eta_7 & \eta_8 \end{pmatrix}$$

The riple is not maximal if and only if one of the following conditions is satisfied:

1. $\eta_8 = 0$.
2. $\eta_4 = 0$.

2.20.8.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	e_1+u_1	0
e_2	0	0	0	e_1+e_2	u_1
u_1	0	0	0	$2\,u_1$	0
u_2	$-e_1-u_1$	$-e_1-e_2$	$-2\,u_1$	0	e_2-u_3
u_3	0	$-u_1$	0	$-e_2+u_3$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_7 & \eta_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_7 & \eta_3 & \eta_4 & 0 & 2\,\eta_7+1 & \eta_8 & 0 & 1 & \eta_7+1 \\ \eta_8 & \eta_4 & \eta_1 & 0 & \eta_8 & 0 & 0 & \eta_7 & 2\,\eta_8 \end{pmatrix}$$

2.20.9.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$\alpha\,e_1$
e_2	0	0	0	0	$(\alpha+1)\,e_2+u_1$
u_1	0	0	0	0	$2\,\alpha\,u_1$
u_2	$-u_1$	0	0	0	$e_1+\alpha\,u_2$
u_3	$-\alpha\,e_1$	$(-\alpha-1)\,e_2-u_1$	$-2\,\alpha\,u_1$	$-e_1-\alpha\,u_2$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_4 & \eta_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_4 & \eta_7 & \eta_8 & 0 & 2\,\eta_4 & \eta_6 & 0 & 0 & \eta_4 \\ \eta_6-2 & \eta_8 & \eta_1 & 0 & \eta_6+\alpha-2 & 0 & 0 & \eta_4 & 2\,\eta_6+\alpha-1 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_8=\eta_4=0$.

2.20.10.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$\alpha\,e_1$
e_2	0	0	0	0	$(\alpha+1)\,e_2+u_1$
u_1	0	0	0	0	$(2\,\alpha+1)\,u_1$
u_2	$-u_1$	0	0	0	$e_2+(\alpha+1)\,u_2$
u_3	$-\alpha\,e_1$	$(-\alpha-1)\,e_2-u_1$	$*$	$*$	0

where $\alpha\in\mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_2 & \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_2 & \eta_7 & \eta_8 & 0 & 2\,\eta_2 & \eta_4 & 0 & 0 & \eta_2 \\ \eta_4-1 & \eta_8 & \eta_6 & 0 & \eta_4+\alpha-1 & 0 & 0 & \eta_2 & 2\,\eta_4+\alpha \end{pmatrix}$$

The triple is not maximal if and only if $\eta_7=\eta_2=0$.

2.20.11.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$-e_1$
e_2	0	0	0	e_1	u_1
u_1	0	0	0	$-e_1$	$-u_1$
u_2	$-u_1$	$-e_1$	e_1	0	$-e_2$
u_3	e_1	$-u_1$	u_1	e_2	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_3 & \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_3 & \eta_7 & \eta_8 & 0 & 2\,\eta_3 & \eta_4 & 0 & 1 & \eta_3 \\ \eta_4 - 1 & \eta_8 & \eta_5 & 0 & \eta_4 - 2 & 0 & 0 & \eta_3 & 2\,\eta_4 - 1 \end{pmatrix}$$

2.20.12.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$-2\,e_1$
e_2	0	0	0	e_1	$-e_2 + u_1$
u_1	0	0	0	0	$-3\,u_1$
u_2	$-u_1$	$-e_1$	0	0	$e_2 - u_2$
u_3	$2\,e_1$	$e_2 - u_1$	$3\,u_1$	$-e_2 + u_2$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_3 & \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_3 & \eta_8 & \eta_7 & 0 & 2\,\eta_3 & \eta_4 & 0 & 1 & \eta_3 \\ \eta_4 - 1 & \eta_7 & \eta_5 & 0 & \eta_4 - 3 & 0 & 0 & \eta_3 & 2\,\eta_4 - 2 \end{pmatrix}$$

2.20.13.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$-e_1$
e_2	0	0	0	$-e_1$	u_1
u_1	0	0	0	e_1	$-u_1$
u_2	$-u_1$	e_1	$-e_1$	0	e_2
u_3	e_1	$-u_1$	u_1	$-e_2$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_3 & \frac{\eta_4}{2} + 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_3 & \eta_5 & \eta_7 & 0 & 2\,\eta_3 & \frac{\eta_4}{2} + 1/2 & 0 & -1 & \eta_3 \\ \frac{\eta_4}{2} - 1/2 & \eta_7 & \eta_8 & 0 & \frac{\eta_4}{2} - 3/2 & 0 & 0 & \eta_3 & \eta_4 \end{pmatrix}$$

2.20.14.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$-2\,e_1$
e_2	0	0	0	$-e_1$	$-e_2 + u_1$
u_1	0	0	0	0	$-3\,u_1$
u_2	$-u_1$	e_1	0	0	$e_2 - u_2$
u_3	$2\,e_1$	$e_2 - u_1$	$3\,u_1$	$-e_2 + u_2$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_3 & \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_3 & \eta_8 & \eta_7 & 0 & 2\,\eta_3 & \eta_4 & 0 & -1 & \eta_3 \\ \eta_4 - 1 & \eta_7 & \eta_6 & 0 & \eta_4 - 3 & 0 & 0 & \eta_3 & 2\,\eta_4 - 2 \end{pmatrix}$$

2.20.15.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	e_1+u_1	e_2
e_2	0	0	0	e_2	u_1
u_1	0	0	0	u_1	0
u_2	$-e_1-u_1$	$-e_2$	$-u_1$	0	0
u_3	$-e_2$	$-u_1$	0	0	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_7 & \frac{\eta_8}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_7-1 & \eta_1 & \eta_3 & 0 & 2\,\eta_7 & \frac{\eta_8}{2} & 0 & 0 & \eta_7 \\ \frac{\eta_8}{2} & \eta_3 & \eta_4 & 0 & \frac{\eta_8}{2} & 1 & 0 & \eta_7 & \eta_8 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_3=0$, $\eta_4=\eta_1$.

2.20.16.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	$-e_1+u_1$	e_2
e_2	0	0	0	$-e_2$	u_1
u_1	0	0	0	$-u_1$	0
u_2	e_1-u_1	e_2	u_1	0	0
u_3	$-e_2$	$-u_1$	0	0	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_3 & \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_3+1 & \eta_8 & \eta_7 & 0 & 2\,\eta_3 & \eta_4 & 0 & 0 & \eta_3 \\ \eta_4 & \eta_7 & \eta_5 & 0 & \eta_4 & 1 & 0 & \eta_3 & 2\,\eta_4 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_7=0$, $\eta_5=-\eta_8$.

2.20.17.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	$\alpha\,e_1+u_1$	e_1+e_2
e_2	0	0	0	$\alpha\,e_2$	e_2+u_1
u_1	0	0	0	$\alpha\,u_1$	u_1
u_2	$-\alpha\,e_1-u_1$	$-\alpha\,e_2$	$-\alpha\,u_1$	0	0
u_3	$-e_1-e_2$	$-e_2-u_1$	$-u_1$	0	0

where $\alpha\in\mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_7 & \eta_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_7-\alpha & \eta_4 & \eta_3 & 0 & 2\,\eta_7 & \eta_8 & 0 & 0 & \eta_7 \\ \eta_8-1 & \eta_3 & \eta_2 & 0 & \eta_8 & 1 & 0 & \eta_7 & 2\,\eta_8 \end{pmatrix}$$

The riple is not maximal if and only if one of the following conditions is satisfied:

1. $\alpha\neq 0$, $\eta_4-\alpha\eta_3=0$, $\eta_3-\eta_2\,\alpha+\eta_4=0$
2. $\alpha=0$, $\eta_4=\eta_7=0$.
2. $\alpha=0$, $\eta_4=\eta_3=0$.

2.20.18.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$\alpha\, e_1$
e_2	0	0	0	0	$\alpha\, e_2 + u_1$
u_1	0	0	0	0	$(\alpha+1)\, u_1$
u_2	$-u_1$	0	0	0	u_2
u_3	$-\alpha\, e_1$	$-\alpha\, e_2 - u_1$	$(-\alpha-1)\, u_1$	$-u_2$	0

where $\alpha \neq 1$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_3 & \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_3 & \eta_8 & \eta_7 & 0 & 2\,\eta_3 & \eta_4 & 0 & 0 & \eta_3 \\ \eta_4 + 1 - \alpha & \eta_7 & \eta_5 & 0 & \eta_4 + 1 & 0 & 0 & \eta_3 & 2\,\eta_4 + 1 \end{pmatrix}$$

The riple is not maximal if and only if one of the following conditions is satisfied:

1. $\eta_3 = 0$.
2. $\eta_8 = \eta_7 = 0$.

2.20.19.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$(\beta+1)\, e_1$
e_2	0	0	0	0	$\beta\, e_2 + u_1$
u_1	0	0	0	0	$(\alpha+\beta)\, u_1$
u_2	$-u_1$	0	0	0	$(\alpha-1)\, u_2$
u_3	$(-\beta-1)\, e_1$	$-\beta\, e_2 - u_1$	$(-\alpha-\beta)\, u_1$	$(1-\alpha)\, u_2$	0

where $\alpha, \beta \in \mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_7 & \eta_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_7 & \eta_4 & \eta_3 & 0 & 2\,\eta_7 & \eta_8 & 0 & 0 & \eta_7 \\ \eta_8 + \alpha - \beta & \eta_3 & \eta_1 & 0 & \eta_8 + \alpha + 1 & 0 & 0 & \eta_7 & 2\,\eta_8 + \alpha \end{pmatrix}$$

The riple is not maximal if and only if one of the following conditions is satisfied:

1. $\beta = \alpha - 1$, $\eta_4 = 0$.
2. $\beta = \alpha - 1$, $\eta_7 = 0$.
3. $\beta = \alpha - 2$, $\eta_3 = 0$.
4. $\beta = \alpha - 2$, $\eta_7 = 0$.
5. $\beta \neq \alpha - 2$, $\beta \neq \alpha - 1$, $\eta_7 = \eta_4 = 0$.
6. $\beta \neq \alpha - 2$, $\beta \neq \alpha - 1$, $\eta_7 = \eta_3 = 0$.
7. $\beta \neq \alpha - 2$, $\beta \neq \alpha - 1$, $\eta_4 = \eta_3 = 0$.

2.20.20.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	e_1+u_1	$(\beta+1)\,e_1$
e_2	0	0	0	e_2	$\beta\,e_2+u_1$
u_1	0	0	0	u_1	$(\beta+1)\,u_1$
u_2	$-e_1-u_1$	$-e_2$	$-u_1$	0	0
u_3	$(-\beta-1)\,e_1$	$-\beta\,e_2-u_1$	$(-\beta-1)\,u_1$	0	0

where $\beta\in\mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_7 & \frac{\eta_8}{2}-1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_7-1 & \eta_1 & \eta_3 & 0 & 2\,\eta_7 & \frac{\eta_8}{2}-1/2 & 0 & 0 & \eta_7 \\ \frac{\eta_8}{2}+1/2-\beta & \eta_3 & \eta_4 & 0 & \frac{\eta_8}{2}+3/2 & 0 & 0 & \eta_7 & \eta_8 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_4=\beta\,\eta_3$, $\eta_3=\eta_1\,(1+\beta)$.

2.20.21.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	$\beta\,e_1+u_1$	$\alpha\,e_1$
e_2	0	0	0	$e_1+\beta\,e_2$	$(\alpha+1)\,e_2+u_1$
u_1	0	0	0	$-e_1+\beta\,u_1$	$\alpha\,u_1$
u_2	$-\beta\,e_1-u_1$	$-e_1-\beta\,e_2$	$e_1-\beta\,u_1$	0	0
u_3	$-\alpha\,e_1$	$*$	$-\alpha\,u_1$	0	0

where $\alpha,\,\beta\in\mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_3 & \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_3-\beta & \eta_7 & \eta_8 & 0 & 2\,\eta_3 & \eta_4 & 0 & 1 & \eta_3 \\ \eta_4-2-\alpha & \eta_8 & \eta_5 & 0 & \eta_4-2 & 0 & 0 & \eta_3 & 2\,\eta_4-1 \end{pmatrix}$$

The triple is not maximal if and only if

$$\eta_7\alpha-\eta_8\beta-\eta_5=0,\ \eta_8(\beta^2+\alpha+1)-\eta_7\alpha\,\beta=0$$

.

2.20.22.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$\alpha\,e_1$
e_2	0	0	0	e_1	$(\alpha+1)\,e_2+u_1$
u_1	0	0	0	0	$(\alpha-1)\,u_1$
u_2	$-u_1$	$-e_1$	0	0	$-u_2$
u_3	$-\alpha\,e_1$	$(-\alpha-1)\,e_2-u_1$	$(1-\alpha)\,u_1$	u_2	0

where $\alpha,\,\beta\in\mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_4 & \eta_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_4 & \eta_1 & \eta_8 & 0 & 2\,\eta_4 & \eta_5 & 0 & 1 & \eta_4 \\ \eta_5-3-\alpha & \eta_8 & \eta_6 & 0 & \eta_5-3 & 0 & 0 & \eta_4 & 2\,\eta_5-2 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_8\,(\alpha-2)=0$, $\eta_6=\eta_1\,(1+\alpha)$

2.20.23.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	$\alpha\, e_1 + u_1$	$\beta\, e_1$
e_2	0	0	0	$-e_1 + \alpha\, e_2$	$(\beta+1)\, e_2 + u_1$
u_1	0	0	0	$e_1 + \alpha\, u_1$	$\beta\, u_1$
u_2	$-\alpha\, e_1 - u_1$	$e_1 - \alpha\, e_2$	$-e_1 - \alpha\, u_1$	0	0
u_3	$-\beta\, e_1$	$*$	$-\beta\, u_1$	0	0

where $\beta \in \mathbb{R}$, $\alpha \geq 0$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_6 & \frac{\eta_7}{2} + 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_6 - \alpha & \eta_8 & \eta_2 & 0 & 2\,\eta_6 & \frac{\eta_7}{2} + 1/2 & 0 & -1 & \eta_6 \\ \frac{\eta_7}{2} - 3/2 - \beta & \eta_2 & \eta_3 & 0 & \frac{\eta_7}{2} - 3/2 & 0 & 0 & \eta_6 & \eta_7 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_2\,\beta + \eta_2 - \eta_3\,\alpha = 0$, $\eta_8\,\beta - \eta_2\,\alpha + \eta_3 = 0$.

2.20.24.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$\alpha\, e_1$
e_2	0	0	0	$-e_1$	$(\alpha+1)\, e_2 + u_1$
u_1	0	0	0	0	$(\alpha-1)\, u_1$
u_2	$-u_1$	e_1	0	0	$-u_2$
u_3	$-\alpha\, e_1$	$(-\alpha-1)\, e_2 - u_1$	$(1-\alpha)\, u_1$	u_2	0

where $\alpha \in \mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_8 & \frac{\eta_7}{2} + 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_8 & \eta_6 & \eta_5 & 0 & 2\,\eta_8 & \frac{\eta_7}{2} + 1 & 0 & -1 & \eta_8 \\ \frac{\eta_7}{2} - 2 - \alpha & \eta_5 & \eta_3 & 0 & \frac{\eta_7}{2} - 2 & 0 & 0 & \eta_8 & \eta_7 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_3 = -\eta_6\,(\alpha+1)$, $\eta_5\,(\alpha-2) = 0$.

2.20.25.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	$-\alpha\, e_1 + u_1$	$\left(\frac{\alpha}{3} - 1/3\right) e_1$
e_2	0	0	0	$-e_1 - \alpha\, e_2$	$\left(\frac{\alpha}{3} + 2/3\right) e_2 + u_1$
u_1	0	0	0	$\frac{3\, e_1}{2} - e_2 + (-3/2 - \alpha)\, u_1$	$-\frac{e_1}{2} + \left(\frac{\alpha}{3} + 1/6\right) u_1$
u_2	$*$	$*$	$*$	0	$\frac{u_2}{2} + \frac{3\, u_3}{2}$
u_3	$*$	$*$	$*$	$*$	0

where $\alpha \in \mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_7 + 3/2 & \frac{2\,\eta_8 + 1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_7 + \alpha & \eta_1 & \eta_3 & 0 & 2\,\eta_7 + 3/2 & \frac{2\,\eta_8 + 1}{4} & 0 & -1 & \eta_7 \\ \frac{\eta_8}{2} - \frac{11}{12} - \frac{\alpha}{3} & \eta_3 & \eta_4 & 0 & \frac{\eta_8}{2} - 5/4 & 0 & 0 & \frac{2\,\eta_7 + 3}{2} & \eta_8 \end{pmatrix}$$

The triple is not maximal if and only if $\frac{\eta_3\,\alpha}{3} + \frac{\eta_3}{6} + \eta_4\,\alpha - \frac{3\,\eta_4}{2} = 0$, $\frac{\eta_1\,\alpha}{3} - \frac{5\,\eta_1}{6} + \eta_3\,\alpha + \eta_4 - \frac{3\,\eta_3}{2} = 0$

2.20.26.

$[\,,]$	e_1	e_2	u_1	u_2	u_3
e_1	0	0	0	u_1	$\alpha\, e_1 + e_2$
e_2	0	0	0	0	$\alpha\, e_2 + u_1$
u_1	0	0	0	0	$(\alpha+1)\, u_1$
u_2	$-u_1$	0	0	0	u_2
u_3	$-\alpha\, e_1 - e_2$	$-\alpha\, e_2 - u_1$	$(-\alpha-1)\, u_1$	$-u_2$	0

where $\alpha \in \mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_7 & \frac{\eta_8}{2} - 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_7 & \eta_1 & \eta_3 & 0 & 2\,\eta_7 & \frac{\eta_8}{2} - 1/2 & 0 & 0 & \eta_7 \\ \frac{\eta_8}{2} + 1/2 - \alpha & \eta_3 & \eta_4 & 0 & \frac{\eta_8}{2} + 1/2 & 1 & 0 & \eta_7 & \eta_8 \end{pmatrix}$$

The riple is not maximal if and only if one of the following conditions is satisfied:

1. $\alpha \neq 1$, $\eta_1 = \eta_3 = 0$.
2. $\alpha \neq 1, \eta_1 = \eta_7 = 0$.
3. $\alpha = 1$, $\eta_1 = 0$.
4. $\alpha = 1$, $\eta_7 = 0$.

4.4. **General case,** $\dim \mathfrak{g} = 3$.

3.3.2.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$2\, e_2$	$-2\, e_3$	u_1	$-u_2$	0
e_2	$-2\, e_2$	0	e_1	0	u_1	0
e_3	$2\, e_3$	$-e_1$	0	u_2	0	0
u_1	$-u_1$	0	$-u_2$	0	u_3	0
u_2	u_2	$-u_1$	0	$-u_3$	0	0
u_3	0	0	0	0	0	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_4 & 0 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_4 & 1/2 & 0 & 0 \\ \eta_4 & 0 & 0 & 0 & \eta_4 & 0 & 0 & 0 & \eta_1 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_1 = \eta_4 = 0$.

3.13.6. $(\mu = 0)$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_3	u_1	0	0
e_2	0	0	0	e_3	$2\, e_2$	u_2
e_3	$-e_3$	0	0	0	e_3	u_1
u_1	$-u_1$	$-e_3$	0	0	$-u_1$	0
u_2	0	$-2\, e_2$	$-e_3$	u_1	0	$2\, u_3$
u_3	0	$-u_2$	$-u_1$	0	$-2\, u_3$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & -1 & \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \eta_4 & 0 & 0 & 1 \\ \eta_4 & 0 & 0 & 0 & \eta_4 & \eta_2 & 0 & -1 & 2\,\eta_4 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_2 = 0$.

3.13.26.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$e_2 + \alpha\, e_3$
u_2	$-u_2$	0	0	0	0	$\beta\, e_3 + u_1$
u_3	0	$-u_2$	$-u_1$	$-e_2 - \alpha\, e_3$	$-\beta\, e_3 - u_1$	0

where α, $\beta \in \mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \frac{\eta_5}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & \frac{\eta_5}{2} & 0 & 0 & 0 \\ \frac{\eta_5}{2} & 1/2 & 0 & 0 & \frac{\eta_5}{2} & 0 & 0 & 0 & \eta_5 \end{pmatrix}$$

3.13.27.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$\alpha\, e_3$
u_2	$-u_2$	0	0	0	0	$e_2 + u_1$
u_3	0	$-u_2$	$-u_1$	$-\alpha\, e_3$	$-e_2 - u_1$	0

where $\alpha \in \mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & \eta_4 & 0 & 0 & 0 \\ \eta_4 & 1/2 & 0 & 0 & \eta_4 & 0 & 0 & 0 & 2\,\eta_4 \end{pmatrix}$$

The triple is not maximal if and only if $\alpha - 1 = 0$

3.13.28.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$\alpha\, e_3$
u_2	$-u_2$	0	0	0	0	$-e_2 + u_1$
u_3	0	$-u_2$	$-u_1$	$-\alpha\, e_3$	$e_2 - u_1$	0

where $\alpha \in \mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & \eta_5 & 0 & 0 & 0 \\ \eta_5 & 1/2 & 0 & 0 & \eta_5 & 0 & 0 & 0 & 2\,\eta_5 \end{pmatrix}$$

The triple is not maximal if and only if $\alpha + 1 = 0$

3.13.29.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	e_3
u_2	$-u_2$	0	0	0	0	u_1
u_3	0	$-u_2$	$-u_1$	$-e_3$	$-u_1$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & \eta_5 & 0 & 0 & 0 \\ \eta_5 & 1/2 & 0 & 0 & \eta_5 & 0 & 0 & 0 & 2\,\eta_5 \end{pmatrix}$$

3.13.30.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$-e_3$
u_2	$-u_2$	0	0	0	0	u_1
u_3	0	$-u_2$	$-u_1$	e_3	$-u_1$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & \eta_4 & 0 & 0 & 0 \\ \eta_4 & 1/2 & 0 & 0 & \eta_4 & 0 & 0 & 0 & 2\,\eta_4 \end{pmatrix}$$

3.13.32.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$\alpha\, e_3 + u_1$
u_2	$-u_2$	0	0	0	0	$\beta\, e_2 + e_3$
u_3	0	$-u_2$	$-u_1$	$-\alpha\, e_3 - u_1$	$-\beta\, e_2 - e_3$	0

where $\alpha,\, \beta \in \mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_5 - 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_5 & 0 & 0 & 0 \\ \eta_5 + 1/2 & 0 & 0 & 0 & \eta_5 & 0 & 0 & 0 & 2\,\eta_5 \end{pmatrix}$$

3.13.33.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$e_2+\alpha\, e_3+u_1$
u_2	$-u_2$	0	0	0	0	$\beta\, e_2+\gamma\, e_3$
u_3	0	$-u_2$	$-u_1$	$-e_2-\alpha\, e_3-u_1$	$-\beta\, e_2-\gamma\, e_3$	0

where $\alpha,\ \beta,\ \gamma \in \mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_4+1/2 & 0 & 0 & 0 \\ \eta_4+1 & 0 & 0 & 0 & \eta_4+1/2 & 0 & 0 & 0 & 2\,\eta_4+1 \end{pmatrix}$$

3.13.34.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	u_2	0
e_2	$-e_2$	0	0	0	0	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$\gamma\, e_2+\alpha\, e_3-u_2$
u_2	$-u_2$	0	0	0	0	$\alpha\, e_2+\beta\, e_3+u_1$
u_3	0	$-u_2$	$-u_1$	$-\gamma\, e_2-\alpha\, e_3+u_2$	$*$	0

where $|\beta| \le |\gamma|,\ \beta \ne -\gamma$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_5 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & \eta_5 & 0 & 0 & 0 \\ \eta_5 & 1/2 & 0 & -1/2 & \eta_5 & 0 & 0 & 0 & 2\,\eta_5 \end{pmatrix}$$

3.20.2. $(\mu = 1/2)$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$\frac{e_3}{2}$	u_1	0	$\frac{u_3}{2}$
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-\frac{e_3}{2}$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	u_1	0
u_2	0	$-u_1$	0	$-u_1$	0	$-u_3$
u_3	$-\frac{u_3}{2}$	0	$-u_1$	0	u_3	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_3-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & 2\,\eta_3-1 & 0 & 0 & 0 & \eta_3 \\ 0 & 0 & \eta_1 & 0 & 0 & 0 & 0 & \eta_3-1 & 0 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_1 = 0$.

3.20.3. ($\lambda = 0$)

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	u_1
u_2	0	$-u_1$	0	0	0	u_2
u_3	0	0	$-u_1$	$-u_1$	$-u_2$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_3 & \frac{\eta_4}{2} - 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & 2\,\eta_3 & \frac{\eta_4}{2} - 1/2 & 0 & 0 & \eta_3 \\ \frac{\eta_4}{2} + 1/2 & 0 & 0 & 0 & \frac{\eta_4}{2} + 1/2 & 0 & 0 & \eta_3 & \eta_4 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_3 = 0$.

3.20.5. ($\mu = 1/2$)

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$\frac{e_3}{2}$	u_1	0	$\frac{u_3}{2}$
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-\frac{e_3}{2}$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_1$	0	0	0	e_3
u_3	$-\frac{u_3}{2}$	0	$-u_1$	0	$-e_3$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_2 & 0 & 0 & 0 & 2\,\eta_2 & 0 & 0 & 0 & \eta_2 \\ 0 & 0 & \eta_3 & 0 & 0 & 0 & 0 & \eta_2 & 0 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_2 = 0$.

3.20.11.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	e_2	$e_3 + u_1$
u_1	$-u_1$	0	0	0	$-e_2$	0
u_2	0	$-u_1$	$-e_2$	e_2	0	0
u_3	0	0	$-e_3 - u_1$	0	0	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_3 & \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & 2\,\eta_3 & \eta_4 & 0 & 1 & \eta_3 \\ \eta_4 - 2 & 0 & 0 & 0 & \eta_4 - 2 & 0 & 0 & \eta_3 & 2\,\eta_4 - 1 \end{pmatrix}$$

3.20.12.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	e_2	e_3+u_1
u_1	$-u_1$	0	0	0	0	$-u_1$
u_2	0	$-u_1$	$-e_2$	0	0	$-u_2$
u_3	0	0	$-e_3-u_1$	u_1	u_2	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_2 & \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_2 & 0 & 0 & 0 & 2\,\eta_2 & \eta_4 & 0 & 1 & \eta_2 \\ \eta_4-3 & 0 & 0 & 0 & \eta_4-3 & 0 & 0 & \eta_2 & 2\,\eta_4-2 \end{pmatrix}$$

3.20.13.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	$-e_2$	e_3+u_1
u_1	$-u_1$	0	0	0	e_2	0
u_2	0	$-u_1$	e_2	$-e_2$	0	0
u_3	0	0	$-e_3-u_1$	0	0	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_4 & \eta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_4 & 0 & 0 & 0 & 2\,\eta_4 & \eta_3 & 0 & -1 & \eta_4 \\ \eta_3-2 & 0 & 0 & 0 & \eta_3-2 & 0 & 0 & \eta_4 & 2\,\eta_3-1 \end{pmatrix}$$

3.20.14.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	$-e_2$	e_3+u_1
u_1	$-u_1$	0	0	0	0	$-u_1$
u_2	0	$-u_1$	e_2	0	0	$-u_2$
u_3	0	0	$-e_3-u_1$	u_1	u_2	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_3 & \frac{\eta_4}{2}+1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & 2\,\eta_3 & \frac{\eta_4}{2}+1 & 0 & -1 & \eta_3 \\ \frac{\eta_4}{2}-2 & 0 & 0 & 0 & \frac{\eta_4}{2}-2 & 0 & 0 & \eta_3 & \eta_4 \end{pmatrix}$$

3.20.15.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-e_3$	0	0	0	$-e_2$	e_3+u_1
u_1	$-u_1$	0	0	0	$\frac{3\,e_2}{2}-e_3-\frac{3\,u_1}{2}$	$-\frac{e_2}{2}+\frac{u_1}{2}$
u_2	0	$-u_1$	e_2	$*$	0	$\frac{e_1}{2}+\frac{u_2}{2}+\frac{3\,u_3}{2}$
u_3	0	0	$-e_3-u_1$	$*$	$-\frac{e_1}{2}-\frac{u_2}{2}-\frac{3\,u_3}{2}$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_3+3/2 & \frac{\eta_4}{2}+1/4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & 2\,\eta_3+3/2 & \frac{\eta_4}{2}+1/4 & 0 & -1 & \eta_3 \\ \frac{2\,\eta_4-5}{4} & 0 & 0 & 0 & \frac{\eta_4}{2}-5/4 & 0 & 0 & \frac{2\,\eta_3+3}{2} & \eta_4 \end{pmatrix}$$

3.20.20. ($\lambda = 0$)

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	e_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	$\alpha\,u_1$
u_2	0	$-u_1$	0	0	0	$(\alpha-1)\,u_2$
u_3	0	$-e_2$	$-u_1$	$-\alpha\,u_1$	$(1-\alpha)\,u_2$	0

where $\alpha \in \mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_3 & \frac{\eta_4}{2}-\frac{\alpha}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & 2\,\eta_3 & \frac{\eta_4}{2}-\frac{\alpha}{2} & 0 & 0 & \eta_3 \\ \frac{\eta_4}{2}+\frac{\alpha}{2} & 0 & 0 & 0 & \frac{\eta_4}{2}+\frac{\alpha}{2}+1 & 0 & 0 & \eta_3 & \eta_4 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_3 = 0$.

3.20.21. ($\mu = 1/2$)

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$\frac{e_3}{2}$	u_1	0	$\frac{u_3}{2}$
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-\frac{e_3}{2}$	0	0	0	e_3	u_1
u_1	$-u_1$	0	0	0	$(1-\alpha)\,u_1$	0
u_2	0	$-u_1$	$-e_3$	$(\alpha-1)\,u_1$	0	$\alpha\,u_3$
u_3	$-\frac{u_3}{2}$	0	$-u_1$	0	$-\alpha\,u_3$	0

where $\alpha \in \mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_3-\alpha-1 & 0 & 0 & 0 & 2\,\eta_3-\alpha-1 & 0 & 0 & 0 & \eta_3-\alpha \\ 0 & 0 & \eta_2 & 0 & 0 & 0 & 0 & \eta_3 & 0 \end{pmatrix}$$

The riple is not maximal if and only if one of the following conditions is satisfied:

1. $\eta_2 = 0$.
2. $\alpha = 0$, $\eta_3 = 0$.
3. $\alpha + 1 = 0$.

3.20.22.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	$\frac{e_3}{2}$	u_1	0	$\frac{u_3}{2}$
e_2	$-e_2$	0	0	0	u_1	0
e_3	$-\frac{e_3}{2}$	0	0	0	e_3	u_1
u_1	$-u_1$	0	0	0	$2\,u_1$	0
u_2	0	$-u_1$	$-e_3$	$-2\,u_1$	0	$e_3 - u_3$
u_3	$-\frac{u_3}{2}$	0	$-u_1$	0	$-e_3 + u_3$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_2 & 0 & 0 & 0 & 2\,\eta_2 & 0 & 0 & 0 & \eta_2 + 1 \\ 0 & 0 & \eta_3 & 0 & 0 & 0 & 0 & \eta_2 & 0 \end{pmatrix}$$

3.20.24.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	e_3
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	u_1
u_2	0	$-u_1$	0	0	0	u_2
u_3	0	$-e_3$	$-u_1$	$-u_1$	$-u_2$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_3 & \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & 2\,\eta_3 & \eta_4 & 0 & 0 & \eta_3 \\ \eta_4 + 1 & 0 & 0 & 0 & \eta_4 + 1 & 1 & 0 & \eta_3 & 2\,\eta_4 + 1 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_3 = 0$.

3.20.25. $(\mu = 0)$

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	e_2	e_3	u_1	0	0
e_2	$-e_2$	0	0	0	u_1	e_3
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	0	$-u_1$	0	0	0	0
u_3	0	$-e_3$	$-u_1$	0	0	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & \eta_3 & \eta_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & 2\,\eta_3 & \eta_4 & 0 & 0 & \eta_3 \\ \eta_4 & 0 & 0 & 0 & \eta_4 & 1 & 0 & \eta_3 & 2\,\eta_4 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_3 = 0$.

3.20.27.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{4e_2}{5}$	$\frac{3e_3}{5}$	u_1	$\frac{u_2}{5}$	$\frac{2u_3}{5}$
e_2	$-\frac{4e_2}{5}$	0	0	0	u_1	0
e_3	$-\frac{3e_3}{5}$	0	0	0	e_2	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\frac{u_2}{5}$	$-u_1$	$-e_2$	0	0	e_3
u_3	$-\frac{2u_3}{5}$	0	$-u_1$	0	$-e_3$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3.23.3.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{e_2}{2}$	e_3	u_1	$\frac{u_2}{2}$	0
e_2	$-\frac{e_2}{2}$	0	0	0	u_1	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\frac{u_2}{2}$	$-u_1$	0	0	0	e_2
u_3	0	$-u_2$	$-u_1$	0	$-e_2$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_2 & 0 & 0 & 0 \\ \eta_2 & 0 & 0 & 0 & \eta_2 & 0 & 0 & 0 & 2\eta_2 \end{pmatrix}$$

3.23.4.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{e_2}{2}$	e_3	u_1	$\frac{u_2}{2}$	0
e_2	$-\frac{e_2}{2}$	0	0	0	u_1	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0
u_2	$-\frac{u_2}{2}$	$-u_1$	0	0	0	$-e_2$
u_3	0	$-u_2$	$-u_1$	0	e_2	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_2 & 0 & 0 & 0 \\ \eta_2 & 0 & 0 & 0 & \eta_2 & 0 & 0 & 0 & 2\eta_2 \end{pmatrix}$$

3.23.5.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	$\frac{e_2}{2}$	e_3	u_1	$\frac{u_2}{2}$	0
e_2	$-\frac{e_2}{2}$	0	0	0	u_1	u_2
e_3	$-e_3$	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	u_1
u_2	$-\frac{u_2}{2}$	$-u_1$	0	0	0	$\alpha\, e_2 + u_2$
u_3	0	$-u_2$	$-u_1$	$-u_1$	$-\alpha\, e_2 - u_2$	0

where $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_2 & 0 & 0 & 0 \\ \eta_2 + 1 & 0 & 0 & 0 & \eta_2 + 1 & 0 & 0 & 0 & 2\,\eta_2 + 1 \end{pmatrix}$$

3.25.30.

$[\,,]$	e_1	e_2	e_3	u_1	u_2	u_3
e_1	0	0	e_2	0	u_1	0
e_2	0	0	0	0	e_2	u_1
e_3	$-e_2$	0	0	e_2	$2\,e_3$	u_2
u_1	0	0	$-e_2$	0	$-u_1$	0
u_2	$-u_1$	$-e_2$	$-2\,e_3$	u_1	0	$2\,u_3$
u_3	0	$-u_1$	$-u_2$	0	$-2\,u_3$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & -1 & \eta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \eta_2 & 0 & 0 & 1 \\ \eta_2 & 0 & \eta_4 & 0 & \eta_2 & 0 & 0 & -1 & 2\,\eta_2 \end{pmatrix}$$

The riple is not maximal if and only if one of the following conditions is satisfied:

1. $\eta_2 = 0$.
2. $\eta_4 = 0$.

4.5. **General case,** $\dim \mathfrak{g} = 4$.

4.8.2.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3
e_1	0	0	e_3	0	u_1	0	0
e_2	0	0	0	e_4	0	u_2	0
e_3	$-e_3$	0	0	0	0	0	u_1
e_4	0	$-e_4$	0	0	0	0	u_2
u_1	$-u_1$	0	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	0	e_4
u_3	0	0	$-u_1$	$-u_2$	0	$-e_4$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\eta_3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\eta_3}{2} & 0 & 0 & 0 \\ \frac{\eta_3}{2} & 0 & 0 & 0 & \frac{\eta_3}{2} & 0 & 0 & 0 & \eta_3 \end{pmatrix}$$

4.8.3.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3
e_1	0	0	e_3	0	u_1	0	0
e_2	0	0	0	e_4	0	u_2	0
e_3	$-e_3$	0	0	0	0	0	u_1
e_4	0	$-e_4$	0	0	0	0	u_2
u_1	$-u_1$	0	0	0	0	0	0
u_2	0	$-u_2$	0	0	0	0	$-e_4$
u_3	0	0	$-u_1$	$-u_2$	0	e_4	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 & 2\,\eta_3 \end{pmatrix}$$

4.8.4.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3
e_1	0	0	e_3	0	u_1	0	0
e_2	0	0	0	e_4	0	u_2	0
e_3	$-e_3$	0	0	0	0	0	u_1
e_4	0	$-e_4$	0	0	0	0	u_2
u_1	$-u_1$	0	0	0	0	0	e_3
u_2	0	$-u_2$	0	0	0	0	$\alpha\, e_4$
u_3	0	0	$-u_1$	$-u_2$	$-e_3$	$-\alpha\, e_4$	0

where $0 < \alpha < 1$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 & 2\,\eta_3 \end{pmatrix}$$

4.8.5.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3
e_1	0	0	e_3	0	u_1	0	0
e_2	0	0	0	e_4	0	u_2	0
e_3	$-e_3$	0	0	0	0	0	u_1
e_4	0	$-e_4$	0	0	0	0	$\alpha\, u_2$
u_1	$-u_1$	0	0	0	0	0	e_3
u_2	0	$-u_2$	0	0	0	0	$-\alpha\, e_4$
u_3	0	0	$-u_1$	$-\alpha\, u_2$	$-e_3$	$\alpha\, e_4$	0

where $0 < \alpha$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 & 2\,\eta_3 \end{pmatrix}$$

4.8.6.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3
e_1	0	0	e_3	0	u_1	0	0
e_2	0	0	0	e_4	0	u_2	0
e_3	$-e_3$	0	0	0	0	0	u_1
e_4	0	$-e_4$	0	0	0	0	$\alpha\, u_2$
u_1	$-u_1$	0	0	0	0	0	$-e_3$
u_2	0	$-u_2$	0	0	0	0	$-\alpha\, e_4$
u_3	0	0	$-u_1$	$-\alpha\, u_2$	e_3	$\alpha\, e_4$	0

where $0 < \alpha < 1$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 & 2\,\eta_3 \end{pmatrix}$$

4.9.4.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3
e_1	0	0	e_4	$-e_3$	$-u_2$	u_1	0
e_2	0	0	e_3	e_4	u_1	u_2	0
e_3	$-e_4$	$-e_3$	0	0	0	0	u_2
e_4	e_3	$-e_4$	0	0	0	0	u_1
u_1	u_2	$-u_1$	0	0	0	0	$e_3 + \alpha\, e_4$
u_2	$-u_1$	$-u_2$	0	0	0	0	$\alpha\, e_3 - e_4$
u_3	0	0	$-u_2$	$-u_1$	$-e_3 - \alpha\, e_4$	$-\alpha\, e_3 + e_4$	0

where $\alpha \geq 0$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 & 2\,\eta_3 \end{pmatrix}$$

4.9.5.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3
e_1	0	0	e_4	$-e_3$	$-u_2$	u_1	0
e_2	0	0	e_3	e_4	u_1	u_2	0
e_3	$-e_4$	$-e_3$	0	0	0	0	u_2
e_4	e_3	$-e_4$	0	0	0	0	u_1
u_1	u_2	$-u_1$	0	0	0	0	$-e_3+\alpha\, e_4$
u_2	$-u_1$	$-u_2$	0	0	0	0	$\alpha\, e_3+e_4$
u_3	0	0	$-u_2$	$-u_1$	$e_3-\alpha\, e_4$	$-\alpha\, e_3-e_4$	0

where $\alpha \geq 0$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \frac{\eta_3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\eta_3}{2} & 0 & 0 & 0 \\ \frac{\eta_3}{2} & 0 & 0 & 0 & \frac{\eta_3}{2} & 0 & 0 & 0 & \eta_3 \end{pmatrix}$$

4.21.2. $(\lambda = 1)$

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3
e_1	0	0	e_3	e_4	u_1	u_2	0
e_2	0	0	e_4	0	0	u_1	0
e_3	$-e_3$	$-e_4$	0	0	0	0	u_2
e_4	$-e_4$	0	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0	0
u_2	$-u_2$	$-u_1$	0	0	0	0	e_4
u_3	0	0	$-u_2$	$-u_1$	0	$-e_4$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 & 2\,\eta_3 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_3 = 0$.

4.21.3.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3
e_1	0	0	e_3	e_4	u_1	u_2	0
e_2	0	0	e_4	0	0	u_1	0
e_3	$-e_3$	$-e_4$	0	0	0	0	u_2
e_4	$-e_4$	0	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0	e_4
u_2	$-u_2$	$-u_1$	0	0	0	0	e_3+e_4
u_3	0	0	$-u_2$	$-u_1$	$-e_4$	$-e_3-e_4$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 & 2\,\eta_3 \end{pmatrix}$$

4.21.4.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3
e_1	0	0	e_3	e_4	u_1	u_2	0
e_2	0	0	e_4	0	0	u_1	0
e_3	$-e_3$	$-e_4$	0	0	0	0	u_2
e_4	$-e_4$	0	0	0	0	0	u_1
u_1	$-u_1$	0	0	0	0	0	$-e_4$
u_2	$-u_2$	$-u_1$	0	0	0	0	$-e_3+e_4$
u_3	0	0	$-u_2$	$-u_1$	e_4	e_3-e_4	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 \\ \eta_3 & 0 & 0 & 0 & \eta_3 & 0 & 0 & 0 & 2\,\eta_3 \end{pmatrix}$$

4.21.11. ($\mu = 0$)

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3
e_1	0	e_2	0	e_4	u_1	0	0
e_2	$-e_2$	0	e_4	0	0	e_2+u_1	0
e_3	0	$-e_4$	0	0	0	$-2\,e_3$	u_2
e_4	$-e_4$	0	0	0	0	$-e_4$	e_2+u_1
u_1	$-u_1$	0	0	0	0	0	0
u_2	0	$-e_2-u_1$	$2\,e_3$	e_4	0	0	$-2\,u_3$
u_3	0	0	$-u_2$	$-e_2-u_1$	0	$2\,u_3$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \eta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \eta_2 & 0 & 0 & -1 \\ \eta_2 & 0 & 0 & 0 & \eta_2 & 0 & 0 & 1 & 2\,\eta_2 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_2 = 0$.

4.21.11. ($\mu = 1/2$)

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3
e_1	0	e_2	$-\frac{e_3}{2}$	$\frac{e_4}{2}$	u_1	0	$\frac{u_3}{2}$
e_2	$-e_2$	0	e_4	0	0	e_2+u_1	0
e_3	$\frac{e_3}{2}$	$-e_4$	0	0	0	$-2\,e_3$	u_2
e_4	$-\frac{e_4}{2}$	0	0	0	0	$-e_4$	e_2+u_1
u_1	$-u_1$	0	0	0	0	0	0
u_2	0	$-e_2-u_1$	$2\,e_3$	e_4	0	0	$-2\,u_3$
u_3	$-\frac{u_3}{2}$	0	$-u_2$	$-e_2-u_1$	0	$2\,u_3$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & \eta_1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_1 = 0$.

4.21.25.

$[\,,]$	e_1	e_2	e_3	e_4	u_1	u_2	u_3
e_1	0	0	e_3	e_4	u_1	u_2	0
e_2	0	0	e_4	0	0	u_1	e_2
e_3	$-e_3$	$-e_4$	0	0	0	0	u_2
e_4	$-e_4$	0	0	0	0	0	e_4+u_1
u_1	$-u_1$	0	0	0	0	0	$\alpha\, e_4$
u_2	$-u_2$	$-u_1$	0	0	0	0	$\alpha\, e_3+e_4-u_2$
u_3	0	$-e_2$	$-u_2$	$-e_4-u_1$	$-\alpha\, e_4$	$*$	0

where $\alpha \in \mathbb{R}$.

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \frac{\eta_3}{2}+1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\eta_3}{2}+1/2 & 0 & 0 & 0 \\ \frac{\eta_3}{2}-3/2 & 0 & 0 & 0 & \frac{\eta_3}{2}-1/2 & 0 & 0 & 0 & \eta_3 \end{pmatrix}$$

4.6. **General case,** $\dim \mathfrak{g} = 6$.

6.1.1.

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3
e_1	0	$2\,e_2$	$-2\,e_3$	0	e_5	$-e_6$	u_1	$-u_2$	0
e_2	$-2\,e_2$	0	e_1	0	0	e_5	0	u_1	0
e_3	$2\,e_3$	$-e_1$	0	0	e_6	0	u_2	0	0
e_4	0	0	0	0	e_5	e_6	u_1	u_2	0
e_5	$-e_5$	0	$-e_6$	$-e_5$	0	0	0	0	u_1
e_6	e_6	$-e_5$	0	$-e_6$	0	0	0	0	u_2
u_1	$-u_1$	0	$-u_2$	$-u_1$	0	0	0	0	0
u_2	u_2	$-u_1$	0	$-u_2$	0	0	0	0	0
u_3	0	0	0	0	$-u_1$	$-u_2$	0	0	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \frac{\eta_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\eta_2}{2} & 0 & 0 & 0 \\ \frac{\eta_2}{2} & 0 & 0 & 0 & \frac{\eta_2}{2} & 0 & 0 & 0 & \eta_2 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_2 = 0$.

6.1.2.

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3
e_1	0	$2\,e_2$	$-2\,e_3$	0	e_5	$-e_6$	u_1	$-u_2$	0
e_2	$-2\,e_2$	0	e_1	0	0	e_5	0	u_1	0
e_3	$2\,e_3$	$-e_1$	0	0	e_6	0	u_2	0	0
e_4	0	0	0	0	e_5	e_6	u_1	u_2	0
e_5	$-e_5$	0	$-e_6$	$-e_5$	0	0	0	0	u_1
e_6	e_6	$-e_5$	0	$-e_6$	0	0	0	0	u_2
u_1	$-u_1$	0	$-u_2$	$-u_1$	0	0	0	0	e_5
u_2	u_2	$-u_1$	0	$-u_2$	0	0	0	0	e_6
u_3	0	0	0	0	$-u_1$	$-u_2$	$-e_5$	$-e_6$	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \frac{\eta_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\eta_2}{2} & 0 & 0 & 0 \\ \frac{\eta_2}{2} & 0 & 0 & 0 & \frac{\eta_2}{2} & 0 & 0 & 0 & \eta_2 \end{pmatrix}$$

The triple is not maximal if and only if $\eta_2^2 - 4 = 0$.

6.1.3.

$[\,,]$	e_1	e_2	e_3	e_4	e_5	e_6	u_1	u_2	u_3
e_1	0	$2\,e_2$	$-2\,e_3$	0	e_5	$-e_6$	u_1	$-u_2$	0
e_2	$-2\,e_2$	0	e_1	0	0	e_5	0	u_1	0
e_3	$2\,e_3$	$-e_1$	0	0	e_6	0	u_2	0	0
e_4	0	0	0	0	e_5	e_6	u_1	u_2	0
e_5	$-e_5$	0	$-e_6$	$-e_5$	0	0	0	0	u_1
e_6	e_6	$-e_5$	0	$-e_6$	0	0	0	0	u_2
u_1	$-u_1$	0	$-u_2$	$-u_1$	0	0	0	0	$-e_5$
u_2	u_2	$-u_1$	0	$-u_2$	0	0	0	0	$-e_6$
u_3	0	0	0	0	$-u_1$	$-u_2$	e_5	e_6	0

The connection with zero torsion has the form:

$$\begin{pmatrix} 0 & 0 & \frac{\eta_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\eta_2}{2} & 0 & 0 & 0 \\ \frac{\eta_2}{2} & 0 & 0 & 0 & \frac{\eta_2}{2} & 0 & 0 & 0 & \eta_2 \end{pmatrix}$$

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