A Remark on the Equivalence between Poisson and Gaussian Stochastic Partial Differential Equations

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Abstract

We discuss the connection between Gaussian and Poisson Wick-type stochastic partial differential equations.

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§1 Introduction

In this paper we look at the connection between Gaussian and Poisson Wick-type stochastic partial differential equations. We will show that for several cases it is possible to obtain the solution of a Poisson stochastic partial differential equation through an unitary mapping of the solution of the corresponding Gaussian equation.

We will make use of the white noise analysis based on the Poisson probability measure recently developed by Kondratiev et al. [KSWY]. Already in [IK] the existence of a unitary mapping between the space of square integrable functions with respect to the Poisson measure and the space of square integrable functions with respect to the Gaussian measure, was shown. From the analysis of Kondratiev et al., it is easily seen that this mapping extends to spaces of generalized random variables, such as the Kondratiev distribution spaces.

For applications to Wick-type stochastic partial differential equations, this unitary mapping has two attractive features: It maps the Gaussian Wick product into the Poisson Wick product and Gaussian white noise into compensated Poisson noise. It is now clear that if we know the solution of a Gaussian Wick-type equation, we obtain the solution of the corresponding compensated Poisson equation by simply applying the unitary mapping on the solution of the Gaussian equation. Of course, this also works the other way around. We note that this connection seems to hold only for linear stochastic partial differential equations and Wick-type stochastic partial differential equations.

Gaussian Wick-type equations have already been treated by several authors. We here mention some works in this direction: [DP], [GHØUZ], [HLØUZ], [HLØUZ2], [HLØUZ3], [HLØUZ4], [LØU], [LØU2], [LØU3], [P]. On the other hand, [BS] has considered Burgers equation with non-Gaussian additive force and [D] has treated Burgers equation with an additive compensated Poisson noise force.

We organize this paper as follows: In section §2 we give an introduction of basic facts concerning Poisson and Gaussian analysis, based on results of [KSWY], [U] and [IK]. We apply in section §3 known results from the Gaussian case to solve two stochastic partial differential equations disturbed by compensated Poisson noise.

§2 Gaussian and Poisson analysis

Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of all rapidly decreasing C^∞- functions on $\mathbb{R}^d$. It is well known that $\mathcal{S}$ is a Fréchet space under the family of seminorms

$$\|f\|_{N,\alpha} := \sup_{x \in \mathbb{R}^d} (1 + |x|^N) |\partial^\alpha f(x)|$$

where $\partial^\alpha := \frac{\partial^{|\alpha|}}{\partial_1^{\alpha} x_1 \cdots \partial_d^{\alpha} x_d}$, $N \geq 0$ is an integer and $\alpha = (\alpha_1, \cdots, \alpha_d)$ is a multi-index of non-negative integers α_j. The space of tempered distributions is the dual $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^d)$ of $\mathcal{S}$. We equip $\mathcal{S}'$ with the weak star topology. $\mathcal{B} := \mathcal{B}(\mathcal{S}')$ denotes the Borel σ-algebra of $\mathcal{S}'$.

Consider the probability measures μ_G and μ_P on $(\mathcal{S}', \mathcal{B})$ with characteristic functionals

$$\int_{\mathcal{S}'} e^{i\langle \omega, \phi \rangle} \, d\mu_G(\omega) = \exp\{-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R}^d)}\} \, , \quad \forall \phi \in \mathcal{S}$$

and

$$\int_{\mathcal{S}'} e^{i\langle \omega, \phi \rangle} \, d\mu_P(\omega) = \exp\{\int_{\mathbb{R}^d} (e^{i\phi(x)} - 1) \, dx\} \, , \quad \forall \phi \in \mathcal{S}$$

respectively. The existence of such measures μ_G and μ_P are given by the Bochner-Minlos theorem ([H],[M],[GV]).

DEFINITION 2.1 The triple $(\mathcal{S}, \mathcal{B}, \mu_G)$ is called the **Gaussian white noise probability space** and the triple $(\mathcal{S}, \mathcal{B}, \mu_P)$ is called the **Poisson white noise probability space.**

An application of the characteristic functionals for μ_G and μ_P gives us the following relations(see [HKPS],[I2]): For $\phi \in \mathcal{S}$,

1. $\int\limits_{\mathcal{S}'} \langle \omega, \phi \rangle^2 \, d\mu_G = \int\limits_{\mathbb{R}^d} \phi(x)^2 \, dx$

2. $\int\limits_{\mathcal{S}'} \langle \omega, \phi \rangle \, d\mu_G = 0$

3. $\int\limits_{\mathcal{S}'} \langle \omega, \phi \rangle^2 \, d\mu_P = \int\limits_{\mathbb{R}^d} \phi(x)^2 \, dx + (\int\limits_{\mathbb{R}^d} \phi(x) \, dx)^2$

4. $\int\limits_{\mathcal{S}'} \langle \omega, \phi \rangle \, d\mu_P = \int\limits_{\mathbb{R}^d} \phi(x) \, dx$

Hence, the mappings

$$\mathcal{S} \ni \phi \mapsto \langle \omega, \phi \rangle \in L^2(\mu_G)$$

and

$$\mathcal{S} \ni \phi \mapsto \langle \omega, \phi \rangle - \int_{\mathbb{R}^d} \phi(x) \, dx \in L^2(\mu_P)$$

may be extended to isometries from $L^2(\mathbb{R}^d)$ into $L^2(\mu_G)$ and $L^2(\mu_P)$ respectively.

DEFINITION 2.2 We define the following processes ([GHLØUZ],[U],[I2])

1. The continuous version of the mapping $\mathbb{R}^d \ni x = (x_1, \cdots, x_d) \mapsto B_x(\omega) \in L^2(\mu_G)$ defined by
$$B_x(\omega) = \langle \omega, \Xi(x_1) \times \cdots \times \Xi(x_d) \rangle$$
is called a d-parameter **Brownian motion**, where $\mathbb{R} \ni s \mapsto \Xi(s) \in L^2(\mathbb{R})$ is defined by

$$\Xi(s) := \begin{cases} \chi_{(0,s]} & s \in [0, \infty) \\ -\chi_{(s,0]} & s \in (-\infty, 0) \end{cases}$$

and χ is the usual indicator function.

2. The right-continuous integer-valued version of the mapping $\mathbb{R}^d \ni x = (x_1, \cdots, x_d) \mapsto P_x(\omega) \in L^2(\mu_P)$ defined by

$$P_x(\omega) := \langle \omega, \Xi(x_1) \times \cdots \times \Xi(x_d) \rangle$$

is called the **Poisson process.**

3. The mapping $\mathbb{R}^d \ni x = (x_1, \cdots, x_d) \mapsto P_x(\omega) - \prod_{i=1}^d x_i \in L^2(\mu_P)$ is called the **compensated Poisson process.**

THEOREM 2.3 (The Wiener-Itô expansion) ([HKPS],[I],[IK])

Every $f \in L^2(\mu_G)$ has the expansion

$$f(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^{nd}} f_n(x) \, dB_x^{\otimes n}(\omega)$$

where $f_n \in \widehat{L^2(\mathbb{R}^{nd})}$ ($\widehat{}$ denotes symmetrization in nd variables) and

$$\|f\|_{L^2(\mu_G)}^2 = \sum_{n=0}^{\infty} n! \|f_n\|_{L^2(\mathbb{R}^{nd})}^2$$

Similarly, every $g \in L^2(\mu_P)$ has the expansion

$$g(\omega) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^{nd}} g_n(x) \, d(P_x - \prod_{i=1}^d x_i)^{\otimes n}(\omega)$$

where $g_n \in \widehat{L^2(\mathbb{R}^{nd})}$ and

$$\|g\|_{L^2(\mu_P)}^2 = \sum_{n=0}^{\infty} n! \|g_n\|_{L^2(\mathbb{R}^{nd})}^2$$

The integrals on the right hand side are in both cases the multiple Wiener integrals.

There exist equivalent expansions of $f \in L^2(\mu_G)$ and $g \in L^2(\mu_P)$ in terms of Hermite polynomials and Charlier polynomials respectively, see ([GHLØUZ],[I]) and below:

For $n = 1, 2, \cdots$ let $\xi_n(x)$ be the Hermite functions of order n defined by

$$\mathcal{S} \ni \xi_n(x) := \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{n-1}(\sqrt{2}x); \quad x \in \mathbb{R}$$

where h_n is the n'th Hermite polynomial defined by

$$h_n(x) := (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}) \quad x \in \mathbb{R}, \quad n = 0, 1, 2, \cdots$$

It is well known that $\{\xi_n\}_{n=1}^{\infty}$ forms an orthonormal basis for $L^2(\mathbb{R})$. Therefore the family $\{e_\alpha\}$ of tensor products

$$e_\alpha := e_{\alpha_1, \cdots, \alpha_d} := \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_d}$$

forms an orthonormal basis for $L^2(\mathbb{R}^d)$. Now assume that the family of all multi-indices $\beta = (\beta_1, \cdots, \beta_d)$ is given a fixed ordering

$$(\beta^{(1)}, \beta^{(2)}, \cdots, \beta^{(n)}, \cdots)$$

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where $\beta^{(k)} = (\beta_1^{(k)}, \cdots, \beta_d^{(k)})$, and put $e_n := e_{\beta(n)}$; $n = 1, 2, \cdots$. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $n \in \mathbb{N}$, define the Hermite polynomial functionals as

$$H_\alpha(\omega) := \prod_{j=1}^n h_{\alpha_j}(\langle \omega, e_j \rangle)$$

and the Charlier polynomial functionals as

$$C_\alpha(\omega) := C_{|\alpha|}(\omega; \overbrace{e_1, \cdots, e_1}^{\alpha_1}, \cdots, \overbrace{e_k, \cdots, e_k}^{\alpha_k})$$

We use the convention

$$C_n(\omega; \eta_1, \cdots, \eta_n) := \frac{\partial}{\partial w_1 \cdots \partial w_n} e^{\{\langle \omega, \log(1 + \sum_{j=1}^n w_j \eta_j) \rangle\} - \sum_{j=1}^n w_j \int_{\mathbb{R}^d} \eta_j(y) \, dy\}} \bigg|_{w_1 = \cdots = w_n = 0}$$

The following two equalities hold true:

$$H_\alpha(\omega) = \int_{\mathbb{R}^{nd}} e^{\otimes |\alpha|} \, dB_x^{\otimes |\alpha|}(\omega)$$

and

$$C_\alpha(\omega) = \int_{\mathbb{R}^{nd}} e^{\otimes |\alpha|} \, d(P_x - \prod_{i=1}^d x_i)^{\otimes |\alpha|}(\omega)$$

It follows that any $f \in L^2(\mu_G)$ also has the representation

$$f(\omega) = \sum_\alpha a_\alpha H_\alpha(\omega)$$

The sum is taken over all multi-indices α of non-negative integers. Moreover, it can be shown that

$$\|f\|_{L^2(\mu_G)}^2 = \sum_\alpha \alpha! a_\alpha^2$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$. Similarly, any $g \in L^2(\mu_P)$ has the representation

$$g(\omega) = \sum_\alpha b_\alpha C_\alpha(\omega)$$

with

$$\|g\|_{L^2(\mu_P)}^2 = \sum_\alpha \alpha! b_\alpha^2$$

REMARK 2.4 With the above results, it is clear that the mapping $\mathcal{U} : L^2(\mu_G) \to L^2(\mu_P)$ defined by

$$\mathcal{U}(\sum_\alpha c_\alpha H_\alpha) := \sum_\alpha c_\alpha C_\alpha$$

is unitary.

DEFINITION 2.5 ([HLØUZ3]) Let $0 \le \rho \le 1$.

- The space of **Gaussian test functions**, $(\mathcal{S})_G^\rho$, consists of all

$$f = \sum_\alpha c_\alpha H_\alpha \in L^2(\mu_G)$$

such that

$$\|f\|_{\rho,k}^2 := \sum_\alpha c_\alpha^2 (\alpha!)^{1+\rho}(2\mathbb{N})^{\alpha k} < \infty \quad \forall k \in \mathbb{N}$$

where

$$(2\mathbb{N})^\alpha := \prod_{i=1}^k (2j)^{\alpha_j} \quad \text{if } \alpha = (\alpha_1, \cdots, \alpha_k).$$

- The space of **Gaussian distributions**, $(\mathcal{S})_G^{-\rho}$, consist of all formal expansions

$$F = \sum_\alpha b_\alpha H_\alpha$$

such that

$$\sum_\alpha b_\alpha^2 (\alpha!)^{1-\rho}(2\mathbb{N})^{-\alpha q} < \infty \quad \text{for some } q \in \mathbb{N}$$

The **Poisson test functions** and **distributions**, denoted by $(\mathcal{S})_P^\rho$ and $(\mathcal{S})_P^{-\rho}$ respectively, are defined similarly (H_α will in this case be replaced by C_α).

We know that $(\mathcal{S})_G^{-\rho}$ and $(\mathcal{S})_P^{-\rho}$ are the topological duals of $(\mathcal{S})_G^\rho$ and $(\mathcal{S})_P^\rho$ respectively (when the latter spaces have the topology given by the seminorms $\|\cdot\|_{\rho,k}$). With $F = \sum b_\alpha H_\alpha \in (\mathcal{S})_G^{-\rho}$ ($G = \sum b_\alpha C_\alpha \in (\mathcal{S})_P^{-\rho}$) and $f = \sum c_\alpha H_\alpha \in (\mathcal{S})_G^\rho$ ($g = \sum c_\alpha C_\alpha \in (\mathcal{S})_P^\rho$)

$$\langle F, f \rangle = \langle G, g \rangle = \sum_\alpha b_\alpha c_\alpha \alpha!.$$

Moreover, it is obvious that we have the inclusions

$$(\mathcal{S})_X^1 \subset (\mathcal{S})_X^\rho \subset L^2(\mu_X) \subset (\mathcal{S})_X^{-\rho} \subset (\mathcal{S})_X^{-1} , \quad \rho \in [0,1]$$

where X stands for G or P. In the rest of this section we will consider the larger spaces $(\mathcal{S})_G^{-1}$ and $(\mathcal{S})_P^{-1}$.

REMARK 2.6 It is obvious how to extend the mapping $\mathcal{U}$ to an isomorphism $\mathcal{U} : (\mathcal{S})_G^{-\rho} \to (\mathcal{S})_P^{-\rho}$, $\rho \in [0,1]$. It is also easy to see that the restriction of $\mathcal{U}$ to $(\mathcal{S})_G^\rho$ is a unititary mapping from $(\mathcal{S})_G^\rho$ to $(\mathcal{S})_P^\rho$, $\rho \in [0,1]$.

DEFINITION 2.7 ([HLØUZ3]) The **Wick product** of two elements in $(\mathcal{S})_G^{-1}$ given by

$$F = \sum_\alpha a_\alpha H_\alpha \ , \quad G = \sum_\beta b_\beta H_\beta$$

is defined by

$$F \diamond G = \sum_\gamma c_\gamma H_\gamma$$

where

$$c_\gamma = \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta.$$

The Wick product of two elements in $(\mathcal{S})_P^{-1}$ is similarly defined.

LEMMA 2.8 ([HLØUZ3],[KLS]) The following holds true $(X = G, P)$

1. $F, G \in (\mathcal{S})_X^{-1} \Rightarrow F \diamond G \in (\mathcal{S})_X^{-1}$

2. $f, g \in (\mathcal{S})_X^{1} \Rightarrow f \diamond g \in (\mathcal{S})_X^{1}$

REMARK 2.9 Note that $\mathcal{U}$ preserves the Wick product but not the ordinary product.

DEFINITION 2.10 Let $F = \sum b_\alpha H_\alpha \in (\mathcal{S})_G^{-1}$. Then the **Hermite transform** of F, denoted by $\mathcal{H}_G F$, is defined to be (whenever convergent)

$$\mathcal{H}_G F := \sum_\alpha b_\alpha z^\alpha$$

where $z = (z_1, z_2, \cdots)$ and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_k^{\alpha_k}$ if $\alpha = (\alpha_1, \cdots, \alpha_k)$.

The Hermite transform $\mathcal{H}_P$ for Poisson distributions is defined similarly.

LEMMA 2.11 If $F, G \in (\mathcal{S})_X^{-1}, \ (X = G, P)$ then

$$\mathcal{H}_X(F \diamond G)(z) = \mathcal{H}_X F(z) \cdot \mathcal{H}_X G(z)$$

for all z such that $\mathcal{H}_X F(z)$ and $\mathcal{H}_X G(z)$ exist.

LEMMA 2.12 Suppose $g(z_1, z_2, \cdots)$ is a bounded analytic function on $\mathbf{B}_q(\delta)$ for some $\delta > 0, q < \infty$ where

$$\mathbf{B}_q(\delta) := \left\{ \eta = (\eta_1, \eta_2, \cdots) \in \mathbb{C}_0^{\mathbb{N}}; \sum_{\alpha \neq 0} |\eta^\alpha|^2 (2\mathbf{N})^{\alpha q} < \delta^2 \right\}$$

Then there exist $F \in (\mathcal{S})_G^{-1}$ and $D \in (\mathcal{S})_P^{-1}$ such that $\mathcal{H}_G F = g = \mathcal{H}_P D$.

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LEMMA 2.13 ([KLS]) Suppose $Z \in (\mathcal{S})_X^{-1}$ $(X = G, P)$ and that f is an analytic function in a neighborhood of $\mathcal{H}_X Z(0)$ in $\mathbb{C}$. Then there exists $Q \in (\mathcal{S})_X^{-1}$ $(X = G, P)$ such that $\mathcal{H}_X Q = f \circ \mathcal{H}_X Z$.

EXAMPLE 2.14 Let us calculate the Wiener-Itô expansions of the d-parameter Brownian motion $B_x(\omega)$ and the d-parameter compensated Poisson process $P_x(\omega) - \prod_{i=1}^{d} x_i$:

First note that we have the expansion

$$\Xi(x_1) \times \cdots \times \Xi(x_n) = \sum_{k=1}^{\infty} (\Xi(x_1) \times \cdots \times \Xi(x_n), e_k)_{L^2(\mathbb{R}^d)} e_k$$

where the convergence is in $L^2(\mathbb{R}^d)$. From this it follows that

$$
\begin{aligned}
B_x(\omega) &= \langle \omega, \Xi(x_1) \times \cdots \times \Xi(x_n) \rangle \\
&= \sum_{k=1}^{\infty} (\Xi(x_1) \times \cdots \times \Xi(x_n), e_k)_{L^2(\mathbb{R}^d)} \langle \omega, e_k \rangle \\
&= \sum_{k=1}^{\infty} (\Xi(x_1) \times \cdots \times \Xi(x_n), e_k)_{L^2(\mathbb{R}^d)} H_{\epsilon_k}(\omega)
\end{aligned}
$$

where $\epsilon_k = (0, \cdots, 0, 1)$ with 1 on the k'th place. The convergence is in $L^2(\mu_G)$.

Similarly, we obtain

$$
\begin{aligned}
P_x(\omega) - \prod_{i=1}^{d} x_i &= \langle \omega, \Xi(x_1) \times \cdots \times \Xi(x_n) \rangle - \prod_{i=1}^{d} x_i \\
&= \lim_{j \to \infty} \left(\sum_{k=1}^{j} (\Xi(x_1) \times \cdots \times \Xi(x_n), e_k)_{L^2(\mathbb{R}^d)} \langle \omega, e_k \rangle \right. \\
&\qquad \left. - \int_{\mathbb{R}^d} \sum_{k=1}^{j} (\Xi(x_1) \times \cdots \times \Xi(x_n), e_k)_{L^2(\mathbb{R}^d)} e_k(y) \, dy \right) \\
&= \lim_{j \to \infty} \sum_{k=1}^{j} (\Xi(x_1) \times \cdots \times \Xi(x_n), e_k)_{L^2(\mathbb{R}^d)} \left(\langle \omega, e_k \rangle - \int_{\mathbb{R}^d} e_k(y) \, dy \right) \\
&= \sum_{k=1}^{\infty} (\Xi(x_1) \times \cdots \times \Xi(x_n), e_k)_{L^2(\mathbb{R}^d)} C_{\epsilon_k}(\omega)
\end{aligned}
$$

The convergence is in $L^2(\mu_P)$.

REMARK 2.15 From the above example it is clear that $\mathcal{U}(B_x) = P_x - \prod_{i=1}^{d} x_i$.

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DEFINITION 2.16 ([GHLØUZ],[U][I2]) We define the following generalized processes

1. The d-parameter **white noise process** $W_x(\omega)$ is defined by the formal expansion

$$W_x(\omega) := \sum_{k=1}^{\infty} e_k(x) H_{\epsilon_k}(\omega).$$

2. The d-parameter **compensated Poisson noise**, denoted by $\dot{P}_x(\omega) - 1$, is defined by the formal expansion

$$\dot{P}_x(\omega) - 1 := \sum_{k=1}^{\infty} e_k(x) C_{\epsilon_k}(\omega).$$

LEMMA 2.17 ([GHLØUZ],[U],[KSWY]) We have the following relations

1. $W_x(\omega) \in (\mathcal{S})_G^{-\rho}$; $\rho \in [0,1]$

2. $\dot{P}_x(\omega) - 1 \in (\mathcal{S})_P^{-\rho}$; $\rho \in [0,1]$

3. $\frac{\partial^n}{\partial x_1 \cdots \partial x_d} B_x(\omega) = W_x(\omega)$

4. $\frac{\partial^n}{\partial x_1 \cdots \partial x_d} \left(P_x(\omega) - \prod_{i=1}^{d} x_i \right) = \dot{P}_x(\omega) - 1$

where differentiation is coefficient-wise.

For the rest of this section we assume that the parameter d is equal to 1. We study stochastic integration:

DEFINITION 2.18 The process $\mathbb{R} \ni t \mapsto F(t) = \sum_{\alpha} c(t)_{\alpha} H_{\alpha}(\omega)$ is said to be **generalized Skorohod integrable** with respect to the Gaussian measure μ_G if

$$\int_{\mathbb{R}} F(t) \delta B_t(\omega) := \sum_{\alpha} \int_{\mathbb{R}} c_{\alpha}(t)\, dB_t(\omega) \diamond H_{\alpha}(\omega)$$

exists in $(\mathcal{S})_G^{-1}$ ([HØUZ]).

Similarly, the process $\mathbb{R} \ni t \mapsto D(t) = \sum_{\alpha} b(t)_{\alpha} C_{\alpha}(\omega)$ is said to be **generalized Skorohod integrable** with respect to the Poisson measure μ_P if

$$\int_{\mathbb{R}} D(t) \delta(P_t(\omega) - t) := \sum_{\alpha} \int_{\mathbb{R}} b_{\alpha}(t)\, d(P_t(\omega) - t) \diamond C_{\alpha}(\omega)$$

exists in $(\mathcal{S})_P^{-1}$.

LEMMA 2.19 Let $\mathcal{F}_t^{X_t}$ be the σ-algebra generated by the stochastic process X_t. Suppose that we are given a $\mathcal{F}_t^{B_t}$-adapted process $F(t)$ and a $\mathcal{F}_t^{P_{t-t}}$-adapted process $D(t)$ such that

$$\int_{\mathbb{R}} E_{\mu_G}[F^2(t)]\, dt < \infty \quad ; \quad \int_{\mathbb{R}} E_{\mu_P}[D^2(t)]\, dt < \infty.$$

Then we have existence and equalities of the following integrals:

1. $\int\limits_{\mathbb{R}} F(t)\, \delta B_t(\omega) = \int\limits_{\mathbb{R}} F(t)\, dB_t(\omega)$

2. $\int\limits_{\mathbb{R}} D(t)\, \delta(P_t(\omega) - t) = \int\limits_{\mathbb{R}} D(t)\, d(P_t(\omega) - t)$

PROOF:

Statement 1 can be found in [HØUZ] and statement 2 follows by applying the unitary mapping $\mathcal{U}$ on statement 1. ∎

The next theorem states a useful connection between the Skorohod integrals and the Wick product with noise processes.

THEOREM 2.20 We have the following

1. $\int\limits_{\mathbb{R}} Y(t)\, \delta B_t = \int\limits_{\mathbb{R}} Y(t) \diamond W_t\, dt$

2. $\int\limits_{\mathbb{R}} Z(t)\, \delta(P_t - t) = \int\limits_{\mathbb{R}} Y(t) \diamond (\dot{P}_t - 1)\, dt$

The integrals on the right hand side are understood in the sense of Pettis integrals.

PROOF:

Statement 1 can be found in [HØUZ] and statement 2 follows again by applying the unitary mapping $\mathcal{U}$ on statement 1. ∎

§3 Stochastic partial differential equations with Gaussian or Poisson noise

In this section we consider Burgers equation with additive noise and a transport equation with gradient coupled noise. Both equations have been treated in the Gaussian case by several authors. See [BCL], [HLØUZ2], [HLØUZ4] for a consideration of Burgers equation, and [Ch], [GHØUZ],

[P], [DP], [BDP] for a treatment of different types of transport equations with gradient coupled noise. We also mention the work by [BS] on Burgers equation disturbed by a non-Gaussian additive force and [D] on Burgers equation with an additive compensated Poisson noise force.

The main idea we want to illustrate in this section is that on the $\mathcal{H}$-transform level, both Gaussian and Poisson stochastic equations are identical. That means, after $\mathcal{H}$-transforming a Poisson or Gaussian stochastic equation, we end up with considering the same deterministic equation. This since the $\mathcal{H}$-transform in both cases transforms the Wick product into ordinary pointwise multiplication. Moreover $\mathcal{H}_P$-transform of the compensated Poisson noise is equal to the $\mathcal{H}_G$-transform of white noise.

§3.1 The 1-dimensional Burgers equation with compensated Poisson noise and Gaussian noise

We consider the Burgers equation

$$u_t + u \diamond u_x = \nu u_{xx} + W_{t,x} \tag{1}$$
$$u(0,x) = -h_x(x) \diamond h(x)^{\diamond(-1)} \tag{2}$$

where h is a continuous differentiable $(\mathcal{S})_G^{-1}$-process. This has a solution in $(\mathcal{S})_G^{-1}$ (see [HLØUZ4]) given by

$$u(t,x) = -\nu \Phi_x(t,x) \diamond \Phi(t,x)^{\diamond(-1)}$$

where

$$\Phi(t,x) = \hat{E}[h(b_{\alpha t}^{\diamond\frac{1}{2\nu}} \diamond Exp\{\frac{1}{2\nu} \int_0^t N(s, b_{\alpha t}\, ds\}]$$

and

$$N(s,x) = \sum_{k=0}^{\infty} (\int_0^x e_k(s,y)\, dy) H_{\epsilon_k}(\omega)$$

b_t is a Brownian motion and $\alpha = \sqrt{2\nu}$. By $\mathcal{H}_G$-transforming, we obtain the deterministic Burgers equation

$$\hat{u} + \hat{u} \cdot \hat{u}_x = \nu \hat{u}_{xx} + \sum_k e_k(t,x) z_k$$

$$\hat{u}(0,x) = -\hat{h}_x/\hat{h}$$

which has solution

$$\hat{u}(t,x) = -\nu \frac{\hat{\Phi}_x(t,x)}{\hat{\Phi}(t,x)}$$

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Here

$$\hat{\Phi}(t,x) = \hat{E}^x[h(b_{\alpha t})^{\frac{1}{2\nu}} \cdot \exp\{\frac{1}{2\nu}\int_0^t \hat{N}(s,b_{\alpha s})\,ds\}]$$

and

$$\hat{N}(s,x) = \sum_k \int_0^x e_k(s,y)\,dy \cdot z_k$$

But if we $\mathcal{H}_P^{-1}$-transform this equation, it is easily seen that

$$v(t,x) = -\nu\Psi_x(t,x) \diamond \Psi(t,x)^{\diamond(-1)}$$

where

$$\Psi(t,x) = \hat{E}[h(b_{\alpha t}^{\diamond\frac{1}{2\nu}}) \diamond \text{Exp}\{\frac{1}{2\nu}\int_0^t M(s,b_{\alpha s})\,ds\}]$$

Moreover, $v(t,x)$ is a $(\mathcal{S})_P^{-1}$-solution of

$$v_t + v \diamond v_x = \nu v_{xx} + (\dot{P}_{tx} - 1) \tag{3}$$

We have set

$$M(s,x) = \sum_k (\int_0^x e_k(s,y)\,dy)C_{\epsilon_k}$$

Hence, we see that the solution in $(\mathcal{S})_P^{-1}$ of (3) is given as

$$v(t,x) = \mathcal{H}_P^{-1}\mathcal{H}_G u(t,x).$$

where u is the $(\mathcal{S})_G^{-1}$ solution of (1).

§3.2 The transport equation with gradient coupled noise

The transport equation with a gradient term disturbed by white noise

$$u_t = \frac{1}{2}\nu u_{xx} + u_x \diamond W_t \ , \ \nu > 0$$
$$u(0,x) = f(x) \in C_b^2(\mathbb{R})$$

has a solution in $(\mathcal{S})_G^{-0}$ given by ([GHØUZ], [P])

$$u(t,x,\omega) = \frac{1}{2\pi\nu t}\int_{\mathbb{R}} \text{Exp}\{-\frac{1}{2\nu t}(y - B_t)^2\}f(x-y)\,dy$$

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Taking $\mathcal{H}_G$-transform, we obtain the partial differential equation

$$\hat{u}_t = \frac{1}{2}\nu\hat{u}_{xx} + \hat{u}_x \cdot \sum_k e_k(t)z_k$$

with solution

$$\hat{u}(y, x; z) = \frac{1}{2\pi\nu t}\int_{\mathbb{R}} \exp\{-\frac{1}{2\nu t}(y - \sum_k \int_0^t e_k(s)z_k \, ds)^2\}f(x - y)\, dy$$

Invoking the $\mathcal{H}_P^{-1}$-transform, we obtain the transport equation with a gradient term disturbed by compensated Poisson noise

$$\nu_t = \frac{1}{2}\nu\nu_{xx} + \nu_x \diamond (\dot{P}_t - 1)$$
$$\nu(0, x) = f(x)$$

with solution in $(\mathcal{S})_P^{-0}$ given by

$$\nu(t, x, \omega) = \frac{1}{2\pi\nu t}\int_{\mathbb{R}} \text{Exp}\{-\frac{1}{2\nu t}(y - (P_t - t))^2\}f(x - y)\, dy$$

Also in this case we see that the solution of the equation disturbed by compensated Poisson noise is related to the solution of the equation disturbed by white noise through

$$\nu(t, x, \omega) = \mathcal{H}_P^{-1}\mathcal{H}_G u(t, x, \omega)$$

REMARK 3.1 In [P] it is shown that

$$u(t, x, \cdot) \in L^2(\mu_G)$$

whenever $\nu > 1$. Since $\mathcal{U}$ is unitary with $\mathcal{U}(u) = \nu$, we obtain similarly

$$\nu(t, x, \cdot) \in L^2(\mu_P)$$

whenever $\nu > 1$.

REMARK 3.2 Note that the unitary mapping $\mathcal{U}$ is given as $\mathcal{H}_P^{-1}\mathcal{H}_G$.

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