

# **A stochastic model for slow-drift motions of offshore structures**

by

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## **Abstract**

The slow-drift motions of an offshore structure is modelled by a second order stochastic differential equation. This equation is transformed into a stochastic Volterra equation, which is solved by means of stochastic calculus and the Wick product. Special emphasis is paid to the role of time-dependent wave-drift damping. The solution is used to deduce probabilistic properties of the slow-drift motions.

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- §1 Introduction. Formulation of a stochastic model for slow-drift motions.
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- §5 Concluding remarks.
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## §1. Introduction

Floating and moored offshore structures of large volume, like oil platforms and ships, can perform oscillatory motions in the surface of a sea. Such motions are determined by the environmental loads due to wind, waves and currents, by the damping forces, and by the properties of the moorings. The motions can often be considered as composed of two parts; one part oscillating with the frequencies of the incoming waves and the wind gusts, and a second part dominated by oscillations with frequencies being close to the lowest resonance frequency of the mechanical system consisting of body and moorings.

This resonance frequency is in several cases small compared to the frequencies of the waves and the wind. The resulting motions of the body are then oscillations with the frequencies of the waves and the wind gusts superposed on oscillations with frequencies centered about the relatively much lower resonance frequency of the body/moorings. The latter part is most often termed slow-drift motions since the force generating this motion has resemblance with the force causing a steady drift of an unmoored body. Horizontal slow-drift motions of an offshore structure may in severe sea states have quite large amplitudes. These motions are important to analyze when designing the structure.

In the present paper we develop a stochastic model for horizontal slow-drift motions of a floating and moored offshore structure in an irregular sea. Only the effect of waves is accounted for. The model is applied to find the probabilistic properties of the motion, and may serve to analyze extreme values of the motion.

One of the purposes of the study is to analyze the effect of time-dependent wave-drift damping, which appears in the equation for the slow-drift motion. The model is simplified sufficiently to obtain convenient formulae for the slow-drift motions, without removing the essence of the problem. Some of the probabilistic properties of the motion may then be obtained by formulae in explicit form, being easy to interpret.

We assume that the slow-drift motions are in one horizontal direction, which may be relevant when the sea is longcrested. Let the coordinate of the slow-drift position of the body be denoted by  $x(t)$  and the slow-drift velocity by  $\dot{x}(t)$ , where a dot denotes time derivative.  $x(t)$  and  $\dot{x}(t)$  are real functions of time  $t$ . The slow-drift force is quadratic in the characteristic wave amplitude of the irregular sea, and is a function of the (small) slow-drift velocity  $\dot{x}$ . The slow-drift force may be written

$$(1.1) \quad F(t) = T(t)a^2(t) - \alpha(t)\dot{x}a^2(t)$$

Here,  $a(t)$  denotes the local slowly-varying amplitude of the irregular sea, and  $T(t)$  and  $\alpha(t)$  denote slowly varying functions of time, which depend on the geometry of the body and the local (slowly-varying) frequency of the incoming waves. Equation (1.1) is based on the references [F, pp. 155-168], [GP], [M], [N], [NGP] and [S], where also methods are described how to obtain  $a(t)$ ,  $T(t)$  and  $\alpha(t)$ .

The first term of (1.1) appears as an excitation force in the equation for the slow-drift motions. The second term, known as the wave-drift damping, appears as a time-dependent damping force. We note that the wave-drift damping represents a positive damping force provided that  $\alpha(t) > 0$ , which usually is the case.

One of the interesting features of (1.1) is that the magnitude of the excitation force and the damping force are correlated, and are competing forces. When the excitation force is small or moderate, then the damping force is correspondingly small or moderate. However, when the exciting force is large, leading to motions of large amplitude, then the damping of the motions is also large. This is a feature which is expected to significantly limit extreme slow-drift motions. The effect of time-dependent wave-drift damping has been suggested earlier, see [ES], [G], and [ZF]. Its role is not yet, however, given a clear interpretation. This is a motivation for the present study.

To further simplify the analysis, we shall approximate  $T(t)$  and  $\alpha(t)$  by the positive constants  $T_0$  and  $\alpha_0$ , respectively. This may be a valid approximation in many examples. The slow-drift force (1.1) then reads

$$(1.2) \quad F(t) = T_0 a^2(t) - \alpha_0 \dot{x} a^2(t)$$

Assuming that the moorings are linear, the mooring force is given by  $-cx(t)$ , where  $c$  is the spring constant. The equation of motion for  $x(t)$  then becomes

$$(1.3) \quad (m_{11} + m)\ddot{x} + cx = F(t)$$

Here,  $m_{11}$  and  $m$  denote the added mass and the mass, respectively, of the offshore structure. If we put

$$(1.4) \quad a_0 = \overline{a^2(t)}/(m_{11} + m), \theta^2 = c/(m_{11} + m)$$

$$(1.5) \quad T_0 = \overline{T(t)}/(m_{11} + m), \alpha_0 = \overline{\alpha(t)}/(m_{11} + m)$$

where a bar denotes time average, we obtain

$$(1.6) \quad \ddot{x} + a_0 \dot{x} + \theta^2 x = T_0 a_0 + (T_0 - \alpha_0 \dot{x})(a^2(t) - \overline{a^2(t)})$$

The term  $T_0 a_0$  is constant, and leads to a trivial constant in the solution for  $x(t)$ , which hereafter is left out of the analysis. A quadratic damping term due to viscous drag forces, proportional to  $\dot{x}|\dot{x}|$ , should also be included in (1.6). By applying equivalent linearization this damping term may be approximated by a linear damping term, see [F, pp. 98-100]. The effect of viscous drag is then in principle included in (1.6) by the term  $a_0 \dot{x}$ .

To find  $x(t)$  we may proceed by generating a numerical approximation of  $a(t)$ , and integrating the differential equation (1.6) by numerical methods.

Here a different approach is followed, as we replace equation (1.6) by a corresponding second order stochastic differential equation, and consider the solution of this equation. The first step is to replace the function

$$(1.7) \quad a^2(t) - \overline{a^2(t)}$$

by

$$(1.8) \quad \eta W_t$$

Here,  $W_t$  denotes singular white noise, see §2, and  $\eta$  denotes the ‘amplitude’ of the noise. The motion  $x(t)$  is then governed by the equation

$$(1.9) \quad \ddot{x} + a_0 \dot{x} + \theta^2 x = (T_0 - \alpha_0 \dot{x})\eta W_t$$

By re-organization we obtain

$$(1.10) \quad \ddot{x} + (a_0 + \alpha_0 \eta W_t) \dot{x} + \theta^2 x = T_0 \eta W_t$$

This equation contains white noise both in the excitation force and in a part of the damping force. The solution of this equation is found and discussed in the subsequent parts of the paper.

## §2. White noise, the Wick product and stochastic integration

Here we give a short introduction to the white noise theory needed to make precise - and to solve - the mathematical model presented in §1.

General references for this section are [H], [HKPS], [HØUZ], [HP], [LØU 1-3], [Ø1], [Ø2] and [GHLØUZ].

We start with the construction of the *white noise probability space*  $(\mathcal{S}', \mathcal{B}, \mu)$ :

Let  $\mathcal{S} = \mathcal{S}(\mathbf{R})$  be the Schwartz space of rapidly decreasing smooth functions on  $\mathbf{R}$  with the usual topology and let  $\mathcal{S}' = \mathcal{S}'(\mathbf{R})$  be its dual (the space of *tempered distributions*). Let  $\mathcal{B}$  denote the family of all Borel subsets of  $\mathcal{S}'(\mathbf{R})$  (equipped with the weak-star topology). If  $\omega \in \mathcal{S}'$  and  $\phi \in \mathcal{S}$  we let

$$(2.1) \quad \omega(\phi) = \langle \omega, \phi \rangle$$

denote the action of  $\omega$  on  $\phi$ . (For example, if  $\omega$  is a measure  $m$  on  $\mathbf{R}$  then

$$\langle \omega, \phi \rangle = \int_{\mathbf{R}} \phi(x) dm(x)$$

and if  $\omega$  is evaluation at  $x_0 \in \mathbf{R}$  then

$$\langle \omega, \phi \rangle = \phi(x_0) \quad \text{etc.})$$

By the Minlos theorem [GV] there exists a probability measure  $\mu$  on  $\mathcal{S}'$  such that

$$(2.2) \quad \int_{\mathcal{S}'} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2} \quad ; \quad \phi \in \mathcal{S}$$

where

$$(2.3) \quad \|\phi\|^2 = \int_{\mathbf{R}} |\phi(x)|^2 dx = \|\phi\|_{L^2(\mathbf{R})}^2.$$

$\mu$  is called the *white noise probability measure* and  $(\mathcal{S}', \mathcal{B}, \mu)$  is called the *white noise probability space*.

**DEFINITION 2.1** The (smoothed) white noise process is the map

$$W : \mathcal{S} \times \mathcal{S}' \rightarrow \mathbf{R}$$

given by

$$(2.4) \quad W(\phi, \omega) = W_\phi(\omega) = \langle \omega, \phi \rangle \quad ; \quad \phi \in \mathcal{S}, \omega \in \mathcal{S}'$$

From  $W_\phi$  we can construct Brownian motion  $B_t$  as follows:

**STEP 1.** (The Ito isometry)

$$(2.5) \quad E_\mu[\langle \cdot, \phi \rangle^2] = \|\phi\|^2 \quad ; \quad \phi \in \mathcal{S}$$

where  $E_\mu$  denotes expectation w.r.t.  $\mu$ , so that

$$E_\mu[\langle \cdot, \phi \rangle^2] = \int_{\mathcal{S}'} \langle \omega, \phi \rangle^2 d\mu(\omega).$$

**STEP 2.** Use Step 1 to define, for arbitrary  $\psi \in L^2(\mathbf{R})$ ,

$$(2.6) \quad \begin{aligned} \langle \omega, \psi \rangle &:= \lim \langle \omega, \phi_n \rangle, \\ \text{where } \phi_n &\in \mathcal{S} \quad \text{and} \quad \phi_n \rightarrow \psi \text{ in } L^2(\mathbf{R}) \end{aligned}$$

**STEP 3.** Use Step 2 to define

$$(2.7) \quad \tilde{B}_t(\omega) := \langle \omega, \chi_{[0,t]}(\cdot) \rangle \quad \text{for } t \geq 0$$

by choosing

$$\psi(s) = \chi_{[0,t]}(s) = \begin{cases} 1 & \text{if } s \in [0, t] \\ 0 & \text{if } s \notin [0, t] \end{cases}$$

which belongs to  $L^2(\mathbf{R})$  for all  $t \geq 0$ .

**STEP 4.** Prove that  $\tilde{B}_t$  has a *continuous modification*  $B_t$ , i.e.

$$P[\tilde{B}_t(\cdot) = B_t(\cdot)] = 1 \quad \text{for all } t.$$

This continuous process  $B_t$  is a Brownian motion.

From this it follows that the relation between white noise  $W_\phi(\omega)$  and Brownian motion  $B_t(\omega)$  is

$$(2.8) \quad W_\phi(\omega) = \int_{\mathbf{R}} \phi(t) dB_t(\omega) \quad ; \quad \phi \in \mathcal{S}$$

where the integral on the right is the Wiener-Itô integral.

### The Wiener-Itô chaos expansion

Define the *Hermite polynomials*  $h_n(x)$  by

$$(2.9) \quad h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} (e^{-\frac{x^2}{2}}); \quad n = 0, 1, 2, \dots$$

This gives for example

$$\begin{aligned} h_0(x) &= 1, h_1(x) = x, h_2(x) = x^2 - 1, h_3(x) = x^3 - 3x \\ h_4(x) &= x^4 - 6x^2 + 3, h_5(x) = x^5 - 10x^3 + 15x, \dots \end{aligned}$$

Let  $e_k$  be the  $k$ 'th *Hermite function* defined by

$$(2.10) \quad e_k(x) = \pi^{-\frac{1}{4}} ((n-1)!)^{-\frac{1}{2}} \cdot e^{-\frac{x^2}{2}} h_{k-1}(\sqrt{2}x); \quad k = 1, 2, \dots$$

Then  $\{e_k\}_{k \geq 1}$  constitutes an orthonormal basis for  $L^2(\mathbf{R})$  and  $e_k \in \mathcal{S}$  for all  $k$ .

Define

$$(2.11) \quad \theta_k(\omega) = \langle \omega, e_k \rangle = W_{e_k}(\omega) = \int_{\mathbf{R}} e_k(x) dB_x(\omega)$$

Let  $\mathcal{J}$  denote the set of all finite multi-indices  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$  ( $m = 1, 2, \dots$ ) of non-negative integers  $\alpha_i$ . If  $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathcal{J}$  we put

$$(2.12) \quad H_\alpha(\omega) = \prod_{j=1}^m h_{\alpha_j}(\theta_j)$$

For example, if  $\alpha = \epsilon_k = (0, 0, \dots, 1)$  with 1 on  $k$ 'th place, then

$$H_{\epsilon_k}(\omega) = h_1(\theta_k) = \langle \omega, e_k \rangle,$$

while

$$H_{3,0,2}(\omega) = h_3(\theta_1)h_0(\theta_2)h_2(\theta_3) = (\theta_1^3 - 3\theta_1) \cdot (\theta_3^2 - 1).$$

The family  $\{H_\alpha(\cdot)\}_{\alpha \in \mathcal{J}}$  is an orthogonal basis for the Hilbert space

$$(2.13) \quad L^2(\mu) = \{X : \mathcal{S}' \rightarrow \mathbf{R} \text{ such that } \|X\|_{L^2(\mu)}^2 := \int_{\mathcal{S}'} X(\omega)^2 d\mu(\omega) < \infty\}.$$

In fact, we have

**THEOREM 2.2 (The Wiener-Ito chaos expansion theorem I)**

For all  $X \in L^2(\mu)$  there exist (uniquely determined) numbers  $c_\alpha \in \mathbf{R}$  such that

$$(2.14) \quad X(\omega) = \sum_{\alpha} c_{\alpha} H_{\alpha}(\omega).$$

Moreover, we have

$$(2.15) \quad \|X\|_{L^2(\mu)}^2 = \sum_{\alpha} \alpha! c_{\alpha}^2$$

where  $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_m!$  if  $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ .

There is an equivalent formulation of this theorem in terms of *multiple Ito integrals*:

If  $\psi(t_1, t_2, \cdots, t_n)$  is a real symmetric function in its  $n$  (real) variables  $t_1, \cdots, t_n$  and  $\psi \in L^2(\mathbf{R}^n)$ , i.e.

$$(2.16) \quad \|\psi\|_{L^2(\mathbf{R}^n)} := \left[ \int_{\mathbf{R}^n} |\psi(t_1, t_2, \cdots, t_n)|^2 dt_1 dt_2 \cdots dt_n \right]^{1/2} < \infty$$

then its  $n$ -tuple *Ito integral* is defined by

$$(2.17) \quad \int_{\mathbf{R}^n} \psi dB^{\otimes n} := n! \int_{-\infty}^{\infty} \left( \int_{-\infty}^{t_n} \left( \int_{-\infty}^{t_{n-1}} \cdots \left( \int_{-\infty}^{t_2} \psi(t_1, t_2, \cdots, t_n) dB_{t_1} \right) dB_{t_2} \cdots \right) dB_{t_n} \right)$$

where the integral on the right consists of  $n$  iterated Ito integrals (note that in each step the corresponding integrand is adapted because of the upper limits of the preceding integrals). Applying the Ito isometry  $n$  times we see that

$$(2.18) \quad E\left[\left(\int_{\mathbf{R}^n} \psi dB^{\otimes n}\right)^2\right] = n! \|\psi\|_{L^2(\mathbf{R}^n)}^2; \quad n \geq 1$$

For  $n = 0$  we adopt the convention that

$$(2.19) \quad \int_{\mathbf{R}^0} \psi dB^{\otimes 0} = \psi = \|\psi\|_{L^2(\mathbf{R}^0)} \quad \text{when } \psi \text{ is constant}$$

Let  $\hat{L}^2(\mathbf{R}^n)$  denote the set of symmetric real functions (on  $\mathbf{R}^n$ ) which are square integrable with respect to Lebesgue measure. Then we have:

**THEOREM 2.3 (The Wiener-Ito chaos expansion theorem II)**

For all  $X \in L^2(\mu)$  there exist (uniquely determined) functions  $f_n \in \hat{L}^2(\mathbf{R}^n)$  such that

$$(2.20) \quad X(\omega) = \sum_{n=0}^{\infty} \int_{\mathbf{R}^n} f_n dB^{\otimes n}(\omega)$$

Moreover, we have

$$(2.21) \quad \|X\|_{L^2(\mu)}^2 = \sum_{n=0}^{\infty} n! \|f_n\|_{L^2(\mathbf{R}^n)}^2$$

**REMARK** The connection between these two expansions in Theorem 2.2 and Theorem 2.3 is given by

$$(2.22) \quad f_n = \sum_{\substack{\alpha \in \mathcal{J} \\ |\alpha|=n}} c_\alpha e_1^{\otimes \alpha_1} \hat{\otimes} e_2^{\otimes \alpha_2} \hat{\otimes} \dots \hat{\otimes} e_m^{\otimes \alpha_m} \quad ; \quad n = 0, 1, 2, \dots$$

where  $|\alpha| = \alpha_1 + \dots + \alpha_m$  if  $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathcal{J}$  ( $m = 1, 2, \dots$ ). The functions  $e_1, e_2, \dots$  are defined in (2.10) and  $\otimes$  and  $\hat{\otimes}$  denote tensor product and symmetrized tensor product, respectively. For example, if  $f$  and  $g$  are real functions on  $\mathbf{R}$  then

$$(f \otimes g)(x_1, x_2) = f(x_1)g(x_2)$$

and

$$(f \hat{\otimes} g)(x_1, x_2) = \frac{1}{2} [f(x_1)g(x_2) + f(x_2)g(x_1)] \quad ; \quad (x_1, x_2) \in \mathbf{R}^2.$$

Analogous to the test functions  $\mathcal{S}(\mathbf{R})$  and the tempered distributions  $\mathcal{S}'(\mathbf{R})$  on the real line  $\mathbf{R}$ , there is a useful space of *stochastic test functions*  $(\mathcal{S})$  and a space of *stochastic distributions*  $(\mathcal{S})^*$  on the white noise probability space:

**DEFINITION 2.4** ([Z])

a) We say that  $f = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha \in L^2(\mu)$  belongs to the *Hida test function space*  $(\mathcal{S})$  if

$$(2.23) \quad \sum_{\alpha \in \mathcal{J}} \alpha! c_\alpha^2 \left\{ \prod_{j=1}^{\infty} (2j)^{\alpha_j} \right\}^k < \infty \quad \text{for all } k < \infty$$

b) A formal sum  $F = \sum_{\alpha \in \mathcal{J}} b_\alpha H_\alpha$  belongs to the *Hida distribution space*  $(\mathcal{S})^*$  if

$$(2.24) \quad \text{there exists } q < \infty \text{ s.t. } \sum_{\alpha \in \mathcal{J}} \alpha! c_\alpha^2 \left\{ \prod_{j=1}^{\infty} (2j)^{\alpha_j} \right\}^{-q} < \infty$$

$(\mathcal{S})^*$  is the dual of  $(\mathcal{S})$ . The action of  $F = \sum_{\alpha} b_\alpha H_\alpha \in (\mathcal{S})^*$  on  $f = \sum_{\alpha} a_\alpha H_\alpha \in (\mathcal{S})$  is given by

$$\langle F, f \rangle = \sum_{\alpha} \alpha! a_\alpha b_\alpha$$

We have the inclusions

$$(\mathcal{S}) \subset L^2(\mu) \subset (\mathcal{S})^*.$$



### EXAMPLE 2.5

- a) The *smoothed white noise*  $W_\phi(\cdot)$  belongs to  $(\mathcal{S})$  if  $\phi \in \mathcal{S}$ , because if  $\phi = \sum_j c_j e_j$  we have

$$(2.25) \quad W_\phi = \sum_j c_j H_{\epsilon_j}$$

so  $W_\phi \in (\mathcal{S})$  if and only if (using (2.23))

$$\sum_j c_j^2 (2j)^k < \infty \quad \text{for all } k,$$

which holds because  $\phi \in \mathcal{S}$ . (See e.g. [RS]).

- b) The *singular (pointwise) white noise*  $W_t(\cdot)$  is defined as follows:

$$(2.26) \quad W_t(\omega) = \sum_k e_k(t) H_{\epsilon_k}(\omega)$$

Using (2.24) one can verify that  $W_t(\cdot) \in (\mathcal{S})^*$  for all  $t$ . This is the precise definition of singular/pointwise white noise!

### The Wick product

In addition to a canonical vector space structure, the spaces  $(\mathcal{S})$  and  $(\mathcal{S})^*$  also have a natural multiplication:

**DEFINITION 2.6** If  $X = \sum_\alpha a_\alpha H_\alpha \in (\mathcal{S})^*$ ,  $Y = \sum_\beta b_\beta H_\beta \in (\mathcal{S})^*$  then the *Wick product*,  $X \diamond Y$ , of  $X$  and  $Y$  is defined by

$$(2.27) \quad X \diamond Y = \sum_{\alpha, \beta} a_\alpha b_\beta H_{\alpha+\beta} = \sum_\gamma \left( \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) H_\gamma$$

Using (2.24) and (2.23) one can now verify the following:

$$(2.28) \quad X, Y \in (\mathcal{S})^* \Rightarrow X \diamond Y \in (\mathcal{S})^*$$

$$(2.29) \quad X, Y \in (\mathcal{S}) \Rightarrow X \diamond Y \in (\mathcal{S})$$

(Note, however, that  $X, Y \in L^2(\mu) \not\Rightarrow X \diamond Y \in L^2(\mu)$ )

## EXAMPLE 2.7

(i) *The Wick square of white noise is*

$$\text{(singular case)} \quad W_t^{\diamond 2} = \sum_{k,m} e_k(t) e_m(t) H_{\epsilon_k + \epsilon_m}$$

$$\text{(smoothed case)} \quad W_\phi^{\diamond 2} = \sum_{u,m} c_u c_m H_{\epsilon_u + \epsilon_m} \quad \text{if} \quad \phi = \sum c_k e_k \in \mathcal{S}$$

Since

$$H_{\epsilon_k + \epsilon_m} = \begin{cases} H_{\epsilon_k} \cdot H_{\epsilon_m} & \text{if } k \neq m \\ H_{\epsilon_k}^2 - 1 & \text{if } k = m \end{cases}$$

we see that

$$W_\phi^{\diamond 2} = W_\phi^2 - \sum_c c_c^2 = W_\phi^2 - \|\phi\|^2.$$

Note, in particular, that  $W_\phi^{\diamond 2}$  is not positive. In fact,  $E[W_\phi^{\diamond 2}] = 0$  by (2.5).

(ii) *The Wick exponential of white noise is defined by*

$$\exp^\diamond W_\phi = \sum_{n=0}^{\infty} \frac{1}{n!} W_\phi^{\diamond n} \quad ; \quad \phi \in \mathcal{S}.$$

It can be shown that

$$(2.30) \quad \exp^\diamond W_\phi = \exp(W_\phi - \frac{1}{2}\|\phi\|^2)$$

so  $\exp^\diamond W_\phi$  is positive. Moreover, we have

$$(2.31) \quad E[\exp^\diamond W_\phi] = 1.$$

## Why the Wick product?

We list some reasons that the Wick product is natural to use in stochastic calculus:

- 1) First, note that if (at least) one of the factors  $X, Y$  is deterministic, then

$$X \diamond Y = X \cdot Y$$

Therefore the two types of products, the Wick product and the ordinary ( $\omega$ -pointwise) product, coincide in the deterministic calculus. So when one extends a deterministic model to a stochastic model by introducing noise, it is not obvious which interpretation to choose for the products involved. The choice should be based on additional modelling and mathematical considerations.

- 2) The Wick product is the only product which is defined for singular white noise  $W_t$ . Pointwise product  $X \cdot Y$  does not make sense in  $(\mathcal{S})^*$ !
- 3) The Wick product has been used for 40 years already in quantum physics as a renormalization procedure.
- 4) Last, but not least: There is a fundamental relation between Ito integrals and Wick products, given by

$$(2.32) \quad \int Y_t(\omega) dB_t(\omega) = \int Y_t \diamond W_t dt$$

(see [LØU 2], [B]).

Here the integral on the right is interpreted as a Pettis integral with values in  $(\mathcal{S})^*$ .

In view of (2.32) one could say that the Wick product is the core of Ito integration, hence it is natural to use in stochastic calculus in general.

Finally we recall the definition of a pair of dual spaces,  $\mathcal{G}$  and  $\mathcal{G}^*$ , which are useful in our model. See [PT] and the references therein for more information.

#### DEFINITION 2.8

- a) Let  $\lambda \in \mathbf{R}$ . Then the space  $\mathcal{G}_\lambda$  consists of all formal expansions

$$(2.33) \quad X = \sum_{n=0}^{\infty} \int_{\mathbf{R}^n} f_n dB^{\otimes n}$$

such that

$$(2.34) \quad \|X\|_\lambda := \left[ \sum_{n=0}^{\infty} n! e^{2\lambda n} \|f_n\|_{L^2(\mathbf{R}^n)}^2 \right]^{\frac{1}{2}} < \infty$$

For each  $\lambda \in \mathbf{R}$  the space  $\mathcal{G}_\lambda$  is a Hilbert space with inner product

$$(2.35) \quad (X, Y)_{\mathcal{G}_\lambda} = \sum_{n=0}^{\infty} n! e^{2\lambda n} (f_n, g_n)_{L^2(\mathbf{R}^n)} \text{ if } X = \sum_{n=0}^{\infty} \int_{\mathbf{R}^n} f_n dB^{\otimes n}, Y = \sum_{m=0}^{\infty} \int_{\mathbf{R}^m} g_m dB^{\otimes m}$$

Note that  $\lambda_1 \leq \lambda_2 \Rightarrow \mathcal{G}_{\lambda_2} \subseteq \mathcal{G}_{\lambda_1}$ . Define

$$(2.36) \quad \mathcal{G} = \bigcap_{\lambda \in \mathbf{R}} \mathcal{G}_\lambda, \quad \text{with projective limit topology.}$$

b)  $\mathcal{G}^*$  is defined to be the dual of  $\mathcal{G}$ . Hence

$$(2.37) \quad \mathcal{G}^* = \bigcup_{\lambda \in \mathbf{R}} \mathcal{G}_\lambda, \quad \text{with inductive limit topology.}$$

**REMARK.** Note that an element  $Y \in \mathcal{G}^*$  can be represented as a formal sum

$$(2.38) \quad Y = \sum_{n=0}^{\infty} \int_{\mathbf{R}^n} g_n dB^{\otimes n}$$

where  $g_n \in \hat{L}^2(\mathbf{R}^n)$  and  $\|Y\|_\lambda < \infty$  for *some*  $\lambda \in \mathbf{R}$ , while an  $X \in \mathcal{G}$  satisfies  $\|X\|_\lambda < \infty$  for *all*  $\lambda \in \mathbf{R}$ .

If  $X \in \mathcal{G}$  and  $Y \in \mathcal{G}^*$  have the representations (2.33), (2.38), respectively, then the action of  $Y$  on  $X$ ,  $\langle Y, X \rangle$ , is given by

$$(2.39) \quad \langle Y, X \rangle = \sum_{n=0}^{\infty} n! (f_n, g_n)_{L^2(\mathbf{R}^n)}$$

where

$$(2.40) \quad (f_n, g_n)_{L^2(\mathbf{R}^n)} = \int_{\mathbf{R}^n} f(x)g(x)dx$$

One can show that

$$(2.41) \quad (\mathcal{S}) \subset \mathcal{G} \subset L^2(\mu) \subset \mathcal{G}^* \subset (\mathcal{S})^*.$$

The space  $\mathcal{G}^*$  is not big enough to contain the singular white noise  $W_t$ . However, we shall see that it does contain the solution  $X_t$  of the stochastic differential equation we consider in this paper. This fact allows us to deduce some useful properties of  $X_t$  (see Sections 3, 4).

Like  $(\mathcal{S})$  and  $(\mathcal{S})^*$  the spaces  $\mathcal{G}$  and  $\mathcal{G}^*$  are closed under Wick product ([PT, Theorem 2.7]):

$$(2.42) \quad X_1, X_2 \in \mathcal{G} \Rightarrow X_1 \diamond X_2 \in \mathcal{G}$$

$$(2.43) \quad Y_1, Y_2 \in \mathcal{G}^* \Rightarrow Y_1 \diamond Y_2 \in \mathcal{G}^*$$

### §3. Solution of the stochastic differential equation

We now proceed to solve the equation

$$(3.1) \quad \ddot{x}(t) + [a_0 + \alpha_0 \eta W_t] \diamond \dot{x}(t) + \theta^2 x(t) = T_0 \eta W_t; \quad x(0), \dot{x}(0) \text{ given}$$

First we will transform the equation into a stochastic Volterra equation. Then we will use Wick calculus as in [ZØ] to solve this equation.

First we Wick multiply both sides by the 'integrating factor'

$$(3.3) \quad \mathcal{E}(t) := \exp^\diamond(J(t)), \quad \text{where } J(t) := a_0 t + \alpha_0 \eta \int_0^t W_s ds = a_0 t + \alpha_0 \eta B_t$$

This gives

$$(3.4) \quad \frac{d}{dt}(\mathcal{E}(t) \diamond \dot{x}(t)) = -\theta^2 \mathcal{E}(t) \diamond x(t) + T_0 \eta \mathcal{E}(t) \diamond W_t$$

From this we deduce that

$$(3.5) \quad \begin{aligned} \mathcal{E}(t) \diamond \dot{x}(t) &= \dot{x}(0) + \int_0^t T_0 \eta \mathcal{E}(s) \diamond W_s ds - \theta^2 \int_0^t \mathcal{E}(s) \diamond x(s) ds \\ &= \dot{x}(0) + \frac{T_0}{\alpha_0} [\mathcal{E}(t) - 1 - a_0 \int_0^t \mathcal{E}(s) ds] - \theta^2 \int_0^t \mathcal{E}(s) \diamond x(s) ds \end{aligned}$$

Now Wick multiply by the Wick inverse of  $\mathcal{E}(t)$ ,

$$\mathcal{E}(t)^{\diamond(-1)} = \exp^\diamond(-J(t))$$

and obtain

$$(3.6) \quad \dot{x}(t) = b(t) - \theta^2 \int_0^t \exp^\diamond(J(s) - J(t)) \diamond x(s) ds$$

where

$$(3.7) \quad b(t) = b(t, \omega) = \dot{x}(0) \diamond \exp^\diamond(-J(t)) + \frac{T_0}{\alpha_0} [1 - \exp^\diamond(-J(t)) - a_0 \int_0^t \exp^\diamond(J(s) - J(t)) ds]$$

From this we deduce that

$$(3.8) \quad x(t) = x(0) + \int_0^t b(s) ds + \int_0^t (-\theta^2 \int_0^r \exp^\diamond(J(s) - J(r)) \diamond x(s) ds) dr$$

Now interchange the order of integration in the last iterated integral and obtain

$$(3.9) \quad x(t) = x(0) + \int_0^t b(s)ds + \int_0^t (-\theta^2 \int_s^t \exp^\diamond(J(s) - J(r))dr) \diamond x(s)ds$$

This is a *stochastic Volterra equation* of the form

$$(3.10) \quad x(t) = A(t, \omega) + \int_0^t K(t, s, \omega) \diamond x(s)ds$$

where

$$(3.11) \quad A(t, \omega) = x(0) + \int_0^t b(s, \omega)ds$$

and

$$(3.12) \quad K(t, s, \omega) = -\theta^2 \int_s^t \exp^\diamond(J(s) - J(r))dr$$

In the deterministic case there is a well known solution method for Volterra equations. It turns out (see [ZØ]) that in the stochastic case one can proceed in a similar way, except that the Wick product replaces the ordinary product and one must work harder to get the estimates needed for convergence. We now explain this in detail.

We proceed as in [ZØ, §3]:

Define

$$(3.13) \quad K_1(t, s) = K(t, s, \omega)$$

and inductively

$$(3.14) \quad K_{n+1}(t, s) = \int_0^t K_n(t, u) \diamond K_1(u, s)du \quad ; \quad n \geq 1$$

By induction we have (see formula (3.12) in [ZØ])

$$(3.15) \quad K_n(t, s) = \int_{s \leq u_{n-1} \leq \dots \leq u_1 \leq t} \dots \int_0^t \prod_{0 \leq k \leq n-1}^\diamond K(u_k, u_{k+1}) du_1 \dots du_{n-1}$$

where  $\overset{\diamond}{\prod}_{0 \leq k \leq n-1}$  denotes the Wick product from  $k = 0$  to  $k = n - 1$  and we have put  $u_0 = t, u_n = s$ .

Now

$$\begin{aligned}
(3.16) \quad \mathcal{K} &:= K(t, u_1) \diamond K(u_1, u_2) \diamond \cdots \diamond K(u_{n-1}, s) \\
&= (-\theta^2)^n \int_{u_1}^t \exp^\diamond(J(u_1) - J(r_1)) dr_1 \diamond \int_{u_2}^{u_1} \exp^\diamond(J(u_2) - J(r_2)) dr_2 \diamond \\
&\quad \cdots \diamond \int_s^{u_{n-1}} \exp^\diamond(J(s) - J(r_n)) dr_n \\
&= (-\theta^2)^n \int \cdots \int_{\substack{u_j \leq r_j \leq u_{j-1} \\ (1 \leq j \leq n)}} \exp^\diamond(-a_0(r_k - u_k) - \alpha_0 \eta(B_{r_k} - B_{u_k})) dr_1 \cdots dr_n
\end{aligned}$$

Therefore

$$(3.17) \quad E[\mathcal{K}^2] \leq \theta^{2n} \left( \int \cdots \int_{\substack{u_j \leq r_j \leq u_{j-1} \\ 1 \leq j \leq n}} E[\mathcal{L}^2] dr_1 \cdots dr_n \right) \cdot \left( \int \cdots \int_{\substack{u_j \leq r_j \leq u_{j-1} \\ 1 \leq j \leq n}} dr_1 \cdots dr_n \right)$$

where

$$(3.18) \quad \mathcal{L} = \exp^\diamond(-a_0(r_k - u_k) - \alpha_0 \eta(B_{r_k} - B_{u_k})).$$

Now by (2.30)

$$(3.19) \quad \exp^\diamond(-\alpha_0 \eta(B_{r_k} - B_{u_k})) = \exp(-\alpha_0 \eta(B_{r_k} - B_{u_k}) - \frac{1}{2} \alpha_0^2 \eta^2(r_k - u_k))$$

and therefore, by (2.31),

$$\begin{aligned}
(3.20) \quad E[\mathcal{L}^2] &= E[\exp\{-2\alpha_0 \eta(B_{r_k} - B_{u_k}) - (\alpha_0^2 \eta^2 + 2a_0)(r_k - u_k)\}] \\
&= E[\exp^\diamond\{-2\alpha_0 \eta(B_{r_k} - B_{u_k}) + (\alpha_0^2 \eta^2 - 2a_0)(r_k - u_k)\}] \\
&= \exp\{(\alpha_0^2 \eta^2 - 2a_0)(r_k - u_k)\}
\end{aligned}$$

Hence, by (3.17),

$$\begin{aligned}
(3.21) \quad E[\mathcal{K}^2] &\leq \left[ \frac{\theta^2}{\alpha_0^2 \eta^2 - 2a_0} \right]^n \prod_{k=1}^n (\exp\{(\alpha_0^2 \eta^2 - 2a_0)(u_{k-1} - u_k)\} - 1) \\
&\quad \cdot \prod_{k=1}^n (u_{k-1} - u_k) \\
&\leq \left[ \frac{\theta^2}{\alpha_0^2 \eta^2 - 2a_0} \right]^n \exp\{(\alpha_0^2 \eta^2 - 2a_0)(t - s)\} \cdot \prod_{k=1}^n (u_{k-1} - u_k)
\end{aligned}$$

Substituted in (3.15) this gives

$$\begin{aligned}
(3.22) \quad E[K_n^2(t, s)] &\leq \left[ \int \cdots \int_{s \leq u_{n-1} \leq \cdots \leq u_1 \leq t} E[\mathcal{K}^2] du_1 \cdots du_{n-1} \right] \cdot \left[ \int \cdots \int_{s \leq u_{n-1} \leq \cdots \leq u_1 \leq t} du_1 \cdots du_{n-1} \right] \\
&\leq \left[ \frac{\theta^2}{\alpha_0^2 \eta^2 - 2a_0} \right]^n \exp\{(\alpha_0^2 \eta^2 - 2a_0)(t - s)\} \\
&\quad \cdot \int \cdots \int_{s \leq u_{n-1} \leq \cdots \leq u_1 \leq t} \prod_{k=1}^n (u_{k-1} - u_k) du_1 \cdots du_{n-1} \\
&\quad \cdot \frac{(t - s)^{n-1}}{(n-1)!} \leq \left[ \frac{\theta^2}{\alpha_0^2 \eta^2 - 2a_0} \right]^n \exp\{(\alpha_0^2 \eta^2 - 2a_0)(t - s)\} \frac{(t - s)^{2n-1}}{[(n-1)!]^2}
\end{aligned}$$

From (3.22) we deduce that

$$(3.23) \quad H(t, s, \omega) := \sum_{n=1}^{\infty} K_n(t, s, \omega)$$

converges in  $L^2(\mu)$ , uniformly for  $0 \leq s, t \leq T$ .

Therefore, by the same argument as in the proof of Theorem 3.7 in [ZØ] we get

### THEOREM 3.1

The unique solution  $x(t) \in (\mathcal{S})^*$  of the stochastic Volterra equation

$$(3.24) \quad x(t) = A(t, \omega) + \int_0^t K(t, s, \omega) \diamond x(s) ds$$

where  $A(t, \omega)$  and  $K(t, s, \omega)$  are given by (3.11) and (3.12) respectively, is given by

$$(3.25) \quad x(t) = A(t, \omega) + \int_0^t H(t, s, \omega) \diamond A(s, \omega) ds,$$

where  $H$  is given by (3.23).

### COROLLARY 3.2

The solution  $x(t)$  of (3.24) belongs to  $\mathcal{G}^*$  for all  $t \geq 0$ .

*Proof.* We know that  $\mathcal{G}^*$  is closed under Wick products. Since  $H(t, s, \cdot) \in L^2(\mu)$  we have  $H(t, s, \cdot) \in \mathcal{G}^*$ , by (2.41). In fact, by (2.34) we see that  $H(t, s, \cdot) \in \mathcal{G}_0$  for all  $t, s$ . So by (3.25) it remains only to prove that there exists  $\lambda > -\infty$  such that  $A(t, \cdot) \in \mathcal{G}_\lambda$  for all  $t$ . But this is straightforward from (3.11) and (3.7).  $\square$



## §4. Probabilistic properties of the motion

A natural question is now: What kind of probabilistic properties does  $x(t)$  have? Before we try to answer this question we emphasize that we have *not* been able to prove that  $x(t)$  is a classical stochastic process. Corollary 3.2 only guarantees that  $x(t)$  exists as a generalized stochastic process with values in  $\mathcal{G}^*$ . The problem is that even though  $A(s, \cdot)$  and  $H(t, s, \cdot)$  both are square integrable random variables for all  $t, s$ , their Wick product  $H(t, s, \omega) \diamond A(s, \omega)$  need not be, because  $L^2(\mu)$  is not closed under Wick products. However, since  $x \in \mathcal{G}_\lambda$  for some  $\lambda > -\infty$  we know by (2.37) that  $x(t)$  has an orthogonal expansion (in the Hilbert space  $\mathcal{G}_\lambda$ ) of the form (see (2.17) and (2.34))

$$(4.1) \quad x(t) = f_0(t) + \int_{\mathbf{R}} f_1(t; s) dB_s + \cdots + \int_{\mathbf{R}^n} f_n(t; s_1, \dots, s_n) dB_s^{\otimes n} + \cdots$$

where  $f_n(t; \cdot) \in \hat{L}^2(\mathbf{R}^n)$  for all  $n$  and

$$(4.2) \quad \|X\|_\lambda^2 = \sum_{n=0}^{\infty} n! e^{2\lambda n} \|f_n\|_{L^2(\mathbf{R}^n)}^2 < \infty \quad \text{for some } \lambda > -\infty.$$

(If  $T < \infty$  is given we can find  $\lambda > -\infty$  which works for all  $t \in [0, T]$ .) Since  $E[\int_{\mathbf{R}^n} f_n(t; s) dB_s^{\otimes n}] = 0$  for all  $n \geq 1$ , we may regard  $f_0(t)$  as a *generalized expectation* of  $x(t)$ . (If  $x(t)$  happens to be in  $L^2(\mu)$ , then  $f_0(t)$  the classical expectation of  $x(t)$ ,  $f_0(t) = E[x(t)]$ .) In view of the orthogonal expansion (4.1) we could also say that  $f_0(t)$  is *the best  $\omega$ -constant approximation* to  $x(t)$  in  $\mathcal{G}_\lambda$ . Similarly, the sum of the first two terms

$$(4.3) \quad z(t, \omega) = f_0(t) + \int_{\mathbf{R}} f_1(t, s) dB_s$$

may be regarded as *the best Gaussian approximation* to  $x(t)$  in  $\mathcal{G}_\lambda$ . It has mean value  $f_0(t)$  and by the Ito isometry its variance is given by

$$(4.4) \quad E[(\int_{\mathbf{R}} f_1(t; s) dB_s)^2] = \int_{\mathbf{R}} f_1^2(t; s) ds = \|f_1(t; \cdot)\|_{L^2(\mathbf{R})}^2$$

To find  $z(t)$  we use the argument from the proof of Theorem 3.1 in [HLØUZ]: We consider the orthogonal expansion of  $x(t)$  by the basis  $\{H_\alpha(\cdot)\}$  of  $\mathcal{G}_\lambda$

$$(4.5) \quad x(t) = \sum_{\alpha} \phi_{\alpha}(t) H_{\alpha}(\omega)$$

Note that  $\phi_0(t) = f_0(t)$ . Substituting (4.5) in (3.1) gives

$$(4.6) \quad \begin{aligned} & \sum_{\alpha} \ddot{\phi}_{\alpha}(t) H_{\alpha} + [a_0 + \alpha_0 \eta \sum_{k=1}^{\infty} e_k(t) H_{\epsilon_k}] \diamond \sum_{\gamma} \dot{\phi}_{\gamma}(t) H_{\gamma} + \theta^2 \sum_{\alpha} \phi_{\alpha}(t) H_{\alpha} \\ & = T_0 \eta \sum_{k=1}^{\infty} e_k(t) H_{\epsilon_k}. \end{aligned}$$

Collecting the coefficients of each  $H_\alpha$  and using that  $H_\beta \diamond H_\gamma = H_{\beta+\gamma}$ , we get

$$(4.7) \quad \begin{aligned} \ddot{\phi}_\alpha(t) + a_0 \dot{\phi}_\alpha(t) + \alpha_0 \eta \left( \sum_{\epsilon_k + \gamma = \alpha} e_k(t) \dot{\phi}_\gamma(t) + \theta^2 \phi_\alpha(t) \right) \\ = T_0 \eta e_k(t) \cdot \chi_{|\alpha|=1} ; \quad \forall \alpha \in \mathcal{J} \end{aligned}$$

In particular, choosing  $\alpha = 0$  we get

$$(4.8) \quad \ddot{\phi}_0(t) + a_0 \dot{\phi}_0(t) + \theta^2 \phi_0(t) = 0.$$

Together with the initial values

$$(4.9), \quad \phi_0(0) = E[x(0)], \quad \dot{\phi}_0(0) = E[\dot{x}(0)]$$

this determines the *generalized expectation*  $\phi_0(t) = f_0(t)$  for all  $t$ .

Next, choosing  $\alpha = \epsilon_j$  we get (see the text following (2.12))

$$(4.10) \quad \ddot{\phi}_{\epsilon_j}(t) + a_0 \dot{\phi}_{\epsilon_j}(t) + \theta^2 \phi_{\epsilon_j}(t) = \eta e_j(t) [T_0 - \alpha_0 \dot{\phi}_0(t)]$$

which is a second order differential equation in  $\phi_{\epsilon_j}$  with constant coefficients and thus easily solved, since  $\dot{\phi}_0(t) = \dot{f}_0(t)$  is already known.

Now the Gaussian random variable  $z(t, \omega)$  is given by

$$(4.11) \quad \begin{aligned} z(t) - E[x(t)] &= \sum_{j=1}^{\infty} \phi_{\epsilon_j}(t) \langle \omega, e_j \rangle = \sum_{j=1}^{\infty} \phi_{\epsilon_j}(t) \int_{\mathbf{R}} e_j(s) dB_s \\ &= \int_{\mathbf{R}} \left( \sum_{j=1}^{\infty} \phi_{\epsilon_j}(t) e_j(s) \right) dB_s = \int_{\mathbf{R}} f_1(t; s) dB_s \end{aligned}$$

and hence the variance of  $z(t)$  is

$$(4.12) \quad E[(z(t) - E[x(t)])^2] = \left\| \sum_{j=1}^{\infty} \phi_{\epsilon_j}(t) e_j(\cdot) \right\|_{L^2(\mathbf{R})}^2 = \sum_{j=1}^{\infty} |\phi_{\epsilon_j}(t)|^2.$$

We have proved

#### THEOREM 4.1

The best Gaussian approximation in  $\mathcal{G}_\lambda$  of the solution  $x(t, \cdot)$  of (3.1) has mean value  $f_0(t) = \phi_0(t)$  given by (4.8) – (4.9) and variance given by (4.12), where the  $\phi'_{\epsilon_j}$ s satisfy (4.10).

Similarly one may use (4.7) to find higher order approximands of  $x(t)$ .

The functions  $\phi_0, \phi_{\epsilon_j}$  may be obtained on explicit form. Consider the second order (deterministic) differential equation

$$(4.13) \quad \ddot{\phi}(t) + a_0 \dot{\phi}(t) + \theta^2 \phi(t) = f(t), \quad j \geq 0$$

Introducing

$$(4.14) \quad \Phi = \begin{pmatrix} \phi(t) \\ \dot{\phi}(t) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -\theta^2 & -a_0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we may write (4.13) as

$$(4.15) \quad d\Phi = A\Phi dt + Mf(t)dt$$

Multiplying by  $e^{-At}$  and integrating, we obtain

$$(4.16) \quad \Phi(t) = e^{-At}\Phi(0) + \int_0^t e^{A(t-s)} Mf(s)ds$$

It is then easily found that

$$(4.17) \quad \phi(t) = \frac{1}{\Omega} \text{Im} \left\{ \left[ \left( \frac{a_0}{2} + i\Omega \right) \phi(0) + \dot{\phi}(0) \right] e^{i\Omega' t} \right\} + \frac{1}{\Omega} \text{Im} \int_0^t e^{i\Omega'(t-s)} f(s) ds$$

where

$$(4.18) \quad \Omega = \left( \theta^2 - \frac{a_0^2}{4} \right)^{\frac{1}{2}}, \quad \Omega' = i \frac{a_0}{2} + \Omega$$

and  $\text{Im}$  denotes imaginary part. (4.17) – (4.18) is the general solution of (4.13) which is completely determined when the initial conditions are given.

The solution  $\phi$  will ‘forget’ the initial conditions when  $t$  increases, due to the damping term in (4.13). Thus, for large values of time,  $\phi$  is determined by the last term of (4.17). This part of the solution is obtained by introducing  $\phi(0) = 0, \dot{\phi}(0) = 0$  in (4.17). Denote this special solution by  $\phi^P(t)$ . Hence

$$(4.19) \quad \phi^P(t) = \frac{1}{\Omega} \int_0^t e^{-\frac{\alpha_0}{2}(t-s)} \sin \Omega(t-s) f(s) ds$$

We now apply this to equations (4.8) and (4.10) for  $\phi_0(t)$  and  $\phi_{\epsilon_j}(t)$ , respectively. In (4.8) we have that  $f(t) = 0$ . (4.19) then gives that

$$(4.20) \quad \phi_0^P(t) = 0.$$

Next, in (4.10) we have

$$(4.21) \quad f(t) = \eta T_0 [e_j(t) - \frac{\alpha_0}{T_0} \dot{\phi}_0(t)]$$

(4.19) then gives

$$(4.22) \quad \phi_{\epsilon_j}^P(t) = \frac{\eta T_0}{\Omega} \int_0^t e^{-\frac{\alpha_0}{2}(t-s)} \sin \Omega(t-s) [e_j(s) - \frac{\alpha_0}{T_0} \dot{\phi}_0(s)] ds, \quad j \geq 1$$

Consider the solution  $\phi_{\epsilon_j}^P(t)$  for large values of the time  $t$ . Then we have that the contribution in (4.22) due to  $\dot{\phi}_0(s)$  for small and moderate values of  $s$  is damped out by the factor  $e^{-\frac{\alpha_0}{2}t}$ . Furthermore, noting that  $\dot{\phi}_0(s) \sim \dot{\phi}_0^P(s) = 0$  for large  $s$ , it is easy to demonstrate that the contribution due to  $\dot{\phi}_0$  in (4.22) is vanishingly small for large values of  $t$ . Since  $\phi_{\epsilon_j}(t) \sim \phi_{\epsilon_j}^P(t)$  for large  $t$ , we obtain

$$(4.23) \quad \phi_{\epsilon_j}(t) \sim \frac{\eta T_0}{\Omega} \int_0^t e^{-\frac{\alpha_0}{2}(t-s)} \sin \Omega(t-s) e_j(s) ds, \quad j \geq 1 \quad \text{for large } t$$

We have then obtained

#### **COROLLARY 4.2**

For large values of time the best Gaussian approximation in  $\mathcal{G}_\lambda$  of the solution  $x(t, \cdot)$  of (3.1) has zero mean value and a variance given by (4.12) and (4.23).

We note that (4.20) and (4.23) do not depend on the value of  $\alpha_0$ . Thus, in order to obtain the mean value and the variance of  $x(t)$  for large  $t$  it is sufficient to consider the solution of (3.1) with  $\alpha_0 = 0$ .

#### **COROLLARY 4.3**

The time-dependent damping of (3.1) does not contribute to the mean value and the variance of the solution  $x(t)$  of (3.1) for large  $t$ .

We remark that Corollary 4.3 is in agreement with numerical simulations by [G] and [ZF].

## §5. Concluding remarks

A stochastic model for slow-drift motions of offshore structures is developed. One of the principal aims has been to interpret the role of the time-dependent part of the damping force, which appears in the equation for the slow-drift motions. It is argued that the slow-drift motion may be modelled by the second order stochastic differential equation (1.10). This equation contains white noise both in the excitation force and in a part of the damping force. Solution of the differential equation is found by means of stochastic Volterra equations, see equation (3.25). The stochastic properties of the motion is considered. The mean value and the variance of the motion are obtained on explicit form. We find that the mean value approaches zero for large value of time. The variance is found to be proportional to the amplitude of the excitation force divided by the mean value of the damping. The time-dependent part of the damping is found to have no effect on the mean value and the variance of the motion for large value of the time. This is in agreement with numerical simulations ([G] and [ZF]). It is believed that the time-dependent damping is of importance to higher order stochastic properties of the motion than the mean value and the variance. The present analysis may serve as basis for obtaining higher order approximands of the stochastic properties of the slow-drift motions.

**ACKNOWLEDGEMENTS** B. Øksendal holds a professorship sponsored by VISTA, a research cooperation between the Norwegian Academy of Science and Letters and Den Norske Stats Oljeselskap A.S. (Statoil). Parts of this work were written while he was visiting the University of Botswana as a part of the Cooperation Program between the University of Oslo and the University of Botswana. This program is sponsored by NUFU through SUM (Senter for Utvikling og Miljø) at the University of Oslo. He wishes to thank VISTA and NUFU/SUM for their support and the University of Botswana for its hospitality.

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