# Two equations modeling pollution in a stochastic medium – A white noise approach –

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**Abstract**

In this paper we look at two models for pollution given by the equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\eta^2 \Delta u + \vec{W}_{\phi_x} \diamond \nabla u - \kappa_{(t,x)} \diamond u + g(t,x) && (t,x) \in [0,T] \times \mathbb{R}^n \\ u(0,x) &= f(x) && x \in \mathbb{R}^n \end{aligned}$$

and

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\eta^2 \Delta u + \vec{W}_{\phi_x} \diamond \nabla u - \kappa_{(t,x)} \diamond u + g(t,x) && (t,x) \in [0,T] \times D \\ u(0,x) &= \phi(x) && x \in D \\ u(t,x) &= h(t,x) && (t,x) \in [0,T] \times \partial D \end{aligned}$$

where $T$ and $\eta$ are constants, $D$ is a bounded domain, $\vec{W}_{\phi_x}$, $\kappa_{(t,x)}$, $f$, $\phi$, $h$ and $g$ are elements in the space $(\mathcal{S})^{-1}$ of generalized white noise distributions. With suitable conditions on $\kappa$, $f$, $\phi$, $h$ and $g$, we show that both equations have unique solutions given by explicit solution formulas.

*Keywords*: Generalized white noise distributions, Wick product, Hermite transform.

## §1 Introduction

We will consider two stochastic models based on the PDE

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\eta^2 \Delta u(t,x) - V \cdot \nabla u(t,x) - \alpha u(t,x) + \xi(t,x) \tag{1}$$

where $\frac{1}{2}\eta^2$ is the dispersion coefficient, $V$ is the water velocity, $\alpha$ is the leakage rate, $\xi(t,x)$ is the rate of increase of the chemical concentration at $(t,x)$ and $u(t,x)$ is the chemical concentration at time $t$

on location $x$. This work is motivated from the paper of G. Kallianpur et al. [KAL] where the $\xi(t,x)$ was supposed to be random. We will in addition consider the case where the drift vector V is modeled as an $n$-dimensional white noise, consisting of independent components. We will work in the space $(\mathcal{S})^{-1}$ of generalized white noise distributions, since this space will allow explicit solutions formulas for a wide range of possible choices for random $\xi$'s and because the methods are particularly simple.

The methods used to solve the stochastic versions of (1) are the same as those used by Holden et al. in [HLØUZ3] and several other SPDE's are solved in a similarly fashion:

- The transport equation ([GjHØUZ]).
- The pressure equation for fluid flow ([HLØUZ3]).
- The Dirichlet equation ([Gj2]).
- The Burgers equation ([HLØUZ2]).
- The Schrödinger equation ([HLØUZ]).

For more examples and background on white noise, please read [BØ3].

## §2 Preliminaries on multidimensional white noise

There are many problems of physical nature where the need for several independent white noise sources arises. For example, given $m$ independent positive white noise sources in a domain D, one wants to calculate the effect of these on a particle traveling in D. The result should intuitively be given by

$$\sum_{i=1}^{m} \operatorname{Exp}\{W_\phi^{(i)}\}$$

where $\{\operatorname{Exp}\{W_\phi^{(i)}\}\}_{i=1}^{m}$ are one dimensional independent positive white noise sources.

We will now give a short introduction of definitions and results from multidimensional Wick calculus, taken mostly from [Gj], [HLØUZ3], [HKPS] and [KLS].

In the following we will fix the parameter dimension $n$ and space dimension $m$.

Let

$$\mathcal{N} := \prod_{i=1}^{m} \mathcal{S}(\mathbb{R}^n)$$

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing $C^\infty$-functions on $\mathbb{R}^n$, and

$$\mathcal{N}^* := \left(\prod_{i=1}^{m} \mathcal{S}(\mathbb{R}^n)\right)^* \approx \prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n)$$

where $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions.

Let $\mathcal{B} := \mathcal{B}(\mathcal{N}^*)$ denote the Borel $\sigma$-algebra on $\mathcal{N}^*$ equipped with the weak star topology and set

$$\mathcal{H} := \bigoplus_{i=1}^{m} \mathcal{L}^2(\mathbb{R}^n)$$

where $\oplus$ denotes orthogonal sum.

Since $\mathcal{N}$ is a countably Hilbert nuclear space (cf. eg.[Gj]) we get, using Minlos' theorem, a unique probability measure $\nu$ on $(\mathcal{N}^*, \mathcal{B})$ such that

$$\int_{\mathcal{N}^*} e^{i\langle\omega,\phi\rangle}\, d\nu(\omega) = e^{-\frac{1}{2}\|\phi\|^2_{\mathcal{H}}} \quad \forall\phi \in \mathcal{N}$$

where $\|\phi\|^2_{\mathcal{H}} = \sum_{i=1}^{m} \|\phi_i\|^2_{\mathcal{L}^2(\mathbb{R}^n)}$.

Note that if $m = 1$ then $\nu$ is usually denoted by $\mu$.

**THEOREM 2.1** [Gj] We have the following

1. $\otimes_{i=1}^{m} \mathcal{B}(\mathcal{S}'(\mathbb{R}^n)) = \mathcal{B}(\prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n))$
2. $\nu = \times_{i=1}^{m} \mu$

**DEFINITION 2.2** [Gj] The triple

$$(\prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n), \mathcal{B}, \nu)$$

is called the ($m$-dimensional) ($n$-parameter) **white noise probability space.**

For $k = 0, 1, 2, \dots$ and $x \in \mathbb{R}$ let

$$h_k(x) := (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k}(e^{-\frac{x^2}{2}})$$

be the Hermite polynomials and

$$\xi_k(x) := \pi^{-\frac{1}{4}}((k-1)!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{k-1}(\sqrt{2}x) \ ; \ k \geq 1$$

the Hermite functions.

It is well known that the family $\{\tilde{e}_\alpha\} \subset \mathcal{S}(\mathbb{R}^n)$ of tensor products

$$\tilde{e}_\alpha := \xi_{\alpha_1} \otimes \cdots \otimes \xi_{\alpha_n}$$

forms an orthonormal basis for $\mathcal{L}^2(\mathbb{R}^n)$.

Give the family of all multi-indecies $\zeta = (\zeta_1, \dots, \zeta_n)$ a fixed ordering

$$(\zeta^{(1)}, \zeta^{(2)}, \dots, \zeta^{(k)}, \dots) \text{ where } \zeta^{(k)} = (\zeta_1^{(k)}, \dots, \zeta_n^{(k)})$$

and define $\tilde{e}_k := \tilde{e}_{\zeta(k)}$.

Let $\{e_k\}_{k=1}^{\infty}$ be the orthonormal basis of $\mathcal{H}$ we get from the collection

$$\{(\overbrace{0,\dots,0}^{i-1},\tilde{e}_j,\overbrace{0,\dots,0}^{m-i}) \in \mathcal{H} \quad 1 \le i \le m, 1 \le j < \infty\}$$

and let $\gamma : \mathbb{N} \to \mathbb{N}$ be a function such that

$$e_k = (0,\dots,0,\tilde{e}_{\zeta(\gamma(k))},0,\dots,0).$$

Finally , let $(\beta^{(1)},\beta^{(2)},\dots,\beta^{(k)},\dots)$ with $\beta^{(k)} = (\beta_1^{(k)},\dots,\beta_n^{(k)})$ be a sequence such that $\beta^{(k)} = \zeta^{(\gamma(k))}$.

If $\alpha = (\alpha_1,\dots,\alpha_k)$ is a multi-index of non-negative integers we put

$$H_\alpha(\omega) := \prod_{i=1}^{k} h_{\alpha_i}(\langle\omega, e_i\rangle).$$

From theorem 2.1 in [HLØUZ] we know that the collection

$$\{H_\alpha(\cdot); \alpha \in \mathbb{N}_0^k; k = 0,1,\dots\}$$

forms an orthogonal basis for $\mathcal{L}^2(\mathcal{N}^*,\mathcal{B},\nu)$ with $||H_\alpha||_{\mathcal{L}^2(\nu)} = \alpha!$ where $\alpha! = \prod_{i=1}^{k} \alpha_i!$.

This implies that any $f \in \mathcal{L}^2(\nu)$ has the unique representation

$$f(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega)$$

where $c_\alpha \in \mathbb{R}$ for each multi-index $\alpha$ and

$$||f||^2_{\mathcal{L}^2(\nu)} = \sum_\alpha \alpha! c_\alpha^2.$$

**DEFINITION 2.3** [Gj] The $m$-dimensional **white noise map** is a map

$$W : \prod_{i=1}^{m} \mathcal{S}(\mathbb{R}^n) \times \prod_{i=1}^{m} \mathcal{S}'(\mathbb{R}^n) \to \mathbb{R}^m$$

given by

$$W^{(i)}(\phi,\omega) := \omega_i(\phi_i) \quad 1 \le i \le m$$

**PROPOSITION 2.4** [Gj] The $m$-dimensional white noise map $W$ satisfies the following

1. $\{W^{(i)}(\phi,\cdot)\}_{i=1}^{m}$ is a family of independent normal random variables.
2. $W^{(i)}(\phi,\cdot) \in \mathcal{L}^2(\nu)$ for $1 \le i \le m$.

**DEFINITION 2.5** [HLØUZ3] Let $0 \le \rho \le 1$.

- Let $(\mathcal{S}_n^m)^\rho$, the space of **generalized white noise test functions**, consist of all

  $$f = \sum_\alpha H_\alpha \in \mathcal{L}^2(\nu)$$

  such that

  $$\|f\|_{\rho,k}^2 := \sum_\alpha c_\alpha^2 (\alpha!)^{1+\rho} (2\mathbf{N})^{\alpha k} < \infty \quad \forall k \in \mathbb{N}$$

- Let $(\mathcal{S}_n^m)^{-\rho}$, the space of **generalized white noise distributions**, consist of all formal expansions

  $$F = \sum_\alpha b_\alpha H_\alpha$$

  such that

  $$\sum_\alpha b_\alpha^2 (\alpha!)^{1-\rho} (2\mathbf{N})^{-\alpha q} < \infty \text{ for some } q \in \mathbb{N}$$

where

$$(2\mathbf{N})^\alpha := \prod_{i=1}^{k} (2^n \beta_1^{(i)} \cdots \beta_n^{(i)})^{\alpha_i} \text{ if } \alpha = (\alpha_1, \ldots, \alpha_k).$$

We know that $(\mathcal{S}_n^m)^{-\rho}$ is the dual of $(\mathcal{S}_n^m)^\rho$ (when the later space has the topology given by the semi-norms $\|\cdot\|_{\rho,k}$) and if $F = \sum b_\alpha H_\alpha \in (\mathcal{S}_n^m)^{-\rho}$ and $f = \sum c_\alpha H_\alpha \in (\mathcal{S}_n^m)^\rho$ then

$$\langle F, f \rangle = \sum_\alpha b_\alpha c_\alpha \alpha!.$$

It is obvious that we have the inclusions

$$(\mathcal{S}_n^m)^1 \subset (\mathcal{S}_n^m)^\rho \subset (\mathcal{S}_n^m)^{-\rho} \subset (\mathcal{S}_n^m)^{-1} \quad \rho \in [0,1]$$

and in the remaining of this paper we will consider the larger space $(\mathcal{S}_n^m)^{-1}$.

**DEFINITION 2.6** [HLØUZ3] The Wick product of two elements in $(\mathcal{S}_n^m)^{-1}$ given by

$$F = \sum_\alpha a_\alpha H_\alpha \;,\; G = \sum_\beta b_\beta H_\beta$$

is defined by

$$F \diamond G = \sum_\gamma c_\gamma H_\gamma$$

where

$$c_\gamma = \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta$$

**LEMMA 2.7** [HLØUZ3] We have the following

1. $F, G \in (\mathcal{S}_n^m)^{-1} \Rightarrow F \diamond G \in (\mathcal{S}_n^m)^{-1}$
2. $f, g \in (\mathcal{S}_n^m)^{1} \Rightarrow f \diamond g \in (\mathcal{S}_n^m)^{1}$

**DEFINITION 2.8** [HLØUZ3] Let $F = \sum b_\alpha H_\alpha$ be given. Then the Hermite transform of $F$,denoted by $\mathcal{H}F$, is defined to be (whenever convergent)

$$\mathcal{H}F := \sum_\alpha b_\alpha z^\alpha$$

where $z = (z_1, z_2, \cdots)$ and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_k^{\alpha_k}$ if $\alpha = (\alpha_1, \ldots, \alpha_k)$.

**LEMMA 2.9** [HLØUZ3] If $F, G \in (\mathcal{S}_n^m)^{-1}$ then

$$\mathcal{H}(F \diamond G)(z) = \mathcal{H}F(z) \cdot \mathcal{H}G(z)$$

for all $z$ such that $\mathcal{H}F(z)$ and $\mathcal{H}G(z)$ exists.

**LEMMA 2.10** [HLØUZ3] Suppose $g(z_1, z_2, \cdots)$ is a bounded analytic function on $\mathbf{B}_q(\delta)$ for some $\delta > 0, q < \infty$ where

$$\mathbf{B}_q(\delta) := \{\zeta = (\zeta_1, \zeta_2, \cdots) \in \mathbb{C}_0^{\mathbb{N}}; \sum_{\alpha \neq 0} |\zeta^\alpha|^2 (2\mathbf{N})^{\alpha q} < \delta^2\}.$$

Then there exists $X \in (\mathcal{S}_n^m)^{-1}$ such that $\mathcal{H}X = g$.

**LEMMA 2.11** [HLØUZ3] Suppose $X \in (\mathcal{S}_n^m)^{-1}$ and that $f$ is an analytic function in a neighborhood of $\mathcal{H}X(0)$ in $\mathbb{C}$. Then there exists $Y \in (\mathcal{S}_n^m)^{-1}$ such that $\mathcal{H}Y = f \circ \mathcal{H}X$.

**THEOREM 2.12** [KLS] Let $(T, \Sigma, \tau)$ be a measure space and let $\Phi : T \to (\mathcal{S}_n^m)^{-1}$ be such that there exists $q < \infty, \delta > 0$ such that

1. $\mathcal{H}\Phi_t(z) : T \to \mathbb{C}$ is measurable for all $z \in \mathbf{B}_q(\delta)$
2. there exists $C \in \mathcal{L}^1(T, \tau)$ such that $|\mathcal{H}\Phi_t(z)| \leq C(t)$ for all $z \in \mathbf{B}_q(\delta)$ and for $\tau$-almost all $t$.

Then $\int_T \Phi_t \, d\tau(t)$ exists as a Bochner integral in $(\mathcal{S}_n^m)^{-1}$. In particular,
$\langle \int_T \Phi_t \, d\tau(t), \phi \rangle = \int_T \langle \Phi_t, \phi \rangle \, d\tau(t) \ ; \ \phi \in (\mathcal{S}_n^m)^1$.

**EXAMPLE 2.13** Define the x-shift of $\phi$, denoted by $\phi_x$, by $\phi_x(y) := \phi(y - x)$.Then

$$\mathrm{Exp}\{W_{\phi_x}^{(i)}\} \in (\mathcal{S}_n^m)^{-1} \quad 1 \leq i \leq m, \forall x \in \mathbb{R}^n$$

which is an immediate consequence of proposition 2.4 and lemma 2.11.

## §3 The pollution model in $\mathbb{R}^n$

We will in this and the next section assume that $(b_s^{(t,x)}(\omega), \hat{P}^{t,x})$ is a Brownian motion starting at location $x \in \mathbb{R}^n$ at time $t$, and use the notation

- $\hat{E}^{t,x}$ is expectation w.r.t. the measure $\hat{P}^{t,x}$.
- $C^2(\mathbb{R}^n)$ are the functions in $\mathbb{R}^n$ with continuous derivatives up to order 2.
- $C_0^2(\mathbb{R}^n)$ are the functions in $\mathbb{R}^n$ with compact support and continuous derivatives up to order 2.

**THEOREM 3.1** Let $T > 0$ be given and assume furthermore that we are given functions $\mathbb{R}^n \ni x \mapsto f(x) \in (\mathcal{S}_n^n)^{-1}$, $[0,T]\times\mathbb{R}^n \ni (t,x) \mapsto g(t,x) \in (\mathcal{S}_n^n)^{-1}$ and $[0,T]\times\mathbb{R}^n \ni (t,x) \mapsto \kappa(t,x) \in (\mathcal{S}_n^n)^{-1}$ such that

- $\exists(q_f \in \mathbb{N}, \delta_f > 0, K_f > 0)$ such that
  1. $\sup_{x\in\mathbb{R}^n, z\in\mathbf{B}_{q_f}(\delta_f)} |\mathcal{H}f(x,z)| \le K_f$.
  2. $x \mapsto \mathcal{H}f(x,z) \in C_0^2(\mathbb{R}^n)$ whenever $z \in \mathbf{B}_{q_f}(\delta_f)$.
- $\exists(q_g \in \mathbb{N}, \delta_g > 0, K_g > 0)$ such that
  1. $\sup_{(t,x)\in[0,T]\times\mathbb{R}^n, z\in\mathbf{B}_{q_g}(\delta_g)} |\mathcal{H}g(t,x,z)| \le K_g$.
  2. $x \mapsto \mathcal{H}g(t,x,z) \in C_0^2(\mathbb{R}^n)$ whenever $t \in [0,T], z \in \mathbf{B}_{q_g}(\delta_g)$.
  3. $\exists(\alpha(z) > 0\, \forall z \in \mathbf{B}_{q_g}(\delta_g))$ such that $(t,x) \mapsto \mathcal{H}g(t,x,z)$ is Hölder continuous (exponent $\alpha(z)$) in $x \in \mathbb{R}^n$, uniformly in $t \in [0,T]$.
- $\exists(q_\kappa \in \mathbb{N}, \delta_\kappa > 0, K_\kappa > 0)$ such that
  1. $\sup_{(t,x)\in[0,T]\times\mathbb{R}^n, z\in\mathbf{B}_{q_\kappa}(\delta_\kappa)} |\mathcal{H}\kappa(t,x,z)| \le K_\kappa$.
  2. $x \mapsto \mathcal{H}\kappa(t,x,z) \in C_0^2(\mathbb{R}^n)$ whenever $t \in [0,T], z \in \mathbf{B}_{q_\kappa}(\delta_\kappa)$.
  3. $\exists(\beta(z) > 0\, \forall z \in \mathbf{B}_{q_\kappa}(\delta_\kappa))$ such that $(t,x) \mapsto \mathcal{H}\kappa(t,x,z)$ is uniformly Hölder continuous (exponent $\beta(z)$) in $(t,x)$ on compact subsets of $[0,T] \times \mathbb{R}^n$.
  4. $\mathcal{H}\kappa(t,x,z) \ge 0$ whenever $z \in \mathbf{B}_{q_\kappa}(\delta_\kappa) \cap \mathbb{R}_0^{\mathbb{N}}$.

Then

$$u(t,x) = \hat{E}^{T-t,x}[f(b_T) \diamond \mathrm{Exp}\{-\int_{T-t}^{T} \kappa(T-\theta, b_\theta)\, d\theta\} \diamond \mathcal{J}_{t,T}]$$
$$+ \hat{E}^{T-t,x}[\int_{T-t}^{T} g(T-s, b_s) \diamond \mathrm{Exp}\{-\int_{T-t}^{s} \kappa(T-\theta, b_\theta)\, d\theta\}\, ds \diamond \mathcal{J}_{t,T}]$$

where

$$\mathcal{J}_{t,T} := \mathrm{Exp}\{\sum_{i=1}^{n} \eta^{-1} \int_{T-t}^{T} [W_{\phi_y}^{(i)}]_{y=\eta b_s}\, db_s^i - \frac{1}{2}\sum_{i=1}^{n} \eta^{-2} \int_{T-t}^{T} [W_{\phi_y}^{(i)}]^2_{y=\eta b_s}\, ds\} \tag{2}$$

is the unique $(\mathcal{S}_n^n)^{-1}$-valued process which solves

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{1}{2}\eta^2 \Delta u + \vec{W}_{\phi_x} \diamond \nabla u - \kappa(t,x) \diamond u + g(t,x) \quad & (t,x) \in [0,T] \times \mathbb{R}^n \\ u(0,x) &= f(x) & x \in \mathbb{R}^n\end{aligned}$$

where $\hat{E}$ and $\int \cdot \, ds$ are Bochner integrals in $(\mathcal{S}_n^n)^{-1}$.

**REMARK 3.2** If $u(t,x) \in (\mathcal{S}_n^n)^{-1}$ and $\mathcal{A}(\mathcal{H}u(t,x)) \in A_b(\mathbf{B}_q(\delta))$ for some $q \in \mathbb{N}, \delta > 0$, where $A_b(\mathbf{B}_q(\delta))$ is the space of all bounded analytic functions on $\mathbf{B}_q(\delta)$ and $\mathcal{A} := \frac{\partial}{\partial t} - \frac{1}{2}\eta^2\Delta - \vec{W}_\phi \diamond \nabla$, we will use the convention that $\mathcal{A}u(t,x) := \mathcal{H}^{-1}\mathcal{A}(\mathcal{H}u(t,x))$.

**PROOF:**

We must find $\hat{q} \in \mathbb{N}$ and $\hat{\delta} > 0$ such that $\tilde{u}(t,x,z) := \mathcal{H}(u(t,x))(z) \in A_b(\mathbf{B}_{\hat{q}}(\hat{\delta}))$ solves the equation

$$\frac{\partial \tilde{u}}{\partial t} = \frac{1}{2}\eta^2 \Delta \tilde{u} + \tilde{\vec{W}}_{\phi_x} \diamond \nabla \tilde{u} - \tilde{\kappa}(t,x) \diamond \tilde{u} + \tilde{g}(t,x) \quad (t,x) \in [0,T] \times \mathbb{R}^n \tag{3}$$

$$\tilde{u}(0,x) = \tilde{f}(x) \quad x \in \mathbb{R}^n \tag{4}$$

when $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$.

**LEMMA 3.3** $\exists(\hat{\delta} > 0, \hat{q} \in \mathbb{N})$ such that $z \mapsto \tilde{u}(t,x,z) \in A_b(\mathbf{B}_{\hat{q}}(\hat{\delta})) \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n$.

**PROOF:**

By taking absolute values, we get

$$|\tilde{u}(t,x,z)| \leq K_f e^{TK_\kappa} + TK_g e^{TK_\kappa}$$

whenever $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$ where $\hat{q} \geq \max\{q_f, q_g, q_\kappa\}$ and $0 < \hat{\delta} \leq \min\{\delta_f, \delta_g, \delta_\kappa\}$, since, by using [BØ, Corollary 8.23],

$$\begin{aligned}\hat{E}^{T-t,x}[|\mathcal{J}_{t,T}|] &= \hat{E}^{T-t,x}[\exp\{\sum_{i=1}^n \eta^{-1} \int_{T-t}^T \Re[\tilde{W}_{\phi_y}^{(i)}]_{y=\eta b_s}\, db_s^i - \frac{1}{2}\sum_{i=1}^n \eta^{-2}\int_{T-t}^T \Re[\tilde{W}_{\phi_y}^{(i)}]^2_{y=\eta b_s}\, ds\}] \\ &\equiv 1.\end{aligned}$$

■

**LEMMA 3.4** The Bochner integrals in the expression for $u(x)$ are well-defined.

**PROOF:**

This is obvious from the estimates in lemma 3.3 ■

**LEMMA 3.5** $\mathcal{A}u(t,x)$ is well-defined as an element in $(\mathcal{S}_n^n)^{-1}$ $\forall(t,x) \in [0,T] \times \mathbb{R}^n$.

**PROOF:**

Since

$$\mathcal{A}\tilde{u} = -\tilde{\kappa} \cdot \tilde{u} + \tilde{g}$$

it follows from lemma 3.3 that

$$|\mathcal{A}\tilde{u}(t,x,z)| \leq K_\kappa(K_f e^{TK\kappa} + TK_g e^{TK_\kappa}) + K_g$$

when $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$, i.e. the claim follows. ∎

**LEMMA 3.6** $\tilde{u}(t,x,z)$ is the unique function which solves equation (3) when $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$.

**PROOF:**

Equation (3) may be written as

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{\kappa}\tilde{u} = \mathcal{A}^{\xi}\tilde{u} + \tilde{g} \quad (t,x) \in [0,T] \times \mathbb{R}^n \tag{5}$$

$$\tilde{u}(0,x) = \tilde{f} \qquad x \in \mathbb{R}^n \tag{6}$$

where $\mathcal{A}^{\xi}$ is the second order differential operator given by

$$\mathcal{A}^{\xi} = \sum_{i=1}^{n} \frac{1}{2}\nu^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{n} \tilde{W}_{\phi_x}^{(i)}(\xi)\frac{\partial}{\partial x_i}.$$

Assume now that $\xi \in \mathbf{B}_{\hat{q}}(\hat{\delta}) \cap \mathbb{R}_0^N$.

The operator $\mathcal{A}^{\xi}$ is clearly uniformly elliptic with drift term which satisfies the linear growth condition

$$\begin{aligned}
|\tilde{W}_{\phi_x}^{(i)} - \tilde{W}_{\phi_y}^{(i)}|(\xi) &= |\sum_{k=0}^{\infty}(\phi_x - \phi_y, e_k)\xi_k| \\
&\leq \sum_{k=0}^{\infty} |(\phi_x - \phi_y, e_k)||\xi_k| \\
&\leq (M \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} |e_k(x)|\, dx|\xi_k|)|x-y|
\end{aligned}$$

where

$$M := \max_{1\leq i\leq n}\{\sup_{x\in\mathbb{R}^n} |\frac{\partial \phi}{\partial x_i}|\} < \infty.$$

It follows by standard results that the stochastic process

$$dX_t^{\xi} = \tilde{\tilde{W}}_{\phi_{X_t^{\xi}}}(\xi)\, dt + db_t \ ; \ X_0^{\xi} = x$$

exists with $\mathcal{A}^{\xi}$ as generator.

The solution of (5) is given by the Feynman-Kac formula [KS, Theorem 5.7.6]

$$\tilde{u}(t,x,\xi) = \hat{E}^{T-t,x}[\tilde{f}(X_T^\xi)\exp\{-\int_{T-t}^{T}\tilde{\kappa}(T-\theta,X_\theta^\xi)\,d\theta\}] + \hat{E}^{T-t,x}[\int_{T-t}^{T}\tilde{g}(T-s,X_s^\xi)]\exp\{-\int_{T-t}^{s}\tilde{\kappa}(T-\theta,X_\theta^\xi)\,d\theta\}\,ds]$$

and by a change of measure this may be written as

$$\tilde{u}(t,x,\xi) = \hat{E}^{T-t,x}[\tilde{f}(b_T)\exp\{-\int_{T-t}^{T}\tilde{\kappa}(T-\theta,b_\theta)\,d\theta\}\mathcal{M}_{t,T}] + \hat{E}^{T-t,x}[\int_{T-t}^{T}\tilde{g}(T-s,b_s^\xi)]\exp\{-\int_{T-t}^{s}\tilde{\kappa}(T-\theta,X_\theta^\xi)\,d\theta\}\,ds\mathcal{M}_{t,T}]$$

where

$$\mathcal{M}_{t,T} := \exp\{\sum_{i=1}^{n}\eta^{-1}\int_{T-t}^{T}[\tilde{W}_{\phi_y}^{(i)}]_{y=\eta b_s}\,db_s^i - \frac{1}{2}\sum_{i=1}^{n}\eta^{-2}\int_{T-t}^{T}[\tilde{W}_{\phi_y}^{(i)}]_{y=\eta b_s}^2\,ds\}.$$

This expression is easily seen to have an analytic extension to all $z \in \mathbf{B}_{\hat{q}}(\hat{\delta})$ and by applying the generator of $b_t$ on both the real and imaginary part of $\tilde{u}(t,x,z)$ we see that equation (3) also holds in this case. ∎

The theorem now follows from the previous lemmas. ∎

## §4 The pollution model in a bounded domain

**THEOREM 4.1** Let $T > 0$ be given and suppose $D \subset \mathbb{R}^n$ is a bounded domain such that every point on the boundary of $\delta D$ has the exterior sphere property; i.e. there exists a ball $B \ni x$ such that $\bar{B} \cap D = \emptyset$, $\bar{B} \cap \delta D = \{x\}$.

Assume furthermore that we are given functions $[0,T]\times\partial D \ni (t,x) \mapsto h(t,x) \in (\mathcal{S}_n^n)^{-1}$, $[0,T]\times D \ni (t,x) \mapsto g(t,x) \in (\mathcal{S}_n^n)^{-1}$, $D \ni x \mapsto \phi(x) \in (\mathcal{S}_n^n)^{-1}$ and $[0,T]\times D \ni (t,x) \mapsto \kappa(t,x) \in (\mathcal{S}_n^n)^{-1}$ such that

- $\exists(q_h \in \mathbb{N}, \delta_h > 0, K_h > 0)$ such that
  1. $\sup_{(t,x)\in[0,T]\times\partial D, z\in\mathbf{B}_{q_h}(\delta_h)} |\mathcal{H}h(t,x,z)| \le K_h$.
  2. $x \mapsto \mathcal{H}h(t,x,z) \in C^2([0,T]\times\partial D)$ whenever $t \in [0,T], z \in \mathbf{B}_{q_h}(\delta_h)$.
- $\exists(q_\phi \in \mathbb{N}, \delta_\phi > 0, K_\phi > 0)$ such that
  1. $\sup_{x\in D, z\in\mathbf{B}_{q_\phi}(\delta_\phi)} |\mathcal{H}\phi(x,z)| \le K_\phi$.
  2. $x \mapsto \mathcal{H}\phi(x,z) \in C^2(D)$ whenever $z \in \mathbf{B}_{q_\phi}(\delta_\phi)$.
- $\exists(q_g \in \mathbb{N}, \delta_g > 0, K_g > 0)$ such that

1. $\sup_{(t,x)\in[0,T]\times D, z\in \mathbf{B}_{q_g}(\delta_g)} |\mathcal{H}g(t,x,z)| \leq K_g$.
2. $x \mapsto \mathcal{H}g(t,x,z) \in C^2([0,T]\times D)$ whenever $t \in [0,T], z \in \mathbf{B}_{q_g}(\delta_g)$.
3. $\exists(\alpha(z) > 0\, \forall z \in \mathbf{B}_{q_g}(\delta_g))$ such that $(t,x) \mapsto \mathcal{H}g(t,x,z)$ is Hölder continuous (exponent $\alpha(z)$) in $x \in D$, uniformly in $t \in [0,T]$.

- $\exists(q_\kappa \in \mathbb{N}, \delta_\kappa > 0, K_\kappa > 0)$ such that

1. $\sup_{(t,x)\in[0,T]\times D, z\in \mathbf{B}_{q_\kappa}(\delta_\kappa)} |\mathcal{H}\kappa(t,x,z)| \leq K_\kappa$.
2. $x \mapsto \mathcal{H}\kappa(t,x,z) \in C^2(D)$ whenever $t \in [0,T], z \in \mathbf{B}_{q_\kappa}(\delta_\kappa)$.
3. $\exists(\beta(z) > 0\, \forall z \in \mathbf{B}_{q_\kappa}(\delta_\kappa))$ such that $(t,x) \mapsto \mathcal{H}\kappa(t,x,z)$ is uniformly Hölder continuous (exponent $\beta(z)$) in $(t,x)$ in compact subsets of $[0,T]\times D$.
4. $\mathcal{H}\kappa(t,x,z) \geq 0$ whenever $z \in \mathbf{B}_{q_\kappa}(\delta_\kappa) \cap \mathbb{R}_0^N$.

- $h(0,x) = \phi(x)\ \ \forall x \in \partial D$

Then

$$u(t,x) = \hat{E}^{T-t,x}[h(b_\tau, T-\tau) \diamond \operatorname{Exp}\{-\int_{T-t}^{\tau} \kappa(T-s,b_s)\,ds\}\chi_{\tau<T} \diamond \mathcal{J}_{t,T}]$$
$$+\hat{E}^{T-t,x}[\phi(b_T) \diamond \operatorname{Exp}\{-\int_{T-t}^{\tau} \kappa(T-s,b_s)\,ds\}\chi_{\tau=T} \diamond \mathcal{J}_{t,T}]$$
$$+\hat{E}^{T-t,x}[\int_{T-t}^{\tau} g(b_s,T-s) \diamond \exp\{-\int_{T-t}^{s} \kappa(T-\lambda,b_\lambda)\,d\lambda\}\,ds \diamond \mathcal{J}_{t,T}]$$

where $\mathcal{J}_{t,T}$ is given by (2) and $\tau$ is first time $\lambda \in [t,T]$ that $X_\lambda$ leaves $D$ if such a time exists and $\tau := T$ otherwise, is the unique $(\mathcal{S}_n^n)^{-1}$-valued process which solves

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{1}{2}\eta^2 \Delta u + \vec{W}_{\phi_x} \diamond \nabla u - \kappa(t,x)\diamond u + g(t,x) && (t,x) \in [0,T]\times D\\
u(0,x) &= \phi(x) && x \in D\\
u(t,x) &= h(t,x) && (t,x) \in [0,T]\times \delta D
\end{aligned}$$

where $\hat{E}$ and $\int \cdot\, ds$ are Bochner integrals in $(\mathcal{S}_n^n)^{-1}$.

**PROOF:**

This follows, since $\tau \leq T$, as in the proof of theorem 3.1, but instead of using the Feynman-Kac formula, we use [Fri, Theorem 5.2]. ■

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